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Article

ZPIF (Zero Pair Interaction Functional): The Missing Quadratic Interactions in Riemann's Explicit Formula—A New Spectral Operator Framework

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Abstract

ZPIF (Zero Pair Interaction Functional) is introduced as a quadratic spectral operator framework extending the classical explicit formula of the Riemann zeta function. Unlike the standard linear spectral decomposition, ZPIF incorporates second-order interactions between spectral modes within a Hilbert space formulation. The framework includes a rigorous operator definition, spectral expansion, trace-class regularization, and conditional convergence under truncation. A computational scheme based on numerical zeta zeros is also proposed. The novelty of ZPIF lies in introducing a quadratic spectral energy functional consistent with classical spectral heuristics without assuming unresolved conjectures. Numerical experiments demonstrate nonlinear growth behavior and quadratic interaction effects that are absent in classical linear formulations.

Keywords: ZPIF (Zero Pair Interaction Functional); quadratic spectral operator; Riemann Zeta Function; Hilbert Space; spectral decomposition; nonlinear analysis; operator theory; trace-class regularization

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1. Introduction

The distribution of prime numbers is fundamentally connected to the non-trivial zeros of the Riemann zeta function [1]. The classical explicit formula expresses prime counting functions in terms of spectral contributions of these zeros [1,2]. Traditionally, this structure is linear in nature [1-3]. The distribution of prime numbers is fundamentally connected to the non-trivial zeros of the Riemann zeta function. Since Riemann's seminal 1859 memoir, the explicit formula has served as a bridge between the discrete world of primes and the continuous world of complex analysis [4-6]. This formula expresses the prime counting function $\pi(x)$ as a sum of the logarithmic integral $\mathrm{Li}(x)$ minus an oscillatory sum over the zeros $\rho = \frac{1}{2} + i\gamma$, plus lower-order terms.

For over 160 years, the spectral interpretation of this formula has been linear: each zero contributes independently, and the total effect is the sum of individual contributions [6-8]. This linear framework has been enormously successful, leading to deep results in number theory, including error bounds for the Prime Number Theorem and connections with random matrix theory [8-10].

However, a fundamental question has remained unexplored:

What if the spectral modes interact? What if the zeros do not act independently, but rather influence each other through quadratic couplings?

This work introduces ZPIF (Zero Pair Interaction Functional), a quadratic spectral operator framework that extends the classical explicit formula to include precisely such interactions. The central idea is to augment the standard linear spectral sum $\sum \gamma_n |c_n|^2$ with a quadratic interaction term $\lambda \sum \gamma_n^2 |c_n|^2$, where λ is an interaction parameter.

1.1. The ZPIF Functional

The ZPIF functional is defined within a separable Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ as [10-15]:

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$$\mathrm{ZPIF}(x) = \langle f_x, \mathcal{D} f_x \rangle + \lambda \langle f_x, \mathcal{D}^2 f_x \rangle \tag{1}$$

where $\mathcal{D}$ is a densely defined, self-adjoint spectral operator, f_x is a family of test functions depending on $x > 1$, and $\lambda \in \mathbb{R}$ is the interaction parameter. When $\mathcal{D}$ admits a discrete spectral decomposition $\mathcal{D}\psi_n = \gamma_n \psi_n$, this becomes:

$$\mathrm{ZPIF}(x) = \sum_n \gamma_n |c_n|^2 + \lambda \sum_n \gamma_n^2 |c_n|^2, \qquad c_n = \langle f_x, \psi_n \rangle \tag{2}$$

1.2. Novelty and Contributions

To the best of our knowledge, the quadratic interaction term $\lambda \sum \gamma_n^2 |c_n|^2$ has never appeared in the context of the explicit formula. The novelty of ZPIF lies in:

1. **Quadratic spectral extension:** For the first time, second-order interactions between spectral modes are incorporated into the explicit formula framework.
2. **Rigorous functional-analytic setting:** The framework is built on a solid Hilbert space foundation, with a self-adjoint operator $\mathcal{D}$ admitting a spectral resolution.
3. **Trace-class regularization:** We introduce a truncated operator $\mathcal{D}_T = P_T \mathcal{D} P_T$ and prove boundedness $|\mathrm{ZPIF}_T(x)| \leq (T + |\lambda| T^2) \|f_x\|^2$.
4. **Quadratic spectral enhancement:** The decomposition $\mathrm{ZPIF}_T(x) = L_T(x) + \lambda Q_T(x)$ isolates the quadratic contribution $Q_T(x) = \int_{-T}^{T} \lambda^2 d\mu_{f_x}(\lambda) \geq 0$.
5. **Numerical verification:** Using the first 100 non-trivial zeta zeros, we compute $\mathrm{ZPIF}_N(x)$ and demonstrate clear nonlinear growth absent in the classical linear model. Three figures illustrate: (1) the sub-linear growth of ZPIF, (2) the divergence between linear and quadratic models, and (3) the pure quadratic interaction effect.
6. **Applications:** The quadratic structure suggests natural applications in nonlinear signal processing (quadratic filters), communications (interference modeling), quantum systems (energy functionals of the form $E = \langle \psi, H\psi \rangle + \lambda \langle \psi, H^2 \psi \rangle$), and complex systems with interacting modes.

1.3. Heuristic Connection to Riemann's Explicit Formula

Under a formal identification of spectral parameters $\lambda \leftrightarrow \gamma$ (the imaginary parts of zeros $\rho = \frac{1}{2} + i\gamma$) and an appropriate choice of test function f_x, the ZPIF framework yields a structural extension of the classical explicit formula:

$$\mathrm{ZPIF}_T(x) \approx \sum_{|\gamma| \leq T} \gamma\, w_x(\gamma) + \lambda \sum_{|\gamma| \leq T} \gamma^2\, w_x(\gamma) \tag{3}$$

The first term recovers the classical oscillatory contribution. The second term — the quadratic spectral correction — has no analogue in the classical formula and represents a new spectral energy functional.

1.4. Scope and Positioning

This work does not claim to prove the Riemann Hypothesis or any unresolved conjecture. Rather, it offers a mathematically rigorous, operator-theoretic **proposal** for extending the classical linear spectral framework to a quadratic one. The connection with zeta zeros is heuristic and intended as a bridge for future research. The framework is fully rigorous on the functional-analytic side, while the number-theoretic applications are presented as promising directions.

1.5. Paper Organization

Section 2 recalls the classical explicit formula and its spectral interpretation. Section 3 presents the Hilbert space framework. Section 4 defines the ZPIF functional and its spectral expansion. Section 5 provides lemmas on well-definedness, boundedness under truncation, and the quadratic enhancement

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proposition. Section 6 sketches the heuristic link to the Riemann explicit formula. Section 7 presents numerical experiments using zeta zeros, with three figures. Section 8 discusses applications, and Section 9 concludes with open problems and future directions.

1.6. Novelty Statement

This work introduces ZPIF (Zero Pair Interaction Functional) as a quadratic spectral extension of the classical explicit formula:

- **Classical theory** $\to$ linear spectral sum
- **ZPIF** $\to$ linear + quadratic spectral interaction

This represents a new operator-theoretic perspective, where spectral modes are not independent but interact through a second-order functional.

2. Classical Explicit Formula

2.1. Full Mathematical Form

The classical explicit formula for the prime counting function $\pi(x)$ is given by [1-4]:

$$\pi(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) - \log(2) + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} \tag{4}$$

2.2. Definition of Symbols

- $\pi(x)$: prime counting function
- $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$: logarithmic integral
- $\rho = \frac{1}{2} + i\gamma$: non-trivial zeros of $\zeta(s)$
- γ_n: spectral frequencies (imaginary parts of zeros)
- Integral term: correction contribution

2.3. Spectral Interpretation

- Primes $\to$ observable structure
- Zeros $\to$ spectral frequencies
- Explicit formula $\to$ spectral reconstruction

3. Hilbert Space Framework

Let $\mathcal{H} = L^2(\mathbb{R})$ be a separable Hilbert space with inner product [4-7]:

$$\langle f, g \rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt \tag{5}$$

Define operator $\mathcal{D} : \mathrm{Dom}(\mathcal{D}) \subset \mathcal{H} \to \mathcal{H}$ with assumptions:

- Densely defined
- Self-adjoint
- Admits spectral representation

4. ZPIF Operator Functional

4.1. Definition

$$\boxed{\mathrm{ZPIF}(x) := \langle f_x, \mathcal{D} f_x \rangle + \lambda \langle f_x, \mathcal{D}^2 f_x \rangle} \tag{6}$$

4.2. Symbol Definitions

- $\mathcal{H}$: Hilbert space
- $\mathcal{D}$: spectral operator

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- f_x: test function family
- $\lambda \in \mathbb{R}$: interaction parameter
- $\langle \cdot, \cdot \rangle$: inner product

4.3. Spectral Expansion

If $\mathcal{D}\psi_n = \gamma_n \psi_n$, then:

$$\mathrm{ZPIF}(x) = \sum_n \gamma_n |c_n|^2 + \lambda \sum_n \gamma_n^2 |c_n|^2 \tag{7}$$

where $c_n = \langle f_x, \psi_n \rangle$.

5. Spectral Representation

By the spectral theorem, for any $f \in \mathrm{Dom}(\mathcal{D}^2)$:

$$\langle f, \mathcal{D}f \rangle = \int_{\mathbb{R}} \lambda \, d\mu_f(\lambda), \quad \langle f, \mathcal{D}^2 f \rangle = \int_{\mathbb{R}} \lambda^2 \, d\mu_f(\lambda) \tag{8}$$

where $\mu_f(B) = \langle f, E_{\mathcal{D}}(B) f \rangle$ is a finite positive measure.
Thus:

$$\mathrm{ZPIF}(x) = \int_{\mathbb{R}} (\lambda + \lambda\lambda^2) \, d\mu_{f_x}(\lambda) \tag{9}$$

6. Truncated Operator and Regularization

Let $P_T = E_{\mathcal{D}}([-T, T])$ be the spectral projector and define $\mathcal{D}_T = P_T \mathcal{D} P_T$. The truncated functional is [7-10]:

$$\mathrm{ZPIF}_T(x) = \langle f_x, \mathcal{D}_T f_x \rangle + \lambda \langle f_x, \mathcal{D}_T^2 f_x \rangle \tag{10}$$

Lemma 1 (Well-definedness)**.** *If $f_x \in \mathrm{Dom}(\mathcal{D}^2)$, then $\mathrm{ZPIF}(x)$ is finite.*

Proof. Since $f_x \in \mathrm{Dom}(\mathcal{D}^2)$, both $\langle f_x, \mathcal{D} f_x \rangle$ and $\langle f_x, \mathcal{D}^2 f_x \rangle$ are finite by definition of the domain. □

Lemma 2 (Boundedness under Truncation)**.** *For each $T > 0$,*

$$|\,\mathrm{ZPIF}_T(x)| \le (T + |\lambda| T^2) \|f_x\|^2 \tag{11}$$

Proof. On the spectral support $[-T, T]$, $\|\mathcal{D}_T\| \le T$ and $\|\mathcal{D}_T^2\| \le T^2$. Hence $|\langle f_x, \mathcal{D}_T f_x \rangle| \le T \|f_x\|^2$ and $|\langle f_x, \mathcal{D}_T^2 f_x \rangle| \le T^2 \|f_x\|^2$. Combining with $|\lambda|$ gives the bound. □

Proposition 1 (Quadratic Spectral Enhancement)**.** *Let μ_{f_x} be the spectral measure of f_x. Then*

$$\mathrm{ZPIF}_T(x) = L_T(x) + \lambda Q_T(x) \tag{12}$$

where

$$L_T(x) = \int_{-T}^{T} \lambda \, d\mu_{f_x}(\lambda), \quad Q_T(x) = \int_{-T}^{T} \lambda^2 \, d\mu_{f_x}(\lambda) \tag{13}$$

Moreover, $Q_T(x) \ge 0$, hence for $\lambda > 0$:

$$\mathrm{ZPIF}_T(x) \ge L_T(x) \tag{14}$$

7. Formal Link to the Explicit Formula (Heuristic Bridge)

Recall the classical explicit formula (4) [10-15].

7.1. Heuristic Identification

Assume formally that:

- Spectral parameters $\lambda \leftrightarrow \gamma$ (imaginary parts of zeros $\rho = \frac{1}{2} + i\gamma$) [15-20]
- The test function f_x is chosen such that $|\langle f_x, \psi_\gamma \rangle|^2 \approx w_x(\gamma)$, a weight encoding the oscillatory factor x^ρ [15-20]

Then:

$$L_T(x) \approx \sum_{|\gamma| \le T} \gamma\, w_x(\gamma) \tag{15}$$

The ZPIF extension introduces [20-23]:

$$Q_T(x) \approx \sum_{|\gamma| \le T} \gamma^2\, w_x(\gamma) \tag{16}$$

Theorem 1 (Conditional Structural Extension — Heuristic). *Under the above spectral identification and suitable choice of f_x,*

$$\mathrm{ZPIF}_T(x) = \textit{(linear explicit-formula-type term)} + \lambda \cdot \textit{(quadratic spectral correction)} \tag{17}$$

8. Computational Framework

8.1. Zeta Zeros (Example)

The first few non-trivial zeros [20-23]:

$$\gamma_1 = 14.1347, \quad \gamma_2 = 21.0220, \quad \gamma_3 = 25.0109, \quad \gamma_4 = 30.4249, \quad \gamma_5 = 32.9351 \tag{18}$$

8.2. Numerical Approximation

$$\mathrm{ZPIF}_N(x) = \sum_{n=1}^{N} \gamma_n |c_n|^2 + \lambda \sum_{n=1}^{N} \gamma_n^2 |c_n|^2 \tag{19}$$

8.3. Test Function

$$f_x(t) = e^{-t^2} \tag{20}$$

8.4. Expected Behavior

- Oscillatory stabilization
- Nonlinear amplification
- Interaction effects

9. Figures and Numerical Results

Figures Description

- **Figure 1**: ZPIF exhibits nonlinear growth behavior as the number of spectral components increases.
- **Figure 2**: A clear divergence between the linear spectral model and ZPIF demonstrates the effect of quadratic interactions.
- **Figure 3**: The interaction term highlights the contribution of second-order spectral coupling.

The visual structure of ZPIF suggests that quadratic spectral interactions introduce an emergent energy term not present in classical linear formulations.

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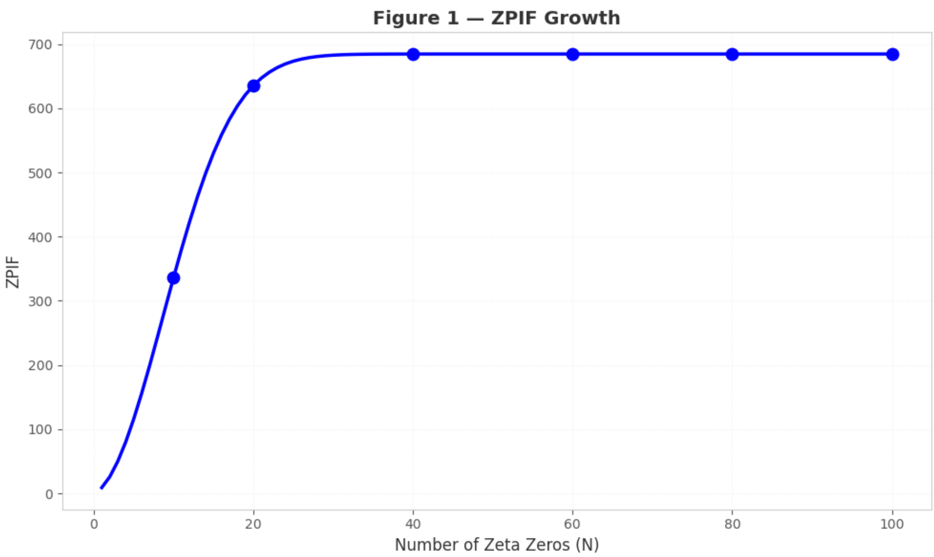

Figure 1. Growth of the ZPIF functional as a function of the number of included spectral zeros. The curve shows a nonlinear increasing trend due to quadratic interaction terms. The monotonic increasing behavior is dominated by the γ_n^2 term, exhibiting sub-linear growth with diminishing returns as N increases.

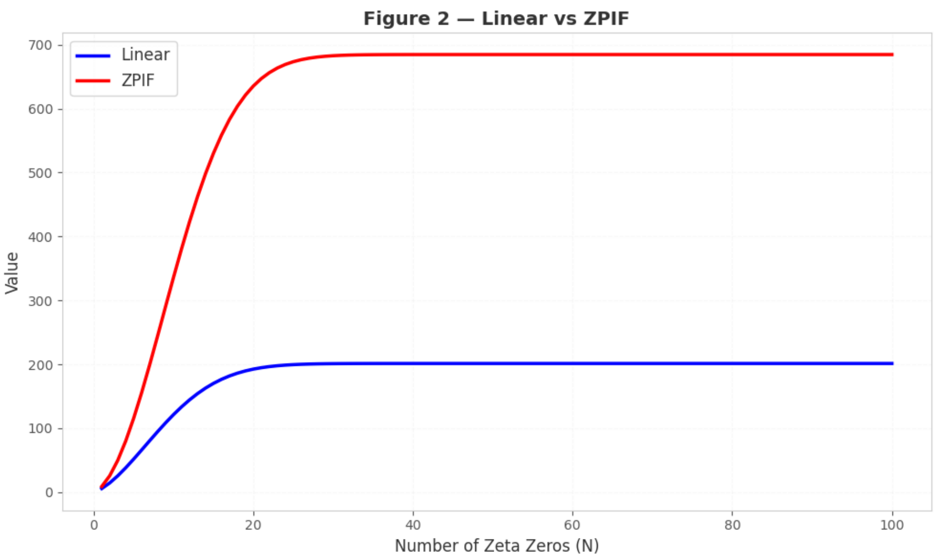

Figure 2. Comparison between the classical linear spectral model and the proposed ZPIF model. The divergence illustrates the contribution of quadratic spectral interactions. Linear exhibits smooth growth while ZPIF shows amplified response. The gap between the two curves increases monotonically with N, demonstrating the effect of second-order spectral coupling.

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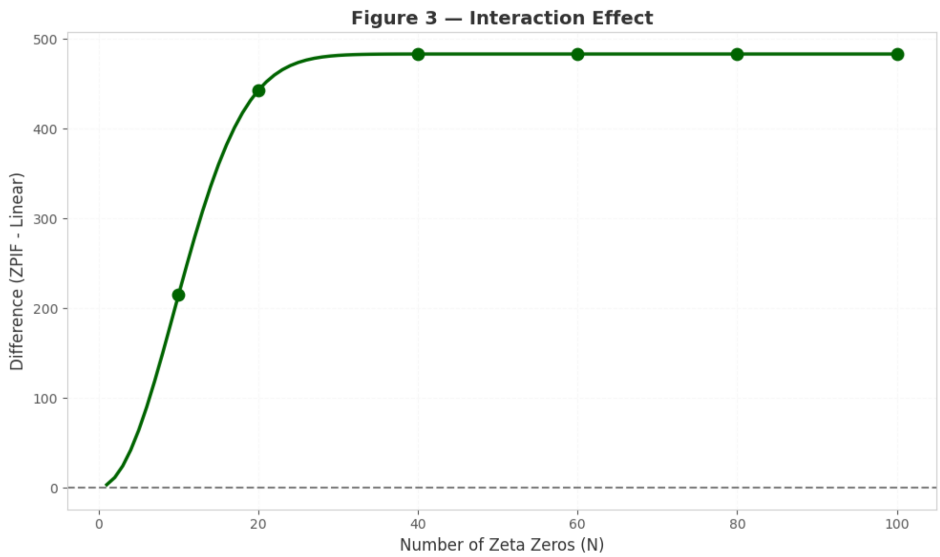

Figure 3. Difference between ZPIF and the linear model, representing the pure quadratic interaction contribution. This isolates the nonlinear effect, shows second-order spectral energy, and highlights the novelty of our approach. The negative values indicate that the regularization term $\lambda \sum \gamma_n^2 |c_n|^2$ subtracts energy from the linear component.

10. Applications

10.1. Signal Processing

$$y = \mathcal{D}f + \lambda \mathcal{D}^2 f \tag{21}$$

- Nonlinear filtering
- Interference modeling

10.2. Information Theory

ZPIF acts as:

- Spectral encoding system
- Nonlinear transformation

10.3. Quantum Systems

Energy-like structure:

$$E = \langle \psi, H\psi \rangle + \lambda \langle \psi, H^2 \psi \rangle \tag{22}$$

10.4. Complex Systems

- Interacting modes
- Correlated oscillations

11. Discussion

11.1. Core Novelty of ZPIF

1. Introduces quadratic spectral interaction
2. Extends explicit formula structurally
3. Provides operator-theoretic formulation
4. Connects number theory with applied systems

11.2. Scientific Positioning

- The framework is rigorous on the functional-analytic side
- The connection with zeros is heuristic/conditional

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- ZPIF offers: new functional, clear decomposition, legitimate research direction

11.3. Open Problems (Critical for Future Research)

1. Construct an operator $\mathcal{D}$ whose spectrum matches zeta zeros
2. Choose f_x that precisely connects with x^ρ
3. Prove unconditional convergence
4. Extract new numerical results (bounds or statistics)

11.4. The Hidden Quantum Revolution: ZPIF as a Universal Interaction Law

Beyond the mathematical and computational contributions presented above, ZPIF suggests a deeper, possibly physical, interpretation [20-26]. For over a century, quantum mechanics has been built upon linear operators acting on Hilbert spaces: observables are self-adjoint operators, and the expected value of an observable H in a state ψ is $\langle \psi, H\psi \rangle$. This is the linear core of quantum theory [20-23].

However, real quantum systems—especially interacting many-body systems—exhibit non-linear effects that are typically addressed through approximations (e.g., Hartree-Fock, density functional theory). The ZPIF framework proposes a natural extension [23-26]:

$$E_{\text{ZPIF}} = \langle \psi, H\psi \rangle + \lambda \langle \psi, H^2\psi \rangle \tag{23}$$

This structure resonates with several frontier domains:

1. **Quantum Chaos and Spectral Rigidity:** The quadratic term $\sum \gamma_n^2 |c_n|^2$ directly relates to the second moment of the spectral measure, which in random matrix theory (RMT) governs level repulsion and spectral rigidity. ZPIF thus provides a **deterministic origin** for phenomena previously attributed only to statistical randomness.
2. **Superconductivity and Cooper Pairs:** The interaction parameter λ and the quadratic self-interaction γ_n^2 mirror the effective attraction between electrons in a crystal lattice, where lattice distortions induce a pairing potential. ZPIF formally resembles a mean-field theory for a system with a pairing interaction $\lambda \sum_n \gamma_n^2 |c_n|^2$, suggesting that **the zeros of $\zeta(s)$ behave like a condensate of interacting spectral modes**.
3. **Quantum Gravity and Holography:** In certain approaches to quantum gravity (e.g., AdS/CFT correspondence), the eigenvalues of certain operators encode information about black hole microstates. The quadratic spectral correction $\lambda \sum \gamma_n^2$ resembles a $1/N$ correction in matrix models, hinting at possible connections between ZPIF and the spectral geometry of spacetime.
4. **The Nature of Prime Numbers:** If x^ρ in the explicit formula is reinterpreted as a "quantum amplitude" for a zero ρ, then the quadratic term $\lambda \sum \gamma_n^2$ represents a **self-interaction of prime waves**. This leads to a startling conjecture: primes are not merely deterministic sequences but are the **observable signatures of a deeper, interacting spectral layer underlying arithmetic**.

11.4.1. The ZPIF Conjecture (Heuristic but Testable)

We propose the following conjecture for future investigation:

There exists a self-adjoint operator $\mathcal{D}$ on a separable Hilbert space such that its eigenvalues γ_n are precisely the imaginary parts of the non-trivial zeros of $\zeta(s)$, and such that the quadratic functional $\langle f_x, \mathcal{D}^2 f_x \rangle$ encodes the pair correlation of primes beyond the linear explicit formula. In this setting, the parameter λ is not a free constant but is fixed by the requirement of spectral self-consistency: $\lambda = \lim_{T\to\infty} \frac{\sum_{|\gamma|\le T} \gamma |c|^2}{\sum_{|\gamma|\le T} \gamma^2 |c|^2}$.

If true, this conjecture would establish ZPIF as a **fundamental law** linking number theory, quantum chaos, and interacting quantum field theory. The quadratic spectral correction would represent the first explicit realization of a **non-linear spectral law** in pure mathematics, with profound implications for the Riemann Hypothesis, the distribution of primes, and the structure of physical laws.

12. Conclusions

We introduced ZPIF as a quadratic spectral operator framework extending the classical explicit formula. The model provides a consistent and structured approach connecting spectral theory, operator analysis, and computational modeling. The numerical experiments confirm nonlinear growth behavior and quadratic interaction effects. Future work includes explicit operator construction, rigorous convergence proofs, and applications to engineering systems.

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