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Article

Fourier Series Indexed by the Indescribables

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Abstract: We propose a new class of solutions to classic partial differential equations using the class of indescribable numbers.

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1. Indescribable Numbers

We now know most numbers cannot be described [1] by any theory but what happens here at the very boundaries of our thoughts. Can we use these numbers? A number α is called first-order definable in the language of set theory without parameters if there exists a formula ϕ in the language of set theory with one free variable such that $\phi(\alpha)$ holds. However this formulation for definability cannot be expressed within that set theory. We will say the numbers which cannot be expressed this way belong to $\mathbb{M}$. How do we talk about numbers that by definition we cannot talk about? How can we prove theorems about them? What are the applications? In this paper we propose a new method for solving classic partial differential equations using these numbers.

2. Countable Fourier Series

Fourier introduced the fourier series to solve the heat equation [2]. We now write:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(2\pi \frac{n}{p} x) + b_n \sin(2\pi \frac{n}{p} x)) \quad (1)$$

With $n \in \mathbb{N}$ and P equal to the wavelength. We know well behaved and even some poorly behaved functions can be rewritten as linear combinations of sines and cosines with different frequencies.

3. Uncountable Fourier Series

Now we want to generalize. We write:

$$f(x) = a_0 + \sum_{\theta \in \mathbb{M} \cap (0, 2\pi)} (a_{\theta} \cos(\theta x) + b_{\theta} \sin(\theta x)) \quad (2)$$

So now we are indexed by the reals. To do this we have to use the axiom of choice.

4. The Heat Equation

The heat equation in one dimension reads as follows [2]:

$$u_t = k u_{xx} \quad (3)$$

Where x is the space variable so $x \in [0, L]$ where L is the length of the rod. t is the time variable so $t \in [0, \infty)$. We can assume the initial condition:

$$u(x, 0) = f(x), \forall x \in [0, L]$$

Where the function f is given and we can have boundary condition:

$$u(0, t) = 0 = u(L, t), \forall t > 0$$

We can separate variables to attempt to find a solution to 3 satisfying these conditions. We write:

$$\begin{aligned} u(x, t) &= X(x)T(t) \\ \frac{T'(t)}{kT(t)} &= \frac{X''(x)}{X(x)} \end{aligned}$$

Since both sides are functions of independent variables we write:

$$T'(t) = -\lambda kT(t)$$

And:

$$X''(x) = -\lambda X(x)$$

Suppose $\lambda < 0$. Then $\exists A, B \in \mathbb{R}$ such that:

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

Substituting into the boundary conditions we receive that $A=B=0$. Suppose $\lambda = 0$. Then $\exists A, B \in \mathbb{R}$ such that $X(x) = Ax + B$. When we substitute boundary conditions we conclude an identically zero result. Suppose $\lambda > 0$. Then $\exists A, B, C \in \mathbb{R}$ such that:

$$T(t) = Ae^{-\lambda kt}$$

and

$$X(x) = B \sin(\sqrt{\lambda}x) + C \cos(\sqrt{\lambda}x)$$

From the boundary conditions we get $C = 0$. And:

$$\sqrt{\lambda} = \frac{\theta}{L}$$

Sums of solutions to this equation which satisfy the boundary conditions also satisfy the initial equation. We write:

$$u(x, t) = \sum_{\theta \in \mathbb{M} \cap (0, 2\pi)} \alpha_{\theta} \sin\left(\frac{\theta}{L}x\right) e^{\frac{-\theta^2 kt}{L^2}}$$

Where

$$\alpha_{\theta} = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\theta x}{L}\right) dx$$

5. The Wave Equation

The wave equation in one dimension reads as follows [3]:

$$u_{tt} = c^2 u_{xx} \quad (4)$$

Where x is the space variable so $x \in [0, L]$ where L is the length of the rod. t is the time variable so $t \in [0, \infty)$. We can assume the initial condition:

$$\begin{aligned}u(x, 0) &= f(x), \forall x \in [0, L] \\u_t(x, 0) &= g(x), \forall x \in [0, L]\end{aligned}$$

Where the function f is given and we can have boundary condition:

$$u(0, t) = 0 = u(L, t), \forall t > 0$$

We can separate variables to attempt to find a solution to 4 satisfying these conditions. We write:

$$\begin{aligned}u(x, t) &= X(x)T(t) \\ \frac{T''(t)}{c^2 T(t)} &= \frac{X''(x)}{X(x)}\end{aligned}$$

Since both sides are functions of independent variables we write:

$$T''(t) = -\lambda c^2 T(t)$$

And:

$$X''(x) = -\lambda X(x)$$

Suppose $\lambda < 0$. Then $\exists A, B \in \mathbb{R}$ such that:

$$X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{-\lambda}x}$$

Substituting into the boundary conditions we receive that $A=B=0$. Suppose $\lambda = 0$. Then $\exists A, B, C, D \in \mathbb{R}$ such that $X(x) = Ax + B$ and $T(t) = Ct + D$. When we substitute boundary conditions we conclude an identically zero result. Suppose $\lambda > 0$. Then $\exists A, B, C, D \in \mathbb{R}$ such that:

$$T(t) = Ae^{c\sqrt{\lambda}t} + Be^{-c\sqrt{\lambda}t}$$

and

$$X(x) = C \sin(\sqrt{\lambda}x) + D \cos(\sqrt{\lambda}x)$$

From the boundary conditions we get $A = B$ and $D = 0$. And:

$$\sqrt{\lambda} = \frac{\theta}{L}$$

Sums of solutions to this equation which satisfy the boundary conditions also satisfy the initial equation. We write:

$$u(x, t) = \sum_{\theta \in \mathbb{M} \cap (0, 2\pi)} \sin\left(\frac{\theta}{L}x\right) [\alpha_{\theta} \cos\left(\frac{c\theta}{L}t\right) + \beta_{\theta} \sin\left(\frac{c\theta}{L}t\right)]$$

Where

$$\begin{aligned}\alpha_{\theta} &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\theta x}{L}\right) dx \\ \beta_{\theta} &= \frac{2}{c\theta\pi} \int_0^L g(x) \sin\left(\frac{\theta x}{L}\right) dx\end{aligned}$$

6. Conclusions

These systems of solutions cannot be checked for and only exist when we are not actually attempting to describe them. As soon as we make the attempt to describe them they collapse into the trivial solution.

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