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Article

Global Efficiency

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Abstract

In this paper the global behavior of an estimator is studied in framework of Intrinsic Analysis, [1]. Two indices of performance of an estimator in a bounded region are analyzed: the average of the intrinsic risk (the loss function is the squared Rao distance) and the maximum risk. The Riemannian volume, provided by the Fisher metric on the manifold associated with the parametric model, allows us to take an average of the intrinsic risk. Cramér–Rao type integral inequalities for the integrated mean squared Rao distance of estimators are derived using variational methods, extending the previous work of several authors see, in particular, [2–5]. Additionally, lower bounds for the maximum risk are also derived, by using integral expressions.

Keywords: Riemannian metric; information metric; Rao distance; intrinsic analysis; Cramér–Rao type integral lower bounds; mini-max risk

1. Introduction

The study of estimation efficiency within the framework of information geometry has evolved significantly since the pioneering work of Rao [6] and some years later by other [2,7] or [8], and may be studied in more recent papers and books like [9,10]. The Fisher information metric, providing a canonical Riemannian structure on parametric statistical models, allows the intrinsic quantification of statistical distinguishability and the derivation of sharp risk bounds. This was developed in the article [1], where intrinsic classical results, such as Cramér–Rao inequalities, under regularity conditions, were established. Building on these foundations, the notion of global efficiency has recently attracted renewed attention, emphasizing the behavior of estimators not only locally but across regions of the parameter space, particularly if we take into account that the interplay between geometry and physics has been further enriched by applications of Fisher information to variational principles in classical and quantum mechanics; see [11–13].

The specific aim of this paper is to derive lower bounds for two global risk measures of an estimator over a subset of the parameter space, under the intrinsic geometry induced by the Fisher information: the average risk and the maximum risk. In the Introduction, we outline the setting of the problem and recall some results on local risk bounds, which will be later applied to obtain global bounds. Related work appears in [2,3].

1.1. The Framework

Let $\mathcal{X}$ be a sample space, $\mathcal{A}$ a σ -algebra of subsets of $\mathcal{X}$ and μ a σ -finite positive measure in $(\mathcal{X}, \mathcal{A})$. A *parametric statistical model* is defined as the triple $\{(\mathcal{X}, \mathcal{A}, \mu); \Theta; f\}$, where $(\mathcal{X}, \mathcal{A}, \mu)$ is a measure space, Θ is a topological space, known as the *parameter space*, and f is a non-negative measurable map, $f: \mathcal{X} \times \Theta \rightarrow \mathbb{R}_{\geq 0}$ such that $P_{\theta}(dx) = f(x, \theta)\mu(dx)$ is a probability measure on $(\mathcal{X}, \mathcal{A})$, $\forall \theta \in \Theta$. Here, μ is referred to as the *reference measure* and f as the *model function*.

For simplicity, in this paper we shall focus on the case in which Θ is an open connected subset of $\mathbb{R}^n$. In this setting, it is customary to use the same symbol θ to denote both the points in Θ and their

coordinate representations. Adopting this convention, we shall present the results in this familiar form hereafter, even though the statements can be formulated in greater generality.

Additionally, we assume that the model function f satisfies certain regularity conditions:

1. When x is fixed, the real function $\xi \mapsto f(x; \xi)$ is a C^∞ function on the manifold Θ .
2. The functions in x , $\partial \ln f(x; \theta) / \partial \theta^i$ $i = 1, \dots, n$, are linearly independent, and belong to $L^\alpha(f(\cdot; \theta) d\mu)$ for a suitable $\alpha > 0$.
3. The partial derivatives of the required orders

$$\partial / \partial \theta^i, \quad \partial^2 / \partial \theta^i \partial \theta^j, \quad \partial^3 / \partial \theta^i \partial \theta^j \partial \theta^k, \quad \dots, \quad i, j, k = 1, \dots, n,$$

and the integration of $f(x; \theta)$ with respect to $d\mu$ can always be interchanged.

4. The model is identifiable: the map $\theta \mapsto P_\theta$, with $dP_\theta = f(\cdot; \theta) d\mu$ is one-to-one.

Within this framework, the probabilistic mechanism that generates the data under analysis can be equivalently described as a probability measure, a density function or a parameter, that is, a point in the parametric manifold Θ . When all these conditions are satisfied, we shall say that the parametric statistical model is *regular*. We shall start by considering that Θ is a Riemannian manifold with an arbitrary fundamental tensor h in Θ with components h_{ij} , although in this case it is well known that the parameter space has a natural Riemannian structure, induced by probability measures, called the *information metric*, whose fundamental tensor components, g_{ij} , are the components of the Fisher information matrix. For further details, see [1,6–8,10] among many others.

In this context, given a sample size k , an *estimator* $\mathcal{U}$ for the true parameter $\theta \in \Theta$, that is, the parameter θ corresponding to the true probabilistic mechanism that has generated the sample data, is a measurable map $\mathcal{U} : \chi^k \mapsto \Theta$, assuming that the probability measure on χ^k is $(P)_k(dx) = f_{(k)}(x; \theta) \mu_k(dx) = \prod_{i=1}^k f(x_i; \theta) \mu(dx_i)$.

1.2. Local Bounds

Let $h_{\alpha\beta}$ be the components of the metric tensor of the Riemannian metric on Θ and $g_{\alpha\beta}$ the components of the information metric on Θ . Then consider the Levi-Civita connection associated with $h_{\alpha\beta}$ and

$$A = \exp_\theta^{-1}(\mathcal{U}), \quad B = E_\theta(\exp_\theta^{-1}(\mathcal{U})), \quad (1)$$

and estimators $\mathcal{U}$ such that B is a C^∞ field on Θ . Let $\mathfrak{S}_\theta = \{\xi \in \Theta_\theta, |\xi| = 1\}$, Θ_θ be the tangent space at θ ; for each $\xi \in \mathfrak{S}_\theta$ we define

$$\mathcal{C}_\theta(\xi) = \sup\{s > 0 : d(\theta, \gamma_\xi(s)) = s\}, \quad (2)$$

where d is the Riemannian distance and γ_ξ is a geodesic defined in an open interval that contains zero, such that $\gamma_\xi(0) = \theta$ and with a tangent vector at θ equal to ξ . Now, if we set the following,

$$\mathfrak{D}_\theta = \{s\xi \in \Theta_\theta : 0 \leq s < \mathcal{C}_\theta(\xi) ; \xi \in \mathfrak{S}_\theta\}$$

and

$$D_\theta = \exp_\theta(\mathfrak{D}_\theta),$$

it is known that $\exp_\theta$ is a diffeomorphism which maps $\mathfrak{D}_\theta$ onto D_θ (see Hicks [14]). We have

Theorem 1 (Riemannian Cramér-Rao lower bound). *Let $\mathcal{U}$ be an estimator for a sample size k , corresponding to an n -dimensional regular parametric family of density functions. Assume that the manifold Θ is simply connected and that $(P)_k \circ \mathcal{U}^{-1}(\Theta \setminus D_\theta) = 0$, $\forall \theta \in \Theta$. Let us assume that the mean squared Riemannian*

distance given by $h_{\alpha\beta}$, between the true parameter and an estimate, $E_\theta(d^2(\mathcal{U}, \theta))$, exists and that the covariant derivative of B can be obtained by differentiating under the integral sign. Then

$$E_\theta(d^2(\mathcal{U}, \theta)) \geq \frac{(\text{div}(B) - E_\theta(\text{div}(A)))^2}{kc} + \|B\|^2, \quad (3)$$

where $c = \sum_{\alpha, \beta} h^{\alpha\beta} g_{\alpha\beta}$ and $\text{div}(\cdot)$ represent the divergence operator.

Proof. Let C be any vector field. Then, applying the Cauchy–Schwartz inequality twice,

$$E_\theta(|\langle A - B, C \rangle|) \leq E_\theta(\|A - B\| \|C\|) \leq \sqrt{E_\theta(\|A - B\|^2)} \sqrt{E_\theta(\|C\|^2)},$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote, respectively, the inner product and the norm defined on each tangent space.

Let $C(x; \theta) = \text{grad}(\ln f_{(k)}(x; \theta))$, where $\text{grad}(\cdot)$ is the gradient operator. Taking expectations and using the repeated index convention,

$$E_\theta(\|C\|^2) = E_\theta\left(h_{\alpha\beta} h^{\beta\gamma} \frac{\partial \ln f_{(k)}}{\partial \theta^\gamma} h^{\alpha\lambda} \frac{\partial \ln f_{(k)}}{\partial \theta^\lambda}\right) = k h_{\alpha\beta} h^{\beta\gamma} h^{\alpha\lambda} g_{\gamma\lambda} = k h^{\gamma\lambda} g_{\gamma\lambda}.$$

Furthermore, we also have

$$|E_\theta(\langle A, C \rangle)| = |E_\theta(\langle A - B, C \rangle)| \leq E_\theta(|\langle A - B, C \rangle|)$$

and

$$E_\theta(\|A - B\|^2) = E_\theta(\|A\|^2) - \|B\|^2.$$

Thus,

$$|E_\theta(\langle A, C \rangle)| \leq \sqrt{E_\theta(\|A\|^2) - \|B\|^2} \sqrt{k c},$$

but $\|A\|^2 = d^2(\mathcal{U}, \theta)$. Moreover,

$$\text{div}(B) = E_\theta(\text{div}(A)) + E_\theta(\langle A, C \rangle).$$

Then the theorem follows. $\square$

Remark 1. We can choose a geodesic spherical coordinate system with origin $\mathcal{U}(x)$; under this coordinate system, we have the following.

$$\frac{\partial A^\alpha}{\partial \theta^\alpha} = -1 \quad \text{and} \quad \Gamma_{\alpha j}^\alpha A^j = -\rho \Gamma_{\alpha 1}^\alpha = -\frac{\partial \ln \sqrt{g}}{\partial \rho} \rho,$$

where g is the determinant of the metric tensor. Then

$$\text{div}(A) = -1 - \rho \frac{\partial \ln \sqrt{g}}{\partial \rho}.$$

Now we can use Bishop's comparison theorems (see [15, pp. 71-73]) to estimate $\frac{\partial \ln \sqrt{g}}{\partial \rho}$.

In the Euclidean case,

$$\frac{\partial \ln \sqrt{g}}{\partial \rho} = \frac{n-1}{\rho},$$

and thus $\text{div}(A) = -n$.

When the sectional curvatures are non-positive, we obtain

$$\frac{\partial \ln \sqrt{g}}{\partial \rho} \geq \frac{n-1}{\rho},$$

and therefore $\operatorname{div}(A) \leq -n$.

Finally, when the supreme of the sectional curvatures, $\mathcal{K}$, is positive and the diameter of the manifold satisfies $d(M) < \pi/2\sqrt{\mathcal{K}}$, we have

$$\frac{\partial \ln \sqrt{g}}{\partial \rho} \geq 0,$$

and then we obtain $\operatorname{div}(A) \leq -1$.

In any case, $\operatorname{div}(A) \leq -a$, with $a = n$ or $a = 1$, depending on the sectional curvature sign.

Corollary 1. Suppose there is a global chart such that $h_{\alpha\beta} = \delta_{\alpha\beta}$. Identifying the points with their coordinates, we have

$$\operatorname{MSE}(\mathcal{U}) \geq \frac{(\operatorname{div}(E_{\theta}(\mathcal{U})))^2}{kg_{\beta\beta}} + (\operatorname{Bias}(\mathcal{U}))^2,$$

where MSE is the mean squared error, and we are using the repeated index summation convention.

Proof. It follows straightforwardly from the previous theorem and the facts that d is the Euclidean distance, $A = \mathcal{U} - \theta$ and $\operatorname{div}(A) = -n$. $\square$

Corollary 2 (Intrinsic Cramér-Rao lower bound). If $h_{\alpha\beta} = g_{\alpha\beta}$, we have

$$E_{\theta}(\rho^2(\mathcal{U}, p)) \geq \frac{(\operatorname{div}(B) - E_{\theta}(\operatorname{div}(A)))^2}{kn} + \|B\|^2,$$

where ρ is the Rao distance.

Proof. If the Riemannian metric is the Fisher metric, the distance is known by the Rao distance and $c = g^{\alpha\beta}g_{\alpha\beta} = \delta_{\alpha}^{\alpha} = n$. $\square$

1.3. Global Bounds

Whatever loss function is considered, it is well known that, in general, there is no estimator whose risk function is uniformly smaller than any other. Therefore, given an estimator, it seems reasonable, in order to measure its performance over a certain region of the statistical model, to compute the integral of the risk and then to divide this quantity by the Riemannian volume of the region considered. In the following, we take the Rao distance as a loss function and the Riemannian metric as the Fisher metric. This is the Intrinsic Analysis framework.

Let $\mathcal{B} \subset \Theta$ be a measurable subset with $0 < V(\mathcal{B}) < \infty$, where V is the Riemannian measure. We denote the Riemannian average of the mean squared Rao distance by

$$\mathcal{R}_{\mathcal{U}}^2(\mathcal{B}) = \frac{\int_{\mathcal{B}} E_{\theta}(\rho^2(\mathcal{U}, p)) dV}{\int_{\mathcal{B}} dV}.$$

The performance index obtained is a weighted average of the mean squared distance. This approach is compatible with a Bayesian point of view: a uniform prior with respect to the Riemannian volume is a kind of noninformative prior (see [16]). It can be shown (see [17]) that when the parameter space is a locally compact topological group, this Riemannian volume is a left-invariant Haar measure and is unique up to a multiplicative constant. In any case, this volume is invariant under any group that leaves the parametric family of densities invariant. In the first part of the paper we derive lower bounds for this global index on balls of radius R .

Another way to measure the global behavior of an estimator is to consider the maximum risk in a region of the parameter space. This is a mini-max approach. The last part of the paper is devoted to obtaining lower bounds for the maximum risk.

2. Variational Methods to Obtain Global Bounds

As we shall show, variational methods can be used to obtain global bounds. A previous study in this direction can be found in [2]. The idea is to consider the integral of the local bounds for the Rao distance given above when the Riemannian metric is the Fisher metric and on a submanifold $W \subset \Theta$ with boundary $\partial W \subset \Theta$. That is,

$$\mathcal{Y}(B) = \int_W \left\{ \|B\|^2 + \frac{1}{kn} (\operatorname{div}(B) + a)^2 \right\} dV,$$

where we take $a = n$ if the sectional curvatures are non-positive and $a = 1$ in the other case. The above functional depends only on B , and we can attempt to find the C^∞ vector field B that minimizes it. Since the minimum we obtain is in a class of vector fields larger than that of C^∞ bias vector fields, this method gives a lower bound for the *average of the mean squared Rao distance*.

Lemma 1. *The C^∞ field B minimizes the functional*

$$\mathcal{Y}(B) = \int_W \left\{ \|B\|^2 + \frac{1}{kn} (\operatorname{div}(B) + a)^2 \right\} dV$$

iff it verifies

$$B - \frac{1}{kn} \operatorname{grad}(\operatorname{div}(B)) = 0, \quad \forall \theta \in W, \quad (4)$$

$$\operatorname{div}(B) + a = 0, \quad \forall \theta \in \partial W,$$

and the minimum value is given by

$$\mathcal{Y}^* = \frac{a^2}{kn} \operatorname{vol}(W) - \frac{a}{kn} \int_{\partial W} \|B^*\| d\sigma = \frac{a^2}{kn} \operatorname{vol}(W) + \frac{a}{kn} \int_W \operatorname{div}(B^*) dV, \quad (5)$$

where B^ verifies (4).*

Proof. Consider the first variation $\delta\mathcal{Y}(B, \eta)$, where η is an arbitrary field. Then it is easy to see that

$$\lim_{\epsilon \rightarrow 0} \frac{\mathcal{Y}(B + \epsilon\eta) - \mathcal{Y}(B)}{\epsilon} \equiv \delta\mathcal{Y}(B, \eta) = \int_W \left(2\langle B, \eta \rangle + \frac{2}{kn} \operatorname{div}(\eta)(\operatorname{div}(B) + a) \right) dV$$

and

$$\mathcal{Y}(B + \eta) - \mathcal{Y}(B) = \delta\mathcal{Y}(B; \eta) + \int_W \left(\|\eta\|^2 + \frac{1}{kn} (\operatorname{div}(\eta))^2 \right) dV.$$

Thus, the functional is strictly convex, and the stationary point is a global minimum. Now, condition $\delta\mathcal{Y}(B; \eta) = 0$ is equivalent to

$$\int_W \left(\langle B, \eta \rangle + \frac{1}{kn} \operatorname{div}(\eta)(\operatorname{div}(B) + a) \right) dV = 0.$$

We take into account that

$$\operatorname{div}(fX) = f \operatorname{div}(X) + \langle X, \operatorname{grad}(f) \rangle, \quad (6)$$

we obtain

$$\begin{aligned} \frac{1}{kn} \operatorname{div}(\eta)(\operatorname{div}(B) + a) &= \operatorname{div}\left(\frac{1}{kn} \eta(\operatorname{div}(B) + a)\right) - \left\langle \operatorname{grad}\left(\frac{1}{kn} (\operatorname{div}(B) + a)\right), \eta \right\rangle \\ &= \frac{1}{kn} \operatorname{div}(\eta(\operatorname{div}(B) + a)) - \frac{1}{kn} \langle \operatorname{grad}(\operatorname{div}(B)), \eta \rangle. \end{aligned}$$

We are able to write the stationary condition as

$$\int_W \left(\langle B, \eta \rangle + \frac{1}{kn} \operatorname{div}(\eta(\operatorname{div}(B) + a)) - \frac{1}{kn} \langle \operatorname{grad}(\operatorname{div}(B)), \eta \rangle \right) dV = 0$$

and by the Gauss divergence theorem,

$$\int_W \left\langle B - \frac{1}{kn} \operatorname{grad}(\operatorname{div}(B)), \eta \right\rangle dV + \frac{1}{kn} \int_{\partial W} \langle (\operatorname{div}(B) + a)\eta, \nu \rangle d\sigma = 0,$$

where $d\sigma$ is the Riemannian measure on ∂W . Then (4) follows from the fact that the previous equality must be verified for any η .

Now we see the second part of the proposition. By the first stationary condition (4) and by (6), we obtain the following,

$$\|B^*\|^2 = \frac{1}{kn} \left(\operatorname{div}(B^* \operatorname{div}(B^*)) - (\operatorname{div}(B^*))^2 \right),$$

and, putting it in $\mathcal{Y}(B)$,

$$\begin{aligned} \mathcal{Y}^* &= \int_W \frac{1}{kn} \left(\operatorname{div}(B^* \operatorname{div}(B^*)) + a^2 + 2a \operatorname{div}(B^*) \right) dV \\ &= \frac{a^2}{kn} \operatorname{vol}(W) + \frac{1}{kn} \int_{\partial W} \langle B^* \operatorname{div}(B^*) + 2aB^*, \nu \rangle d\sigma. \end{aligned}$$

Now, by the second stationary condition in (4),

$$\mathcal{Y}^* = \frac{a^2}{kn} \operatorname{vol}(W) + \frac{a}{kn} \int_{\partial W} \langle B^*, \nu \rangle d\sigma. \quad (7)$$

It is clear that $0 \geq \operatorname{div}(B^*) \geq -a$, then since $\operatorname{div}(B^*) = -a$ on ∂W and $B^* = \operatorname{grad}(\operatorname{div}(B^*))$ it turns out that $\langle B^*, \nu \rangle = -\|B^*\|$. Finally, by the Gauss divergence theorem we obtain the second equality in (5). $\square$

Remark 2. Note that the minimum value of $\mathcal{Y}(B)$ depends only on $\operatorname{div}(B^*)$, and that $f^* \equiv \operatorname{div}(B^*)$ verifies the partial differential equation.

$$\Delta f = knf, \quad \text{with } f(\theta) = -a, \quad \forall \theta \in \partial W, \quad (8)$$

as is easy to check from (4).

We have solved this boundary value problem in the case where $W = S_R$, a ball of radius R , and constant sectional curvatures $\mathcal{K}$.

Theorem 2. When the parametric statistical model is a manifold of constant sectional curvature $\mathcal{K}$, we have the following lower bound for the average of the mean squared Rao distance, in balls of radius R such that $|\mathcal{K}|S_{\mathcal{K}}^2(R) < 1$:

$$\mathcal{R}_{\mathcal{U}}^2(S_R) \geq \frac{a^2}{kn} \left(1 - \frac{f'(R)S_{\mathcal{K}}^{n-1}(R)}{knf(R) \int_0^R S_{\mathcal{K}}^{n-1}(r) dr} \right), \quad (9)$$

where

$$f(R) = a_0 \sum_{j=0}^{\infty} \frac{\prod_{s=1}^j \{kn + 2\mathcal{K}(s-1)(n+2s-3)\}}{j! (\frac{n}{2})_j 4^j} S_{\mathcal{K}}^{2j}(R)$$

and

$$S_{\mathcal{K}}(t) = \begin{cases} \frac{\sin(\sqrt{\mathcal{K}}t)}{\sqrt{\mathcal{K}}} & \text{if } \mathcal{K} > 0, \\ t & \text{if } \mathcal{K} = 0, \\ \frac{\sinh(\sqrt{-\mathcal{K}}t)}{\sqrt{-\mathcal{K}}} & \text{if } \mathcal{K} < 0. \end{cases} \tag{10}$$

Proof. By symmetry and uniqueness, the solution of the boundary problem in S_R ,

$$\Delta f = knf, \quad \text{with } f(\theta) = -a \quad \forall \theta \in \partial S_R, \tag{11}$$

depends only on the distance to the center of S_R . Then, taking geodesic spherical coordinates (r, u) with origin in the center of S_R , since $\sqrt{g}(r, u) = S_{\mathcal{K}}^{n-1} \Omega(u)$ (see the Appendix), we have

$$\Delta f = \frac{1}{\sqrt{g}} \frac{d}{dr} \left(\sqrt{g} \frac{d}{dr} f \right) = \frac{1}{S_{\mathcal{K}}^{n-1}} \frac{d}{dr} \left(S_{\mathcal{K}}^{n-1} \frac{d}{dr} f \right).$$

We can then write

$$(n-1) \frac{S'_{\mathcal{K}}}{S_{\mathcal{K}}} f' + f'' = knf.$$

Let $v = S_{\mathcal{K}}(r)$ and $h(v) = f(S_{\mathcal{K}}^{-1}(v))$; taking into account that

$$f'(r) = S'_{\mathcal{K}}(r) h'(v) \quad \text{and} \quad f''(r) = S''_{\mathcal{K}}(r) h'(v) + S_{\mathcal{K}}^2(r) h''(v),$$

we obtain

$$(n-1) \frac{S_{\mathcal{K}}^2(r) h'(v)}{S_{\mathcal{K}}(r)} + S''_{\mathcal{K}}(r) h'(v) + S_{\mathcal{K}}^2(r) h''(v) = kn h(v).$$

Moreover, since

$$S_{\mathcal{K}}^2(r) + \mathcal{K} S_{\mathcal{K}}^2(r) = 1 \quad \text{and} \quad S''_{\mathcal{K}}(r) + \mathcal{K} S_{\mathcal{K}}(r) = 0,$$

it turns out that

$$v(1 - \mathcal{K}v^2) h''(v) + (n(1 - \mathcal{K}v^2) - 1) h'(v) - knvh(v) = 0.$$

If we try $h(v) = \sum_{j=0}^{\infty} a_j v^j$, since

$$h'(v) = \sum_{j=1}^{\infty} a_j j v^{j-1}, \quad h''(v) = \sum_{j=2}^{\infty} a_j j(j-1) v^{j-2},$$

we have

$$\begin{aligned} \sum_{j=2}^{\infty} a_j j(j-1) v^{j-1} - \mathcal{K} \sum_{j=2}^{\infty} a_j j(j-1) v^{j+1} + (n-1) \sum_{j=1}^{\infty} a_j j v^{j-1} \\ - \mathcal{K} n \sum_{j=1}^{\infty} a_j j v^{j+1} - kn \sum_{j=0}^{\infty} a_j v^{j+1} = 0, \end{aligned}$$

that is,

$$(n-1)a_1 > + (2(n-1)a_2 + 2a_2 - kna_0)v \\ + \sum_{j=1}^{\infty} (a_{j+2}(j+2)(n+j) - a_j(kn + \mathcal{K}j(n+j-1)))v^{j+1} = 0.$$

Now, if $n \neq 1$, then

$$a_1 = 0 \quad \text{and} \quad a_{j+2} = \frac{kn + \mathcal{K}j(n+j-1)}{(n+1)(j+2)} a_j, \quad j \geq 0.$$

Thus,

$$h(v) = a_0 \sum_{j=0}^{\infty} \frac{\prod_{s=1}^j \{kn + 2\mathcal{K}(s-1)(n+2s-3)\}}{j! \left(\frac{n}{2}\right)_j 4^j} v^{2j}$$

and

$$f(r) = a_0 \sum_{j=0}^{\infty} \frac{\prod_{s=1}^j \{kn + 2\mathcal{K}(s-1)(n+2s-3)\}}{j! \left(\frac{n}{2}\right)_j 4^j} S_{\mathcal{K}}^{2j}(r),$$

where a_0 is determined by the condition $f(R) = -a$. It is easy to see that this series is convergent iff $|\mathcal{K}|S_{\mathcal{K}}^2(r) < 1$. This is always true in the case of non-negative sectional curvatures.

Furthermore, we have to evaluate $\int_{S_R} f dV$. In spherical coordinates (see Appendix)

$$\int_{S_R} f dV = \text{ar}(S) \int_0^R S_{\mathcal{K}}^{n-1} f(r) dr,$$

Since $(S_{\mathcal{K}}^{n-1}(r)f'(r))' = knS_{\mathcal{K}}^{n-1}(r)f(r)$, we find that

$$\int_{S_R} f dV = \frac{2\pi^{n/2}}{kn\Gamma(n/2)} S_{\mathcal{K}}^{n-1}(R)f'(R)$$

and

$$\mathcal{Y}^* = \frac{a^2}{kn} \text{vol}(S_R) \left(1 - \frac{f'(R)S_{\mathcal{K}}^{n-1}(R)}{knf(R) \int_0^R S_{\mathcal{K}}^{n-1}(r) dr} \right).$$

□

Corollary 3. When the parametric statistical model is an Euclidean manifold, we have the following lower bound for the Riemannian average of the mean squared Rao distance, on a ball of radius R :

$$\mathcal{R}_{\mathcal{U}}^2(S_R) \geq \frac{n}{k} \left(1 - \frac{{}_0F_1\left(\frac{n}{2} + 1; \frac{knR^2}{4}\right)}{{}_0F_1\left(\frac{n}{2}; \frac{knR^2}{4}\right)} \right), \quad (12)$$

where ${}_0F_1(a; z)$ is a generalized hypergeometric function (see (A2) of the Appendix).

If the Euclidean manifold M is complete and simply connected, we obtain the following lower bound in the manifold:

$$\mathcal{R}_{\mathcal{U}}^2(M) \equiv \lim_{R \rightarrow \infty} \mathcal{R}_{\mathcal{U}}^2(S_R) \geq \frac{n}{k}.$$

Proof. It is a particular case of the previous theorem with $\mathcal{K} = 0$. The second part of the proposition follows by taking the limit when $R \rightarrow \infty$ in (12). □

Example 1. As an example, consider the n -variate normal distribution with known covariance matrix Σ . Given a sample of size k , the Riemannian density of the mean squared Rao distance corresponding to the sample mean $\bar{X}_k$ is $\mathcal{R}_{\mathcal{U}}^2(S_R) = n/k$, which coincides with the previous bound.

In the case $n = 1$, the manifold is Euclidean and we can apply the previous result; we obtain

$$\frac{\mathcal{Y}^*}{2R} = \frac{1}{k} \left(1 - \frac{\tanh(\sqrt{k}R)}{\sqrt{k}R} \right),$$

which coincides with the result already obtained by Chentsov [2].

In fact, if we take a Cartesian coordinate system with origin θ and try to solve the variational problem for a cube with center θ , $C_R = \{x : |x^i| \leq R, i = 1, \dots, n\}$, we have to solve the Dirichlet problem:

$$\sum_{i=1}^n \frac{\partial^2}{\partial (x^i)^2} f = knf, \text{ with } f(x) = -n \text{ if } |x^i| = R \text{ for some } i = 1, \dots, n.$$

If we try $f(x) = \sum_{i=1}^n f_i(x^i)$ for convenient real-valued functions f_i , we obtain

$$\sum_{i=1}^n \frac{\partial^2}{\partial (x^i)^2} f_i(x_i) = k \sum_{i=1}^n f_i(x_i), \quad \sum_{i=1}^n f_i(\pm R) = -n.$$

Obviously, a solution is given by $f(x) = \sum_{i=1}^n g(x^i)$, with g such that

$$\frac{\partial^2}{\partial z^2} g(z) = kng(z), \quad g(\pm R) = -1.$$

The solution of the last equation is

$$g(z) = -\frac{\cosh(\sqrt{kn}z)}{\cosh(\sqrt{kn}R)},$$

and then

$$f(x) = -\sum_{i=1}^n \frac{\cosh(\sqrt{kn}x^i)}{\cosh(\sqrt{kn}R)}.$$

This provides the bound

$$\begin{aligned} \frac{\mathcal{Y}^*}{\text{vol}(C_R)} &= \frac{n}{k} \left(1 - \frac{g'(R) \text{ar}(C_R)}{kn^2 g(R) \text{vol}(C_R)} \right) \\ &= \frac{n}{k} \left(1 - \frac{2nR\sqrt{kn} \tanh(\sqrt{kn}) R}{2kn^2 R} \right) \\ &= \frac{n}{k} \left(1 - \frac{\tanh(\sqrt{kn}R)}{\sqrt{kn}R} \right), \end{aligned}$$

which improves upon the result given by Chentsov [2]. By Corollary (1), we can also give, in the general non-Euclidean case (fixed coordinate system), for the mean squared error (M.S.E.), a bound of the form

$$\mathcal{R}_{\mathcal{U}}^2(C_R) \geq \frac{n^2}{k\eta} \left(1 - \frac{\tanh(\sqrt{k\eta}) R}{\sqrt{k\eta} R} \right),$$

where η is an upper bound of $\sum_{\beta} g_{\beta\beta}$ in C_R .

We can also give lower bounds to the general case.

Theorem 3. *When the parametric statistical model is a manifold with sectional curvatures bounded from above by $\mathcal{K}$, then we have the following lower bound for the average of the mean squared Rao distance:*

$$\mathcal{R}_U^2(S_R) \geq \frac{a^2}{kn} \left(1 - \frac{f'_K(R) \text{ar}(S_R)}{kn f_K(R) \text{vol}(S_R)} \right) > 0, \quad (13)$$

where $\text{ar}(S_R)$ is the area of the n -dimensional sphere of radius R , $\text{vol}(S_R)$ its volume and $f_K(r)$ is the solution of the boundary problem in S_R , (11), on a manifold of constant sectional curvature K .

Proof. Consider geodesic spherical coordinates (r, u) . Let $f(r, u)$ be the solution to the boundary problem (11) on a manifold with sectional curvatures bounded from above by K . Let $f_K(r)$ be the solution to the same problem but on a manifold of constant sectional curvature K , which, as we know, depends only on the radial coordinate. Then

$$\Delta f_K = \frac{1}{\sqrt{g}} \frac{\partial}{\partial r} \left(\sqrt{g} \frac{\partial}{\partial r} f_K \right) = \frac{\partial^2}{\partial r^2} f_K + \frac{\partial}{\partial r} (\ln \sqrt{g}) \frac{\partial}{\partial r} f_K.$$

By Bishop's comparison theorems, we have

$$(n-1) \frac{S'_K}{S_K} \leq \frac{\partial}{\partial r} (\ln \sqrt{g}),$$

and, since

$$\frac{\partial}{\partial r} f_K \leq 0,$$

we have

$$\Delta f_K - kn f_K \leq \Delta_K f_K - kn f_K = 0,$$

with Δ_K the Laplacian for the constant sectional curvature case. Thus,

$$\Delta f_K - kn f_K \leq \Delta f - kn f = 0.$$

Now, since $f_K(\theta) = f(\theta) = -a$, $\theta \in \partial S_R$, we can apply the comparison theorem to elliptic differential equations, ([18], Theorem 6, p. 243). We find that

$$f(\theta) \leq f_K(\theta), \quad \theta \in S_R,$$

and, since equality holds on the boundary,

$$\frac{\partial}{\partial r} f(\theta) \geq \frac{\partial}{\partial r} f_K(\theta) \quad \theta \in \partial S_R.$$

Finally, by (7) and (4),

$$\mathcal{Y}^* = \frac{a^2}{kn} \text{vol}(S_R) + \frac{a}{kn} \int_{\partial S_R} \frac{\partial}{\partial r} f d\sigma \geq \frac{a^2}{kn} \text{vol}(S_R) \left(1 - \frac{f'_K(R) \text{ar}(S_R)}{kn f_K(R) \text{vol}(S_R)} \right),$$

and the proposition follows. $\square$

Remark 3. Estimates for the volumes of balls given in the Appendix are useful to give a final expression for these bounds. Note that if the sectional curvatures are bounded from below by κ and from above by K , by the proposition (A3), we have

$$\frac{\text{ar}(S_R)}{\text{vol}(S_R)} \leq \frac{S_{\kappa}^{n-1}(R)}{\int_0^R S_K^{n-1}(r) dr}.$$

3. Lower Bounds for the Maximum Risk

Even though we could use the Riemannian average of the risk to derive bounds for the maximum risk, we can obtain sharper minimax bounds and more directly.

Lemma 2. Let X be a smooth field on Θ such that $\operatorname{div}(X) \leq -a$, let f be a non-negative function on Θ and let W be a submanifold with boundary in Θ , then

$$a \int_W f dV \leq \int_{\partial W} f \|X\| d\sigma + \int_W \|X\| \|\operatorname{grad}(f)\| dV$$

Proof.

$$\begin{aligned} a \int_W f dV &\leq \int_W (\langle X, \operatorname{grad}(f) \rangle - \operatorname{div}(fX)) dV \\ &\leq \int_W \|X\| \|\operatorname{grad}(f)\| dV + \int_{\partial W} f \|X\| d\sigma \end{aligned} \quad (14)$$

(15)

□

Theorem 4. We have the following lower bound for the local minimax risk of an estimator $\mathcal{U}$ on W

$$\sup_{\theta \in W} E(\rho^2(\mathcal{U}, \theta)) \geq \frac{a^2}{\left(\frac{\operatorname{ar}(\partial W)}{\operatorname{vol}(W)} + \sqrt{kn} \right)^2}.$$

Proof. By the previous lemma if $X = \exp^{-1}(\mathcal{U})$ by integrating (14) with respect to $f_{(k)}(x; \theta) d\mu$ and by Fubini's Theorem, we have

$$\begin{aligned} a \operatorname{vol}(W) &\leq \int_W \sqrt{E(\|A\|^2)} \sqrt{E(\|C\|^2)} dV + \int_{\partial W} E(\|A\|) d\sigma \\ &\leq \sqrt{kn} \int_W \sqrt{E(\|A\|^2)} dV + \int_{\partial W} \sqrt{E(\|A\|^2)} d\sigma. \end{aligned} \quad (16)$$

Thus,

$$\sup_{\theta \in W} E(\|A\|^2) \geq \left(\frac{a \operatorname{vol}(W)}{\operatorname{ar}(\partial W) + \sqrt{kn} \operatorname{vol}(W)} \right)^2.$$

□

Corollary 4. When the parametric statistical model is an n -dimensional Euclidean manifold, we have the following lower bound for the local minimax risk:

$$\mathcal{M}_{\mathcal{U}}^2(S_R) \geq \left(\frac{n R}{n + \sqrt{kn} R} \right)^2,$$

where $\mathcal{M}_{\mathcal{U}}^2(S_R) = \sup_{\theta \in S_R} E(\rho^2(\mathcal{U}, \theta))$. If the Euclidean manifold, M , is complete and is simply connected, we obtain the following lower bound over the manifold:

$$\mathcal{M}_{\mathcal{U}}^2(\Theta) \geq \frac{n}{k}.$$

Proof. Since

$$\operatorname{vol}(S_r) = \frac{2\pi^{n/2} r^n}{n \Gamma(n/2)},$$

we have

$$\int_0^R \operatorname{vol}(S_r) dr = \frac{2\pi^{n/2} R^{n+1}}{n(n+1) \Gamma(n/2)};$$

thus

$$\mathcal{M}_{\mathcal{U}}^2(S_R) \geq \left(\frac{n R}{n + \sqrt{kn}R} \right)^2.$$

We derive the second statement taking the limit as $R \rightarrow \infty$. $\square$

We can also use the previous lemma to derive the bounds of the average of the mean squared Rao distance.

Theorem 5. *We have the following lower bound for the Riemannian average of the mean squared Rao distance in S_R .*

$$\mathcal{R}_{\mathcal{U}_k}^2(S_R) \geq \left\{ \frac{a \int_0^R \text{vol}(S_r) dr}{\text{vol}(S_R) + \sqrt{kn} \sqrt{\text{vol}(S_R)} \int_0^R \sqrt{\text{vol}(S_r)} dr} \right\}^2, \quad (17)$$

where $a = n$ if the sectional curvatures are non-positive and $a = 1$ if the supreme of the sectional curvatures, $\mathcal{K}$, is positive

Proof. Consider (16) for $W = S_r$, $0 < r \leq R$ and integrate it with respect to dr from 0 to R .

$$\begin{aligned} a \int_0^R \text{vol}(S_r) dr &\leq \sqrt{kn} \int_0^R \int_{S_r} \sqrt{E(\|A\|^2)} dV dr + \int_0^R \int_{\partial S_r} \sqrt{E(\|A\|^2)} d\sigma dr \\ &\leq \sqrt{kn} \int_0^R \sqrt{\text{vol}(S_r)} \sqrt{\int_{S_r} E(\|A\|^2) dV} dr + \text{vol}(S_R) \sqrt{\frac{1}{\text{vol}(S_R)} \int_{S_R} E(\|A\|^2) dV} \end{aligned}$$

Now, we take into account that

$$r \mapsto \sqrt{\int_{S_r} E(\|A\|^2) dV}$$

is a positive monotonic increasing function of r , and $E(\|A\|^2) = E(\rho^2(\mathcal{U}_k, \theta))$, we obtain

$$\begin{aligned} \frac{a}{\text{vol}(S_R)} \int_0^R \text{vol}(S_r) dr &\leq \\ &\leq \left(\frac{\sqrt{kn}}{\sqrt{\text{vol}(S_R)}} \int_0^R \sqrt{\text{vol}(S_r)} dr + 1 \right) \sqrt{\mathcal{R}_{\mathcal{U}}^2(S_R)}. \end{aligned}$$

and the proposition follows. $\square$

Corollary 5. *When the parametric statistical model is a Euclidean manifold, we have the following lower bound for the Riemannian average of the mean squared Rao distance on S_R :*

$$\mathcal{R}_{\mathcal{U}}^2(S_R) \geq \left\{ \frac{n(n+2)R}{(n+1)(n+2+2\sqrt{kn}R)} \right\}^2.$$

If the Euclidean manifold Θ is complete and is simply connected, we obtain the following lower bound over the manifold:

$$\mathcal{R}_{\mathcal{U}}^2(\Theta) \equiv \lim_{R \rightarrow \infty} \mathcal{R}_{\mathcal{U}}^2(S_R) \geq \frac{n(n+2)^2}{4k(n+1)^2}.$$

Proof. Since

$$\text{vol}(S_r) = \frac{2\pi^{n/2} r^n}{n \Gamma(n/2)},$$

we have

$$\int_0^R \sqrt{\text{vol}(S_r)} dr = \left(\frac{8\pi^{n/2} R^{n+2}}{n(n+2)^2 \Gamma(n/2)} \right)^{1/2}$$

and

$$\int_0^R \text{vol}(S_r) dr = \frac{2\pi^{n/2} R^{n+1}}{n(n+1) \Gamma(n/2)};$$

then

$$0 < \left\{ \frac{n(n+2)R}{(n+1)(n+2+2\sqrt{kn}R)} \right\}^2 \leq \mathcal{R}_{\mathcal{U}_k}^2(S_R).$$

We derive the second statement taking the limit as $R \rightarrow \infty$. $\square$

Remark 4. Example (1) shows that the bound obtained here is worse than the variational one if R goes to infinity, but it is better if R goes to zero:

$$\lim_{R \rightarrow 0} \left\{ \frac{\frac{n}{k} \left(1 - \frac{{}_0F_1\left(\frac{n}{2} + 1; \frac{knR^2}{4}\right)}{{}_0F_1\left(\frac{n}{2}; \frac{knR^2}{4}\right)} \right)}{\frac{n^2(n+2)^2 R^2}{(n+1)^2(n+2+2\sqrt{kn}R)^2}} \right\} = \frac{(n+1)^2}{n(n+2)} > 1.$$

Remark 5. Note that global bounds for the average of the mean squared Rao distance also provide bounds for the local minimax risk in an obvious way. It can be shown that these last bounds are sharper than the bounds provided by the variational methods.

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Appendix A. Comparison Theorems and Volumes

We can use Bishop's theorems to obtain the volume of a ball of radius r in a Riemannian manifold whose sectional curvatures are constant and to give bounds for this volume when the sectional curvatures are bounded. We have the following propositions:

Proposition A1. If the sectional curvatures are constant and equal to $\mathcal{K}$, the volume of a Riemannian ball of radius r and center $\theta \in \Theta$ is given by

$$\text{vol}(S_r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r S_{\mathcal{K}}^{n-1}(t) dt.$$

Proof. We have

$$\text{vol}(S_r) = \int_0^r \int_{\xi^{-1}(S_n)} \sqrt{g} du d\rho,$$

where S_n is the unit sphere in Θ_θ . On the other hand, by Bishop's comparison theorems, when the sectional curvatures are constant,

$$\frac{\partial}{\partial \rho} \ln \sqrt{g(\rho, u)} = (n-1) \frac{S'_K(\rho)}{S_K(\rho)},$$

with

$$S_K(t) = \begin{cases} \frac{\sin(\sqrt{K}t)}{\sqrt{K}} & \text{if } K > 0, \\ t & \text{if } K = 0, \\ \frac{\sinh(\sqrt{-K}t)}{\sqrt{-K}} & \text{if } K < 0. \end{cases}$$

Then, integrating this expression, we have

$$\sqrt{g(\rho, u)} = S_K^{n-1} \Omega_K(u).$$

However, Ω_K does not depend on K . In fact,

$$\lim_{\rho \rightarrow 0} \frac{\sqrt{g(\rho, u)}}{\rho^{n-1} \Omega(u)} = 1,$$

where $\Omega(u) du$ is the surface element of the unit sphere in a Euclidean manifold and, since

$$\lim_{\rho \rightarrow 0} \frac{S_K}{\rho} = 1,$$

we conclude that $\Omega_K = \Omega$. Thus, we may write

$$\text{vol}(S_r) = \int_{\xi^{-1}(S_n)} \Omega(u) du \int_0^r S_K^{n-1}(\rho) d\rho,$$

and finally,

$$\int_{\xi^{-1}(S_n)} \Omega(u) du = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

□

Proposition A2. When the sectional curvatures are constant and equal to K and $|K|S_K^2(r) < 1$, we have the following expression for the volume of a Riemannian ball of radius r :

$$\text{vol}(S_r) = \frac{2\pi^{n/2}}{n\Gamma(n/2)} S_K^n(r) \left\{ 1 + \sum_{j=1}^{\infty} \frac{n\Gamma(j + \frac{1}{2})}{\sqrt{\pi}(n+2j)} \frac{K^j S_K^{2j}(r)}{j!} \right\}. \quad (\text{A1})$$

Proof. From the previous proposition,

$$\text{vol}(S_r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^r S_K^{n-1}(t) dt.$$

Then, since by the definition of S_K ,

$$\{S'_K(t)\}^2 + K\{S_K(t)\}^2 = 1,$$

and, making the change $y = S_{\mathcal{K}}^2(t)/S_{\mathcal{K}}^2(r)$, we have

$$\int_0^r S_{\mathcal{K}}^{n-1}(t)dt = \frac{1}{2} S_{\mathcal{K}}^n(r) \int_0^1 y^{\frac{n-2}{2}} (1 - \mathcal{K} S_{\mathcal{K}}^2(r)y)^{-\frac{1}{2}} dy.$$

Moreover, there is a relationship between integrals of this kind and generalized hypergeometric functions. These functions are defined by

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) \equiv \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_p)_j}{(b_1)_j \cdots (b_q)_j} \frac{z^j}{j!}, \quad (\text{A2})$$

where $(a)_j = a(a+1) \cdots (a+j-1)$ and z is any complex number if $p \leq q$, $\|z\| < 1$ if $p = q+1$ and they diverge for all $z \neq 0$ if $p > q+1$ (see Abramowitz [19]). The relationship is as follows:

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt, \quad \operatorname{Re}(c) > \operatorname{Re}(b) > 0.$$

This leads to

$$\begin{aligned} \int_0^r S_{\mathcal{K}}^{n-1}(t)dt &= \frac{1}{2} S_{\mathcal{K}}^n(r) \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+2}{2})} {}_2F_1\left(\frac{1}{2}, \frac{n}{2}; \frac{n+2}{2}; \mathcal{K} S_{\mathcal{K}}^2(r)\right) = \\ &= \frac{1}{2} S_{\mathcal{K}}^n(r) \frac{1}{\Gamma(\frac{1}{2})} \sum_{j=0}^{\infty} \frac{\Gamma(\frac{1}{2}+j)}{\frac{n}{2}+j} \frac{\mathcal{K}^j S_{\mathcal{K}}^{2j}(r)}{j!}, \end{aligned}$$

and the proposition is proved. $\square$

Proposition A3. Let $\operatorname{vol}(S_r(\theta))$ be the volume of a ball with center θ and radius r , on a manifold with sectional curvatures bounded from below by κ and from above by $\mathcal{K}$. Then

$$\operatorname{vol}_{\kappa}(S_r) \geq \operatorname{vol}(S_r(p)) \geq \operatorname{vol}_{\mathcal{K}}(S_r),$$

where $\operatorname{vol}_{\kappa}(S_r)$ and $\operatorname{vol}_{\mathcal{K}}(S_r)$ are, respectively, the volumes of balls of radius r and arbitrary centers in manifolds with constant sectional curvatures κ and $\mathcal{K}$.

Proof. If we integrate, from ρ_0 to ρ , the inequalities in Bishop's comparison theorems, we obtain

$$\frac{S_{\kappa}^{n-1}(\rho)}{S_{\kappa}^{n-1}(\rho_0)} \geq \frac{\sqrt{g(\rho, u)}}{\sqrt{g(\rho_0, u)}} \geq \frac{S_{\mathcal{K}}^{n-1}(\rho)}{S_{\mathcal{K}}^{n-1}(\rho_0)}.$$

Moreover,

$$\lim_{\rho_0 \rightarrow 0} \sqrt{g(\rho_0, u)} \frac{S_{\kappa}^{n-1}(\rho)}{S_{\kappa}^{n-1}(\rho_0)} \geq \sqrt{g(\rho, u)} \geq \lim_{\rho_0 \rightarrow 0} \sqrt{g(\rho_0, u)} \frac{S_{\mathcal{K}}^{n-1}(\rho)}{S_{\mathcal{K}}^{n-1}(\rho_0)},$$

and, since

$$\lim_{\rho_0 \rightarrow 0} \frac{\sqrt{g(\rho_0, u)}}{S_{\kappa}^{n-1}(\rho_0)} = \lim_{\rho_0 \rightarrow 0} \frac{\sqrt{g(\rho_0, u)}}{S_{\mathcal{K}}^{n-1}(\rho_0)} = \Omega(u),$$

we conclude that

$$S_{\kappa}^{n-1}(\rho)\Omega(u) \geq \sqrt{g(\rho, u)} \geq S_{\mathcal{K}}^{n-1}(\rho)\Omega(u).$$

$\square$

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