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Article

Two-Disjoint-Cycle-Cover Pancyclicity of Dragonfly Networks

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Abstract

Interconnection networks, often modeled as graphs, are critical for high-performance computing systems due to their impact on performance metrics like latency and bandwidth. The dragonfly network, denoted as $D(n, r)$, is a promising topology owing to its modularity, low diameter, and cost-effectiveness. Ensuring reliability and efficiency in these networks requires robust cycle embedding properties. The two-disjoint-cycle-cover pancyclicity ensures that the network can be partitioned into two vertex-disjoint cycles of any feasible length, which has practical implications for fault-tolerant routing and load balancing. Formally, a graph G is called two-disjoint-cycle-cover $[a_1, a_2]$ -pancyclic if for any integer ℓ satisfying $a_1 \leq \ell \leq a_2$, there exist two vertex-disjoint cycles C_1 and C_2 in G such that $|V(C_1)| = \ell$ and $|V(C_2)| = |V(G)| - \ell$. While prior work has established Hamiltonicity and pancyclicity for $D(n, r)$, the two-disjoint-cycle-cover problem remains unexplored. This paper fills this gap by proving that $D(n, r)$ is two-disjoint-cycle-cover $[3, \lfloor \frac{|V(D(n, r))|}{2} \rfloor]$ -pancyclic with $n \geq 3$ and $r \geq 2$, generalizing existing knowledge. Moreover, it can be obtained that $D(n, r)$ is vertex-disjoint-cycle-cover. Our proof employs a constructive method with case analysis, ensuring the existence of such cycles.

Keywords: dragonfly networks; pancyclicity; vertex-disjoint cycles; disjoint-cycle-cover; Hamiltonian

1. Introduction

Interconnection networks are critical for high-performance computing (HPC) systems because they directly impact performance metrics such as latency, bandwidth, and scalability. Among various topologies, the dragonfly network $D(n, r)$, introduced by Kim et al. [1], is a symmetric graph that partitions vertices into $g = nr + 1$ groups, each containing n vertices fully connected by internal edges, with each vertex having r external edges connecting to other groups according to a defined rule. Figure 1 provides an example of $D(n, r)$ with $n = 4, r = 2, g = 9$. This topology enables efficient data exchange and has gained attention due to its modularity, low diameter, and cost-effectiveness, making it a promising candidate for large-scale HPC applications.

Graph theory provides a foundation for studying interconnection networks, where cycle embedding properties are essential for assessing structural robustness and reliability. Given a graph $G = (V(G), E(G))$, $V(G)$ and $E(G)$ denotes all the vertices and edges in G , respectively. The number of elements in $V(G)$, written as $|V(G)|$, is known as the *order* of G . Given a vertex $x \in V(G)$, let $N_G(x)$ denote the *neighborhood* of x in G . A *complete graph* having n vertices, indicated by K_n , is such that each vertex is adjacent to every other vertex within the graph. For a cycle C (path P), the length of a cycle (path), denoted by $|V(C)|$ ($|V(P)|$), refers to the number of vertices within the cycle (path). A cycle in graph G that includes every single vertex of G is known as a *Hamiltonian cycle*. A graph that possesses a Hamiltonian cycle is called *Hamiltonian*. A graph with an order of n is regarded to be *pancyclic* if it has cycles of all possible lengths ranging from 3 to n . In particular, we term G as *vertex-pancyclic* (respectively, *edge-pancyclic*) when every vertex (respectively, every edge) is included in a cycle whose

length ℓ satisfies $3 \leq \ell \leq n$. In graph G , when there is a set of ℓ cycles $C_1, C_2, \dots, C_\ell$ whose vertex sets are disjoint and $V(G) = \cup_{i=1}^{\ell} V(C_i)$, we say the set covers G and is called an ℓ -disjoint-cycle-cover. Prior research has established fundamental properties of $D(n, r)$, such as Hamiltonian and pancyclicity [2].

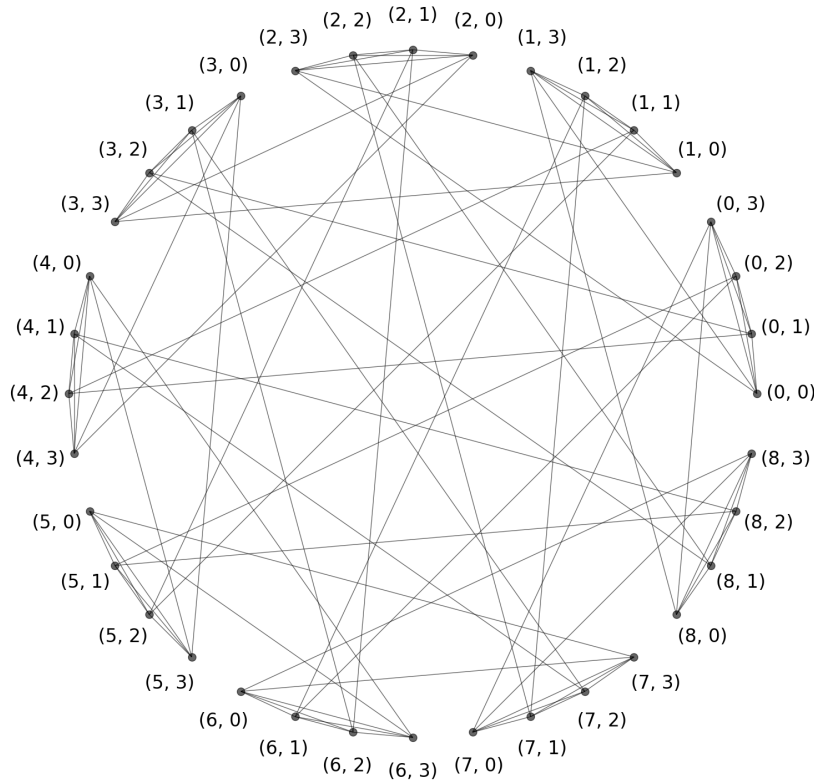


Figure 1. The dragonfly network $D(4, 2)$ ((x, y) denotes vertex y in group x)

A significant extension in graph theory is the concept of *two-disjoint-cycle-cover pancyclicity* introduced by Kung and Chen [3], which integrates the concepts of disjoint-cycle-cover and pancyclicity and ensures that a graph can be partitioned into two vertex-disjoint cycles of variable lengths. Formally, a graph G is two-disjoint-cycle-cover $[a_1, a_2]$ -pancyclic if for any integer ℓ with $a_1 \leq \ell \leq a_2$, there exist two vertex-disjoint cycles C_1 and C_2 in G such that $|V(C_1)| = \ell$ and $|V(C_2)| = |V(G)| - \ell$. This property has been extensively investigated in other network topologies (e.g., crossed cubes [3], locally twisted cubes [4], bipartite generalized hypercube [5], bipartite hypercube-like networks [6], balanced hypercubes [7], augmented cubes [8,9], alternating group graph [10], data center networks [11], bubble-sort star graphs [12], split-star networks [13], star graphs [14]), as it enhances fault-tolerant routing and load balancing capabilities. However, for dragonfly networks, the two-disjoint-cycle-cover pancyclicity problem remains unexplored.

This paper fills the above theoretical gap by proving that $D(n, r)$ is two-disjoint-cycle-cover $[3, \lfloor \frac{|V(D(n, r))|}{2} \rfloor]$ -pancyclic, where $n \geq 3$ and $r \geq 2$. In other words, we can find two cycles C_1 and C_2 in $D(n, r)$ with disjoint vertex sets, where $|V(C_1)| = \ell$ and $|V(C_2)| = |V(D(n, r))| - \ell$ with $3 \leq \ell \leq \lfloor \frac{|V(D(n, r))|}{2} \rfloor$. Our result expands the existing knowledge in [2] regarding the Hamiltonicity and pancyclicity of dragonfly networks. Moreover, we can obtain $D(n, r)$ is a vertex-disjoint-cycle-cover.

The rest of the paper is organized as follows. Section 2 presents the definitions and preliminary knowledge that will be employed across the whole paper. Section 3 and 4 prove the main theorem for $D(4, r)$ and $D(n, r)$ with $n \geq 5$, respectively. Finally, Section 5 concludes the paper.

2. Preliminaries

The dragonfly network $D(n, r)$ is defined as follows.

Definition 1. [15] Let n, r, g be any positive integers that satisfy the condition $g = nr + 1$. The definition of $D(n, r)$ is presented in the following manner.

1. The vertex set of $D(n, r)$ is given by $V(D(n, r)) = \cup_{i=0}^{g-1} V(G_i)$, where for each i from 0 to $g - 1$, $G_i = (V_i, E_i)$ is a complete subgraph isomorphic to K_n , representing the i -th group within $D(n, r)$. Moreover, $V(G_i)$ and $V(G_j)$ are disjoint, i.e., $V(G_i) \cap V(G_j) = \emptyset$, where $0 \leq i, j \leq g - 1$ and $i \neq j$;
2. Any vertex of $D(n, r)$ can be labeled by (x, y) , which denotes vertex y belonging to group x , with $0 \leq x \leq g - 1$ and $0 \leq y \leq n - 1$;
3. The vertex $u = (x_1, y_1)$ is adjacent to vertex $v = (x_2, y_2)$ with $x_1 \neq x_2$ if and only if $y_2 = n - 1 - y_1$, $x_2 = (ry_1 + x_1 + k) \bmod g$, where k ranges from 1 to r .

Obviously, $D(n, r)$ is a graph with a regularity of $(n + r - 1)$ and contains ng vertices. For any $u \in V(G_i)$, we call $V(G_i) - \{u\}$ the internal neighbor set of u . In contrast, the external neighbor set refers to the set of r vertices from other groups adjacent to u . Similarly, for any edge $e \in E(D(n, r))$, we call e an internal edge if $e \in E(G_i)$; otherwise, e is called an external edge.

There are some results in [2] about the Hamiltonicity and pancyclicity of dragonfly networks as follows:

Theorem 1. [2] For $n \geq 1$ and $r \geq 2$, $D(n, r)$ is Hamiltonian.

Theorem 2. [2] For $n \geq 4$ and $r \geq 2$, $D(n, r)$ is vertex-pancyclic.

By Definition 1, precisely one external edge exists between any two different groups. The endpoints of this edge are determined by the following theorem [15]. This theorem is of great significance and serves a crucial function in the proof of our primary result.

Theorem 3. [15] The external edge connecting group i and group j is $((i, \lfloor \frac{i-i-1}{r} \rfloor), (j, n - 1 - \lfloor \frac{j-i-1}{r} \rfloor))$, with $i < j$.

Lemma 1. [2] For $0 \leq i \leq g - 1$, $0 \leq y \leq n - 1$ and $1 \leq l \leq r - 1$, we have $((yr + i + d) \bmod g, n - y - 1) \in N((i, y)) \cap N((i + l, y))$, where $l + 1 \leq d \leq r$.

Lemma 2. For $n \geq 4, r \geq 2$, a Hamiltonian cycle can be found in $V(D(n, r)) - \cup_{j=0}^{\alpha} V(G_j)$ where $\alpha = 0, 1, 2, \dots, (n - 1)r - 2$.

Proof of Lemma 2. For $\alpha \in \{0, 1, \dots, (n - 1)r - 2\}$, by Theorem 3, there is an external edge $e = ((\alpha + 1, \lfloor \frac{g-\alpha-3}{r} \rfloor), (g - 1, n - 1 - \lfloor \frac{g-\alpha-3}{r} \rfloor))$ and $\lfloor \frac{g-\alpha-3}{r} \rfloor \neq 0$. Since $G_i \cong K_n$ with $0 \leq i \leq g - 1$, between any pair of vertices in $V(G_i)$, we can find a Hamiltonian path. Let $P_{\alpha+1}$ be a Hamiltonian path between $(\alpha + 1, 0)$ and $(\alpha + 1, \lfloor \frac{g-\alpha-3}{r} \rfloor)$ in $G_{\alpha+1}$, P_{g-1} be a Hamiltonian path between $(g - 1, n - 1)$ and $(g - 1, n - 1 - \lfloor \frac{g-\alpha-3}{r} \rfloor)$ in G_{g-1} , P_a be a Hamiltonian path between $(a, n - 1)$ and $(a, 0)$ in G_a for any $a = \alpha + 2, \alpha + 3, \dots, g - 2$.

By Theorem 3, $((i, 0), (i + 1, n - 1)) \in E(D(n, r))$ with $0 \leq i \leq g - 1$. Thus, in $V(D(n, r)) - \cup_{j=0}^{\alpha} V(G_j)$, we can find a Hamiltonian cycle:

$$C = \bigcup_{b=\alpha+1}^{g-1} P_b \cup \left(\bigcup_{i=\alpha+1}^{g-2} ((i, 0), (i + 1, n - 1)) \right) \cup \{e\}.$$

□

Since G_i is a complete graph with $i = 0, 1, \dots, g-1$, then between $(i, 0)$ and $(i, n-1)$, there is a Hamiltonian path P_i . By Theorem 3, $((i, 0), (i+1, n-1)) \in E(D(n, r))$. By Theorem 1, there is a Hamiltonian cycle (see Figure 2)

$$C_0 = \bigcup_{j=0}^{g-1} P_j \cup \left(\bigcup_{i=1}^{g-1} ((i, 0), (i+1, n-1)) \right).$$

This cycle C_0 will play a crucial role in the proof of our result.

Let us consider the two-disjoint-cycle-cover pancyclicity problem of $D(n, r)$.

If $n = 1$, by Definition 1, $D(1, r)$ is a complete graph, thus it is clear to show the two-disjoint-cycle-cover pancyclicity in $D(1, r)$.

If $n = 2$, there is no cycle of length 3 in $D(2, r)$.

If $n = 3$ and $r = 1$, there are no cycles of length ℓ with $\ell = 4, 5$.

If $n \geq 3$ and $r \geq 2$, we obtained Theorem 4, which is the main result of this paper.

Theorem 4. $D(n, r)$ is two-disjoint-cycle-cover $[3, \lfloor \frac{|V(D(n, r))|}{2} \rfloor]$ -pancyclic, where $n \geq 3$ and $r \geq 2$.

The proof of Theorem 4 is primarily structured into two parts: Section 3 addresses the case $n = 4$, while Section 4 generalizes the result to $n \geq 5$. For $n = 3$, the proof approach is analogous to that for $n = 4$, but to avoid redundancy, we computationally verify the two-disjoint-cycle-cover pancyclicity of $D(3, r)$ (with $r \geq 2$) using a Python implementation. This code, available at <https://github.com/guanlin-he/disjoint-cycle-pancyclicity-dragonfly>, leverages depth-first search and backtracking algorithms to exhaustively validate the property.

Our finding generalizes prior results in [2] (Theorems 1 and 2) about the Hamiltonicity and pancyclicity of $D(n, r)$. And we have the following corollary.

Corollary 1. $D(n, r)$ is vertex-disjoint-cycle-cover, where $n \geq 3$ and $r \geq 2$.

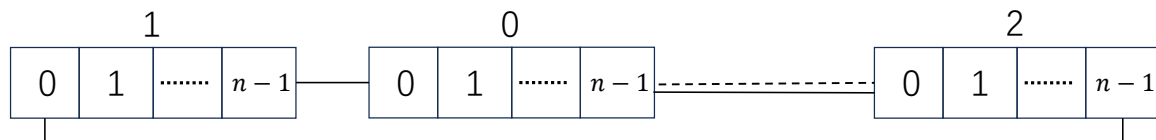


Figure 2. The Hamiltonian cycle C_0 .

3. Two-Disjoint-Cycle-Cover Pancyclicity in $D(4, r)$ with $r \geq 2$

This section focuses on proving the two-disjoint-cycle-cover $[3, \lfloor \frac{|V(D(4, r))|}{2} \rfloor]$ -pancyclicity of $D(4, r)$ with $r \geq 2$ (formally presented by Theorem 5), which serves as a foundational case for the general result in Theorem 4. The proof proceeds by case analysis, covering different ranges of ℓ . Although the approach involves multiple cases, we strive to group similar scenarios and leverage the symmetric properties of $D(4, r)$ to reduce redundancy. Each case constructs cycles explicitly, using paths and edges derived from the graph's topology. The detailed analysis here will provide a blueprint for extending the result to general $n \geq 5$ in Section 4.

Theorem 5. For $r \geq 2$, $D(4, r)$ is two-disjoint-cycle-cover $[3, \lfloor \frac{|V(D(4, r))|}{2} \rfloor]$ -pancyclic.

Proof of Theorem 5. Note that, by Definition 1, $|V(D(4, r))| = 4g$ with $g = (4r + 1)$. Thus, $\lfloor \frac{|V(D(4, r))|}{2} \rfloor = 2g$. We only need to prove that $D(4, r)$ has two cycles C_1 and C_2 where $|V(C_1)| = \ell$ and $|V(C_2)| = ng - \ell$ with $3 \leq \ell \leq 2g$.

For each ℓ within $3 \leq \ell \leq 8$, we found the two cycles C_1 and C_2 presented in Table 1 and the corresponding figures, where P_i denoted a hamiltonian path between any two vertices in G_i for $0 \leq i \leq g-1$. For a cycle $C = c_1 c_2 \cdots c_p c_1$, we denote by $C[c_i, c_j]$ be a path between c_i and c_j in C . Let $\bar{C} = c_p c_{p-1} \cdots c_1 c_p$. Similarly, we denote by $P[p_1, p_q] = p_1 p_2 \cdots p_q$ and $\bar{P}[p_q, p_1] = p_q p_{q-1} \cdots p_1$.

Table 1. The vertex-disjoint cycles C_1 and C_2 with $3 \leq \ell \leq 8$ in $D(4, r)$.

ℓ	C_1	C_2	Figure
3	$((0,0)(0,1)(0,2)(0,0))$	$(P_1[(1,2), (1,0)]C_0[(2,3)(g-3,0)])$ $P_{g-1}[(g-1,3), (g-1,0)](0,3)$ $P_{g-2}[(g-2,0), (g-2,1)](1,2)$	Figure 3
4	$((0,0)(0,1)(0,2)(0,3)(0,0))$	$(G_0[(1,3), (g-1,0)](1,3))$	
5	$((0,0)(1,3)(1,2)(1,0)(2,3)(0,0))$	$(P_2[(2,2), (2,0)]C_0[(2,0), (r+1,3)]$ $P_{r+1}[(r+1,3)(r+1,2)](0,1)(0,2)(0,3)$ $\bar{C}_0[(0,3), (3r+2,0)]P_{3r+2}[(3r+2,0), (3r+2,2)]$ $P_{r+2,1}[(r+2,1), (r+2,2)](1,1)$ $P_{r+3}[(r+3,2), (r+3,0)]$ $C_0[(R+3,0), (3R+1,1)](2,2))$	Figure 4
6	$((0,0)(r,3)(r,0)(2r,3)(2r,2)(0,1)(0,0))$	$(C_0[(1,3), (r-1,0)](r+1,3)C_0[(r+1,3), (2r-1,0)]$ $(2r+1,3)C_0[(2r+1,3), (3r-1,3)]$ $P_{3r-1}[(3r-1,3), (3r-1,1)]$ $(0,2)(0,3)(4r-1,0)\bar{C}_0[(4r-1,0), (3r+1,0)]$ $P_{3r+1}[(3r+1,0), (3r+1,1)](r,2)(r,1)(3r,2)$ $P_{3r}[(3r,2), (3r,3)](2r,0)(2r,1)$ $(g-1,2)P_{g-1}[(g-1,2), (g-1,0)](1,3))$	Figure 5
7	$(0,0)(r,3)(r,0)(2r,3)(2r,2)(0,1)(0,2)(0,0)$	$(C_0[(1,3), (r-1,0)](r+1,3)C_0[(r+1,3), (2r-1,0)]$ $(2r+1,3)P_{2r+1}[(2r+1,3), (2r+1,2)](r,1)$ $(r,2)(g-1,1)(g-1,2)(g-1,3)(4r-2,0)$ $\bar{C}_0[(4r-2,0), (2r+2,3)](2r,0)(2r,1)(4r-1,2)$ $P_{4r-1}[(4r-1,2), (4r-1,0)](0,3)(g-1,0)(1,3))$	Figure 6
8	$((0,0)(r,3)(r,2)(r,1)(r,0)(2r,3)(2r,2)(0,1)(0,0))$	$(C_0[(1,3), (r-1,0)](r+1,3)C_0[(r+1,3), (2r-1,0)]$ $(2r+1,3)P_{2r+1}[(2r+1,3), (2r+1,1)](0,2)(0,3)$ $(4r-1,0)\bar{C}_0[(4r-1,0), (2r+2,3)](2r,0)(2r,1)$ $(g-1,2)P_{g-1}[(g-1,2), (g-1,0)](1,3))$	Figure 7

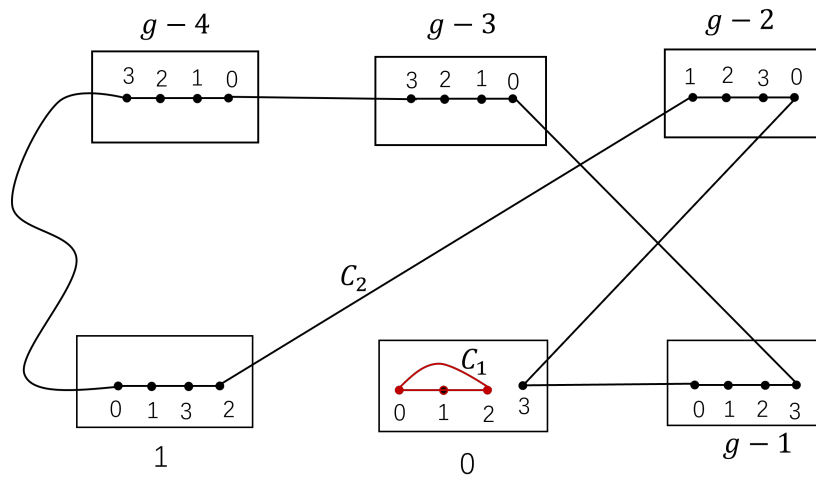


Figure 3. The cycles C_1 and C_2 with $\ell = 3$ in $D(4, r)$.

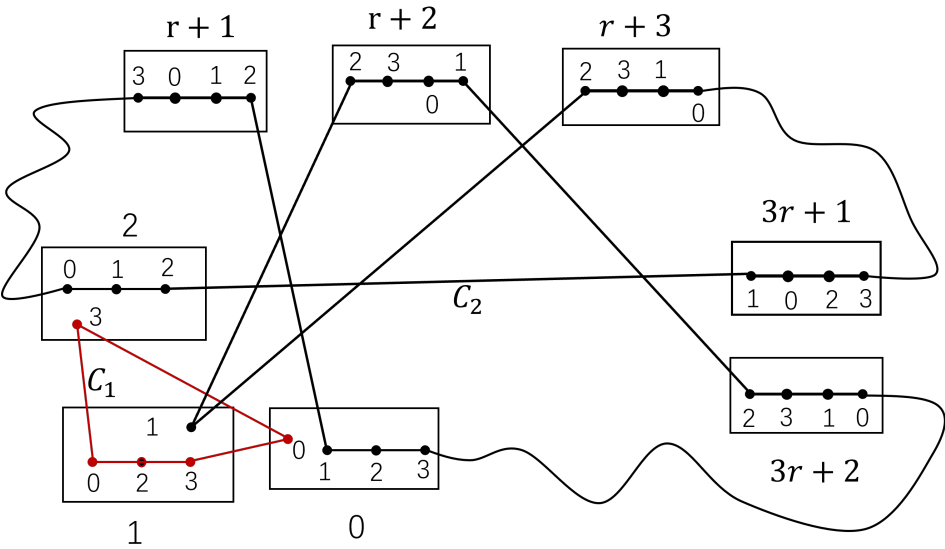


Figure 4. The cycles C_1 and C_2 with $\ell = 5$ in $D(4, r)$.

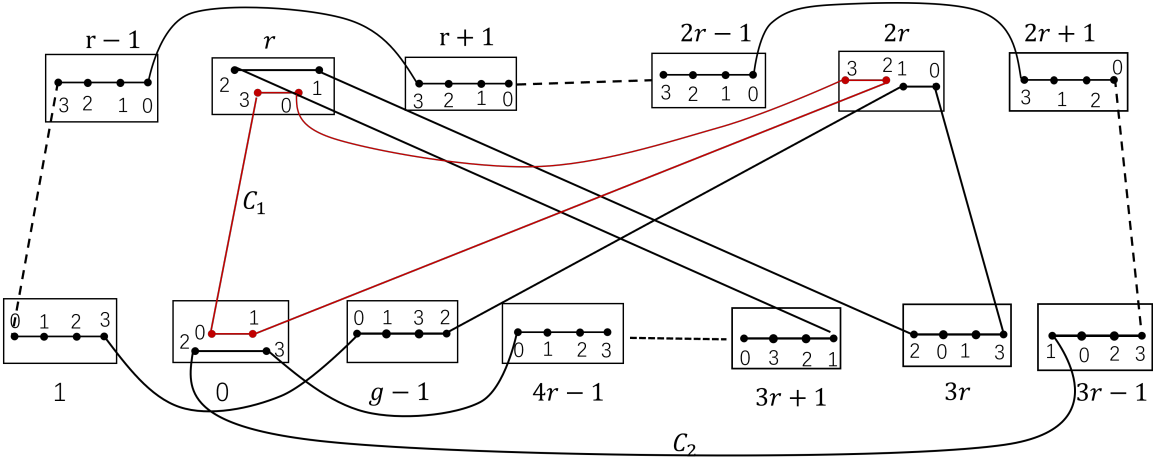


Figure 5. The cycles C_1 and C_2 with $\ell = 6$ in $D(4, r)$.

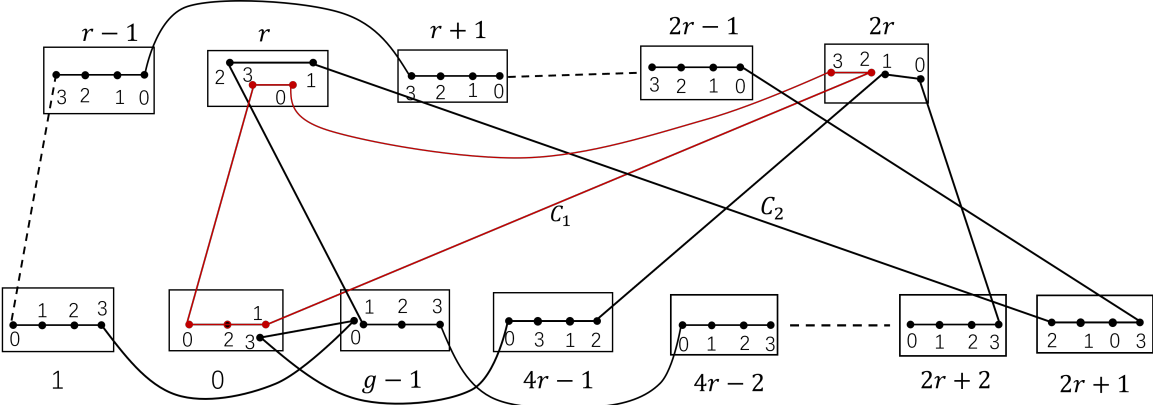


Figure 6. The cycles C_1 and C_2 with $\ell = 7$ in $D(4, r)$.

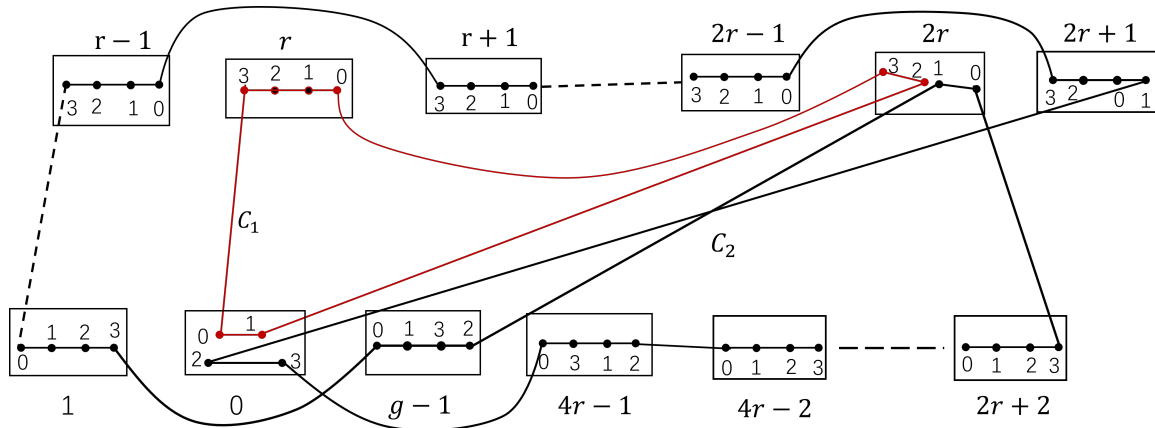


Figure 7. The cycles C_1 and C_2 with $\ell = 8$ in $D(4, r)$.

For $9 \leq \ell \leq 2g$, we present the following claims and prove their validity in turn.

Claim 1. For $\ell = 9 + 4i, i = 0, 1, \dots, \lfloor \frac{(2g-8)}{4} \rfloor$, two vertex-disjoint cycles C_1, C_2 can be found in $D(4, r)$, where $|V(C_1)| = \ell, |V(C_2)| = 4g - \ell$.

Proof of Claim 1. It follows $i \leq 2r - 2$ from $l \leq 2g$ in $D(4, r)$. For $j = 0, 1, \dots, g - 1$, since G_j is a complete graph, there is a Hamiltonian path P_j between $(j, 0)$ and $(j, n - 1)$ in G_j . Let $B_3 = \{1, 2, \dots, 2r - 1\} \setminus \{r + 1, r + 2, \dots, 2r - (i - (r - 1)) - 1\}$, where $|B_3| = i + 1$.

There are two paths: $P = (r - i, 3)(0, 0)(0, 2)(0, 1)(2r, 2)(2r, 3)(r, 0)$ with $|V(P)| = 7$ and $P' = (1, 3)(0, 0)(0, 2)(0, 1)(2r, 2)(2r, 3)(2r - 1, 0)$ with $|V(P')| = 7$ such that $|V(P \cap (\bigcup_{m=r-i}^r P_m))| = 2$ and $|V(P' \cap (\bigcup_{m \in B_3} P_m))| = 2$. By Theorem 3, let $e^3 = ((r, 0), (2r - [i - (r - 1)], 3))$ such that $V(e^3) \subseteq \bigcup_{m \in B_3} P_m$, there is a cycle of length ℓ :

$$C_1 = \begin{cases} (\bigcup_{m=r-i}^r P_m) \cup P \cup (\bigcup_{a=r-i}^r \{(a, 0), (a + 1, 3)\}), & \text{if } i \leq r - 1; \\ (\bigcup_{m \in B_3} P_m) \cup P' \cup (\bigcup_{a \in B_3} \{(a, 0), (a + 1, 3)\}) \cup \{e^3\}, & \text{if } i \geq r. \end{cases}$$

where $|V(C_1)| = 4i + 9$ since

$$|V(C_1)| = \begin{cases} \sum_{m=r-i}^r |V(P_m)| + |V(P)| - |V(P \cap (\bigcup_{m=r-i}^r P_m))|, & \text{if } i \leq r - 1; \\ |B_3| |V(P_m)| + |V(P')| - |V(P' \cap (\bigcup_{m \in B_3} P_m))|, & \text{if } i \geq r. \end{cases}$$

According to the range of i , we can obtain a cycle C_2 of length $4g - \ell$:

When $i \leq r - 2$, let $B_4 = \{0, 4r, 2r + 1, 2r, 4r - 1\}$ with $|B_4| = 5$ and $e^4 = ((r - i - 1, 0), (r + 1, 3))$. There are paths $P^1 = (1, 3)(g - 1, 0)(0, 3)(4r - 1, 0)(4r - 1, 3)(4r - 1, 1)(4r - 1, 2)(2r, 1)(2r, 0)(2r + 2, 3)$ and $P^2 = (2r - 1, 0)(2r + 1, 3)(2r + 1, 2)(2r + 1, 0)(2r + 1, 1)(g - 1, 2)(g - 1, 1)(g - 1, 3)(4r - 2, 0)$ in $D(4, r)$ such that $|V(P^1)| = 10$ and $|V(P^2)| = 9$. Then

$$C_2 = (C_0 - (\bigcup_{a \in B_4} V(G_a) \cup \bigcup_{m=r-i}^r V(G_m))) \cup P^1 \cup P^2 \cup \{e^4\}.$$

Let $F_1 = C_0 - (\bigcup_{a \in B_4} V(G_a) \cup \bigcup_{m=r-i}^r V(G_m))$. We have $V(e^4) \subseteq V(F_1)$ and $|V(P^\alpha) \cap V(F_1)| = 2$ with $\alpha = 1, 2$. So,

$$\begin{aligned} |V(C_2)| &= |V(F_1)| + |V(P^1)| + |V(P^2)| - \sum_{\alpha=1}^2 |V(P^\alpha \cap F_1)| \\ &= 4g - |B_4| |V(G_j)| - \sum_{m=r-i}^r |V(G_m)| + 10 + 9 - 4 \\ &= 4g - 5 \times 4 - 4(i+1) + 15 = 4g - 4i - 9. \end{aligned}$$

When $r-1 \leq i \leq 2r-3$, there are paths $P^3 = (2r, 1)(4r-1, 2)(4r-1, 1)(4r-1, 3)(4r-1, 0)(0, 3)(g-1, 0)(g-1, 1)(r+1, 2)(r+1, 1)(r+1, 0)(r+1, 3)(r+2, 0)$ with $|V(P^3)| = 13$ and $P^4 = (2r - [i - (r-1)] - 1, 0)(2r+1, 3)(2r+1, 2)(2r+1, 0)(2r+1, 1)(g-1, 2)(g-1, 3)(4r-2, 0)$ with $|V(P^4)| = 8$ in $D(4, r)$. By Theorem 3, let $e^5 = ((2r, 0), (2r+2, 3))$. Thus,

$$C_2 = (C_0 - (\bigcup_{a=2r-[i-(r-1)]}^{2r+1} V(G_a) \cup \bigcup_{m=0}^{r+1} V(G_m)) \cup V(G_{g-1}) \cup V(G_{g-2})) \cup P^3 \cup P^4 \cup \{e^5\}$$

Let $F_2 = C_0 - (\bigcup_{a=2r-[i-(r-1)]}^{2r+1} V(G_a) \cup \bigcup_{m=0}^{r+1} V(G_m)) \cup V(G_{g-1}) \cup V(G_{g-2})$. We have $|V(P^3 \cap F_2)| = 1$, $|V(P^4 \cap F_2)| = 2$ and $|V(e^5 \cap F_2)| = 1$. Then,

$$\begin{aligned} |V(C_2)| &= |V(F_2)| + |V(P^1)| + |V(P^2)| - \sum_{\alpha=1}^2 |V(P^\alpha \cap F_2)| \\ &= 4g - |B_4| |V(G_j)| - \sum_{m=r-i}^r |V(G_m)| + 10 + 9 - 4 \\ &= 4g - 5 \times 4 - 4(i+1) + 15 = 4g - 4i - 9. \end{aligned}$$

When $i = 2r-2$, there is a path $P^5 = (2r, 0)(2r, 1)(g-1, 2)(g-1, 3)(g-1, 1)(g-1, 0)(0, 3)(4r-1, 0)$ with $|V(P^5)| = 8$ and $|V(P^5 \cap (C_0 - (\bigcup_{m=0}^{2r} V(G_m)) \cup V(G_{g-1})))| = 1$ in $D(4, r)$. Thus,

$$C_2 = (C_0 - (\bigcup_{m=0}^{2r} V(G_m)) \cup V(G_{g-1})) \cup P^5,$$

where

$$\begin{aligned} |V(C_2)| &= |V((C_0 - (\bigcup_{m=0}^{2r} V(G_m)) \cup V(G_{g-1})))| \\ &\quad + |V(P^1)| + |V(P^2)| \\ &\quad - |V(P^5 \cap (C_0 - (\bigcup_{m=0}^{2r} V(G_m)) \cup V(G_{g-1})))| \\ &= 4g - (2r+2) |V(G_j)| + 8 - 1 \\ &= 4g - 8r - 1 = 4g - 4i - 9. \end{aligned}$$

So, C_1, C_2 are two disjoint cycles of $D(4, r)$, where $|C_1| = \ell$, $|C_2| = 4g - \ell$ (see Figures 8–10). $\square$

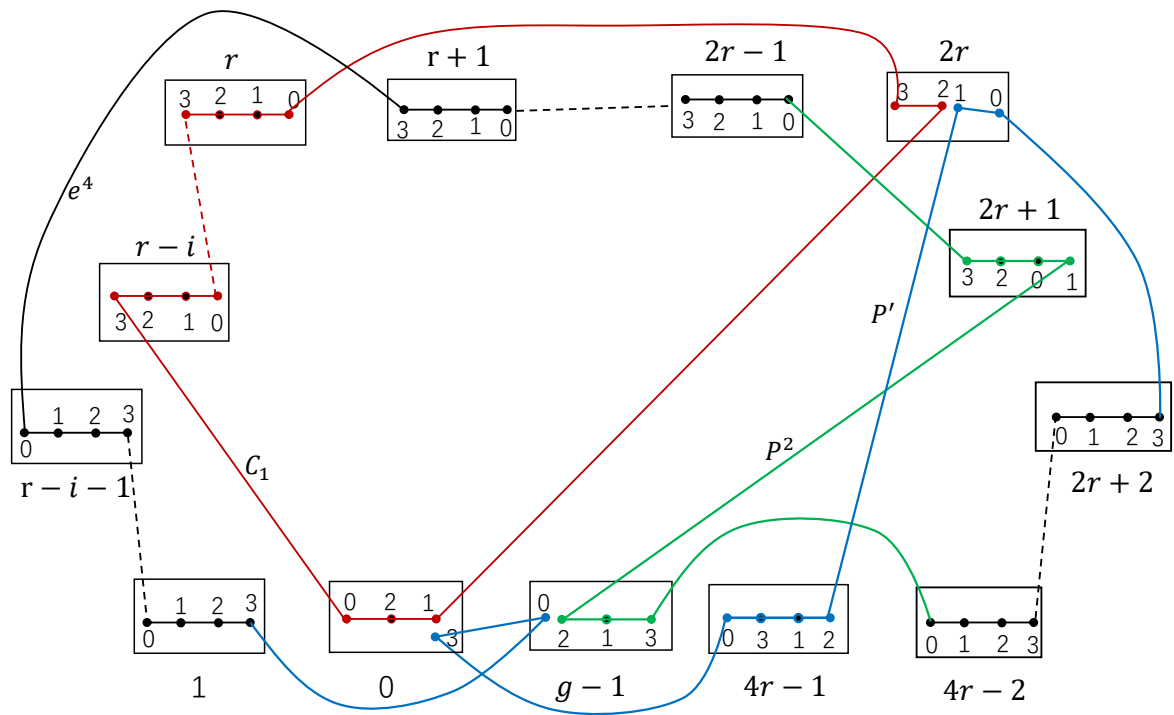


Figure 8. The cycles C_1 and C_2 with $i \leq r - 2$ in Claim 1.

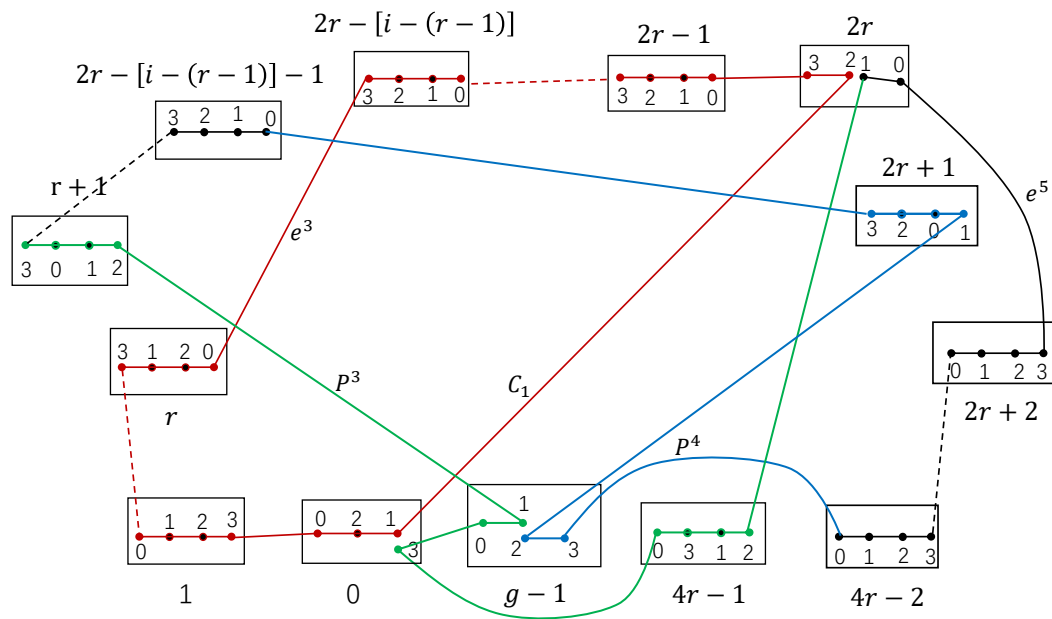


Figure 9. The cycles C_1 and C_2 with $r - 1 \leq i \leq 2r - 3$ in Claim 1.

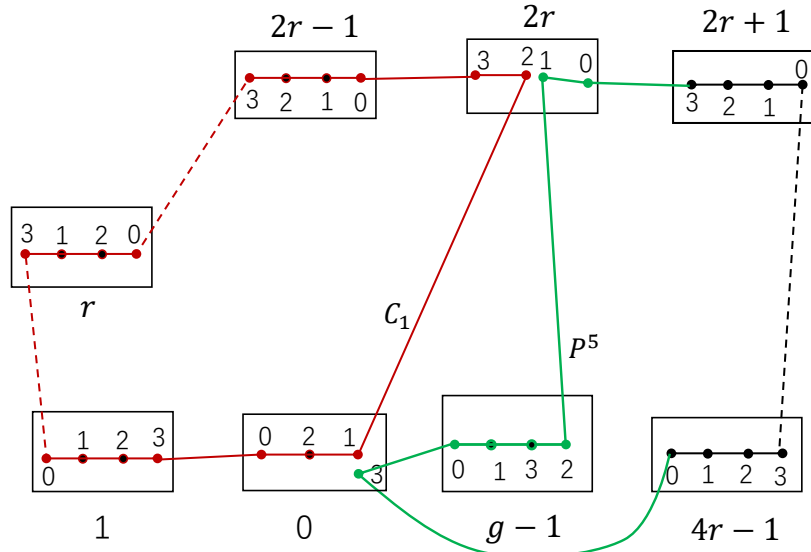


Figure 10. The cycles C_1 and C_2 with $i = 2r - 2$ in Claim 1.

Using the same argument as for claim 1, we can get the following claims.

Claim 2. For $\ell = 10 + 4i, i = 0, 1, \dots, \lfloor \frac{2g-9}{4} \rfloor$, two vertex-disjoint cycles C_1, C_2 can be found in $D(4, r)$, where $|V(C_1)| = \ell, |V(C_2)| = 4g - \ell$.

Proof of Claim 2. Since G_j is a complete graph, for $j = 0, 1, \dots, g - 1$, there is a Hamiltonian path P_j between $(j, 0)$ and $(j, n - 1)$ in G_j .

We have two paths: $P = (r - i, 3)(0, 0)(0, 2)(0, 3)(0, 1)(2r, 2)(2r, 3)(r, 0)$ with $|V(P)| = 8$, and $P' = (1, 3)(0, 0)(0, 2)(0, 3)(0, 1)(2r, 2)(2r, 3)(2r - 1, 0)$ with $|V(P')| = 8$. We have $|V(P \cap \bigcup_{m=r-i}^r P_m)| = 2$ and $|V(P' \cap \bigcup_{m=r-i}^r P_m)| = 2$.

Let $B_3 = \{1, 2, \dots, 2r - 1\} \setminus \{r + 1, r + 2, \dots, 2r - (i - (r - 1)) - 1\}$ with $|B_3| = i + 1$. By Theorem 3, let $e^3 = ((r, 0), (2r - [i - (r - 1)], 3))$ be such that $V(e^3) \subseteq V(\bigcup_{m=r-i}^r P_m)$. There is a cycle of length ℓ :

$$C_1 = \begin{cases} (\bigcup_{m=r-i}^r P_m) \cup P \cup (\bigcup_{a=r-i}^r \{(a, 0), (a + 1, 3)\}), & \text{if } i \leq r - 1; \\ (\bigcup_{m \in B_3} P_m \cup P' \cup (\bigcup_{a \in B_3} \{(a, 0), (a + 1, 3)\}) \cup \{e^3\}, & \text{if } i \geq r. \end{cases}$$

where $|V(C_1)| = 4i + 10$ since

$$|V(C_1)| = \begin{cases} \sum_{m=r-i}^r |V(P_m)| + |V(P)| - |V(P \cap (\bigcup_{m=r-i}^r P_m))|, & \text{if } i \leq r - 1; \\ |B_3| |V(P_m)| + |V(P')| - |V(P' \cap (\bigcup_{m \in B_3} P_m))|, & \text{if } i \geq r. \end{cases}$$

According to the range of i , we can obtain a cycle C_2 of length $4g - \ell$:

If $i \leq r - 2$, there is a path $P^1 = (2r - 1, 0)(2r + 1, 3)(2r + 1, 2)(2r + 1, 0)(2r + 1, 1)(4r - 1, 2)(2r, 1)(2r, 0)(2r + 2, 3)$ with $|V(P^1)| = 9$ in $D(4, r)$, by Theorem 3. Let $B_4 = \{0, 2r + 1, 2r\}$ and $e^4 = ((r - i - 1, 0), (r + 1, 3)), e^5 = ((1, 3), (g - 1, 0)), e^6 = ((4r - 1, 1), (4r - 1, 3))$. Thus,

$$C_2 = (C_0 - (\bigcup_{a \in B_4} V(G_a) \cup \bigcup_{m=r-i}^r V(G_m) \cup \{(4r - 1, 2)\})) \cup P^1 \cup \{e^4, e^5, e^6\}.$$

Let $F_1 = C_0 - (\bigcup_{a \in B_4} V(G_a) \cup \bigcup_{m=r-i}^r V(G_m) \cup \{(4r-1, 2)\})$. We have $|V(P^1 \cap F_1)| = 2$ and $V(\{e^4, e^5, e^6\}) \subseteq V(F_1)$. So,

$$\begin{aligned} |V(C_2)| &= |V(F_1)| + |V(P^1)| - |V(P^1 \cap F_1)| \\ &= 4g - |B_4| |V(G_j)| - \sum_{m=r-i}^r |V(G_m)| - 1 + 9 - 2 \\ &= 4g - 3 \times 4 - 4(i+1) + 6 = 4g - 4i - 10. \end{aligned}$$

If $r-1 \leq i \leq 2r-3$, there are paths $P^2 = (2r - [i - (r-1)] - 1, 0)(2r+1, 3)(2r+1, 2)(2r+1, 0)(2r+1, 1)(4r-1, 2)(2r, 1)(2r, 0)(2r+2, 3)$ and $P^3 = (g-1, 1)(r+1, 2)(r+1, 1)(r+1, 0)(r+1, 3)$ in $D(4, r)$ with $|V(P^2)| = 9$ and $|V(P^3)| = 5$. Then

$$C_2 = (C_0 - (\bigcup_{a=2r-[i-(r-1)]}^{2r+1} V(G_a) \cup \bigcup_{m=0}^r V(G_m)) \cup V(G_{g-2})) \cup P^2 \cup P^3 \cup \{e^6\}$$

Let $F_2 = (C_0 - (\bigcup_{a=2r-[i-(r-1)]}^{2r+1} V(G_a) \cup \bigcup_{m=0}^r V(G_m)) \cup V(G_{g-2}))$. We have $|V(P^2 \cap F_2)| = 3$ and $|V(P^3 \cap F_2)| = 1$, $V(e^6) \subseteq V(F_2)$. So,

$$\begin{aligned} |V(C_2)| &= |V(F_2)| + |V(P^2)| + |V(P^3)| - \sum_{\alpha=2}^3 |V(P^\alpha \cap F_2)| \\ &= 4g - (i-r+3)|V(G_a)| - \sum_{m=0}^r |V(G_m)| \\ &\quad - |V(G_{g-2})| + 9 + 5 - 4 \\ &= 4g - 4(i+5) + 10 = 4g - 4i - 10. \end{aligned}$$

If $i = 2r-2$, there is a path $P^4 = (2r, 0)(2r, 1)(g-1, 2)(g-1, 1)(g-1, 3)(g-1, 0)$ with $|V(P^4)| = 6$ in $D(4, r)$ by Theorem 3. Thus,

$$C_2 = (C_0 - (\bigcup_{m=0}^{2r-1} V(G_m)) \cup V(G_{g-1})) \cup P^4.$$

Let $F_3 = (C_0 - (\bigcup_{m=0}^{2r-1} V(G_m)) \cup V(G_{g-1}))$. We have $|V(P^4 \cap F_3)| = 4$. Thus,

$$\begin{aligned} |V(C_2)| &= |V(F_3)| + |V(P^4)| - |V(P^4 \cap F_3)| \\ &= 4g - (2r+2)|V(G_a)| + 6 - 4 \\ &= 4g - 8r - 2 = 4g - 4i - 10. \end{aligned}$$

Then C_1 and C_2 are two disjoint cycles of $D(4, r)$, where $|C_1| = \ell$, $|C_2| = 4g - \ell$. $\square$

Claim 3. For $\ell = 11 + 4i, i = 0, 1, \dots, \lfloor \frac{2g-10}{4} \rfloor - 1$, two vertex-disjoint cycles C_1, C_2 can be found in $D(4, r)$, where $|V(C_1)| = \ell$, $|V(C_2)| = 4g - \ell$.

Proof of Claim 3. It follows $i \leq 2r-3$ from $l \leq 2g$ in $D(4, r)$. Since G_j is a complete graph, for $j = 0, 1, \dots, g-1$, there is a Hamiltonian path P_j between $(j, 0)$ and $(j, n-1)$ in G_j . Let $B_3 = \{1, 2, \dots, 2r-1\} \setminus \{r+1, r+2, \dots, 2r - (i - (r-1)) - 1\}$. By Theorem 3, let $e^3 = ((r, 0), (2r - [i - (r-1)], 3))$ be an edge such that $V(e^3) \subseteq V(\bigcup_{m \in B_3} P_m)$.

There are two paths: $P = (r-i, 3)(0, 0)(0, 2)(0, 1)(2r, 2)(2r, 1)(2r, 0)(2r, 3)(r, 0)$, and $P' = (1, 3)(0, 0)(0, 2)(0, 1)(2r, 2)(2r, 1)(2r, 0)(2r, 3)(2r-1, 0)$.

We have $|V(P)| = 9$, $|V(P')| = 9$, $|V(P \cap \bigcup_{m=r-i}^r P_m)| = 2$ and $|V(P' \cap \bigcup_{m \in B_3} P_m)| = 2$. There is a cycle of length ℓ :

$$C_1 = \begin{cases} (\bigcup_{a=r-i}^r \{(a, 0), (a+1, 3)\}) \cup (\bigcup_{m=r-i}^r P_m) \cup P, & \text{if } i \leq r-1; \\ (\bigcup_{m \in B_3} P_m \cup P' \cup \{e^3\}) \cup (\bigcup_{a \in B_3} \{(a, 0), (a+1, 3)\}), & \text{if } i \geq r. \end{cases}$$

where $|V(C_1)| = 4i + 11$ since

$$|V(C_1)| = \begin{cases} \sum_{m=r-i}^r |V(P_m)| + |V(P)| - |V(P \cap (\bigcup_{m=r-i}^r P_m))|, & \text{if } i \leq r-1; \\ |B_3| |V(P_m)| + |V(P')| - |V(P' \cap (\bigcup_{m \in B_3} P_m))|, & \text{if } i \geq r. \end{cases}$$

According to the range of i , we can obtain a cycle C_2 of length $4g - \ell$:

When $i \leq r-2$, there are paths $P^1 = (1, 3)(g-1, 0)(0, 3)(4r-1, 0)(4r-1, 1)(4r-1, 2)(4r-1, 3)(3r-1, 0)(3r, 3)$ with $|V(P^1)| = 9$ and $P^2 = (3r-2, 0)(3r-1, 3)(3r-1, 2)(3r-1, 1)(g-1, 2)(g-1, 1)(g-1, 3)(4r-2, 0)$ with $|V(P^2)| = 8$ in $D(4, r)$. By Theorem 3, let $e_1 = ((r-i-1, 0), (r+1, 3))$ and $e_2 = ((2r-1, 0), (2r+1, 3))$ be two edges. Thus,

$$C_2 = (C_0 - (\bigcup_{a \in \{0, 4r, 3r-1, 2r, 4r-1\}} V(G_a) \cup \bigcup_{m=r-i}^r V(G_m))) \cup P^1 \cup P^2 \cup \{e_1, e_2\}.$$

Let $F_1 = C_0 - (\bigcup_{a \in \{0, 4r, 3r-1, 2r, 4r-1\}} V(G_a) \cup \bigcup_{m=r-i}^r V(G_m))$. Then $|V(P^\beta \cap F_1)| = 2$ with $\beta = 1, 2$ and $V(e_1, e_2) \subseteq V(F_1)$. We have

$$\begin{aligned} |V(C_2)| &= |V(F_1)| + |V(P^1)| + |V(P^2)| - \sum_{\beta=1}^2 |V(P^\beta \cap F_1)| \\ &= 4g - 5|V(G_j)| - \sum_{m=r-i}^r |V(G_m)| + 9 + 8 - 2 - 2 \\ &= 4g - 4(i+6) + 17 - 4 = 4g - 4i - 11. \end{aligned}$$

When $r-1 \leq i \leq 2r-3$, there is a path $P^3 = (4r-1, 0)(0, 3)(g-1, 0)(g-1, 3)(g-1, 2)(g-1, 1)(r+1, 2)(r+1, 0)(r+1, 1)(r+1, 3)(r+2, 0)$ with $|V(P^3)| = 11$ in $D(4, r)$. By Theorem 3, let $e_3 = ((2r - [i - (r-1)] - 1, 0), (2r+1, 3)) \in E(D(4, r))$. Thus,

$$C_2 = (C_0 - (\bigcup_{a=2r-[i-(r-1)]}^{2r} V(G_a) \cup \bigcup_{m=0}^r V(G_m) \cup V(G_{g-1}))) \cup P^3 \cup \{e_3\}.$$

Let $F_2 = C_0 - (\bigcup_{a=2r-[i-(r-1)]}^{2r} V(G_a) \cup \bigcup_{m=0}^r V(G_m) \cup V(G_{g-1}))$. We have $|V(P^3 \cap F_2)| = 2$ and $V(e_3) \subseteq V(F_2)$. So,

$$\begin{aligned} |V(C_2)| &= |V(F_2)| + |V(P^3)| + |V(P^3 \cap F_2)| \\ &= 4g - (i+5)|V(G_j)| + 11 - 2 \\ &= 4g - 4i - 11. \end{aligned}$$

So, C_1 and C_2 are two disjoint cycles of $D(4, r)$, where $|V(C_1)| = \ell$, $|V(C_2)| = 4g - \ell$. $\square$

Claim 4. For $\ell = 12 + 4i, i = 0, 1, \dots, \lfloor \frac{2g-11}{4} \rfloor - 1$, two vertex-disjoint cycles C_1, C_2 can be found in $D(4, r)$, where $|V(C_1)| = \ell$, $|V(C_2)| = 4g - \ell$.

Proof of Claim 4. It follows $i \leq 2r - 3$ from $l \leq 2g$ in $D(4, r)$. Since G_j is a complete graph, for $j = 0, 1, \dots, g - 1$, there is a Hamiltonian path P_j between $(j, 0)$ and $(j, n - 1)$ in G_j . Let $B_3 = \{1, 2, \dots, 2r - 1\} \setminus \{r + 1, r + 2, \dots, 2r - (i - (r - 1)) - 1\}$ with $|B_3| = i + 1$.

There are two paths: $P = (r - i, 3)(0, 0)(0, 2)(0, 3)(0, 1)(2r, 2)(2r, 1)(2r, 2)(2r, 3)(r, 0)$, and $P' = (1, 3)(0, 0)(0, 2)(0, 3)(0, 1)(2r, 2)(2r, 1)(2r, 0)(2r, 3)(2r - 1, 0)$.

We have $|V(P)| = |V(P')| = 10$ and $|V(P \cap (\bigcup_{m=r-i}^r P_m))| = |P' \cap (\bigcup_{m \in B_3} P_m)| = 2$. By Theorem 3, there is an edge $e^3 = ((r, 0), (2r - [i - (r - 1)], 3))$ such that $V(e^3) \subseteq (\bigcup_{m \in B_3} P_m)$. So, there is a cycle of length ℓ :

$$C_1 = \begin{cases} (\bigcup_{a=r-i}^r \{(a, 0), (a + 1, 3)\}) \cup (\bigcup_{m=r-i}^r P_m) \cup P, & \text{if } i \leq r - 1 \\ (\bigcup_{a \in B_3} \{(a, 0), (a + 1, 3)\}) \cup \{e^3\} \cup (\bigcup_{m \in B_3} P_m) \cup P' & \text{if } i \geq r. \end{cases}$$

where $|V(C_1)| = 4i + 12$ since

$$|V(C_1)| = \begin{cases} \sum_{m=r-i}^r |V(P_m)| + |V(P)| - |V(P \cap (\bigcup_{m=r-i}^r P_m))|, & \text{if } i \leq r - 1 \\ |B_3| |V(P_m)| + |V(P')| - |V(P' \cap (\bigcup_{m \in B_3} P_m))|, & \text{if } i \geq r. \end{cases}$$

According to the range of i , we can obtain a cycle C_2 of length $4g - \ell$:

When $i \leq r - 2$, by Theorem 3, let $e_2 = ((r - i - 1, 0), (r + 1, 3))$, $e_3 = ((2r - 1, 0), (2r + 1, 3))$ and $e_4 = ((g - 1, 0), (1, 3))$. Thus,

$$C_2 = (C_0 - (\bigcup_{a \in \{0, 2r\}} V(G_a) \cup \bigcup_{m=r-i}^r V(G_m))) \cup \{e_2, e_3, e_4\}.$$

Let $F_1 = C_0 - (\bigcup_{a \in \{0, 2r\}} V(G_a) \cup \bigcup_{m=r-i}^r V(G_m))$. Then $V(\{e_2, e_3, e_4\}) \subseteq F_1$. We have

$$|V(C_2)| = |V(F_1)| = 4g - 2|V(G_j)| - \sum_{m=r-i}^r |V(G_m)| = 4g - 4(i + 6) = 4g - 4i - 12.$$

When $r - 1 \leq i \leq 2r - 3$, by Theorem 3, let $e_5 = ((2r - [i - (r - 1)] - 1, 0), (2r + 1, 3))$ and $e_6 = (g - 1, 1)(r + 1, 2)$. We have $V(\{e_5, e_6\}) \subseteq (C_0 - (\bigcup_{a=2r-[i-(r-1)]}^{2r} V(G_a)))$. Thus,

$$C_2 = (C_0 - (\bigcup_{a=2r-[i-(r-1)]}^{2r} V(G_a) \cup \bigcup_{m=0}^r V(G_m))) \cup \{e_5, e_6\}.$$

where

$$\begin{aligned} |V(C_2)| &= 4g - \sum_{a=2r-[i-(r-1)]}^{2r} |V(G_a)| - \sum_{m=r-i}^r |V(G_m)| \\ &= 4g - 4(i - r + 2) - 4(r + 1) = 4g - 4i - 12. \end{aligned}$$

So, C_1 and C_2 are two disjoint cycles of $D(4, r)$, where $|V(C_1)| = \ell$, $|V(C_2)| = 4g - \ell$. $\square$

By the above case analysis, two vertex-disjoint cycles C_1, C_2 can be found in $D(4, r)$ with $|V(C_1)| = \ell$, $|V(C_2)| = ng - \ell$, where $3 \leq \ell \leq 2g$. $\square$

4. Two-Disjoint-Cycle-Cover Pancyclicity in $D(n, r)$ with $n \geq 5$ and $r \geq 2$

In this section, we focus on proving the two-disjoint-cycle-cover $[3, \lfloor \frac{|V(D(n, r))|}{2} \rfloor]$ -pancyclicity of $D(n, r)$ with $n \geq 5$ and $r \geq 2$, extending the result from Section 3 to prove Theorem 4. The proof builds on the techniques developed for $D(4, r)$ but adapts them to handle the cases of larger n . We partition the range of ℓ into three main cases (Case 1: $3 \leq \ell \leq 5$; Case 2: $6 \leq \ell \leq 3n - 2$; Case

3: $3n - 1 \leq \ell \leq \lfloor \frac{ng}{2} \rfloor$) to systematically address all possible cycle lengths. Each case leverages the Hamiltonian cycle C_0 (defined in Section 2) and the complete graph structure of the groups G_i . By combining path constructions, edge manipulations, and inductive arguments, we demonstrate the existence of the required cycles C_1 and C_2 for every ℓ . While the proof involves case analysis, we emphasize the unifying principles, such as the use of symmetry and recursive group-based constructions, to provide insight into the general problem. This structured approach ensures rigor while minimizing ad-hoc computations.

Case 1. $3 \leq \ell \leq 5$.

Since $G_i \cong K_n$ with $0 \leq i \leq g - 1$, so G_i contains a cycle of length ℓ , where $n \geq 5$. As $D(n, r)$ possesses the property of rotational symmetry, we can pick group 0 as a representative case without loss of generality. Let $C_1 = ((0, 1)(0, 2)(0, 3)(0, 1))$ such that $|C_1| = 3$ and $C_2 = (C_0 \setminus V(C_1)) \cup \{((0, 0), (0, 4))\}$ such that $|C_2| = ng - 3$. Now, we show $\ell = 4$. There is a cycle of length 4:

$$C'_1 = \begin{cases} ((0, 0)(0, 2)(0, 3)(0, 4)(0, 0)), & \text{if } n = 5 \\ ((0, 1)(0, 2)(0, 3)(0, 4)(0, 1)), & \text{if } n \geq 6. \end{cases}$$

Since $G_i \cong K_n$ with $0 \leq i \leq g - 1$, there is a Hamiltonian path P_1 between $(r + 1, n - 1)$ and $(r + 1, n - 2)$ in G_{r+1} and a Hamiltonian path P_2 between $(r + 2, 0)$ and $(r + 2, n - 2)$ in G_{r+2} . When $n \geq 6$, there is a Hamiltonian path P_3 between $(0, 0)$ and $(0, n - 1)$ in $V(G_0) \setminus V(C'_1)$. By Lemma 1, let $P = (r + 1, n - 2)(0, 1)(r + 2, n - 2)$ be a path such that $|V(P \cap P_1)| = |V(P \cap P_2)| = 1$, and $e_1 = ((r, 0), (r + 1, n - 1))$, $e_2 = ((r + 2, 0), (r + 3, n - 1))$, $e_3 = ((1, n - 1), (g - 1, 0))$ be two edges.

By Theorem 3 and Lemma 1, there is a cycle of length $ng - 4$:

$$C'_2 = \begin{cases} (C_0 - (\bigcup_{a \in \{0, r+1, r+2\}} V(G_a)) \cup P \cup P_1 \cup P_2 \cup \{e_1, e_2, e_3\}), & \text{if } n = 5 \\ (C_0 - V(G_0)) \cup P_3 & \text{if } n \geq 6. \end{cases}$$

where

$$|V(C'_2)| = \begin{cases} |V(C_0)| - 3|V(G_a)| + |V(P)| + |V(P_1)| + |V(P_2)| \\ - |V(P \cap P_1)| - |V(P \cap P_2)| = 5g - 4, & \text{if } i \leq r - 1 \\ |V(C_0)| - |V(G_0)| + |V(P_3)| = ng - n + (n - 4) = ng - 4, & \text{if } i \geq r. \end{cases}$$

So C'_1, C'_2 are proved to be two disjoint cycles of $D(n, r)$, where $|C'_1| = 4$, $|C'_2| = ng - 4$.

If $n = 5$ and $\ell = 5$, by Lemma 2, then there are two disjoint cycles, one of which has a length of 5 and the other has a length of $ng - 5$.

If $n \geq 6$ and $\ell = 5$, there is a cycle of length 5:

$$C''_1 = \begin{cases} ((0, 0)(0, 2)(0, 3)(0, 4)(0, 5)(0, 0)), & \text{if } n = 6. \\ ((0, 1)(0, 2)(0, 3)(0, 4)(0, 5)(0, 1)), & \text{if } n \geq 7. \end{cases}$$

When $n \geq 7$, there is a Hamiltonian path P_4 between $(0, 0)$ and $(0, n - 1)$ in $V(G_0) - V(C''_1)$.

There is a cycle of length $ng - \ell$:

$$C'_2 = \begin{cases} (C_0 - (\bigcup_{a \in \{0, r+1, r+2\}} V(G_a)) \cup P \cup P_1 \cup P_2 \cup \{e_1, e_2, e_3\}), & \text{if } n = 6 \\ (C_0 - V(G_0)) \cup P_4 & \text{if } n \geq 7. \end{cases}$$

where

$$|V(C'_2)| = \begin{cases} |V(C_0)| - 3|V(G_a)| + |V(P)| + |V(P_1)| + |V(P_2)| \\ -|V(P \cap P_1)| - |V(P \cap P_2)| = 6g - 5, & \text{if } i \leq r-1 \\ |V(C_0)| - |V(G_0)| + |V(P_3)| = ng - n + (n-5) = ng - 5, & \text{if } i \geq r. \end{cases}$$

Case 2. $6 \leq \ell \leq 3n-2$.

By Theorem 3, let $f_1 = ((0, n-3), ((n-2)r-1, 2))$, $f_2 = (((n-1)r, n-2), ((n-2)r-1, 1))$ and $f_3 = ((0, n-2), ((n-1)r, 1))$. Then there is a cycle of length 6: $C = (0, n-3)((n-2)r-1, 2)((n-2)r-1, 1)((n-1)r, n-2)((n-1)r, 1)(0, n-2)(0, n-3)$ in $G_0 \cup G_{(n-1)r} \cup G_{(n-2)r-1}$ containing f_1, f_2 and f_3 .

Since G_i is a complete graph, there are some edges $e_1 = ((0, n-3), (0, n-2))$, $e'_1 = ((0, 0), (0, n-1))$, $e_2 = (((n-2)r-1, 2), ((n-2)r-1, 1))$, $e'_2 = (((n-2)r-1, 0), ((n-2)r-1, n-1))$, $e_3 = (((n-1)r, n-2), ((n-1)r, 1))$ and $e'_3 = (((n-1)r, n-1), ((n-1)r, 0))$. Let $V' = \bigcup_{i \in \{0, (n-1)r, (n-2)r-1\}} V(G_i) \setminus \{(i, n-1), (i, 0)\}$. For convenience, let $C' = (C_0 - V') \cup \{e'_1, e'_2, e'_3\}$ such that $|C'| = ng - 3n + 6$.

Since $G_i \cong K_n$, there is a path P_0 between $(0, n-3)$ and $(0, n-2)$ in $G_0 - \{(0, 0), (0, n-1)\}$ with $2 \leq |V(P_0)| = m_1 \leq n-2$, a path $P_{(n-2)r-1}$ between $((n-2)r-1, 1)$ and $((n-2)r-1, 2)$ in $G_{(n-2)r-1} - \{((n-2)r-1, 0), ((n-2)r-1, n-1)\}$ with $2 \leq |V(P_{(n-2)r-1})| = m_2 \leq n-2$, a path $P_{(n-1)r}$ between $((n-1)r, 1)$ and $((n-1)r, n-2)$ in $G_{(n-1)r} - \{((n-1)r, 0), ((n-1)r, n-1)\}$ with $2 \leq |V(P_{(n-1)r})| = m_3 \leq n-2$, a path P'_s between $(s, 0)$ and $(s, n-1)$ in $G_s - (V(G_s) \cap \{e_1, e_2, e_3\})$ with $2 \leq |P'_s| \leq n-2$, where $s \in \{0, (n-2)r-1, (n-1)r\}$. Let

$$C_1 = (C - \{e_1, e_2, e_3\}) \cup P_0 \cup P_{(n-2)r-1} \cup P_{(n-1)r}$$

and

$$C_2 = (C' - \{e'_1, e'_2, e'_3\}) \cup \left(\bigcup_{s \in \{0, (n-2)r-1, (n-1)r\}} P'_s \right).$$

Then C_1 and C_2 (see Figure 11 (a)) are proved to be two disjoint cycles of $D(n, r)$, where $|V(C_1)| = \ell'$, $|V(C_2)| = ng - \ell'$ and $V(C_1) \cup V(C_2) = V(D(n, r))$. Let $\ell' = 6 - |V(\bigcup_{i=1,2,3} e_i)| + \sum_{j=1}^3 m_j$. Then $6 \leq \ell' \leq 3n-6$.

Let $Q_{(n-2)r-1}$ between $((n-2)r-1, 1)$ and $((n-2)r-1, 2)$ in $G_{(n-2)r-1}$ with $2 \leq |V(Q_{(n-2)r-1})| \leq n$, a path $Q_{(n-1)r}$ between $((n-1)r, 1)$ and $((n-1)r, n-2)$ in $G_{(n-1)r}$ with $2 \leq |V(Q_{(n-1)r})| \leq n$, $g_1 = (((n-2)r-2, 0), ((n-2)r, n-1))$ and $g_2 = (((n-1)r-1, 0), ((n-1)r+1, n-1))$. Then $C'_1 = (C_1 \setminus P_{(n-2)r-1}) \cup Q_{(n-2)r-1}$ and $C'_2 = (C_2 \setminus P'_{(n-2)r-1}) \cup \{g_1\}$ are proved to be two disjoint cycles of $D(n, r)$, where $|V(C'_1)| = \ell''$, $|V(C'_2)| = ng - \ell''$ and $V(C'_1) \cup V(C'_2) = V(D(n, r))$, with $n+4 \leq \ell'' \leq 3n-4$, see Figure 11 (b).

Then $C''_1 = (C'_1 - P_{(n-1)r}) \cup Q_{(n-1)r}$ and $C''_2 = (C'_2 - P_{(n-1)r}) \cup \{g_2\}$ are proved to be two disjoint cycles of $D(n, r)$, where $|V(C''_1)| = \ell'''$, $|V(C''_2)| = ng - \ell'''$ and $V(C''_1) \cup V(C''_2) = V(D(n, r))$, with $2n+2 \leq \ell''' \leq 3n-2$, see Figure 11 (c).

Since $n \geq 5$, then $n+4 \leq 3n-6$ and $2n+2 \leq 3n-3$. Thus, there exist two disjoint cycles, one of which has a length of ℓ and the other has a length of $ng - \ell$ with $6 \leq \ell \leq 3n-2$.

Case 3. $3n - 1 \leq \ell \leq \lfloor \frac{ng}{2} \rfloor$.

By Theorem 3, let $f = ((0, \lfloor \frac{(n-2)r-i-2}{r} \rfloor), ((n-2)r-i-1, n-1 - \lfloor \frac{(n-2)r-i-2}{r} \rfloor))$ and $f' = (((n-2)r, n-1 - \lfloor \frac{i+1}{r} \rfloor), ((n-2)r-i-2, \lfloor \frac{i+1}{r} \rfloor))$. Since $r \geq 2$ and $i \in \{1, 2, \dots, (n-3)r-2\}$, then

$$1 \leq \lfloor \frac{(n-2)r - i - 2}{r} \rfloor \leq n-3.$$

So

$$S_i^1 = (C_1' - f_1) \cup \{f\} \cup T \bigcup_{a \in A} T_a \cup E',$$

and

$$S_i^{1'} = (C_2 \setminus \bigcup_{a \in A \cup \{(n-2)r-1\}} (V(G_a))) \cup \{f'\}.$$

In addition, we have two disjoint cycles in $D(n, r)$: $S_i^2 = (S_i^1 - V(G_{(n-1)r})) \cup T'$, and $S_i^{2'} = (S_i^{1'} \setminus V(G_{(n-1)r}) \cup \{g_2\})$, where $|V(S_i^2)| = \ell_i^2$ and $|V(S_i^{2'})| = ng - \ell_i^2$ with $(i+2)n+2 \leq \ell_i^1 \leq (i+3)n-2$, see Figure 12 (b). Since $n \geq 5$, then $(i+3)n-3 \geq (i+2)n+2$. Thus, there are two disjoint cycles S_i, S_i' of $D(n, r)$, where $|V(S_i)| = \ell_i, |V(S_i')| = ng - \ell_i$ with $(i+1)n+4 \leq \ell_i \leq (i+3)n-2$ and $i = 1, 2, \dots, (n-3)r-2$.

Since $n \geq 5$ and $i = 1, 2, \dots, (n-3)r-2$, $2n+4 \leq 3n-1$ and $(i+3)n-1 \geq (i+2)n+4$. Also,

$$(i+3)n-2 \leq [(n-3)r-2+3]n-2 = [(n-3)r+1]n-2.$$

So, $2n + 4 \leq \ell \leq [(n-3)r+1]n-2$.

If $n \geq 6$, then

$$\lfloor \frac{ng}{2} \rfloor - 3rn - 2 = (\frac{n}{2} - 3)nr + \frac{n}{2} - 2 \geq 0,$$

that is, $\lfloor \frac{ng}{2} \rfloor \geq 3nr + 2$. So $ng - \lfloor \frac{ng}{2} \rfloor \leq ng - 3nr - 2$, that is, $\lfloor \frac{ng}{2} \rfloor \leq [(n-3)r+1]n-2$.

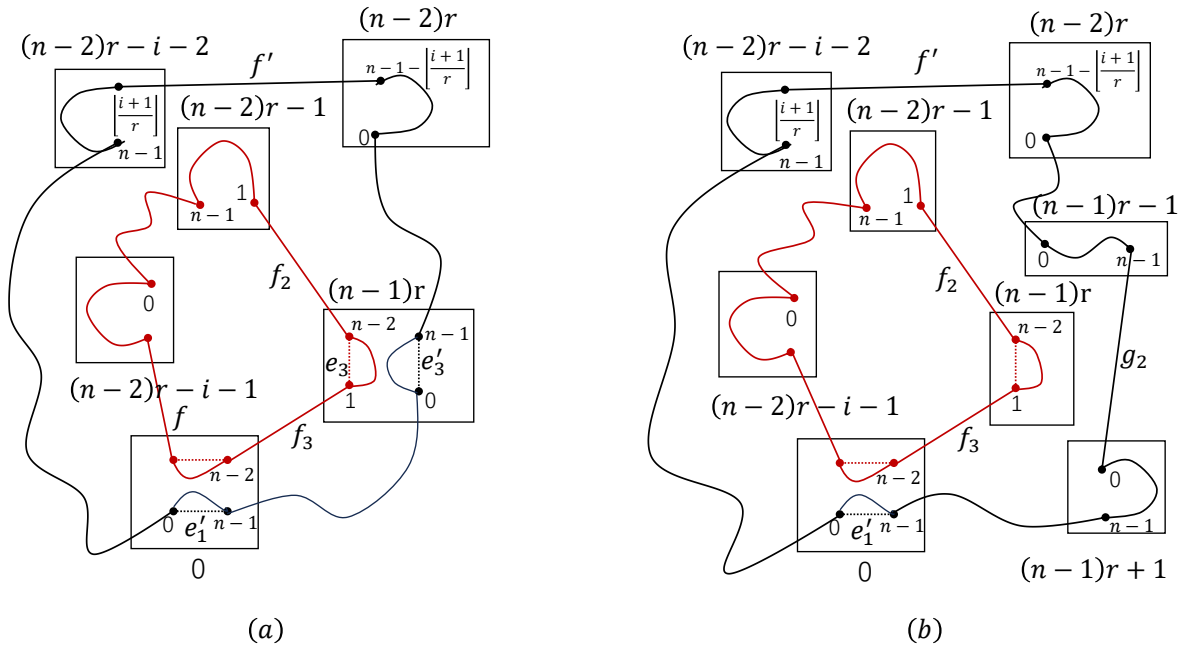


Figure 12. The illustration for Case 3 in Theorem 4.

Thus, if $n \geq 6$, $3n - 1 \leq \ell \leq \lfloor \frac{ng}{2} \rfloor$.

Now we show when $n = 5$, Case 3 holds.

Since $n = 5$ and $(i+1)n+4 \leq \ell_i \leq (i+3)n-2$ with $i \in \{1, 2, \dots, (n-3)r-2\}$, then $\ell \leq 10r+3$.

Now we will prove that two disjoint cycles R and R' can be found in $D(n, r)$, where $|V(R)| = \ell$, $|V(R')| = ng - \ell$ with $10r+4 \leq \ell \leq \lfloor \frac{ng}{2} \rfloor = \lfloor \frac{5(5r+1)}{2} \rfloor$.

By Theorem 3, let $b = ((0, 1), (r+1, 3))$, $b' = ((0, 3), (3r+k, 1))$, $b'' = ((r, 2), (3r+k, 2))$ with $k = 1, 2, \dots, r$. Let $B = \{r+2, r+3, \dots, 3r+k-1\}$, $E'' = \cup_{x \in B} \{(x, 0), (x+1, n-1)\}$. There exist Hamiltonian paths T_x between $(x, 0)$ and $(x, 4)$ in G_x with $x \in B$. And we have a Hamiltonian path W between $((r+1, 0)$ and $((r+1, 3)$ in G_{r+1} and a Hamiltonian path W' between $(3r+k, 4)$ and $(3r+k, 1)$ in G_{3r+k} , a Hamiltonian path W'' between $(r, 2)$ and $(r, 0)$ in G_r .

Let W'_1 be a path between $(0, 0)$ and $(0, 4)$ in $V(G_0) \setminus \{(0, 1), (0, 3)\}$, W_1 be a path between $(0, 1)$ and $(0, 3)$ in $V(G_0) \setminus \{(0, 0), (0, 4)\}$, W'_2 be a path between $(3r+k, 0)$ and $(3r+k, 2)$ in $G_{3r+k} \setminus \{(3r+k, 1), (3r+k, 4)\}$, W_2 be a path between $(3r+k, 1)$ and $(3r+k, 4)$ in $G_{3r+k} \setminus \{(3r+k, 0), (3r+k, 2)\}$, where $2 \leq |V(W_j)| \leq 3$ and $2 \leq |V(W'_j)| \leq 3$ with $j = 1, 2$.

There exist two disjoint cycles

$$R_1 = \bigcup_{x \in B} T_x \cup \bigcup_{j=1,2} W_j \cup W \cup E'' \cup \{b, b'\}$$

and

$$R'_1 = (C_0 \setminus (V(R_1) \cup V(G_r))) \cup \bigcup_{j=1,2} W_j \cup W'' \cup \{b''\}.$$

Then R_1 and R'_1 are two disjoint cycles of $D(n, r)$, where $|V(R_1)| = \alpha$, $|V(R'_1)| = ng - \alpha$ with $10r+4+(k-1)n \leq \alpha \leq 10r+6+(k-1)n$.

Also

$$R_2 = (R_1 - V(G_{3r+k})) \cup W'$$

and

$$R'_2 = (R'_1 \setminus V(G_{3r+k}) - \{b''\}) \cup \{(r, 2), (3r + k + 1, 2)\}.$$

Then R_2 and R'_2 are proved to be two disjoint cycles of $D(n, r)$, where $|V(R_1)| = \beta$, $|V(R'_1)| = ng - \beta$ with $10r + 2 + kn \leq \beta \leq 10r + 3 + kn$.

Since $k \in \{1, 2, \dots, r\}$ and $r \geq 2$, so

$$15r + 3 \geq \frac{5(5r + 1)}{2}.$$

And $10r + 2 + kn = 10r + 6 + (k - 1)n + 1$.

Thus, there are two disjoint cycles R and R' of $D(n, r)$, where $|V(R)| = \ell$, $|V(R')| = ng - \ell$ with $10r + 4 \leq \ell \leq \lfloor \frac{ng}{2} \rfloor = \lfloor \frac{5(5r+1)}{2} \rfloor$.

Considering all the cases discussed above, two vertex-disjoint cycles C_1 and C_2 can be found in $D(n, r)$, where $|V(C_1)| = \ell$, $|V(C_2)| = ng - \ell$ with $3 \leq \ell \leq \lfloor \frac{ng}{2} \rfloor$. Therefore, $D(n, r)$ has two-disjoint-cycle-cover $[3, \lfloor \frac{|V(D(n, r))|}{2} \rfloor]$ -pancyclicity for $n \geq 5$ and $r \geq 2$.

5. Conclusions

This paper has successfully established the two-disjoint-cycle-cover pancyclicity of dragonfly networks. Specifically, we proved that $D(n, r)$ is two-disjoint-cycle-cover $[3, \lfloor \frac{|V(D(n, r))|}{2} \rfloor]$ -pancyclic with $n \geq 3$ and $r \geq 2$, meaning that for any integer ℓ within $[3, \lfloor \frac{|V(D(n, r))|}{2} \rfloor]$, there exist two vertex-disjoint cycles C_1 and C_2 such that $|V(C_1)| = \ell$ and $|V(C_2)| = |V(D(n, r))| - \ell$. This result generalizes the known Hamiltonicity and vertex-pancyclicity properties of $D(n, r)$ demonstrated in prior work [2], thereby deepening the understanding of cycle embedding in interconnection networks. The two-disjoint-cycle-cover pancyclicity property is of theoretical significance as it ensures that dragonfly networks can be partitioned into cycles of variable lengths, which has implications for fault-tolerant routing and load balancing in high-performance computing systems.

The proof relied on a constructive method that exploited the symmetric topology of , though it involved a case-based analysis to cover the range of cycle lengths. While this approach ensured rigor, future work could focus on developing more unified proof techniques, such as mathematical induction or parameterized constructions, to reduce the reliance on exhaustive cases and enhance insight. Additionally, exploring the two-disjoint-cycle-cover pancyclicity in other network topologies would be a natural extension of this research. Algorithmic aspects, such as the efficient computation of these cycles in practical scenarios, also merit investigation.

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Data Availability Statement: No data was used for the research described in this paper. The Python code, which proves the two-disjoint-cycle-cover pancyclicity of dragonfly network $D(n, r)$ in the case of $n = 3$ and $r \geq 2$, is available at <https://github.com/guanlin-he/disjoint-cycle-pancyclicity-dragonfly>.

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