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Binary Sequences with Low Aperiodic Autocorrelations: Small Peak Sidelobe Levels

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Binary Sequences with Low Aperiodic Autocorrelations: Small Peak Sidelobe Levels

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Abstract

Binary sequences (binary codes), where the elements are -1 or $+1$, are useful in many fields, including communications, radar, sonar, mathematics, physics, and cryptography. This paper considers binary sequences with low aperiodic autocorrelations and focuses on the small peak sidelobe levels. Two families of binary sequences are considered, namely Rudin–Shapiro and Legendre sequences. Two conjectures regarding Legendre sequences are proposed: (1) The obtained binary sequences with the best-known peak sidelobe levels have merit factor ≈ 5.0, (2) The number of elements that differ between the resulting binary sequences and the initial Legendre sequences follows a linear dependence on the sequence length (n), namely $\approx 0.01n$. The Rudin–Shapiro sequences do not exhibit these properties, as worse peak sidelobe level and merit factor values were obtained. The number of elements that differ between the resulting binary sequences and the initial Rudin–Shapiro sequences is also much higher compared to that of the Legendre sequences.

Keywords: binary sequence; aperiodic autocorrelation; peak sidelobe level; Legendre sequence; Rudin–Shapiro sequence; merit factor

MSC: 94A55

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1. Introduction

A *binary sequence* of length n is an n-tuple $A = (a_0, a_1, \ldots, a_{n-1})$, where each a_i, for $i \in \{0, 1, \cdots, n-1\}$ takes the value -1 or $+1$. The *aperiodic autocorrelation* function of the binary sequence A at shift k is defined as

$$C_A(k) := \sum_{i=0}^{n-k-1} a_i a_{i+k} \qquad \text{for } k = 0, 1, \ldots, n-1. \tag{1}$$

The second type of autocorrelation function is periodic. The *periodic autocorrelations* are usually much easier to study than their aperiodic counterparts [1]. In this work, we consider aperiodic autocorrelations.

Several authors have studied different measures for the collective smallness of the aperiodic autocorrelations of sequences. Two measures of "smallness" have been used:

1. merit factor, and
2. peak sidelobe level.

The *merit factor* was introduced by Golay [2,3], and it is defined as

$$F(A) := \frac{n^2}{2\sum_{k=1}^{n-1}(C_A(k))^2} \qquad \text{for } n > 1. \tag{2}$$

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The *Peak Sidelobe Level* (PSL) is defined as

$$\mathrm{PSL}(A) := \max_{1 \le k < n} |C_A(k)|. \tag{3}$$

The autocorrelation function of A at shift 0, $C_A(0)$, is called the *main-lobe* level, and it is not included in (3). The rest, $C_A(k), k = 1, 2, \ldots, n-1$, are called *side-lobe* levels. The aperiodic autocorrelation function in (1) is even, since $C_A(-k) = C_A(k)$. Therefore, it is sufficient to consider only the shifts $k = 0, 1, \ldots, (n-1)$.

Sequence design methods in the literature can be broadly classified into two categories: algebraic constructions and algorithmic design approaches [4]. This work focuses mainly on algorithmic approaches.

In practice, we prefer binary sequences that have a lower PSL value and a higher F value. From a computational standpoint, optimization based on the merit factor (2) differs significantly from optimization based on PSL (3). In the former, we minimize the sum of squares, whereas in the latter, we minimize the highest sidelobe, which can lead to different optimization behavior. Some algorithms minimize a combination of metrics to create binary sequences with low aperiodic autocorrelation. Such a combination of metrics can be beneficial in the early stages of the optimization process, especially when starting with a randomly initialized sequence. In contrast, the combination is usually less effective in the late stages. A sequence with the optimal PSL usually has a merit factor which is much lower than the optimal merit factor, and vice versa [5,6]. Various algorithms have been proposed recently to solve this challenging optimization problem for F metric. For example, massively parallelized implementation of the memetic tabu search algorithm [7], the quantum approximate optimization algorithm [8], quantum-enhanced memetic tabu search [9], and a competitive NISQ and qubit-efficient solver for the LABS (low autocorrelation binary sequences) problem. The reader interested in the F metric is referred to the survey [1] (Chapter 3.5) and recent works [10–12]. The LABS problem is included in the recently proposed Quantum Optimization Benchmarking Library [13].

Let $\mathcal{A}_n$ denote the set of all binary sequences of length n. We would like to understand the behavior of

$$\mu(n) := \min_{A \in \mathcal{A}_n} \mathrm{PSL}(A), \tag{4}$$

as $n \to \infty$. We define $\mu(n)$ to be optimal value of the PSL over the set $\mathcal{A}_n$. If someone wishes to compute $\mu(n)$ for a given length n, then they need, using the most naive approach, to probe 2^n different binary sequences. The exponential term of the complexity can be reduced from $O(2^n)$ to approximately $O(1.4^n)$ by more efficient algorithms [14,15]. The considerable numerical computation effort in finding binary sequences with small PSL values has been put in by various authors, and, nowadays, $\mu(n)$ for $n \le 61$ and $n = 64$ are known, while for larger lengths, the best-known PSL values are reported in the literature. For $71 \le n \le 105$, the PSL values are presented in [16]. Currently, results for all lengths $106 \le n \le 300$ are reported in [5,17,18], PSL values for selected lengths for $301 \le n \le 4000$ are published by several authors (see [19] and references therein), for some n up to 8191 in [20,21], and for even longer n (up to 10^6) in [5,19,22–24] and up to $2^{25} - 1$ in [15].

1.1. Contributions of the Paper

In this paper, two families of binary sequences are considered, namely Rudin–Shapiro and Legendre sequences. Based on numerical experimentation, two conjectures associated with Legendre sequences are proposed:

- The obtained resulting binary sequences with the best-known peak sidelobe levels have merit factor ≈ 5.0. It appears to be close to an integer value.
- The number of elements that differ between the resulting binary sequences and the initial Legendre sequences follows a linear dependence on the sequence length (n), namely $\approx 0.01n$.

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Additional experimentation also indicates that no such results can be observed with the Rudin–Shapiro sequences, where worse peak sidelobe level and merit factor values were obtained.

1.2. Structure of the Paper

The paper is organized as follows. Section 2 provides the necessary background knowledge to understand the main ideas of the paper. Related work is discussed in Section 3. Section 4 outlines the methodology used for the experimental part. Extensive experiments conducted to find binary sequences with small peak sidelobe levels and the obtained results are presented in Section 5. Based on the results, we also propose two conjectures associated with the Legendre sequences in this section. Section 6 highlights some additional notes. Finally, Section 7 summarizes the findings and concludes the paper.

2. Preliminary

In this section, we give a brief overview of some families of binary sequences, how we generate them, their properties, and some known results.

The *Rudin–Shapiro* sequence requires a four-letter alphabet with the following substitution rule [25]:

$$(1) \quad \mathrm{S} \mapsto \mathrm{ST}, \quad \mathrm{T} \mapsto \mathrm{SU}, \quad \mathrm{U} \mapsto \mathrm{VT}, \quad \mathrm{V} \mapsto \mathrm{VU},$$

$$(2) \quad \mathrm{S}, \mathrm{T} \mapsto 0, \quad \mathrm{U}, \mathrm{V} \mapsto 1.$$

The first symbolic generations are:

$$\begin{aligned} w^{(0)} &= \mathrm{S}, \\ w^{(1)} &= \mathrm{ST}, \\ w^{(2)} &= \mathrm{STSU}, \\ w^{(3)} &= \mathrm{STSUSTVT}, \\ &\cdots \end{aligned}$$

Applying the binary conversion rules to the Rudin–Shapiro symbolic generations yields:

$$\begin{aligned} w^{(0)} &= 0, \\ w^{(1)} &= 00, \\ w^{(2)} &= 0001, \\ w^{(3)} &= 00010010, \\ &\cdots \end{aligned}$$

Changing each symbol 0 to -1 yields the binary sequence in our format. The length of the Rudin–Shapiro sequence is $n = 2^m$, where $m \in \{1, 2, \cdots\}$.

Calculations by Littlewood show that the Rudin–Shapiro sequences have low mean square autocorrelation [26]. We now examine the lower and upper bounds for PSL values of Rudin–Shapiro sequences. Here we use the abbreviated text from [26] [Theorem 3, p. 3457].

Theorem 1 (Katz and van der Linden [26])**.** *If n is a nonnegative integer and x_n and y_n the n-th Rudin–Shapiro sequence and its companion, then $\mathrm{PSL}(x_n) = \mathrm{PSL}(y_n)$, and we have the lower and upper bound*

$$0.382159... \leq \limsup_{n\to\infty} \frac{\mathrm{PSL}(x_n)}{\alpha_0^n} \leq 0.660113...$$

where $\alpha_0 = 1.658967...$ is the real root of $X^3 + X^2 - 2X - 4$.

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The *difference* between binary sequences A and B, both of length n, can be written as the Hamming distance:

$$d = d(A, B) = \sum_{i=0}^{n-1} d_i, \tag{5}$$

where

$$d_i = \begin{cases} 1, & A_i \neq B_i, \\ 0, & A_i = B_i. \end{cases} \tag{6}$$

In this paper, we are interested in how different the two binary sequences are, where the first sequence is an initial sequence and the second sequence has an optimized value of PSL.

Ein-Dor et al. [27] proposed an "educated guess" about the growth rate of the function $\mu(n)$.

Conjecture 1 (Ein-Dor et al. [27]). *As $n \to \infty$, we have*

$$\frac{\mu(n)}{\sqrt{n}} \to d, \qquad \text{where } d = 0.435\ldots.$$

Theorem 2 (Moon and Moser [28]).

(i) *For any fixed $\epsilon > 0$, the proportion of sequences $A \in \mathcal{A}_n$ such that $\mu(n) \leq (2+\epsilon)\sqrt{n \log n}$ approaches 1 as $n \to \infty$.*

(ii) *If $K(n)$ is any function of n such that $K(n) = o(\sqrt{n})$, then the proportion of sequences $A \in \mathcal{A}_n$ for which $\mu(n)$ approaches 1 as $n \to \infty$.*

Theorem 2, the only proven PSL result for general binary sequences, examines the growth rate of the peak sidelobe level of almost all binary sequences and this growth rate lies between order $\sqrt{n}$ and order $\sqrt{n \log n}$.

A *Legendre sequence* A of length p, where p is an odd prime number, is defined as:

$$A(k) = \left(\frac{k}{p}\right), \qquad \text{for } 0 \leq k < p, \tag{7}$$

where $\left(\frac{k}{p}\right)$ is the Legendre symbol [29,30]. The Legendre symbol is defined as:

$$\left(\frac{k}{p}\right) = \begin{cases} +1, & \textbf{if } p \mid k \textbf{ or } k \text{ is a quadratic residue modulo } p, \\ -1, & \text{otherwise.} \end{cases}$$

Note that we use the convention that $\left(\frac{k}{p}\right) = 1$ if $k = 0$.

Dimitrov [23] gave a complete list of all PSL-optimal Legendre sequences, with or without rotations, for lengths up to 432100. The numerical experiments suggest that the PSL values of all PSL-optimal Legendre sequences, with or without rotations, and with lengths n greater than 235723, are strictly greater than $\sqrt{n}$.

Let us consider yet another family of binary sequences, namely *Galois sequences*, also called m-sequences. Galois sequences can be efficiently generated using linear feedback shift registers [31]. There is a direct correspondence between m-sequences of degree q and primitive polynomials of degree q over $GF(2)$ [32].

Dmitriev and Jedwab [15] performed experiments on m-sequences up to length $2^{25} - 1$ and they showed numerically:

Conjecture 2 (Dmitriev and Jedwab [15]). *The PSL of <u>all</u> m-sequences of length n appears to grow like $O(\sqrt{n} \cdot \log \log n)$.*

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In the same work, Dmitriev and Jedwab also numerically showed:

Conjecture 3 (Dmitriev and Jedwab [15]). *The PSL of almost all m-sequences of length n appears to grow like* $\Theta(\sqrt{n})$.

Numerical evidence in Conjecture 3 relies on an algorithm for calculating the maximum PSL over all cyclic shifts of an m-sequence for $m \leq 25$. The mean (taken over all primitive elements α of GF(n+1) [15,33]) of the maximum PSL over all cyclic shifts appears to be approximately $1.31\sqrt{n}$ for all values of m between 13 and 25.

3. Related Work

In this section, we give an overview of related work. Finding sequences with low autocorrelation sidelobes [34] is a very challenging optimization problem of the digital sequence design [19]. Recently, several heuristic algorithms were proposed to tackle this optimization problem, especially for longer lengths.

Mow et al. [35] performed an experiment for finding long LABS with low PSL. Four different fitness functions were used in their evolutionary algorithm for lengths up to 4096. Coxson et al. [36] applied a multi-thread evolutionary algorithm to search for binary sequences (codes) with small PSL values for n up to 3000. Coxson and Russo [14] used exhaustive search for minimum PSL (see Eq. (4)) binary sequences. Lin et al. [22] proposed the 1bCAN algorithm, used for aperiodic binary sequences design. The proposed algorithm is FFT (Fast Fourier Transform) based, and it can be used to design long sequences with lengths up to $n \sim 10^6$. Dimitrov et al. [17] applied an efficient mechanism for single bit flipping calculation. Binary sequences of length up to $m = 17$ ($n = 131071$) were considered. The hybrid strategies for constructing binary sequences with near-optimal PSL values are presented in [18]. Bošković and Brest [37] proposed the two-phase optimization algorithm. It utilized a concurrent computing on the graphic processing units for finding binary sequences on lengths $n = 2^m - 1$ for $14 \leq m \leq 20$. Lin et al. [24] proposed an efficient gradient based algorithm, called 1bG-PSL, to minimize PSL for binary sequence designs with or without low correlation zone requirements. The result for $n = 2^{17} - 1$ is reported, i.e., PSL $= 284$ and $F = 4.59$. It can be observed that the computational methods presented above used a random binary sequence for initialization of the optimization process.

Brest et al. [19] developed a heuristic algorithm that combines the Legendre sequence to seed an initial binary sequence and an efficient stochastic search method with a dynamic fitness function mechanism to generate a final binary sequence up to length $2^{20} - 1$.

Jedwab and Yoshida [38], and Dmitriev and Jedwab [15] studied the growth rate of PSL for binary sequences and their families, numerically. In [15], the authors reported results for $n = 2^m - 1$ for m up to 25.

4. Methodology

The initial binary sequences, which served as seeds for the optimization algorithm, were selected from two well-known families of binary sequences, namely the Rudin–Shapiro and Legendre sequences.

The length of the Rudin–Shapiro sequence is $n = 2^m$ and the optimization process is performed on the same length, i.e., without any truncation. We denote the initial and optimized Rudin–Shapiro sequences as B^{RS}_{init} and B^{RS}_{opt}, respectively.

A Legendre sequence is defined for length p, where p is an odd prime number. We wish to use a binary sequence of length $n = 2^m - 1$, so we applied the following procedure to ensure the length (the same procedure was used in [19]). We used the first prime p that satisfies $n \leq p$. A Legendre binary sequence of length p is generated using (7) and rotated by 1/4. It is then truncated, if needed, to ensure that the length of the initial binary sequence is equal to n. We denote the initial, 1/4-rotated, and truncated Legendre sequence as B^{Leg}_{init} and the optimized Legendre sequence B^{Leg}_{opt}. Note that the

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length of the Rudin–Shapiro sequence is 2^m, while the Legendre sequences have the length of $2^m - 1$ in this work.

In the experimental study, we employed the heuristic optimization algorithm proposed in [19], which minimizes the PSL value throughout the search process. Details of the algorithm and its parameter settings can be found in [19]. To analyze the relationship between the initial and optimized sequences, we repeated all experiments reported in [19] for Legendre sequences and additionally recorded the number of flipped elements required to obtain the optimized sequences. We then conducted an analogous set of experiments for Rudin–Shapiro sequences to enable a comparative analysis, as we are interested in the differences between the initial and the optimized binary sequence for the Legendre and Rudin–Shapiro sequences, respectively.

For the sequence obtained after optimization, we also calculate the merit factor and compare it with the merit factor of the initial sequence, whose merit factor is known. The obtained results are compared with results from the literature.

5. Experiments

This section presents the results of our search for low PSL values in two families of binary sequences, provides a compilation of the best-known PSL values for binary sequences, and introduces two conjectures motivated by these findings.

5.1. Results on Rudin–Shapiro Sequences

The construction of the Rudin–Shapiro sequences is described in Section 2. For selected lengths, the lower (LB) and upper (UB) bounds alongside the PSL value of the Rudin–Shapiro sequences are shown in Table 1. The bounds are calculated using Theorem 1.

Table 1. The lower and upper bounds, and the PSL value for Rudin–Shapiro sequences.

m	$n = 2^m$	LB	UB	PSL
10	1024	60	104	85
11	2048	100	172	153
12	4096	166	286	217
13	8192	275	475	373
14	16,384	457	789	557
15	32,768	768	1309	961
16	65,536	1257	2172	1717
17	131,072	2086	3604	2445
18	262,144	3461	5779	4285
19	524,288	5743	9920	6257
20	1,048,576	9527	16,457	11,153

[1] LB $= 0.382159\alpha_0^m$. [2] UB $= 0.660113\alpha_0^m$.

Next, we present experimental results of optimized Rudin–Shapiro sequences. For this purpose, we used the heuristic algorithm [19]. The algorithm started with the initial Rudin–Shapiro sequence (B^{RS}_{init}) with length n, and during the optimization process, the algorithm made improvements to the binary sequence by changing a number of elements of the sequence, yielding the optimized Rudin–Shapiro sequence (B^{RS}_{opt}).

Table 2 shows tha PSL of the optimized sequence (B^{RS}_{opt}), and difference between the initial Rudin–Shapiro sequence (B^{RS}_{init}) and the optimized sequence (B^{RS}_{opt}) for $10 \leq m \leq 20$. Comparing the PSL values in Tables 1 and 2, it can be seen that the PSL values of the initial sequences are much higher than those of the optimized sequences. For example, for $m = 16$, the PSL values are 1717 and 209, respectively. Differences between the Rudin–Shapiro sequences (initial) and the optimized sequences divided by their length, d/n, are between 20% and 50%. This indicates that the heuristic algorithm had to flip many elements of the binary sequence.

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Table 2. PSL and F of the optimized sequence (B_{opt}^{RS}), and difference between the Rudin–Shapiro sequence (B_{init}^{RS}) and the optimized sequence (B_{opt}^{RS}).

m	$n = 2^m$	PSL	PSL$/\sqrt{n}$	F	d	d/n [%]
10	1024	24	0.7500	3.84815	519	50.68
11	2048	34	0.7513	4.26986	1002	49.93
12	4096	49	0.7656	4.13834	1952	47.66
13	8192	69	0.7623	4.29800	3590	43.82
14	16,384	101	0.7891	4.11236	5535	33.78
15	32,768	146	0.8065	3.95976	10,872	33.18
16	65,536	209	0.8164	3.94681	21,477	32.77
17	131,072	300	0.8286	3.8971	39,714	30.30
18	262,144	438	0.8555	3.6729	78,608	26.93
19	524,288	631	0.8715	3.52789	138,839	26.48
20	1,048,576	1000	0.9766	2.8829	224,237	21.38

Table 3. PSL and F of the optimized sequence (B_{opt}^{Leg}), and difference between the initial Legendre sequence (B_{init}^{Leg}) and the optimized sequence (B_{opt}^{Leg}).

m	$n = 2^m - 1$	PSL	PSL$/\sqrt{n}$	F	d	d/n [%]
10	1023	22	0.6878	4.78203	13	1.27
11	2047	32	0.7073	4.82918	31	1.51
12	4095	45	0.7032	5.09739	40	0.98
13	8191	66	0.7292	5.02281	86	1.05
14	16,383	93	0.7266	5.03535	165	1.01
15	32,767	133	0.7347	4.99564	353	1.08
16	65,535	189	0.7383	5.06485	700	1.07
17	131,071	269	0.7430	5.00051	1394	1.06
18	262,143	380	0.7422	4.99135	2855	1.09
19	524,287	540	0.7457	5.02356	5319	1.01
20	1,048,575	766	0.7480	4.99350	10,873	1.04

When we compare the corresponding merit factors, we notice that the merit factors in the optimized sequences increased from the initial $F = 3.0$, which is known for Rudin–Shapiro sequences, to values between 2.8 and 4.3. In general, we can see that as the PSL value improves (lower is better), the merit factor also improves (higher is better). The only exception is the sequence of length $m = 20$, where PSL is improved, but the merit factor decreases from 3.0 to 2.8.

5.2. Results on Legendre Sequences

Table 3 shows PSL and merit factor values of the optimized sequence starting from the initial Legendre sequence for lengths $n = 2^m - 1, m \in \{10, 11, \cdots, 20\}$. The values of PSL$/\sqrt{n}$ show an increasing trend, but all of them are lower than 0.75. Merit factor values are very close to 5.0, except for the two shortest lengths (1023 and 2047), where F is ≈ 4.8. Note that our heuristic algorithm applied the minimization criteria on the PSL. The difference (d) between the initial Legendre sequence (B_{init}^{Leg}) and the optimized sequence (B_{opt}^{Leg}), and the ratio d/n in percent are presented in the last two columns. One can see that the ratio d/n follows a constant value that is close to 1% for $m \in \{12, 13, \ldots, 20\}$.

Comparing the optimized PSL values of Rudin–Shapiro (Table 2) and the optimized Legendre (Table 3) sequences, we can see that the PSL values of optimized Rudin–Shapiro sequences are much higher. Also, the differences between the initial and optimized binary sequences are much higher for Rudin–Shapiro compared to Legendre ones, i.e., more than 20% for Rudin–Shapiro and only approximately 1% for Legendre.

Figures 1, 2, and 3 illustrate the differences between the initial Legendre sequence (B_{init}^{Leg}) and the resulting optimized sequence (B_{opt}^{Leg}) for length 1023, 2047, and 4095, respectively. Each binary sequence

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is represented as a square, the sequence starts in the first row in the upper left corner, followed by the second row, etc., the sequence ends in the last row in the lower-right corner. The elements that represent the difference between the initial and resulting sequences are marked in red. For example, there are 40 flipped components, i.e., differences, between the compared sequences (marked with red color) in Figure 3, which is $40/4095 = 0.009768 \approx 1\%$.

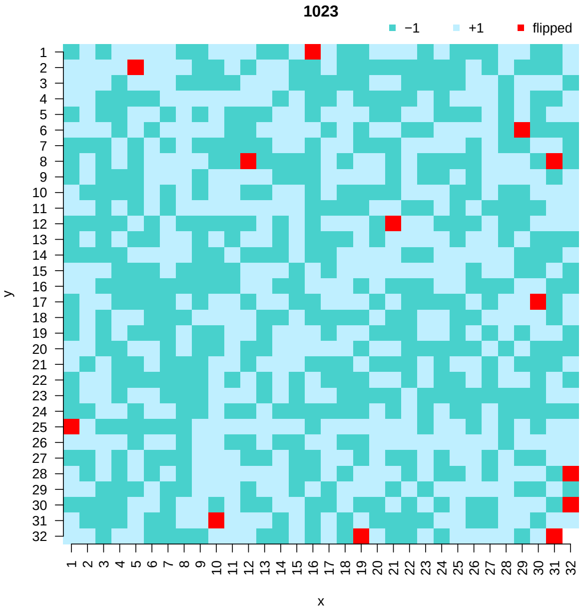

Figure 1. Binary sequence of $n = 1023$ arranged and depicted as a square of 32×32. Red color indicates the differences between the rotated Legendre sequence and the optimized sequence with PSL $= 22$, $d = 13$, and $d/n = 1.27\%$.

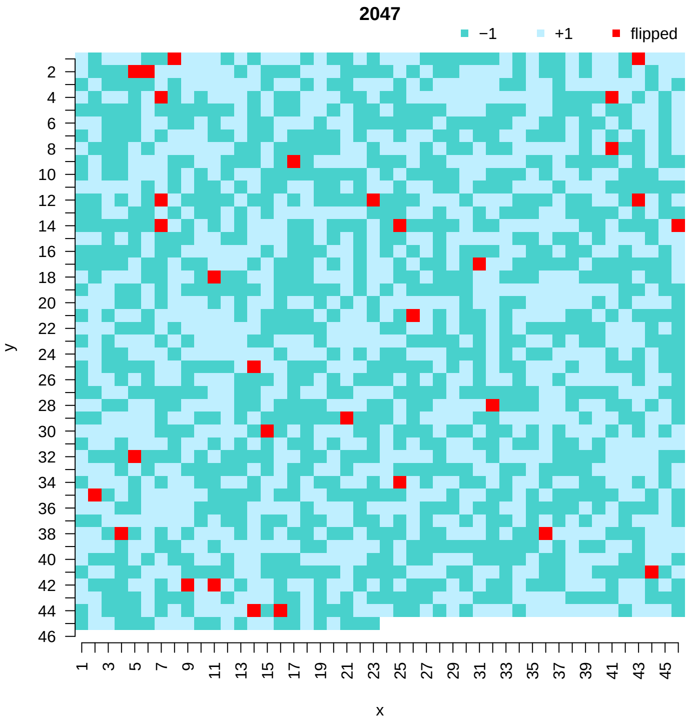

Figure 2. Binary sequence of $n = 2047$ arranged and depicted as a square of 46×46. Red color indicates the differences between the rotated Legendre sequence and the optimized sequence with PSL $= 32$, $d = 31$, and $d/n = 1.51\%$.

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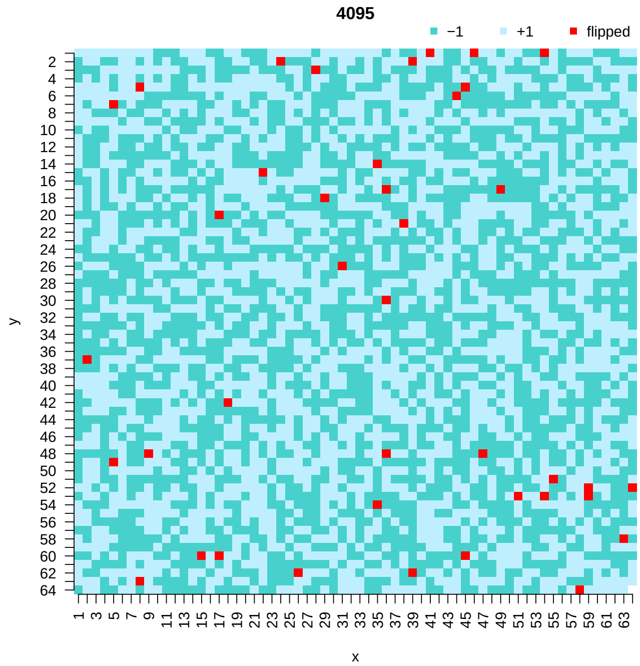

Figure 3. Binary sequence of $n = 4095$ arranged and depicted as a square of 64×64. Red color indicates the differences between the rotated Legendre sequence and the optimized sequence with PSL $= 45$, $d = 40$, and $d/n = 0.98\%$.

Figure 4 shows PSL as a function of time t [s] for 25 runs using different initial seeds for a binary sequence of the length $n = 131071$. Note that a logarithmic scale is used on the x-axis. All runs were initialized with the same Legendre sequence, differing only in the random seed. Similar improvement trends in PSL values are observed across all runs. Regardless of the seed, the algorithm has found binary sequences with a good PSL values in all runs. The PSL values were 271–274, the corresponding F values were 4.96–5.14, and d values were 0.88%–1.04%.

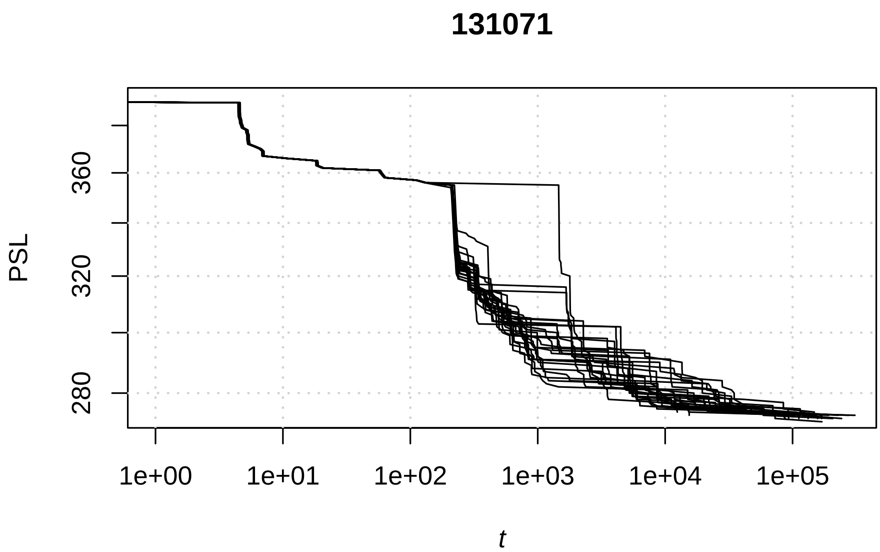

Figure 4. PSL as a function of time t [s] for 25 runs using different initial seeds for a binary sequence of the length $n = 131071$. A logarithmic scale is used on the x-axis.

Figure 5 shows how the difference d and merit factor F changed versus the number of PSL value improvements during the optimization process on the sequence of length 131071 ($2^{17}-1$). The merit factor initially starts at 6.0, then decreases to values below 4.4, followed by a more pronounced increase, eventually approaching approximately 5.0 toward the end. One can also observe the increasing and decreasing trends in the number of flipped elements d from the initial sequence. At the end, $d = 1394$ and $d/n = 0.0106 \approx 1\%$.

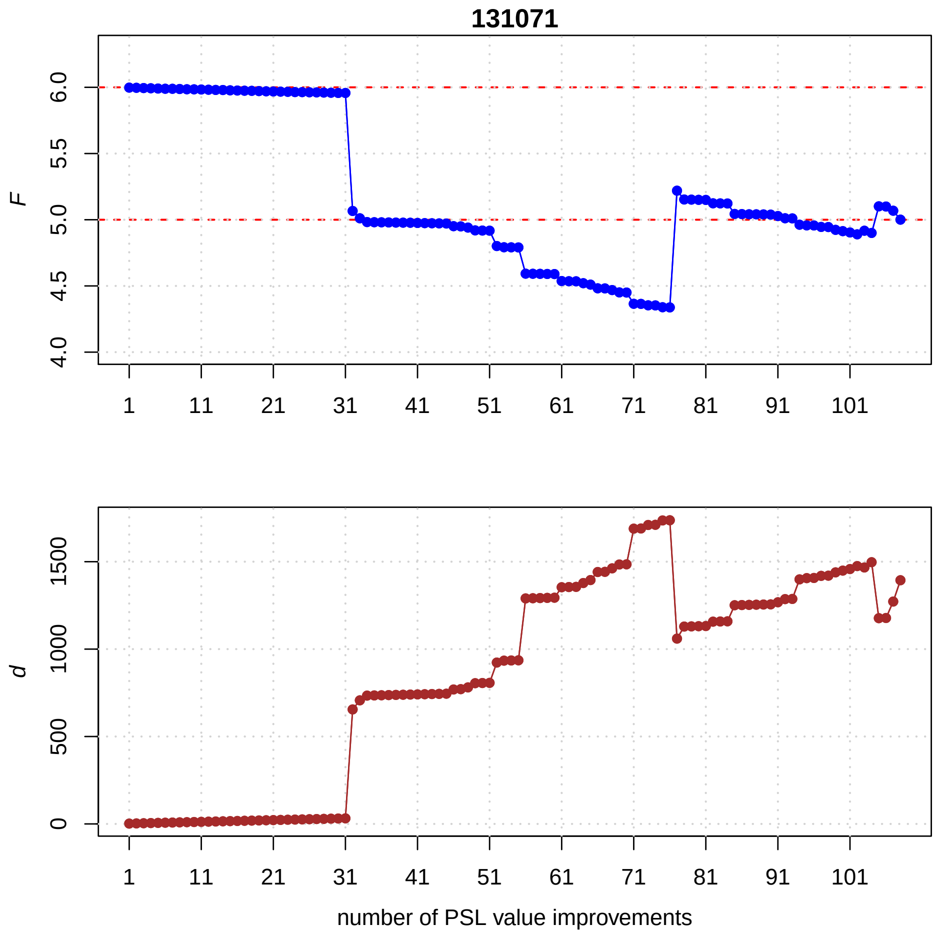

Figure 5. Difference d and merit factor F as a function of the number of PSL value improvements (x-axes) for the sequence of length $n = 131071$ ($2^{17}-1$).

The algorithm changes (i.e., flips) a single bit during the optimization process, and one or two bits during perturbation. Figure 5 shows the number of improvements in PSL on the x-axis. The algorithm can change one or more bits between two consecutive PSL improvements. More changed bits in a binary sequence means that d and F also jump, since a landscape of the LABS problem is very rigid.

Figure 6 shows the values of PSL/$\sqrt{n \log n}$, PSL/$\sqrt{n \log \log n}$, and PSL/$\sqrt{n}$, respectively. Note, log denotes the natural logarithm. The values of PSL/$\sqrt{n \log n}$ and PSL/$\sqrt{n \log \log n}$ show a non-increasing trend for $m \geq 10$, while PSL/$\sqrt{n}$ shows an increasing trend for $m \geq 10$. This suggests, based on experimental investigation up to $2^{20}-1$, that currently best-known PSL appears to grow like $O(\sqrt{n \log \log n})$.

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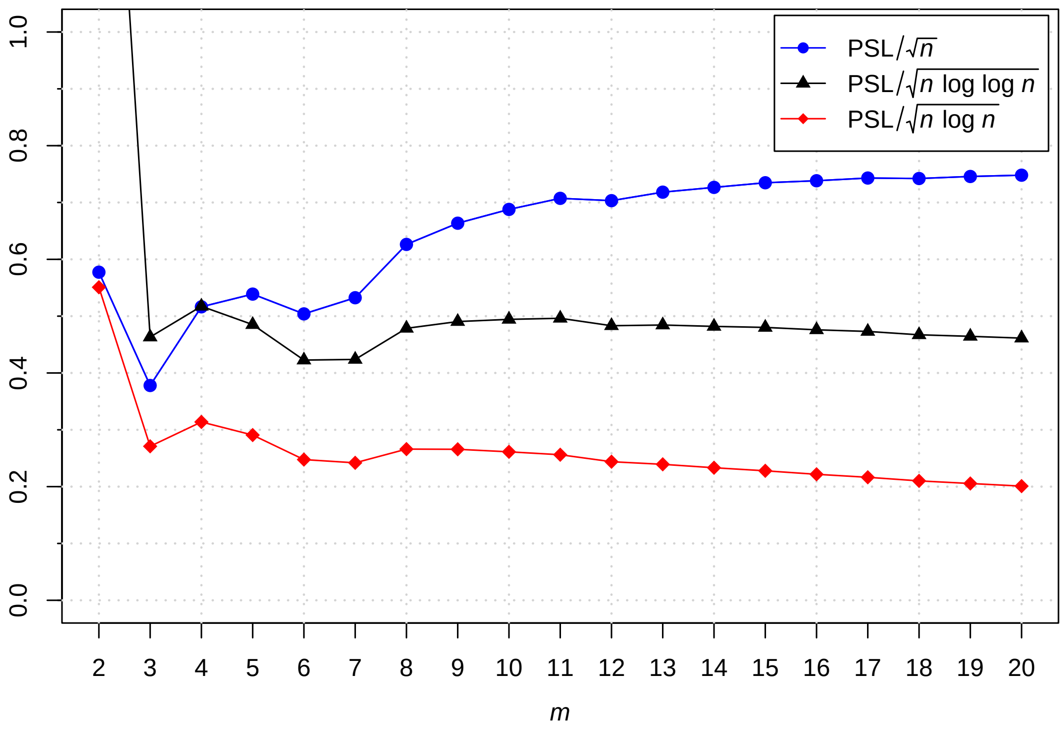

Figure 6. Comparison of the growth rate of: $\mathrm{PSL}/\sqrt{n \log n}$, $\mathrm{PSL}/\sqrt{n \log\log n}$, and $\mathrm{PSL}/\sqrt{n}$, where log denotes the natural logarithm.

Recall that our procedure generates a Legendre sequence via a rotation by $1/4$. Let us now examine the effect of alternative rotations. We will vary the rotation, r, of the initial sequence from $1/4$ to approximately $1/2$ in 50 steps and perform optimization on the resulting sequences, as shown in Equation (8).

$$r = q + \frac{qi}{s} \quad i \in \{1, 2, \cdots, s\}, \tag{8}$$

where quarter size $q = 0.25n$, and the number of steps $s = 50$. For sequence of length $n = 131071$, we have $q = 32767$ and rotations $r = 33423, 34078, \cdots, 65535$. The results obtained after optimization, starting from sequences with these 50 rotations, are presented in Figure 7. The best PSL values (approximately 275) are obtained for r around 35000, i.e., first four rotations with $i = 1, 2, \cdots, 4$. After that, the PSL values are increased, and several values are approximately 320. The initial F starts at 6.0 and decreases to 1.5, which is known for the rotation of the Legendre sequences. The F values of the optimized Legendre sequences are close to 5 for rotations where the best PSL values were obtained. After that, F is decreasing for half of the rotations, and several F values are between 3 and 3.5. The results in Figure 7 indicate that a $1/4$ rotation is the most suitable for the initial binary sequence (or a $3/4$ rotation, as the highest F values occur with $1/4$ and $3/4$ rotations).

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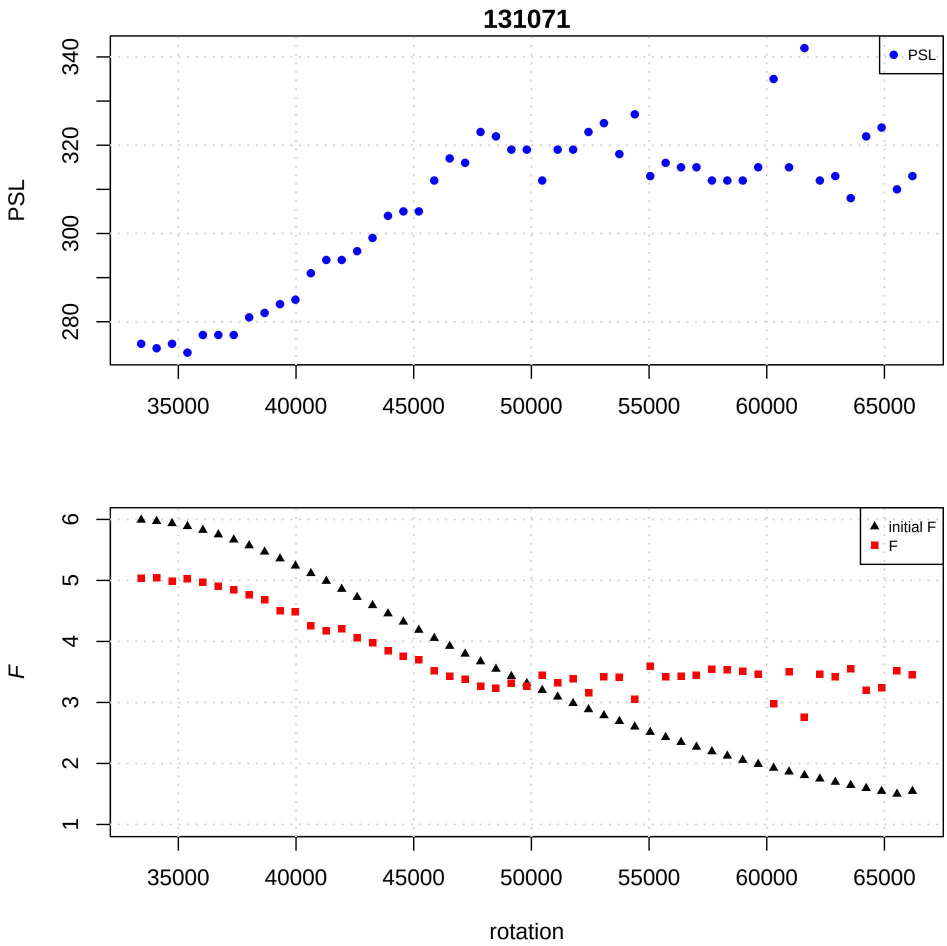

Figure 7. The PSL, F and initial F values versus different rotation of the initial Legendre sequence (B_{init}^{Leg}).

5.3. Best-Known PSL Values

In this subsection, we present the currently best-known PSL values gathered from the literature. Table 4 shows best-known PSL values for binary sequences of length $2^m - 1$ ($m = 10, 11, \cdots, 20$). In this work, three binary sequences with new best-known PSL values have been found. PSL values for $m = 18, 19,$ and 20 were improved in this work by 1, 1, and 4, respectively.

Let us briefly compare the PSL results of optimized Legendre sequences with selected results from the literature. Mow et al. [35] reported PSL $= 61$ and $F = 3.4589$ for $n = 4096$, while Zhang et al. [39] obtained PSL $= 56$ and $F = 4.8197$. Dimitrov et al. [17] gave PSL $= 77$ for $n = 8191$. PSL $= 71$ and $F = 4.00397$ for $n = 8191$, and PSL $= 50$ and $F = 4.0049$ for $n = 4095$ are reported by Bošković and Brest [37]. Brest and Bošković [5] gave PSL $= 70$ and PSL $= 48$ for $n = 8191$ and 4095, respectively. In all five works just mentioned, the algorithms were run from a random binary sequence, and from the comparison of their results, we can see slightly higher PSLs (i.e., worse) than those shown in Table 3. Also, the merit factors are usually lower than 5.0. Comparing the PSL values of optimized Legendre sequences obtained in this article with those found in the literature, we observe very good PSL values for Legendre sequences.

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Table 4. Best known PSL values.

m	$n = 2^m - 1$	Best-known PSL	Reference
10	1023	22	[19]
11	2047	32	[19]
12	4095	45	[19]
13	8191	65	[19]
14	16,383	93	[19]
15	32,767	133	[19]
16	65,535	189	[19]
17	131,071	269	[19]
18	262,143	**380**	this work
19	524,287	**540**	this work
20	1,048,575	**766**	this work

5.4. Two Conjectures

Based on the experimental results on Legendre sequences in Section 5.2, we provide the following two conjectures.

Conjecture 4. Let B_{opt}^{Leg} be an optimized Legendre sequence, then $F(B_{opt}^{Leg}) \approx 5.0$.

Conjecture 5. $d(B_{init}^{Leg}, B_{opt}^{Leg}) \approx 0.01n$.

Table 5. Merit factors of the initial Legendre sequence (B_{init}^{Leg}) and the optimized sequence (B_{opt}^{Leg}).

m	$n = 2^m - 1$	Initial F	F
10	1023	5.84503	4.78203
11	2047	5.86668	4.82918
12	4095	5.95911	5.09739
13	8191	5.99084	5.02281
14	16,383	5.97527	5.03535
15	32,767	5.99717	4.99564
16	65,535	5.99832	5.06485
17	131,071	5.99922	5.00051
18	262,143	5.99962	4.99135
19	524,287	5.99988	5.02356
20	1,048,575	5.99989	4.99350

Conjecture 4 is based on the results of optimized binary sequences (B_{opt}^{Leg}) of length up to $2^{20} - 1$, where the merit factors are approximately 5.0. Table 5 shows side-by-side two merit factors of the initial Legendre sequence (B_{init}^{Leg}) and the optimized sequence (B_{opt}^{Leg}). The (asymptotic) merit factor of the initial Legendre sequence B_{init}^{Leg} appears to be close to a positive integer, i.e., 6, although we do not yet have a good explanation as to why. Based on experimental results, a similar observation can be seen for the merit factor of the optimized Legendre sequence B_{opt}^{Leg}, i.e., it is also close to an integer value, specifically 5. Note that the difference between the two merit factors is approximately equal to 1.

Conjecture 5 shows that

$$\frac{d(B_{init}^{Leg}, B_{opt}^{Leg})}{n} \approx 0.01,$$

i.e., ratio d/n appears to be close to a constant value 0.01. Fitting the dependence of d on n with a linear model constrained to pass through the origin yields a slope of approximately $0.01035603 \approx 1.04\,\%$ with p-value $< 2.2e{-}16$ and an excellent fit ($R^2 = 0.9998$). The actual d, predicted d, and residuals are collected in Table 6. Recall that n grows exponentially, and using a log-log regression model will be

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more suitable. Figure 8 shows a fitted slope and residuals using log–log regression model. The log-log regression model also indicates an excellent fit to the linear curve with $R^2 = 0.9977$.

Table 6. Residuals and predictions for the linear model through origin ($d \sim 0 + n$).

m	$n = 2^m - 1$	Actual d	Predicted d	Residual
10	1023	13	10.59	+2.41
11	2047	31	21.20	+9.80
12	4095	40	42.41	-2.41
13	8191	86	84.83	+1.17
14	16,383	165	169.66	-4.66
15	32,767	353	339.34	+13.66
16	65,535	700	678.68	+21.32
17	131,071	1394	1357.38	+36.62
18	262,143	2855	2714.76	+140.24
19	524,287	5319	5429.53	-110.53
20	1,048,575	10,873	10,859.08	+13.92

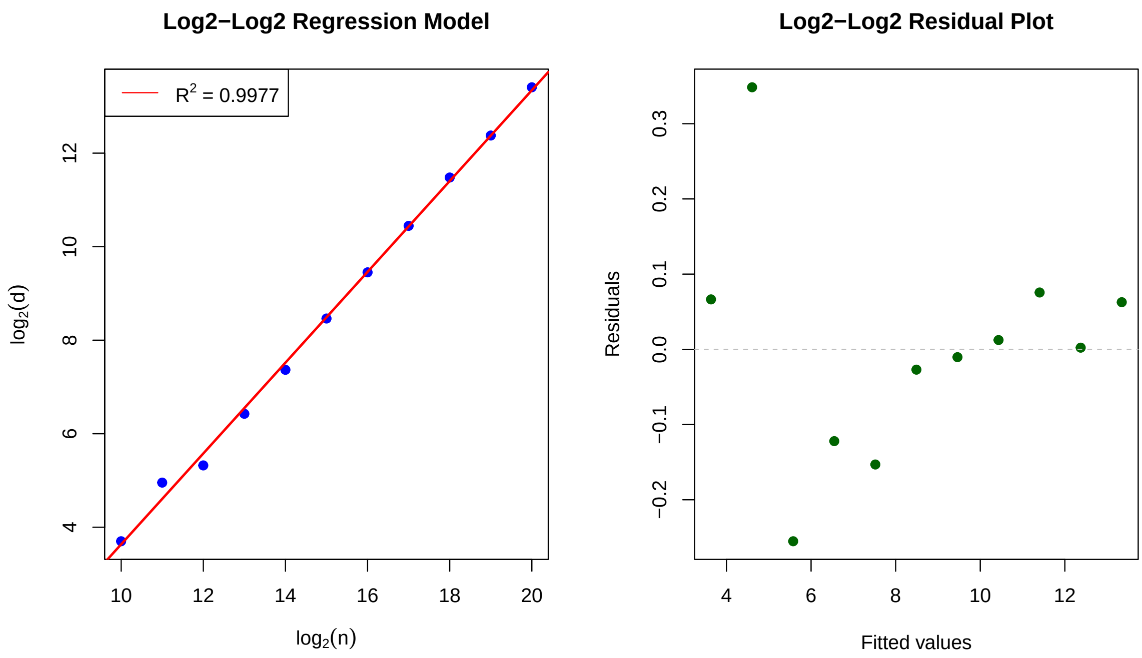

Figure 8. Fitted slope using log-log regression model and residual values.

To verify that this linear behavior is not a consequence of the chosen model, we additionally fitted a more general power-law model:

$$d = C \times n^{\alpha}. \tag{9}$$

The log-log regression yields an excellent fit with $R^2 = 0.9977$, and it is shown in Figure 9. The estimated scaling exponent is $\alpha = 0.9710$, with a 95% confidence interval of $[0.9358, 1.0063]$. Since this interval contains $\alpha = 1$, the hypothesis of linear scaling cannot be rejected at the 95% confidence level. Therefore, the observed relationship between d and n is consistent with a linear dependence. The fitted power-law coefficient is $C = 0.0148$, which is of the same order of magnitude as the directly estimated ratio d/n. Together with the linear model fit ($d/n \approx 0.0104$), these results provide statistical evidence supporting Conjecture 5, suggesting that the number of modified elements between B^{Leg}_{init} and B^{Leg}_{opt} grows approximately proportionally to the sequence length over the investigated range.

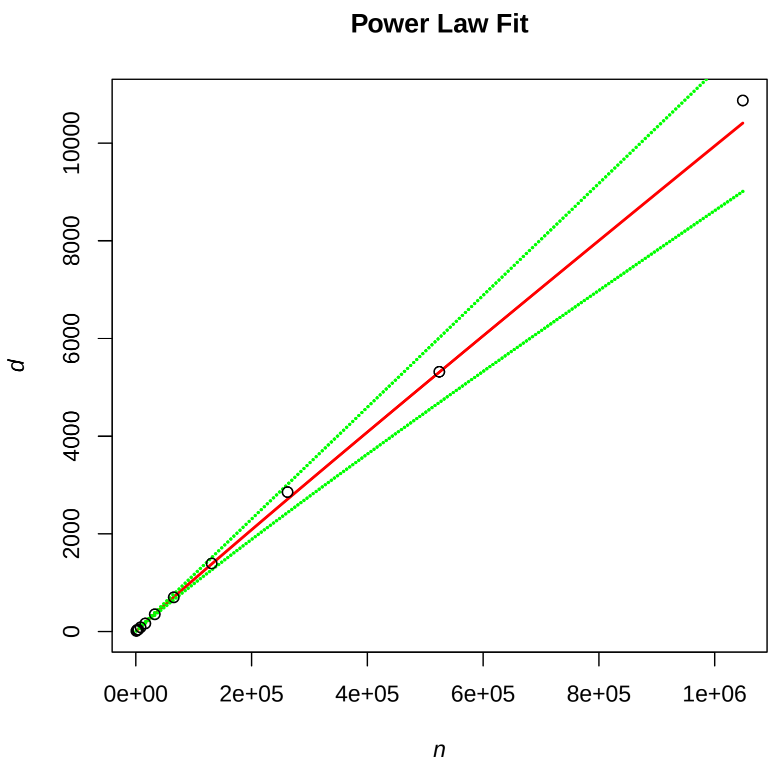

Figure 9. Fitted slope with confidence levels using power-law regression model.

6. Discussion

In the discussion, we highlight a few points. An optimization algorithm can incorporate Conjecture 5 as heuristic knowledge. Let (integer) value $q = 0.01n$. Then the number of possible choices of q elements within a binary sequence of length n is equal to $\binom{n}{q}$. Suppose $n = 1023$, then the number of possible choices is approximately 3.3×10^{23}. The value of possible choices increases sharply as n increases.

This study presents extensive computational experiments and reports the improved PSL values for some longer sequences. The observed properties of optimized Legendre sequences lead to two conjectures that may motivate further theoretical investigation.

Since the search space for finding (optimal) binary sequences is huge, the obtained PSLs in Table 3 are not necessarily optimal. Binary sequences with even better PSLs may be found in the future, but this will likely require considerable computational effort.

Conjectures 2 and 3 proposed by Dmitriev and Jedwab [15] show the excellent growth rate of m-sequences. These sequences may also be good candidates to be used as seed/initial sequences in our heuristic algorithm. This can be a great challenge for further work.

7. Conclusions

In this paper, two families of binary sequences were considered, namely Rudin–Shapiro and Legendre sequences. The observed properties from the experimental results lead to two conjectures associated with Legendre sequences:

- The obtained resulting binary sequences with the best-known peak sidelobe levels have merit factor ≈ 5.0, and
- The number of elements that differ between the resulting binary sequences and the initial Legendre sequences follows a linear dependence on the sequence length (n), namely $\approx 0.01n$.

Both conjectures require further verification and theoretical explanation.

Furthermore, the experimental results yielded three new best-known PSL values for binary sequences of lengths $n = 2^m - 1$ with $m = 18, 19$, and 20.

In this article, we employed a procedure to generate a Legendre sequence using the first prime $p \geq n$, followed by a rotation of one quarter, and truncation to length n. Other prime choices were not considered. This provides a good starting point for further work.

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