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Article

Binomial Convolution of Sequences

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Abstract

Given any two sequences of complex numbers, we establish simple relations between their binomial convolution and the binomial convolution of their individual binomial transforms. We employ these relations to derive new identities involving Fibonacci numbers, Bernoulli numbers, Catalan numbers, harmonic numbers, odd harmonic numbers, and binomial coefficients.

Keywords: binomial transform; binomial coefficient; combinatorial identity; sequence; Bernoulli number; Catalan number; harmonic number

MSC: Primary 05A10; Secondary 05A19, 11B68, 11B39

1. Introduction

Let n be a non-negative integer. Let (s_k) and (σ_k) , $k = 0, 1, 2, \dots$, be sequences of complex numbers. It is known that

$$s_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \sigma_k \iff \sigma_n = \sum_{k=0}^n (-1)^k \binom{n}{k} s_k. \quad (1)$$

Such sequences (s_k) and (σ_k) , as given in (1), will be called a binomial-transform pair of the first kind.

Consider two binomial-transform pairs of the first kind, say $\{(s_k), (\sigma_k)\}$ and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$. We will establish the following binomial convolution identity:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} s_{n-k} t_k = \sum_{k=0}^n (-1)^k \binom{n}{k} \sigma_k \tau_{n-k}.$$

Let (\bar{s}_k) and $(\bar{\sigma}_k)$, $k = 0, 1, 2, \dots$, be sequences of complex numbers. It is also known that

$$\bar{s}_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \bar{\sigma}_k \iff \bar{\sigma}_n = \sum_{k=0}^n \binom{n}{k} \bar{s}_k. \quad (2)$$

Such sequences (\bar{s}_k) and $(\bar{\sigma}_k)$, as given in (2), will be called a binomial-transform pair of the second kind. In this case we also call $(\bar{\sigma}_k)$ the binomial transform of (\bar{s}_k) .

Given two binomial-transform pairs of the second kind, say $\{(\bar{s}_k), (\bar{\sigma}_k)\}$ and $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \dots$, we will derive the following binomial convolution identity:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \bar{s}_k \bar{t}_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \bar{\sigma}_k \bar{\tau}_{n-k}.$$

Throughout this paper, we will distinguish binomial-transform pairs of the second kind with a bar on the letter representing each sequence. We will use a Greek alphabet to denote each binomial transform and the corresponding Latin alphabet to represent the original sequence.

We will also derive the following binomial convolution of one sequence and the binomial transform of the other:

$$\sum_{k=0}^n \binom{n}{k} \bar{s}_k \bar{\tau}_{n-k} = \sum_{k=0}^n \binom{n}{k} \bar{\sigma}_k \bar{t}_{n-k}.$$

Given a binomial-transform pair of the first kind, $\{(s_k), (\sigma_k)\}$ and a binomial-transform pair of the second kind, $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \dots$, we will derive the following binomial convolution identity:

$$\sum_{k=0}^n \binom{n}{k} s_k \bar{t}_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \sigma_k \bar{\tau}_{n-k}.$$

We will give a short proof of the known result [3,5]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} s_{k+m} = \sum_{k=0}^m (-1)^k \binom{m}{k} \sigma_{k+n},$$

and, based partly on it, establish the following identity:

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} s_{n-k+m} \sum_{q=0}^r (-1)^q \binom{r}{q} \tau_{k+q} \\ = \sum_{k=0}^n (-1)^k \binom{n}{k} t_{n-k+r} \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p}. \end{aligned}$$

In fact, we will prove that

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{p=0}^r (-1)^p \binom{r}{p} s_{k+p+m} = \sum_{k=0}^m (-1)^k \binom{m}{k} \sigma_{n+k+r},$$

and hence establish

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{p=0}^r (-1)^p \binom{r}{p} s_{n-k+p+m} \sum_{j=0}^u (-1)^j \binom{u}{j} t_{k+j+v} \\ = \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p+r} \sum_{j=0}^v (-1)^j \binom{v}{j} \tau_{n-k+j+u}, \end{aligned}$$

valid for non-negative integers m, n, r, u and v and any binomial-transform pairs $\{(s_k), (\sigma_k)\}$ and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, of the first kind.

We will develop several results which allow one to construct new binomial-transform pairs from existing ones. For example we will show that if (t_k) and (τ_k) , $k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind, then so are the sequences (a_k) and (α_k) , where

$$a_k = \frac{1}{k+1} \sum_{j=0}^k \tau_j \text{ and } \alpha_k = \frac{t_k}{k+1}, \quad k = 0, 1, 2, \dots$$

Remark 1. Although conversion from one kind of binomial transform to the other is, in general, very straightforward (see Remark 9), it is convenient to retain the distinction between the two kinds because binomial coefficient identities occur naturally in either of the two kinds. Thus, binomial convolution identities can be written directly without the need to convert from one kind to the other.

The binomial convolution identities developed in this paper will facilitate the discovery of an avalanche of new combinatorial identities. We now present a couple of identities, selected from our results, to whet the reader's appetite for reading on.

We derived various general identities involving binomial transform pairs, such as

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{y-k}{x} t_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{y-k}{y-x} \tau_{n-k},$$

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{m H_{k+m}}{k+m} t_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+m}{m}^{-1} (H_{k+m} - H_k) \tau_{n-k},$$

and

$$\sum_{k=0}^n \binom{n}{k} H_{k+m} \bar{t}_{n-k} = H_m \bar{t}_n - \sum_{k=1}^n (-1)^k \binom{n}{k} \binom{k+m}{m}^{-1} \frac{1}{k} \bar{t}_{n-k},$$

where H_k is a harmonic number and m is a complex number that is not a negative integer.

We discovered the following binomial convolution of harmonic and odd harmonic numbers:

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} H_k O_{n-k} = \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} \frac{2^{2(n-k)-1}}{k(n-k)} \binom{2(n-k)}{n-k}^{-1}.$$

We obtained some identities involving Bernoulli numbers and Bernoulli polynomials, including the following:

$$\sum_{k=0}^n \binom{n}{k} k B_k = \begin{cases} n B_n, & \text{if } n \text{ is an even integer or } n = 1; \\ -n B_{n-1}, & \text{if } n > 1 \text{ is an odd integer;} \end{cases}$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{y}{w}\right)^k B_{n-k}(y) B_k(w) = \frac{ny}{2w} \left(1 - \left(\frac{y}{w}\right)^{n-2}\right) B_{n-1}, \quad n \text{ an odd integer,}$$

and

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{x}{k} B_{n-k} = \sum_{k=0}^n \binom{n}{k} \binom{x+k}{k} B_{n-k},$$

where x , y and w are complex numbers.

We found the following binomial convolution of Fibonacci and Bernoulli numbers, valid for n an even integer:

$$\sum_{k=0}^n \binom{n}{k} F_k B_{n-k} = 0.$$

We obtained the following polynomial identities involving, respectively, harmonic numbers and odd harmonic numbers:

$$\sum_{k=0}^n \binom{n}{k} H_k t^k = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{k} t^k (1+t)^{n-k}$$

and

$$\sum_{k=0}^n \binom{n}{k} O_k t^k = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} 2^{2k-1} \binom{2k}{k}^{-1} \frac{1}{k} t^k (1+t)^{n-k},$$

with the special values

$$\sum_{k=0}^n \binom{n}{k} H_k = \sum_{k=1}^n (-1)^{k-1} 2^{n-k} \binom{n}{k} \frac{1}{k}$$

and

$$\sum_{k=0}^n \binom{n}{k} O_k = \sum_{k=1}^n (-1)^{k-1} 2^{n+k-1} \binom{n}{k} \binom{2k}{k}^{-1} \frac{1}{k}.$$

Let

$$C_n = \frac{1}{n+1} \binom{2n}{n}, \quad n = 0, 1, 2, \dots,$$

be the famous Catalan numbers. By employing a result of Mikić [6], we derived the following identity:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{\lfloor k/2 \rfloor} \binom{n-k}{\lfloor (n-k)/2 \rfloor} = \begin{cases} C_{n/2} \binom{n}{n/2}, & n \text{ even;} \\ 0, & n \text{ odd;} \end{cases}$$

where here and throughout this paper, $[x]$ is the greatest integer less than or equal to x .

We derived a couple of binomial coefficient identities, including the following:

$$\sum_{k=0}^n \binom{n}{k} \binom{n-k+m}{u} \binom{r}{j-k} = \sum_{k=0}^n \binom{n}{k} \binom{n-k+r}{j} \binom{m}{u-k},$$

which holds for complex numbers j, m, r , and u .

We close this section by giving a short description of each of the special numbers from which illustrations will be drawn.

Binomial coefficients are defined, for non-negative integers i and j , by

$$\binom{i}{j} = \begin{cases} \frac{i!}{j!(i-j)!}, & i \geq j; \\ 0, & i < j; \end{cases}$$

the number of distinct sets of j objects that can be chosen from i distinct objects.

Generalized binomial coefficients are defined for complex numbers r and s by

$$\binom{r}{s} = \frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)},$$

where the Gamma function, $\Gamma(z)$, is defined for $\Re(z) > 0$ by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt = \int_0^{\infty} (\log(1/t))^{z-1} dt,$$

and is extended to the rest of the complex plane, excluding the non-positive integers, by analytic continuation.

Harmonic numbers, H_n , and odd harmonic numbers, O_n , are defined for non-negative integers n by

$$H_n = \sum_{k=1}^n \frac{1}{k}, \quad O_n = \sum_{k=1}^n \frac{1}{2k-1}, \quad H_0 = 0 = O_0.$$

The Bernoulli numbers, B_k , are defined by the generating function

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}, \quad |z| < 2\pi,$$

and the Bernoulli polynomials by the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{z^k}{k!}, \quad |z| < 2\pi.$$

Clearly, $B_k = B_k(0)$.

The first few Bernoulli numbers are

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, \dots,$$

while the first few Bernoulli polynomials are

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

An explicit formula for the Bernoulli polynomials is

$$\frac{B_n(x)}{x^n} = \sum_{k=0}^n \binom{n}{k} \frac{B_k}{x^k},$$

while a recurrence formula for them is

$$B_n(x+1) = \sum_{k=0}^n \binom{n}{k} B_k(x). \quad (3)$$

The Fibonacci numbers, F_n , and the Lucas numbers, L_n , are defined, for $n \in \mathbb{Z}$, through the recurrence relations

$$F_n = F_{n-1} + F_{n-2}, (n \geq 2), \quad F_0 = 0, F_1 = 1;$$

and

$$L_n = L_{n-1} + L_{n-2}, (n \geq 2), \quad L_0 = 2, L_1 = 1;$$

with

$$F_{-n} = (-1)^{n-1} F_n, \quad L_{-n} = (-1)^n L_n. \quad (4)$$

Explicit formulas (Binet formulas) for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z},$$

where $\alpha = (1 + \sqrt{5})/2$ is the golden ratio and $\beta = (1 - \sqrt{5})/2 = -1/\alpha$.

The generalized Fibonacci sequence, (G_n) , $n = 0, 1, 2, \dots$, the so-called gibbonacci sequence, is a generalization of F_n and L_n ; having the same recurrence relation as the Fibonacci sequence but with arbitrary initial values. Thus

$$G_n = G_{n-1} + G_{n-2}, \quad (n \geq 2);$$

with

$$G_{-n} = G_{-n+2} - G_{-n+1}.$$

where G_0 and G_1 arbitrary numbers (usually integers) not both zero.

2. Identities Involving Binomial-Transform Pairs of the First Kind

Our first main result is stated in Theorem 1; we require the result stated in the next lemma for its proof.

Lemma 1. Let $\{(a_k), (\alpha_k)\}$, $k = 0, 1, 2, \dots$, be a binomial-transform pair of the first kind. Let \mathcal{L}_x be a linear operator defined by $\mathcal{L}_x(x^j) = \alpha_j$ for every complex number x and every non-negative integer j . Then $\mathcal{L}_x((1-x)^j) = \alpha_j$.

Proof. We have

$$\begin{aligned} \mathcal{L}_x((1-x)^j) &= \mathcal{L}_x\left(\sum_{k=0}^j (-1)^k \binom{j}{k} x^k\right) = \sum_{k=0}^j (-1)^k \binom{j}{k} \mathcal{L}_x(x^k) \\ &= \sum_{k=0}^j (-1)^k \binom{j}{k} \alpha_k \\ &= \alpha_j. \end{aligned}$$

□

Theorem 1. Let n be a non-negative integer. If $\{(s_k), (\sigma_k)\}$ and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, are binomial-transform pairs of the first kind, then

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} s_k t_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \sigma_k \tau_{n-k}. \quad (5)$$

Proof. Let n be a non-negative integer. Let x and y be complex numbers. Consider the following identity whose veracity is readily established by the binomial theorem:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} y^{n-k} x^k = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (1-y)^{n-k} (1-x)^k. \quad (6)$$

Let (t_j) , $j = 0, 1, 2, \dots$, be a sequence of complex numbers. Let $\mathcal{L}_x(x^j) = t_j$.

Operate on both sides of (6) with \mathcal{L}_x to obtain

$$\sum_{k=0}^n (-1)^k \binom{n}{k} y^{n-k} t_k = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (1-y)^{n-k} \tau_k,$$

where

$$\tau_k = \mathcal{L}_x((1-x)^k) = \sum_{i=0}^k (-1)^i \binom{k}{i} t_i.$$

Thus,

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} y^k t_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} (1-y)^k \tau_{n-k}. \quad (7)$$

Let (s_j) , $j = 0, 1, 2, \dots$, be a sequence of complex numbers. Let $\mathcal{L}_y(y^j) = s_j$. The action of \mathcal{L}_y on (7) produces

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} s_k t_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \sigma_k \tau_{n-k},$$

where

$$\sigma_k = \mathcal{L}_y((1-y)^k) = \sum_{i=0}^k (-1)^i \binom{k}{i} s_i,$$

and the proof is complete. \square

In the next example we illustrate Theorem 1.

Example 1. Consider the following identity [1, Equation (10.7)], where x and y are complex numbers and n is a non-negative integer:

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \binom{y-k}{x} = \binom{y-n}{y-x}. \quad (8)$$

We can choose

$$s_k = \binom{y-k}{x}, \quad \sigma_k = \binom{y-k}{y-x},$$

and use these in (5) to obtain the following identity

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{y-k}{x} t_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{y-k}{y-x} \tau_{n-k}, \quad (9)$$

which holds for every binomial-transform pair of the first kind $\{(t_k), (\tau_k)\}$.

As an immediate consequence of (9), we have the following polynomial identity in the complex variable t :

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{y-k}{x} t^{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{y-k}{y-x} (1-t)^{n-k},$$

which can also be written as

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{y+k}{x} t^k = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{y+k}{y+n-x} (1-t)^k, \quad (10)$$

which is valid for all complex numbers x, y and t .

Example 2. If we identify

$$s_k = \frac{H_{k+m}}{k+m}, \quad \sigma_k = \frac{H_{k+m} - H_k}{m} \binom{k+m}{k}^{-1},$$

from the following identity [1, Equation (9.46)]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{H_{k+m}}{k+m} = \frac{H_{n+m} - H_n}{m} \binom{n+m}{n}^{-1}, \quad m \neq 0,$$

then from (5), we obtain the following identity

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{m H_{k+m}}{k+m} t_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+m}{m}^{-1} (H_{k+m} - H_k) \tau_{n-k}, \quad (11)$$

which holds for any binomial-transform pair of the first kind, $\{(t_k), (\tau_k)\}$, for every complex number m that is not a non-positive number and every non-negative integer n .

In particular, since

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{G_{tk+r}}{L_t^k} = \frac{(-1)^r}{L_t^r} (G_0 L_{tn-r} - G_{tn-r}),$$

using

$$t_k = \frac{G_{tk+r}}{L_t^k}, \quad \tau_k = \frac{(-1)^r}{L_t^k} (G_0 L_{tk-r} - G_{tk-r}),$$

in (11) yields the following general Fibonacci-harmonic number identity:

$$\begin{aligned} & \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{m H_{k+m}}{k+m} L_t^k G_{t(n-k)+r}, \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+m}{m}^{-1} (H_{k+m} - H_k) L_t^k (G_0 L_{t(n-k)-r} - G_{t(n-k)-r}); \end{aligned} \quad (12)$$

and, in particular,

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{m}{k+m} H_{k+m} F_{n-k} = \sum_{k=0}^n (-1)^{k-1} \binom{n}{k} \binom{k+m}{m}^{-1} (H_{k+m} - H_k) F_{n-k}, \quad (13)$$

with the special value

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{H_{k+1} F_{n-k}}{k+1} = \sum_{k=0}^n (-1)^{k-1} \binom{n}{k} \frac{F_{n-k}}{(k+1)^2}. \quad (14)$$

Example 3. From (8) and [1, Equation (10.1)]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x}{k} \binom{y}{k}^{-1} = \binom{y-x}{n} \binom{y}{n}^{-1}, \quad y-n \notin \mathbb{Z}^-, y \in \mathbb{C}, \quad (15)$$

we can choose

$$s_k = \binom{y-k}{x}, \quad \sigma_k = \binom{y-k}{y-x},$$

and

$$t_k = \binom{u}{k} \binom{v}{k}^{-1}, \quad \tau_k = \binom{v-u}{k} \binom{v}{k}^{-1}.$$

Using these in (5), we obtain the following identity

$$\begin{aligned} & \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{y-k}{x} \binom{u}{n-k} \binom{v}{n-k}^{-1} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{y-k}{y-x} \binom{v-u}{n-k} \binom{v}{n-k}^{-1}, \end{aligned} \quad (16)$$

which is valid for complex numbers x, y, u and v such that $v - n$ is not a negative integer.

In particular, at $v = n$, we get

$$\sum_{k=0}^n (-1)^{n-k} \binom{y-k}{x} \binom{u}{n-k} = \sum_{k=0}^n (-1)^k \binom{y-k}{y-x} \binom{n-u}{n-k}, \quad (17)$$

with the special value

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{y-k}{y-x} = \binom{y-n}{x}, \quad (18)$$

which is the binomial transform of the first kind of (8).

Example 4. From [1, Equation (9.37)]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} H_{k+m} = -\frac{1}{n} \binom{n+m}{m}^{-1}, \quad n \neq 0, \quad (19)$$

and [1, Equation (9.51)]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} O_k = -\binom{2n}{n}^{-1} \frac{2^{2n-1}}{n}, \quad (20)$$

we can identify

$$s_k = H_{k+m}, \quad \sigma_k = \frac{\delta_{k0}(1+H_m) - 1}{k + \delta_{k0}} \binom{k+m}{m}^{-1},$$

and

$$t_k = O_k, \quad \tau_k = -\frac{1 - \delta_{k0}}{k + \delta_{k0}} \binom{2k}{k}^{-1} 2^{2k-1}.$$

Here and throughout this paper, δ_{ij} is Kronecker's delta having the value 1 when $i = j$ and zero otherwise.

Using these in (5) leads to the following harmonic number identity:

$$\begin{aligned} & \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} H_{k+m} O_{n-k} \\ &= -\frac{H_m}{n} \binom{2n}{n}^{-1} 2^{2n-1} + \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} \frac{2^{2(n-k)-1}}{k(n-k)} \binom{k+m}{m}^{-1} \binom{2(n-k)}{n-k}^{-1}, \end{aligned} \quad (21)$$

which is valid for every positive integer n and every complex number m that is not a negative integer.

In particular, we have the following alternating binomial convolution of harmonic number and odd harmonic number:

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} H_k O_{n-k} = \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} \frac{2^{2(n-k)-1}}{k(n-k)} \binom{2(n-k)}{n-k}^{-1}. \quad (22)$$

Corollary 1. Let n be a non-negative integer. If $\{(s_k), (\sigma_k)\}$ and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, are binomial-transform pairs of the first kind, then

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \sigma_k t_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} s_k \tau_{n-k}. \quad (23)$$

Corollary 2. Let n be a non-negative integer. If $\{(s_k), (\sigma_k)\}$, $k = 0, 1, 2, \dots$, is a binomial-transform pair of the first kind, then

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} s_k s_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \sigma_k \sigma_{n-k}. \quad (24)$$

Remark 2. We deduce from (24) that if n is a non-negative integer, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} s_k s_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \sigma_k \sigma_{n-k}, \quad (25)$$

with each sum being 0 for odd n .

Example 5. In view of the following identity [7]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} 2^{-k} \binom{k}{\lfloor k/2 \rfloor} = 2^{-n} C_n, \quad (26)$$

we can choose

$$s_k = 2^{-k} \binom{k}{\lfloor k/2 \rfloor}, \quad \sigma_k = 2^{-k} C_k, \quad (27)$$

and use in (24) to obtain

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{\lfloor k/2 \rfloor} \binom{n-k}{\lfloor (n-k)/2 \rfloor} = (-1)^n \sum_{k=0}^n (-1)^k \binom{n}{k} C_k C_{n-k}. \quad (28)$$

Mikić [6] has shown that if n is a non-negative integer, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} C_k C_{n-k} = \begin{cases} C_{n/2} \binom{n}{n/2}, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases} \quad (29)$$

From (28) and (29), we obtain

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{\lfloor k/2 \rfloor} \binom{n-k}{\lfloor (n-k)/2 \rfloor} = \begin{cases} C_{n/2} \binom{n}{n/2}, & n \text{ even;} \\ 0, & n \text{ odd.} \end{cases} \quad (30)$$

Example 6. Plugging s_k and σ_k from Example 2 into (25) yields, for n a non-negative integer and m a complex number that is not a negative integer,

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{m^2 H_{k+m} H_{n-k+m}}{(k+m)(n-k+m)} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+m}{k}^{-1} \binom{n-k+m}{n-k}^{-1} (H_{k+m} - H_k)(H_{n-k+m} - H_{n-k}), \end{aligned} \quad (31)$$

and, in particular,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{H_{k+1} H_{n-k+1}}{(k+1)(n-k+1)} = \sum_{k=0}^n \frac{(-1)^k}{(k+1)^2 (n-k+1)^2} \binom{n}{k}. \quad (32)$$

Example 7. From (15), with

$$s_k = \binom{x}{k} \binom{y}{k}^{-1}, \quad \sigma_k = \binom{y-x}{k} \binom{y}{k}^{-1},$$

identity (25) gives

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x}{k} \binom{x}{n-k} \binom{y}{k}^{-1} \binom{y}{n-k}^{-1} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{y-x}{k} \binom{y-x}{n-k} \binom{y}{k}^{-1} \binom{y}{n-k}^{-1}, \end{aligned} \quad (33)$$

which, upon setting $x = n$ and $y = n$, in turn, yields

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 \binom{y}{k}^{-1} \binom{y}{n-k}^{-1} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{y-n}{k} \binom{y-n}{n-k} \binom{y}{k}^{-1} \binom{y}{n-k}^{-1} \quad (34)$$

and

$$\sum_{k=0}^n (-1)^k \binom{x}{k} \binom{x}{n-k} \binom{n}{k}^{-1} = \sum_{k=0}^n (-1)^k \binom{n-x}{k} \binom{n-x}{n-k} \binom{n}{k}^{-1}. \quad (35)$$

Setting $y = -1$ in (34) gives, after some simplification,

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \binom{2n-k}{n-k},$$

the left hand side of which is known, Dixon's identity [4]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^3 = \begin{cases} (-1)^{n/2} \binom{n}{n/2} \binom{3n/2}{n}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \quad (36)$$

We therefore obtain

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n+k}{k} \binom{2n-k}{n-k} = \begin{cases} (-1)^{n/2} \binom{n}{n/2} \binom{3n/2}{n}, & \text{if } n \text{ is even;} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \quad (37)$$

Similarly, setting $x = -1$ in (35) gives

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k}^{-1} = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{(n+1)^2}{(n-k+1)(k+1)},$$

from which we get

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{(n-k+1)(k+1)} = \frac{1+(-1)^n}{(n+1)(n+2)}, \quad (38)$$

upon using [10]:

$$\sum_{k=0}^n (-1)^k \binom{n}{k}^{-1} = (1+(-1)^n) \frac{n+1}{n+2}.$$

A sequence (s_k) , $k = 0, 1, 2, \dots$, for which

$$s_n = \sum_{k=0}^n (-1)^k \binom{n}{k} s_k,$$

for every non-negative integer n will be called an invariant sequence, following Sun [8].

A sequence (s_k) , $k = 0, 1, 2, \dots$, for which

$$-s_n = \sum_{k=0}^n (-1)^k \binom{n}{k} s_k,$$

for every non-negative integer n will be called an inverse invariant sequence. Many invariant sequences are known (see e.g. Sun [8], Chen [3], Boyadzhiev [2]). Examples include (L_k) , (kF_{k-1}) , $((\binom{x/2}{k})/(\binom{x}{k}))$, $0 \neq x \notin \mathbb{Z}^+$, $(\binom{2k}{k}/2^{2k})$. Inverse invariant sequences include (F_k) and $(H_k/(k+1))$.

Corollary 3. Let (s_k) and (t_k) , $k = 0, 1, 2, \dots$, be two sequences of complex numbers. Let n be a non-negative integer. If both sequences are invariant or both are inverse invariant, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} s_k t_{n-k} = 0, \text{ if } n \text{ is odd}; \quad (39)$$

while if one of the sequences is invariant and the other is inverse invariant, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} s_k t_{n-k} = 0, \text{ if } n \text{ is even}. \quad (40)$$

Remark 3. The results stated in Corollary 3 were also obtained by Wang [12].

Example 8. If n is a non-negative odd integer and x is a non-zero complex number that $x - n$ is not a negative integer, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x/2}{k} \binom{x}{k}^{-1} L_{n-k} = 0. \quad (41)$$

3. Identities Involving Binomial-Transform Pairs of the Second Kind

Lemma 2. Let $\{(\bar{a}_k), (\bar{\alpha}_k)\}$, $k = 0, 1, 2, \dots$, be a binomial-transform pair of the second kind. Let \mathcal{M}_x be a linear operator defined by $\mathcal{M}_x(x^j) = \bar{a}_j$ for every complex number x and every non-negative integer j . Then $\mathcal{M}_x((1+x)^j) = \bar{\alpha}_j$.

Proof. We have

$$\begin{aligned} \mathcal{M}_x((1+x)^j) &= \mathcal{M}_x\left(\sum_{k=0}^j \binom{j}{k} x^k\right) = \sum_{k=0}^j \binom{j}{k} \mathcal{M}_x(x^k) \\ &= \sum_{k=0}^j \binom{j}{k} \bar{a}_k \\ &= \bar{\alpha}_j. \end{aligned}$$

□

Theorem 2. Let n be a non-negative integer. If $\{(\bar{s}_k), (\bar{\sigma}_k)\}$ and $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \dots$, are binomial-transform pairs of the second kind, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \bar{s}_k \bar{t}_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \bar{\sigma}_k \bar{\tau}_{n-k}. \quad (42)$$

Proof. Write (6) as

$$\sum_{k=0}^n (-1)^k \binom{n}{k} y^{n-k} x^k = \sum_{k=0}^n (-1)^k \binom{n}{k} (1+y)^{n-k} (1+x)^k. \quad (43)$$

Let (\bar{t}_j) , $j = 0, 1, 2, \dots$, be a sequence of complex numbers. Let $\mathcal{M}_x(x^j) = \bar{t}_j$.

Operate on both sides of (43) with \mathcal{M}_x to obtain

$$\sum_{k=0}^n (-1)^k \binom{n}{k} y^{n-k} \bar{t}_k = \sum_{k=0}^n (-1)^k \binom{n}{k} (1+y)^{n-k} \bar{\tau}_k,$$

where

$$\bar{\tau}_k = \mathcal{M}_x((1+x)^k) = \sum_{i=0}^k \binom{k}{i} \bar{t}_i.$$

Thus,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} y^k \bar{t}_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} (1+y)^k \bar{\tau}_{n-k}. \quad (44)$$

Let (\bar{s}_j) , $j = 0, 1, 2, \dots$, be a sequence of complex numbers. Let $\mathcal{M}_y(y^j) = \bar{s}_j$. The action of \mathcal{M}_y on (44) produces

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \bar{s}_k \bar{t}_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \bar{\sigma}_k \bar{\tau}_{n-k},$$

where

$$\bar{\sigma}_k = \mathcal{M}_y((1+y)^k) = \sum_{i=0}^k \binom{k}{i} \bar{s}_i,$$

and the proof is complete. □

Example 9. Consider the general recurrence relation for Bernoulli polynomials:

$$\frac{B_n(x+y)}{y^n} = \sum_{k=0}^n \binom{n}{k} \frac{B_k(x)}{y^k}. \quad (45)$$

If we choose

$$\bar{s}_k = \frac{B_k(x)}{y^k}, \quad \bar{\sigma}_k = \frac{B_k(x+y)}{y^k},$$

in (42), we obtain

$$\sum_{k=0}^n (-1)^k \binom{n}{k} y^k B_{n-k}(x) \bar{t}_k = \sum_{k=0}^n (-1)^k \binom{n}{k} y^k B_{n-k}(x+y) \bar{\tau}_k, \quad (46)$$

which is valid for complex variables x and y and any binomial transform pair $\{(\bar{s}_k), (\bar{\sigma}_k)\}$ of the second kind.

Many results can be derived from (46). To begin with, we have the following polynomial identity:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} y^k B_{n-k}(x) t^k = \sum_{k=0}^n (-1)^k \binom{n}{k} y^k B_{n-k}(x+y) (1+t)^k, \quad (47)$$

which holds for complex numbers x, y and t .

Choosing

$$\bar{t}_k = \frac{B_k(z)}{w^k}, \quad \bar{\tau}_k = \frac{B_k(z+w)}{w^k},$$

in (46) leads to

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} (y/w)^k B_{n-k}(x) B_k(z) \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} (y/w)^k B_{n-k}(x+y) B_k(z+w), \end{aligned} \quad (48)$$

giving, in particular,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} B_{n-k}(x) B_k(z) = \sum_{k=0}^n (-1)^k \binom{n}{k} B_{n-k}(x+w) B_k(z+w), \quad (49)$$

and

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (y/w)^k B_{n-k}(y) B_k(w) = \sum_{k=0}^n (-1)^k \binom{n}{k} (y/w)^k B_{n-k} B_k. \quad (50)$$

Proposition 1. *If n is a positive odd integer and y and w are complex numbers, then*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{y}{w}\right)^k B_{n-k}(y) B_k(w) = \frac{ny}{2w} \left(1 - \left(\frac{y}{w}\right)^{n-2}\right) B_{n-1}. \quad (51)$$

Proof. This is a consequence of the fact that the right-hand side of (50) can be written as

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (y/w)^{2k} B_{n-2k} B_{2k} - \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k-1} (y/w)^{2k-1} B_{n-2k+1} B_{2k-1},$$

and the fact that B_j is zero for odd j greater than unity. \square

Example 10. *Using the binomial-transform pairs*

$$\bar{s}_k = \binom{x}{k}, \quad \bar{\sigma}_k = \binom{x+k}{k}, \quad (52)$$

obtained from [1, Equation (10.10d)]:

$$\sum_{k=0}^n \binom{n}{k} \binom{x}{k} = \binom{n+x}{n} \quad (53)$$

and

$$\bar{t}_k = B_k, \quad \bar{\tau}_k = (-1)^k B_k, \quad (54)$$

found from the recurrence relation

$$\sum_{k=0}^n \binom{n}{k} B_k = (-1)^n B_n,$$

in (42) gives

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{x}{k} B_{n-k} = \sum_{k=0}^n \binom{n}{k} \binom{x+k}{k} B_{n-k}, \quad (55)$$

which holds for a non-negative integer n and every complex number x .

Proposition 2. If n is a non-negative integer and x is a complex number such that $x - n$ is not a negative integer, then

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{x}{k} (H_x - H_{x-k}) B_{n-k} = \sum_{k=0}^n \binom{n}{k} \binom{x+k}{k} (H_{x+k} - H_x) B_{n-k}. \quad (56)$$

In particular,

$$\sum_{k=0}^n \binom{n}{k} \binom{2k}{k} 2^{-2k} O_k B_{n-k} = 0, \quad n \text{ an even integer.} \quad (57)$$

Proof. Differentiate (55) with respect to x , using

$$\frac{d}{dx} \binom{x+a}{k} = \binom{x+a}{k} (H_{x+a} - H_{x+a-k}).$$

In deriving (57), we set $x = -1/2$ in (56) and used

$$H_{k-1/2} = 2O_k - 2 \ln 2, \quad (58)$$

$$\binom{-1/2}{k} = (-1)^k \binom{2k}{k} 2^{-2k}, \quad (59)$$

and

$$\binom{k-1/2}{k} = \binom{2k}{k} 2^{-2k}. \quad (60)$$

□

Corollary 4. Let n be a non-negative integer. If $\{(\bar{s}_k), (\bar{\sigma}_k)\}$ is a binomial-transform pair of the second kind, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \bar{s}_k \bar{s}_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \bar{\sigma}_k \bar{\sigma}_{n-k}. \quad (61)$$

Example 11. Using, in (61), the \bar{s}_k and $\bar{\sigma}_k$ given in (52), we have

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x}{k} \binom{x}{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x+k}{k} \binom{x+n-k}{n-k}; \quad (62)$$

which holds for every non-negative integer n and every complex number x .

Example 12. From [1, Equation (9.62)]:

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k} H_k = \binom{n+m}{m} (H_m + H_n - H_{m+n}), \quad (63)$$

we identify the following binomial-transform pair of the second kind:

$$\bar{s}_k = \binom{m}{k} H_k, \quad \bar{\sigma}_k = \binom{k+m}{m} (H_m + H_k - H_{k+m}), \quad (64)$$

which when used in (61) gives

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m}{k} \binom{m}{n-k} H_k H_{n-k} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k+m}{m} \binom{n-k+m}{m} (H_m + H_k - H_{k+m}) (H_m + H_{n-k} - H_{n-k+m}), \end{aligned} \quad (65)$$

which is valid for every non-negative integer n and every complex number m that is not a negative integer.

Theorem 3. Let n be a non-negative integer. If $\{(\bar{s}_k), (\bar{\sigma}_k)\}$ and $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \dots$, are binomial-transform pairs of the second kind, then

$$\sum_{k=0}^n \binom{n}{k} \bar{s}_k \bar{\tau}_{n-k} = \sum_{k=0}^n \binom{n}{k} \bar{\sigma}_k \bar{t}_{n-k}. \quad (66)$$

Proof. Consider the following variation on (43):

$$\sum_{k=0}^n \binom{n}{k} y^{n-k} (1+x)^k = \sum_{k=0}^n \binom{n}{k} (1+y)^{n-k} x^k, \quad (67)$$

and proceed as in the proof of Theorem 2. \square

Example 13. Using the binomial-transform pairs of the second kind

$$\bar{s}_k = \binom{m}{k} H_k, \quad \bar{\sigma}_k = \binom{k+m}{m} (H_m + H_k - H_{k+m}), \quad \text{Equation (64),}$$

and

$$\bar{t}_k = \binom{x}{k}, \quad \bar{\tau}_k = \binom{x+k}{k}, \quad \text{Equation (52) relabelled,}$$

in (66) yields

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k} \binom{x+n-k}{n-k} H_k = \sum_{k=0}^n \binom{n}{k} \binom{x}{n-k} \binom{k+m}{m} (H_m + H_k - H_{k+m}). \quad (68)$$

Example 14. Consider the following identity [11]:

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{p+k}^{-1} = \frac{m+1}{m-n+1} \binom{m-n}{p}^{-1}, \quad (69)$$

which holds for a non-negative integer n and complex numbers m and p for which $m-n-p$ is not a negative integer.

Choosing

$$\bar{s}_k = \binom{m}{p+k}^{-1}, \quad \bar{\sigma}_k = \frac{m+1}{m-k+1} \binom{m-k}{p}^{-1}, \quad (70)$$

in (66) gives

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{p+k}^{-1} \bar{\tau}_{n-k} = \sum_{k=0}^n \binom{n}{k} \frac{m+1}{m-k+1} \binom{m-k}{p}^{-1} t_{n-k},$$

which can also be written as

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{p+n-k}^{-1} \bar{\tau}_k = \sum_{k=0}^n \binom{n}{k} \frac{m+1}{m-n+k+1} \binom{m-n+k}{p}^{-1} t_k. \quad (71)$$

Example 15. Differentiating (69) with respect to m using

$$\frac{d}{dm} \binom{m}{p+k}^{-1} = \binom{m}{p+k}^{-1} (H_{m-p-k} - H_m)$$

gives

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \binom{m}{p+k}^{-1} (H_{m-p-k} - H_m) \\ &= \frac{-n}{(m-n+1)^2} \binom{m-n}{p}^{-1} - \frac{m+1}{m-n+1} \binom{m-n}{p}^{-1} (H_{m-n} - H_{m-p-n}), \end{aligned}$$

for m and p complex numbers such that $m-n-p \neq \mathbb{Z}^-$, which at $p=0$ gives

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k}^{-1} (H_{m-k} - H_m) = \frac{-n}{(m-n+1)^2}, \quad (72)$$

which yields

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k}^{-1} H_{m-k} = \frac{m+1}{m-n+1} H_m - \frac{n}{(m-n+1)^2},$$

of which the well-known identity

$$\sum_{k=0}^n H_k = (n+1)H_n - n,$$

is a particular case.

Proposition 3. If n is a non-negative integer, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2k}{k}^{-1} 2^{2k} O_k = \frac{-2n}{(2n-1)^2}. \quad (73)$$

Proof. Set $m = -1/2$ in (72) and use (58) and (59). \square

Example 16. Differentiating (69) with respect to p using

$$\frac{d}{dp} \binom{m}{p+k}^{-1} = \binom{m}{p+k}^{-1} (H_{p+k} - H_{m-p-k})$$

yields

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{p+k}^{-1} (H_{m-p-k} - H_{p+k}) = \frac{m+1}{m-n+1} \binom{m-n}{p}^{-1} (H_{m-p-n} - H_p),$$

which can also be written as

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \binom{m}{p+n-k}^{-1} (H_{m-p-n+k} - H_{p+n-k}) \\ &= \frac{m+1}{m-n+1} \binom{m-n}{p}^{-1} (H_{m-p-n} - H_p). \end{aligned} \quad (74)$$

Using

$$\bar{s}_k = \binom{m}{p+n-k}^{-1} (H_{m-p-n+k} - H_{p+n-k}), \quad \bar{o}_k = \frac{m+1}{m-k+1} \binom{m-k}{p}^{-1} (H_{m-p-k} - H_p),$$

in (66) leads to the following identity:

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \binom{m}{p+n-k}^{-1} (H_{m-p-n+k} - H_{p+n-k}) \bar{t}_k \\ &= \sum_{k=0}^n \binom{n}{k} \binom{m-n+k}{p}^{-1} \frac{m+1}{m-n+k+1} (H_{m-p-n+k} - H_p) \bar{t}_k, \end{aligned} \quad (75)$$

which holds for an arbitrary binomial-transform pair of the second kind and complex numbers m and p for which $m-n-p$ is not a negative integer.

Proposition 4. If n is a non-negative integer and $\{(\bar{t}_k), (\bar{\tau}_k)\}, k = 0, 1, 2, \dots$, is a binomial-transform pair of the second kind, then

$$\sum_{k=0}^n \binom{n+1}{k+1} H_k \bar{t}_k = \sum_{k=0}^n (H_k - H_{n-k}) \bar{\tau}_k. \quad (76)$$

In particular,

$$\sum_{k=0}^n \binom{n+1}{k+1} H_k B_k = \sum_{k=0}^n (-1)^k (H_k - H_{n-k}) B_k \quad (77)$$

and

$$\sum_{k=0}^n \binom{n+1}{k+1} H_k x^k = \sum_{k=0}^n (H_k - H_{n-k}) (1+x)^k, \quad (78)$$

for every complex number x .

Proof. Set $m = n$ and $p = 0$ in (75). \square

4. Identities Involving Mixed Binomial Transform Pairs

Theorem 4. Let n be a non-negative integer. If $\{(s_k), (\sigma_k)\}$ is a binomial transform pair of the first kind and $\{(\bar{t}_k), (\bar{\tau}_k)\}, k = 0, 1, 2, \dots$, is a binomial-transform pair of the second kind, then

$$\sum_{k=0}^n \binom{n}{k} s_k \bar{t}_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} \sigma_k \bar{\tau}_{n-k}. \quad (79)$$

Proof. Let $\mathcal{M}_x(x^k) = \bar{t}_k$ and $\mathcal{L}_y(y^k) = s_k$.

Act on the identity

$$\sum_{k=0}^n \binom{n}{k} y^{n-k} x^k = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (1-y)^{n-k} (1+x)^k,$$

with \mathcal{M}_x and \mathcal{L}_y , in succession. \square

Example 17. Use of the binomial-transform pair of the first kind $\{(F_k), (-F_k)\}$, and the binomial-transform pair of the second kind $\{(B_k), ((-1)^k B_k)\}, k = 0, 1, 2, \dots$, in (79) gives

$$\sum_{k=0}^n \binom{n}{k} F_k B_{n-k} = 0, \quad (80)$$

for n an even integer.

Similarly $\{(L_k), (-L_k)\}$ and $\{(B_k), ((-1)^k B_k)\}, k = 0, 1, 2, \dots$, in (79) yields

$$\sum_{k=0}^n \binom{n}{k} L_k B_{n-k} = 0, \quad (81)$$

for n an odd integer.

Example 18. Use of the following binomial-transform pair (see Example 4):

$$s_k = H_{k+m}, \quad \sigma_k = \frac{\delta_{k0}(1 + H_m) - 1}{k + \delta_{k0}} \binom{k+m}{m}^{-1},$$

in (79) leads to the following identity

$$\sum_{k=0}^n \binom{n}{k} H_{k+m} \bar{t}_{n-k} = H_m \bar{t}_n - \sum_{k=1}^n (-1)^k \binom{n}{k} \binom{k+m}{m}^{-1} \frac{1}{k} \bar{t}_{n-k}, \quad (82)$$

which is valid for every binomial-transform pair of the second kind $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \dots$ and every complex number m that is not a negative integer.

In particular, the following polynomial identity holds:

$$\sum_{k=0}^n \binom{n}{k} H_{k+m} t^{n-k} = (1+t)^n H_m - \sum_{k=1}^n (-1)^k \binom{n}{k} \binom{k+m}{m}^{-1} \frac{(1+t)^{n-k}}{k},$$

and can be recast as stated in proposition 5 by writing $1/t$ for t .

Proposition 5. If n is a non-negative integer, m is a complex number that is not a negative integer and t is a complex variable, then

$$\sum_{k=0}^n \binom{n}{k} H_{k+m} t^k = (1+t)^n H_m + \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \binom{k+m}{m}^{-1} \frac{1}{k} t^k (1+t)^{n-k}. \quad (83)$$

In particular, evaluation at $m = 0$ and $m = -1/2$, respectively, gives

$$\sum_{k=0}^n \binom{n}{k} H_k t^k = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \frac{1}{k} t^k (1+t)^{n-k} \quad (84)$$

and

$$\sum_{k=0}^n \binom{n}{k} O_k t^k = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} 2^{2k-1} \binom{2k}{k}^{-1} \frac{1}{k} t^k (1+t)^{n-k}, \quad (85)$$

with the special values

$$\sum_{k=0}^n \binom{n}{k} H_k = \sum_{k=1}^n (-1)^{k-1} 2^{n-k} \binom{n}{k} \frac{1}{k} \quad (86)$$

and

$$\sum_{k=0}^n \binom{n}{k} O_k = \sum_{k=1}^n (-1)^{k-1} 2^{n+k-1} \binom{n}{k} \binom{2k}{k}^{-1} \frac{1}{k}. \quad (87)$$

In deriving (85), we used (58) and (60).

Corollary 5. Let n be a non-negative integer. If $\{(s_k), (\sigma_k)\}$ is a binomial transform pair of the first kind and $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \dots$, is a binomial-transform pair of the second kind, then

$$\sum_{k=0}^n \binom{n}{k} \sigma_k \bar{t}_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} s_k \bar{\tau}_{n-k}. \quad (88)$$

5. Symmetry Properties and Generalizations

In this section we prove two symmetry properties and, based upon them, derive two generalizations of (5).

Theorem 5. Let $\{(s_k), (\sigma_k)\}$, $k = 0, 1, 2, \dots$, be a binomial-transform pair of the first kind. If m and n are non-negative integers, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} s_{k+m} = \sum_{k=0}^m (-1)^k \binom{m}{k} \sigma_{k+n}. \quad (89)$$

Proof. Let $\mathcal{L}_x(x^k) = s_k$ for every complex number x and every non-negative integer k . Operate on both sides of the following variation on the binomial theorem with \mathcal{L}_x :

$$\sum_{k=0}^n (-1)^k \binom{n}{k} x^{k+m} = \sum_{k=0}^m (-1)^k \binom{m}{k} (1-x)^{k+n}.$$

□

Remark 4. The version of (89) involving a binomial-transform pair of the second kind, namely,

$$\sum_{k=0}^n \binom{n}{k} \bar{s}_{k+m} = \sum_{k=0}^m (-1)^{k+m} \binom{m}{k} \bar{\sigma}_{k+n}, \quad (90)$$

corresponds to Chen [3, Proof of Theorem 3.1, generalized in Theorem 3.2]. See also Gould and Quaintance [5] for an alternative proof of (90).

We now present a generalization of (5).

Theorem 6. Let m , n and r be non-negative integers. If $\{(s_k), (\sigma_k)\}$ and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, are binomial-transform pairs of the first kind, then

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} s_{n-k+m} \sum_{q=0}^r (-1)^q \binom{r}{q} \tau_{k+q} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} t_{n-k+r} \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p}. \end{aligned} \quad (91)$$

Proof. From (89) it is clear that s_{k+m} and $\sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p}$ are a binomial-transform pair of the first kind. That is, if $\{(s_k), (\sigma_k)\}$, $k = 0, 1, 2, \dots$, is a binomial-transform pair of the first kind, then so is $\{(a_k), (\alpha_k)\}$, $k = 0, 1, 2, \dots$, where

$$a_k = s_{k+m} \text{ and } \alpha_k = \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p}. \quad (92)$$

Similarly, $\{(b_k), (\beta_k)\}$, $k = 0, 1, 2, \dots$ is a binomial transform pair of the first kind, where

$$b_k = t_{k+r} \text{ and } \beta_k = \sum_{q=0}^r (-1)^q \binom{r}{q} \tau_{k+q}. \quad (93)$$

Now write (5) as

$$\sum_{k=0}^n (-1)^k \binom{n}{k} a_{n-k} b_k = \sum_{k=0}^n (-1)^k \binom{n}{k} \alpha_k \beta_{n-k}, \quad (94)$$

and substitute (92) and (93) to obtain

$$\begin{aligned} & \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} s_{k+m} t_{n-k+r} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p} \sum_{q=0}^r (-1)^q \binom{r}{q} \tau_{n-k+q}, \end{aligned} \quad (95)$$

which can also be written as (91).

□

We illustrate Theorem 6 with the result stated in Proposition 6.

Proposition 6. *If j, m, r and u are complex numbers and n is a non-negative integer, then*

$$\sum_{k=0}^n \binom{n}{k} \binom{n-k+m}{u} \binom{r}{j-k} = \sum_{k=0}^n \binom{n}{k} \binom{n-k+r}{j} \binom{m}{u-k}. \quad (96)$$

Proof. Use

$$s_k = \binom{k}{u}, \quad \sigma_k = (-1)^u \delta_{ku},$$

and

$$t_k = \binom{k}{j}, \quad \tau_k = (-1)^j \delta_{kj},$$

in (91). □

Theorem 7. *Let m and r be non-negative integers. If (s_k) and (σ_k) , $k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind, so are*

$$\sum_{p=0}^r (-1)^p \binom{r}{p} s_{k+p+m} \text{ and } \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p+r}. \quad (97)$$

Thus,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{p=0}^r (-1)^p \binom{r}{p} s_{k+p+m} = \sum_{k=0}^m (-1)^k \binom{m}{k} \sigma_{n+k+r}. \quad (98)$$

Proof. Let (s_k) and (σ_k) , $k = 0, 1, 2, \dots$, be a binomial transform pair of the first kind. From (89), for an arbitrary non-negative integer m , we recognize

$$s_{k+m} \text{ and } \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p}, \quad k = 0, 1, 2, \dots, \quad (99)$$

as a binomial-transform pair of the first kind.

Let (t_k) and (τ_k) , $k = 0, 1, 2, \dots$, be a binomial transform pair of the first kind. By the same token, for an arbitrary non-negative integer r ,

$$t_{k+r} \text{ and } \sum_{p=0}^r (-1)^p \binom{r}{p} \tau_{k+p}, \quad k = 0, 1, 2, \dots, \quad (100)$$

are a binomial-transform pair of the first kind.

In (99), let

$$\tau_k = s_{k+m} \text{ and } t_k = \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p}, \quad k = 0, 1, 2, \dots \quad (101)$$

Thus, using (101), the binomial transform pair given in (100) consists of

$$t_{k+r} = \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p+r} \quad (102)$$

and

$$\sum_{p=0}^r (-1)^p \binom{r}{p} \tau_{k+p} = \sum_{p=0}^r (-1)^p \binom{r}{p} s_{k+p+m}. \quad (103)$$

The proof is complete. \square

In the next theorem, we present another generalization of (5).

Theorem 8. Let m, n, r, u and v be non-negative integers. If $\{(s_k), (\sigma_k)\}$ and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, are binomial-transform pairs of the first kind, then

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{p=0}^r (-1)^p \binom{r}{p} s_{n-k+p+m} \sum_{j=0}^u (-1)^j \binom{u}{j} t_{k+j+v} \\ &= \sum_{k=0}^n (-1)^k \binom{n}{k} \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p+r} \sum_{j=0}^v (-1)^j \binom{v}{j} \tau_{n-k+j+u}. \end{aligned} \quad (104)$$

Proof. In (94) substitute

$$a_k = \sum_{p=0}^r (-1)^p \binom{r}{p} s_{k+p+m}, \quad \alpha_k = \sum_{p=0}^m (-1)^p \binom{m}{p} \sigma_{k+p+r},$$

and

$$b_k = \sum_{j=0}^u (-1)^j \binom{u}{j} t_{k+j+v}, \quad \beta_k = \sum_{j=0}^v (-1)^j \binom{v}{j} \tau_{k+j+u}.$$

\square

6. Relations Between Binomial Transforms

We derive various relations between binomial transform pairs of both kinds. We develop relations which allow one to construct new binomial-transform pairs from known ones.

Proposition 7. Let n be a non-negative integer. If $\{(\bar{t}_k), (\bar{\tau}_k)\}$ is a binomial-transform pair of the second kind, then

$$\sum_{k=0}^n \bar{\tau}_k = \sum_{k=0}^n \binom{n+1}{k+1} \bar{t}_k. \quad (105)$$

In particular, if x is a complex variable, then

$$\sum_{k=0}^n (1+x)^k = \sum_{k=0}^n \binom{n+1}{k+1} x^k. \quad (106)$$

Proof. Set $m = n$ and $p = 0$ in (71). \square

Remark 5. Shifting the summation index, identity (105) can also be written as

$$\sum_{k=1}^n \bar{\tau}_{k-1} = \sum_{k=1}^n \binom{n}{k} \bar{t}_{k-1}; \quad (107)$$

which for a binomial transform pair of the first kind $\{(t_k), (\tau_k)\}, k = 0, 1, 2, \dots$, also means

$$\sum_{k=1}^n \tau_{k-1} = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} t_{k-1}, \quad (108)$$

in view of Remark 9.

Theorem 9. If (t_k) and $(\tau_k), k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind, then so are

$$-\sum_{j=1}^k \tau_{j-1} \text{ and } (1 - \delta_{k0})t_{k-1+\delta_{k0}}.$$

Proof. An immediate consequence of (108). \square

Example 19. Sequences $(L_{k+1} - 1)$ and $(-L_{k-1}), k = 1, 2, \dots$, are a binomial-transform pair of the first kind since, if we take $t_k = L_k = \tau_k$, we have

$$\sum_{j=1}^k L_{j-1} = L_{k+1} - 1.$$

Similarly, $(F_{k+1} - 1)$ and $(F_{k-1}), k = 1, 2, \dots$, are a binomial-transform pair of the first kind.

Theorem 10. If (t_k) and $(\tau_k), k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind, then so are

$$\sum_{j=1}^{k-1+\delta_{k0}} t_{j-1} \text{ and } \sum_{j=1}^{k-1+\delta_{k0}} \tau_{j-1};$$

so that

$$\sum_{k=1}^n (-1)^k \binom{n}{k} \sum_{j=1}^{k-1} t_{j-1} = \sum_{k=1}^{n-1} \tau_{k-1}, \quad n \in \mathbb{Z}^+. \quad (109)$$

In particular, if $(t_k), k = 0, 1, 2, \dots$, is an invariant sequence, so is the sequence $(\sum_{j=1}^{k-1+\delta_{k0}} t_{j-1})$.

Proof. Let $\{(a_k), (\alpha_k)\}, k = 0, 1, 2, \dots$, be a binomial-transform pair of the first kind, where

$$a_k = -\sum_{j=1}^k \tau_{j-1}, \quad \alpha_k = (1 - \delta_{k0})t_{k-1+\delta_{k0}}, \quad (110)$$

and $\{(t_k), (\tau_k)\}, k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind.

By Theorem 9, $-\sum_{j=1}^k \alpha_{j-1}$ and $(1 - \delta_{k0})a_{k-1+\delta_{k0}}$ are a binomial-transform pair of the first kind.

Using (110), we have

$$\begin{aligned} -\sum_{j=1}^k \alpha_{j-1} &= -\sum_{j=1}^k (1 - \delta_{j1})t_{j-2+\delta_{j1}} \\ &= -\sum_{j=1}^k t_{j-2+\delta_{j1}} + \sum_{j=1}^k \delta_{j1}t_{j-2+\delta_{j1}} \\ &= -t_0 - \sum_{j=2}^k t_{j-2} + t_0 \\ &= -\sum_{j=2}^k t_{j-2} = -\sum_{j=1}^{k-1} t_{j-1}. \end{aligned}$$

Since the left side is an empty sum for $k = 0$, in order to preserve equality, we therefore write

$$-\sum_{j=1}^k \alpha_{j-1} = -\sum_{j=1}^{k-1+\delta_{k0}} t_{j-1}.$$

From (110), we also have

$$(1 - \delta_{k0})a_{k-1+\delta_{k0}} = -(1 - \delta_{k0}) \sum_{j=1}^{k-1+\delta_{k0}} \tau_{j-1} = -\sum_{j=1}^{k-1+\delta_{k0}} \tau_{j-1}.$$

□

Theorem 11. If (t_k) and (τ_k) , $k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind, then so are

$$\frac{1}{(k+1)(k+2)} \sum_{j=0}^k t_j \text{ and } \frac{1}{(k+1)(k+2)} \sum_{j=0}^k \tau_j. \quad (111)$$

In particular, if (t_k) , $k = 0, 1, 2, \dots$, is an invariant sequence, so is the sequence

$$\left(\frac{1}{(k+1)(k+2)} \sum_{j=0}^k t_j \right).$$

Proof. By shifting the index, identity (109) can be written as

$$\sum_{k=0}^n (-1)^k \binom{n+2}{k+2} \sum_{j=0}^k t_j = \sum_{k=0}^n \tau_k,$$

from which (111) follows since

$$\binom{n+2}{k+2} = \frac{(n+2)(n+1)}{(k+2)(k+1)} \binom{n}{k}.$$

□

Example 20. The sequence

$$\left(\frac{L_{k+2} - 1}{(k+1)(k+2)} \right), \quad k = 0, 1, 2, \dots$$

is an invariant sequence, since (L_k) is an invariant sequence and

$$\sum_{k=0}^n L_k = L_{n+2} - 1.$$

Theorem 12. If (t_k) and (τ_k) , $k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind, then so are

$$\frac{1}{k+1} \sum_{j=0}^k \tau_j \text{ and } \frac{t_k}{k+1}. \quad (112)$$

Proof. Identity (105) written for a binomial-transform pair of the first kind is

$$\sum_{k=0}^n \tau_k = \sum_{k=0}^n (-1)^k \binom{n+1}{k+1} t_k,$$

which can also be written as

$$\frac{1}{n+1} \sum_{k=0}^n \tau_k = \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{t_k}{k+1}$$

and hence the theorem. \square

Theorem 13. If (t_k) and (τ_k) , $k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind, then so are

$$-\frac{1}{k+1} \sum_{j=1}^k \tau_{j-1} \text{ and } \frac{1}{k+1} \sum_{j=1}^k t_{j-1}. \quad (113)$$

In particular, if (t_k) , $k = 0, 1, 2, \dots$, is an invariant sequence, then the sequence

$$\left(\frac{1}{k+1} \sum_{j=1}^k t_{j-1} \right), \quad k = 0, 1, 2, \dots,$$

is an inverse invariant sequence.

Proof. Let $\{(a_k), (\alpha_k)\}$, $k = 0, 1, 2, \dots$, be a binomial-transform pair of the first kind, where

$$a_k = -\sum_{j=1}^k \tau_{j-1}, \quad \alpha_k = (1 - \delta_{k0})t_{k-1+\delta_{k0}},$$

and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind.

Then

$$\frac{1}{k+1} \sum_{j=0}^k \alpha_j = \frac{1}{k+1} \sum_{j=0}^k (1 - \delta_{j0})t_{j-1+\delta_{j0}} = \frac{1}{k+1} \sum_{j=1}^k t_{j-1}$$

and

$$\frac{a_k}{k+1} = -\frac{1}{k+1} \sum_{j=1}^k \tau_{j-1}$$

are a binomial-transform pair of the first kind, by Theorem 12. \square

Theorem 14. If (t_k) and (τ_k) , $k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind, then so are

$$\frac{1}{k+1} \sum_{j=0}^k \frac{t_j}{j+1} \text{ and } \frac{1}{(k+1)^2} \sum_{j=0}^k \tau_j. \quad (114)$$

Proof. Let $\{(a_k), (\alpha_k)\}$, $k = 0, 1, 2, \dots$, be a binomial-transform pair of the first kind, where

$$a_k = \frac{1}{k+1} \sum_{j=0}^k \tau_j, \quad \alpha_k = \frac{t_k}{k+1},$$

and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, are a binomial-transform pair of the first kind.

Then

$$\frac{1}{k+1} \sum_{j=0}^k \alpha_j = \frac{1}{k+1} \sum_{j=0}^k \frac{t_j}{j+1}$$

and

$$\frac{a_k}{k+1} = \frac{1}{(k+1)^2} \sum_{j=0}^k \tau_j$$

are a binomial-transform pair of the first kind, by Theorem 12. \square

By considering the binomial-transform pair of the second kind $\{(1), (2^k)\}$, $k = 0, 1, 2, \dots$ and making use of Theorem 3, we obtain the next result expressing the binomial transform of a binomial transform.

Theorem 15. *Let n be a non-negative integer. If $\{(\bar{s}_k), (\bar{\sigma}_k)\}$ is a binomial-transform pair of the second kind, then*

$$\sum_{k=0}^n \binom{n}{k} \bar{\sigma}_k = \sum_{k=0}^n \binom{n}{k} 2^{n-k} \bar{s}_k. \quad (115)$$

Remark 6. *Identity (115) corresponds to Example 4 of Boyadzhiev [1].*

Example 21. *With [1, Equation (10.10a)]*

$$\sum_{k=0}^n \binom{n}{k} \binom{x}{k+z} = \binom{n+x}{n+z},$$

in mind, choosing

$$\bar{s}_k = \binom{x}{k+z}, \quad \bar{\sigma}_k = \binom{k+x}{k+z}, \quad (116)$$

in (115) yields

$$\sum_{k=0}^n \binom{n}{k} \binom{k+x}{k+z} = \sum_{k=0}^n \binom{n}{k} \binom{x}{k+z} 2^{n-k}. \quad (117)$$

for n a non-negative integer and x and z complex numbers.

Proposition 8. *If j and n are non-negative integers, then*

$$\sum_{k=0}^n (-1)^{n-k} \binom{n-j}{k-j} t_{n-k} = \tau_{n-j}. \quad (118)$$

Proof. Consider the identity

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{j} = (-1)^j \delta_{nj}, \quad (119)$$

and use $s_k = \binom{k}{j}$ and $\sigma_k = (-1)^j \delta_{kj}$ in (5). \square

Theorem 16. *Let $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, be a binomial-transform pair of the first kind. Then kt_k and $k(\tau_k - \tau_{k-1})$ are a binomial-transform pair of the first kind for $k = 1, 2, \dots$*

Proof. Set $j = 1$ in (118) and use the identity

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$$

to obtain

$$n \sum_{k=0}^n (-1)^k \binom{n}{k} t_k - \sum_{k=0}^n (-1)^k k \binom{n}{k} t_k = n \tau_{n-1};$$

and hence

$$\sum_{k=0}^n (-1)^k \binom{n}{k} kt_k = n(\tau_n - \tau_{n-1}). \quad (120)$$

\square

Example 22. If on the basis of the recurrence relation of Bernoulli numbers, namely,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} (-1)^k B_k = (-1)^n B_n,$$

and use $t_k = (-1)^k B_k = \tau_k$ in (120), we obtain, for n a non-negative integer,

$$\sum_{k=0}^n \binom{n}{k} k B_k = (-1)^n n (B_n + B_{n-1}),$$

that is

$$\sum_{k=0}^n \binom{n}{k} k B_k = \begin{cases} n B_n, & \text{if } n \text{ is even;} \\ -n B_{n-1}, & \text{if } n > 1 \text{ is odd;} \\ -1/2, & \text{if } n = 1. \end{cases} \quad (121)$$

Corollary 6. Let $\{(s_k), (\sigma_k)\}$, k an integer, be a binomial-transform pair of the first kind. Let $S_0(k) = \sigma_k$ and $S_m(k) = k(S_{m-1}(k) - S_{m-1}(k-1))$ for every positive integer m . Then $S_m(k)$ and $k^m s_k$ are a binomial-transform pair of the first kind; that is

$$\sum_{k=0}^n (-1)^k \binom{n}{k} k^m s_k = S_m(n), \quad m = 0, 1, 2, \dots \quad (122)$$

In particular,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} k^2 s_k = n^2 (\sigma_n - \sigma_{n-1}) - n(n-1) (\sigma_{n-1} - \sigma_{n-2}), \quad (123)$$

and

$$\begin{aligned} \sum_{k=0}^n (-1)^k \binom{n}{k} k^3 s_k &= n^3 (\sigma_n - \sigma_{n-1}) - n(n-1)(2n-1) (\sigma_{n-1} - \sigma_{n-2}) \\ &\quad + n(n-1)(n-2) (\sigma_{n-2} - \sigma_{n-3}). \end{aligned} \quad (124)$$

7. Various Extensions

The result stated in the next theorem extends (115).

Theorem 17. Let j and n be non-negative integers such that $j \leq n$. Let $\{(s_k), (\sigma_k)\}$, $k = 0, 1, 2, \dots$, be a binomial-transform pair of the first kind. Then

$$\sum_{k=0}^n (-1)^k \binom{n-j}{k} 2^{n-k} s_k = 2^j \sum_{k=0}^n \binom{n-j}{k} \sigma_k. \quad (125)$$

Proof. By writing (118) in the equivalent form

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{k}{j} 2^k = (-1)^n \binom{n}{j} 2^j,$$

we see that $\binom{k}{j} 2^k$ and $(-1)^k \binom{k}{j} 2^j$ are a binomial pair of the first kind. Using

$$t_k = \binom{k}{j} 2^k \text{ and } \tau_k = (-1)^k \binom{k}{j} 2^j, \quad (126)$$

in (5) gives (125). \square

Theorem 18. Let j and n be non-negative integers. Let $\{(s_k), (\sigma_k)\}, k = 0, 1, 2, \dots$, be a binomial-transform pair of the first kind. Then

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{n-k}{j} 2^k s_k = \sum_{k=0}^n (-1)^{j-k} \binom{n}{k} \binom{n-k}{j} 2^k \sigma_k. \quad (127)$$

Proof. Consider the binomial-transform pair of the first kind version of (115), namely,

$$\sum_{k=0}^n \binom{n}{k} \sigma_k = \sum_{k=0}^n (-1)^k \binom{n}{k} 2^{n-k} s_k. \quad (128)$$

On account of (119), use

$$s_k = \binom{k}{j} \text{ and } \sigma_k = (-1)^j \delta_{kj},$$

in (128) to obtain

$$\sum_{k=0}^n (-1)^k \binom{n}{k} 2^{-k} \binom{k}{j} = (-1)^j 2^{-n} \binom{n}{j},$$

from which we identify the binomial-transform pair of the first kind:

$$t_k = 2^{-k} \binom{k}{j} \text{ and } \tau_k = (-1)^j 2^{-k} \binom{k}{j},$$

which, when plugged into (5) gives (127). \square

Corollary 7. Let j and n be non-negative integers having different parity. If (s_k) is an invariant sequence, that is $\sigma_k = s_k, k = 0, 1, 2, \dots$, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n-k}{j} 2^k s_k = 0. \quad (129)$$

In particular, if n is an odd integer and (s_k) is an invariant sequence, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} 2^k s_k = 0. \quad (130)$$

Corollary 8. Let j and n be non-negative integers having the same parity. If (s_k) is an inverse invariant sequence, that is $\sigma_k = -s_k, k = 0, 1, 2, \dots$, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n-k}{j} 2^k s_k = 0. \quad (131)$$

In particular, if n is an even integer and (s_k) is an inverse invariant sequence, then

$$\sum_{k=0}^n (-1)^k \binom{n}{k} 2^k s_k = 0. \quad (132)$$

Theorem 19. Let n be a non-negative integer. If $\{(\bar{s}_k), (\bar{\sigma}_k)\}$ and $\{(\bar{t}_k), (\bar{\tau}_k)\}, k = 0, 1, 2, \dots$, are binomial-transform pairs of the second kind, then

$$\sum_{k=0}^n \binom{n}{k} 2^{-k} \bar{s}_k \bar{\tau}_{n-k} = \sum_{k=0}^n \binom{n}{k} 2^{-k} \bar{t}_{n-k} \sum_{j=0}^k \binom{k}{j} \bar{\sigma}_j. \quad (133)$$

Proof. From (115), we can identify the following binomial-transform pair of the second kind:

$$\bar{a}_k = 2^{-k} \bar{s}_k \text{ and } \bar{a}_k = 2^{-k} \sum_{j=0}^k \binom{k}{j} \bar{\sigma}_j.$$

Replace \bar{s}_k with \bar{a}_k and $\bar{\sigma}_k$ with \bar{a}_k in (66). This completes the proof. \square

Theorem 20. Let n be a non-negative integer. If $\{(s_k), (\sigma_k)\}$ and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, are binomial-transform pairs of the first kind, then

$$\sum_{k=1}^n (-1)^{n-k-1} \binom{n}{k} s_{k-1} t_{n-k} = \sum_{k=1}^n (-1)^k \binom{n}{k} \tau_{n-k} \sum_{j=1}^k \sigma_{j-1}. \quad (134)$$

Proof. In view of the result stated in Theorem 9, let

$$a_k = (1 - \delta_{k0}) s_{k-1+\delta_{k0}} \text{ and } \alpha_k = - \sum_{j=1}^k \sigma_{j-1}, \quad k = 0, 1, 2, \dots$$

Now, replace s_k with a_k and σ_k with α_k in (5). \square

We bring this section to a close by giving another generalization of (5) based on the result from Corollary 6.

Theorem 21. Let n be a non-negative integer. Let $\{(s_k), (\sigma_k)\}$ and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, are binomial-transform pairs of the first kind. Let $S_0(k) = \sigma_k$ and $S_m(k) = k(S_{m-1}(k) - S_{m-1}(k-1))$ for every positive integer m . Then

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} k^m s_k t_{n-k} = \sum_{k=0}^n (-1)^k \binom{n}{k} S_m(k) \tau_{n-k}. \quad (135)$$

In particular,

$$\sum_{k=1}^n (-1)^{n-k} \binom{n}{k} k s_k t_{n-k} = \sum_{k=1}^n (-1)^k \binom{n}{k} k (\sigma_k - \sigma_{k-1}) \tau_{n-k}, \quad (136)$$

with the special value

$$\sum_{k=1}^n (-1)^{n-k-1} \binom{n}{k} t_{n-k} = \sum_{k=1}^n (-1)^k \binom{n}{k} \tau_{n-k}, \quad (137)$$

obtained with $s_k = -1/k$ and $\sigma_k = H_k$.

Proof. Replace s_k with $k^m s_k$ and σ_k with $S_m(k)$ in (5). \square

8. Polynomial Identities Involving a Binomial Transform Pair

We have already encountered the identities stated in the next theorem; they are variations on (7) and (44).

Theorem 22. If n is a non-negative integer and y is a complex variable, then

$$\sum_{k=0}^n \binom{n}{k} s_{n-k} y^k = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \sigma_{n-k} (1+y)^k, \quad (138)$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \bar{s}_{n-k} y^k = \sum_{k=0}^n (-1)^k \binom{n}{k} \bar{\sigma}_{n-k} (1+y)^k. \quad (139)$$

The implication of Theorem 22 is that every binomial-transform pair has associated with it a polynomial identity.

Remark 7. Identity (139) is equivalent to Boyadzhiev's [1, Corollary 1].

9. Binomial Transform Identities Associated with Polynomial Identities of a Certain Type

Polynomial identities of the following form abound in the literature:

$$\sum_{k=0}^n f(k)t^{p(k)} = \sum_{k=0}^m g(k)(1-t)^{q(k)}; \quad (140)$$

where m and n are non-negative integers, t is a complex variable, $p(k)$ and $q(k)$ are sequences of integers, and $f(k)$ and $g(k)$ are sequences of complex numbers.

The binomial identities stated in Theorems 23 and 24 are readily derived using the operators \mathcal{L}_y and \mathcal{M}_y .

Theorem 23. Let a polynomial identity have the form stated in (140). If $\{(s_k), (\sigma_k)\}$ is a binomial-transform pair of the first kind, then

$$\sum_{k=0}^n f(k)s_{p(k)} = \sum_{k=0}^m g(k)\sigma_{q(k)}. \quad (141)$$

Theorem 24. Let a polynomial identity have the form stated in (140). If $\{(\bar{s}_k), (\bar{\sigma}_k)\}$ is a binomial-transform pair of the second kind, then

$$\sum_{k=0}^n (-1)^{p(k)} f(k)\bar{s}_{p(k)} = \sum_{k=0}^m g(k)\bar{\sigma}_{q(k)}. \quad (142)$$

Lemma 3. Sun [9, Lemma 3.1] If m , n and r are non-negative integers and t is a complex variable, then

$$\sum_{k=0}^n (-1)^{k-r} \binom{n}{k} \binom{k+m}{r} t^{k+m-r} = \sum_{k=0}^m (-1)^k \binom{m}{k} \binom{k+n}{r} (1-t)^{n+k-r}. \quad (143)$$

Example 23. In (143) we can identify

$$f(k) = (-1)^{k-r} \binom{n}{k} \binom{k+m}{r}, \quad g(k) = \binom{m}{k} \binom{k+n}{r}, \quad (144)$$

and

$$p(k) = k + m - r \text{ and } q(k) = n + k - r. \quad (145)$$

Using these in Theorem 23 gives

$$\sum_{k=0}^n (-1)^{k-r} \binom{n}{k} \binom{k+m}{r} s_{k+m-r} = \sum_{k=0}^m (-1)^k \binom{m}{k} \binom{k+n}{r} \sigma_{n+k-r}, \quad (146)$$

for every binomial-transform pair $\{(s_k), (\sigma_k)\}$ of the first kind.

Remark 8. Identity (146) is Chen's Theorem 3.2 [3].

10. Conversion Between Binomial Transform Identities

If a binomial transform identity involves a binomial-transform pair of the first kind, it is straightforward to convert it to an identity involving a binomial-transform pair of the second kind; and vice versa. An identity involving $\{s(k), \sigma(k)\}$, $k = 0, 1, 2, \dots$, can be converted to that involving $\{\bar{s}(k), \bar{\sigma}(k)\}$ by choosing $s(k) = x^k$ and $\sigma(k) = (1-x)^k$, replacing x with $-x$ and then operating on the resulting

identity with \mathcal{M}_x . Similarly, an identity involving $\{\bar{s}(k), \bar{\sigma}(k)\}$ can be converted to that involving $\{s(k), \sigma(k)\}$ by choosing $\bar{s}(k) = x^k$ and $\bar{\sigma}(k) = (1+x)^k$, replacing x with $-x$ and then operating on the resulting identity with \mathcal{L}_x .

Example 24. Let $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \dots$, be a binomial-transform pair of the second kind. Chen's main result [3, Theorem 3.1] (after Gould and Quaintance's simplification [5, Equation (10)]) for non-negative integers m, n and s is

$$\begin{aligned} & \sum_{k=0}^m \binom{m}{k} \binom{n+k+s}{s}^{-1} \bar{t}_{n+k+s} \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{m+k+s}{s}^{-1} \bar{\tau}_{m+k+s} \\ &+ \sum_{k=0}^{s-1} \frac{(-1)^{n+s-k} s}{m+n+s-k} \binom{s-1}{k} \binom{m+n+s-k-1}{n}^{-1} \bar{\tau}_k. \end{aligned} \quad (147)$$

In order to convert (147) to an identity for the binomial-transform pair of the first kind, $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \dots$, we choose $\bar{t}_k = x^k$ and $\bar{\tau}_k = (1+x)^k$ and replace x with $-x$ to obtain

$$\begin{aligned} & \sum_{k=0}^m (-1)^k \binom{m}{k} \binom{n+k+s}{s}^{-1} x^{n+k+s} \\ &= (-1)^s \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+k+s}{s}^{-1} (1-x)^{m+k+s} \\ &+ \sum_{k=0}^{s-1} \frac{(-1)^k s}{m+n+s-k} \binom{s-1}{k} \binom{m+n+s-k-1}{n}^{-1} (1-x)^k. \end{aligned}$$

Operating on the above equation with the linear operator \mathcal{L}_x now gives

$$\begin{aligned} & \sum_{k=0}^m (-1)^k \binom{m}{k} \binom{n+k+s}{s}^{-1} t_{n+k+s} \\ &= (-1)^s \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+k+s}{s}^{-1} \tau_{m+k+s} \\ &+ \sum_{k=0}^{s-1} \frac{(-1)^k s}{m+n+s-k} \binom{s-1}{k} \binom{m+n+s-k-1}{n}^{-1} \tau_k. \end{aligned} \quad (148)$$

Example 25. Another example, Chen [3, Theorem 3.2] is

$$\sum_{k=0}^m \binom{m}{k} \binom{n+k}{s} \bar{t}_{n+k-s} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \binom{m+k}{s} \bar{\tau}_{m+k-s}; \quad (149)$$

whose corresponding version for a binomial-transform pair of the first kind is

$$\sum_{k=0}^m (-1)^{s-k} \binom{m}{k} \binom{n+k}{s} t_{n+k-s} = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{m+k}{s} \tau_{m+k-s}. \quad (150)$$

Example 26. Gould and Quaintance [5, Theorem 3] gave the following identity involving a binomial transform pair of the second kind:

$$\begin{aligned} & \sum_{k=0}^s \binom{s}{k} \binom{m+n+s-k}{m}^{-1} \frac{\bar{t}_k}{m+n+s+1-k} \\ &= \sum_{k=0}^s \binom{s}{k} \binom{m+n+s-k}{n}^{-1} \frac{(-1)^{s-k} \bar{\tau}_k}{m+n+s+1-k'} \end{aligned} \quad (151)$$

which is valid for every non-negative integer s and all complex numbers m and n excluding the set of negative integers.

The corresponding result for a binomial-transform pair of the first kind is

$$\begin{aligned} & \sum_{k=0}^s (-1)^k \binom{s}{k} \binom{m+n+s-k}{m}^{-1} \frac{t_k}{m+n+s+1-k} \\ &= \sum_{k=0}^s \binom{s}{k} \binom{m+n+s-k}{n}^{-1} \frac{(-1)^{s-k} \tau_k}{m+n+s+1-k}. \end{aligned} \quad (152)$$

Remark 9. In view of the foregoing, an identity involving a binomial-transform pair of the first kind $\{s_k, \sigma_k\}$, $k = 0, 1, 2, \dots$, can always be converted to a binomial-transform pair identity of the second kind $\{\bar{s}_k, \bar{\sigma}_k\}$ by doing

$$s_k \rightarrow (-1)^k \bar{s}_k, \quad \sigma_k \rightarrow \bar{\sigma}_k.$$

Similarly, conversion of a second kind transform pair identity to a first kind transform pair is achieved through

$$\bar{s}_k \rightarrow (-1)^k s_k, \quad \bar{\sigma}_k \rightarrow \sigma_k.$$

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