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A Generalisation of Pythagoras' Theorem

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Abstract: We provide a generalisation of Pythagoras' theorem, which enables us to calculate the absolute value of a hypersurface of arbitrary dimensions in terms of its projections. For a one dimensional hyperplane the theorem reduces to Pythagoras' theorem.

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1. A Generalisation of Pythagoras' Theorem

It has been canonical to associate Pythagoras' theorem with the geometry of a triangle and De Gua's theorem and higher dimensional analogues, with the geometry of a simplex. We, however, adopt a more general geometric interpretation. Namely, we state a new theorem, which yields the absolute value of an hyperplane of arbitrary shape in terms of its components,

$$X^2 = \sum_{i_1 < \cdots < i_p}^{d} x^{i_1 \cdots i_p} x_{i_1 \cdots i_p}, \tag{1}$$

where $x^{i_1 \cdots i_p}$, X, p and d are the components of the hyperplane, the absolute value of the hyperplane, the dimension of the hyperplane and the overall dimension of the space respectively. We retrieve Pythagoras' theorem,

$$X^2 = x^2 + y^2$$

for $p = 1$ and $d = 2$. For $p = 2$ and $d = 3$, we obtain a new theorem, which gives the absolute value, X, of a plane of arbitrary shape, in terms of its components

$$X^2 = a^2 + b^2 + c^2,$$

where a, b and c are the projections of the plane, x^{12}, x^{13} and x^{23}, onto the three planes, spanned by the coordinate axis axes. This theorem has geometric meaning, as is illustrated (Figure 1).

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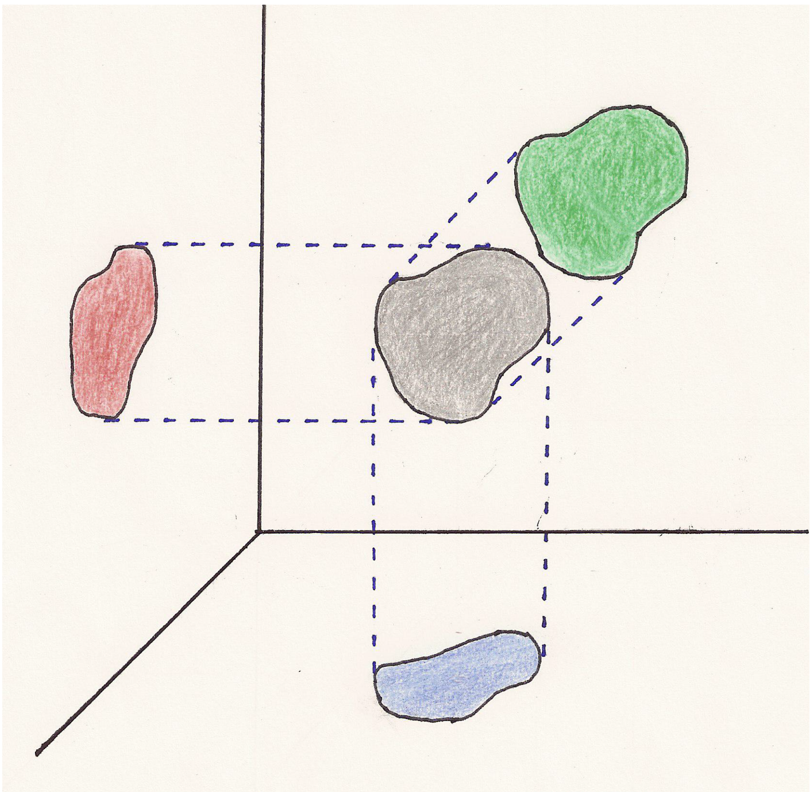

Figure 1. The decomposition of a grey surface, into its projections, x^{12}, x^{13} and x^{23} in blue, red and green is being displayed.

2. Inner Product and Projections

The square of the absolute value of any hyperplane with arbitrary boundary is equal to sum of the squares of the absolute values of the projections of the hyperplane onto its basis elements. We argue that this insight is true in the trivial case, by realizing that we can always rotate any hyperplane, such that it has only one component.

For the sake of clarity, a hyperplane can be decomposed into its projections onto its basis elements as follows

$$x = \sum_{i_1 < \cdots < i_p}^{d} x^{i_1 \cdots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}, \tag{2}$$

where e_i are the basis vectors and where the components are the projections $x^{i_1 \cdots i_p} = \langle e^{i_1} \wedge \cdots \wedge e^{i_p}, x \rangle$. We recognize the theorem for hyperplanes (1) as an inner product of an hyper plane (2) with itself and derive the theorem this way. The inner product of the basis elements is equal to the Gram matrix

$$\langle e_1 \wedge \cdots \wedge e_p, f_1 \wedge \cdots \wedge f_p \rangle = \det \langle e_k, f_l \rangle,$$

where $\langle e_i, e_j \rangle = g_{ij}$ and $e_i = g_{ij} e^j$. This determines the inner product of two p-forms, x, and y, in the following manner

$$\langle x, y \rangle = \sum_{i_1 < \cdots < i_p}^{d} x^{i_1 \cdots i_p} y^{j_1 \cdots j_p} \langle e_{i_1}, f_{j_1} \rangle \cdots \langle e_{i_p}, f_{j_p} \rangle, \tag{3}$$

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where the absence of a sum sign implies Einstein summation convention. We now derive the theorem for hyper planes (1),

$$\langle x, x\rangle = \sum_{i_1<\cdots<i_p}^{d} x^{i_1\cdots i_p} x_{i_1\cdots i_p} = ||x||^2 = X^2$$

Note that the theorem holds in relativistic spaces.

3. Hypersurfaces

We generalize the theorem for hyperplanes to a theorem for hypersurfaces. Letting the components to be infinitesimal, $x^{i_1\cdots i_p} = dx^{[i_1}\cdots dx^{i_p]}$, yields

$$dX^2 = \sum_{i_1<\cdots<i_p}^{d} dx^{[i_1}\cdots dx^{i_p]}dx^{[j_1}\cdots dx^{j_p]}g_{i_1j_1}\cdots g_{i_pj_p}, \tag{4}$$

where dX is the absolute value of an infinitesimal hypersurface, where x^i are arbitrary coordinates and where the square brackets imply antisymmetrization. Whenever the metric is diagonal, we obtain

$$dX^2 = \sum_{i_1<\cdots<i_p}^{d} \left(dx^{[i_1}\cdots dx^{i_p]}\right)^2 g_{i_1i_1}\cdots g_{i_pi_p}.$$

We conclude with three examples for clarification. We retrieve the line element,

$$dX^2 = g_{ij}dx^i dx^j,$$

for $p = 1$.

Taking the root of both sides of the theorem for hypersurfaces, we obtain the d-dimensional volume element

$$dX = \sqrt{|g|}dx^{[1}\cdots dx^{d]},$$

for $p = d$.

The calculation of the surface of a sphere goes as follows. Taking $p = 2$, $d = 3$ and choosing the coordinates to be spherical coordinates, gives us

$$dX^2 = r^2\sin^2\phi(drd\theta)^2 + r^2(drd\phi)^2 + r^4\sin^2\phi(d\theta d\phi)^2,$$

where the metric is given by $g_{rr} = 1, g_{\phi\phi} = r^2$ and $g_{\theta\theta} = r^2\sin^2\phi$. The surface of a sphere is characterized by the radius being constant, $dr = 0$. If we then take the root and put in the appropriate integration boundaries, we get the value for the area

$$X = \int_0^{2\pi}\int_0^{\pi} r^2\sin\phi(d\phi d\theta) = 4\pi r^2.$$

4. Historical Discussion

We believe the theorem of Pythagoras to be a most important theorem and also its generalisation. One might question the novelty of this work though. The Namu-Goto action, for instance,

$$S = -\frac{T_0}{c}\int d\tau d\sigma\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}$$

also yields the absolute value of a two dimensional hypersurface, where $X^\mu = X^\mu(\tau,\sigma)$ is a vector, which determines the shape of the hypersurface. It gives the absolute value, but it is unaware of the projections, the righthand sight of our the theorem for hypersurfaces, a generalisation of Pythagoras' theorem. This is true, because the vector X^μ is paramatrized by σ and τ, and moves only on the surface.

There is actually an older physics equation, the electromagnetic energy, which only captures the righthand sight of the theorem for hypersurfaces. We can recognize $F^{\mu\nu}$ as the projection of an infinitesimal surface on the x^μ, x^ν-plane. The absolute value of F is then given by $F^2 = F^{\mu\nu}F_{\mu\nu}$.

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