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Article

On General Covariance

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Abstract

Two geometrical problems of negative time metric and abuse of distance factors for angular coordinates and other two physical problems of revisit redshift and covariant acceleration were put forward to investigate the traditional frames of general relativity. It is found that sub-indexes of Christoffel symbols in gravitational fields are not really alterable. The concept of trajectory derivative was carried out to clarify the derivatives on motion trajectories which perform far from field derivatives. Calculations on trajectory derivatives of frequency shift and acceleration lead to conclusions that light speed keeps general covariance in gravitational fields but light energy momentum would not, may as well, the motions of massive matters in gravitational fields do not perform general covariance thoroughly. The conservativeness of light angular momentum has been discovered in most surprising forms, as well as that of massive matters. Renovated kinematic equations for light ray propagations and massive matter motions have been carried out that forcefully impact the traditional methodologies on solutions of trajectory and time spending. Dynamic models of fluid planet rings were founded to interpret the evolutions of accretions of quasars and active galactic nuclei. Consequently, the mechanism of relativistic release was raised up based on light speed covariance and energy conservation, although it has not been completely proved. But the equations on relativistic release and relativistic frequency shifts so far as the line widths of emission and absorption could be astonishingly verified in observations, especially on the predictions of the broad line regions and narrow line regions. It could be imagined that the spectrums of relativistic emission and absorption may have been involved with fantastic mystery of matter's intrinsic structures that we know less.

Keywords: general covariance; general relativity; gravitational redshift; gravitational acceleration; differential geometry; covariant derivative; inequality of Christoffel symbols of alternative sub-index; gravitational metric; trajectory derivative; geodesic line; Lagrangian; energy momentum conservativeness; general mass equation; light ray angular momentum equation; planet ring model; relativistic release; relativistic emission; relativistic absorption; relativistic redshift; broad line region; narrow line region

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1. Preface

Einstein carried out the equivalence principle after discussing the equivalence of gravitational mass and inertial mass, and then he generalized the equivalence principle to create general relativity [1], which predicts same physics in curve space of gravity geometrization as that in no gravity space, that could be called general covariance. Theoretically, general covariance should include but not be limited in the performances of motion inertia, energy and momentum conservations as well as equilibrium states in complicated systems. However, we will find that quite amount of observations and evidences perform against general covariance.

It is believed that Riemannian geometry has been employed in general relativity for gravity geometrization [2]. But in fact, a transformation of a space does not really determine physics, what on earth the realities do, except the abuse of geometrics. It is said that not only the geometry but also the general covariance that could be the matters. In fact, we could find out plenty of contradictions in classical theory of general relativity. It could be verified that even the motions of matters freefalling on the Earth cannot be well interpreted in the frames of the classical theory. So that it is time to sponsor a series of inspections and perceptions to insight into the topics on general covariance.

The gravitational redshift and gravitational acceleration are of the two typical effects that gravity acts on the light rays and massive matters respectively. Hence, researches on these topics would be greatly forceful to probe into the investigations and realizations on general covariance.

To discover realities is more important than to carry out new equations and theorems. I do not think these scientific investigations and perceptions will promptly reach the final truth, but any efforts and consistence involved would cast deep insight into realities that might bring about approaching, opportunities and probabilities.

2. Geometrical and Physical Problems in General Relativity

2.1. Two Geometrical Problems

The invariant distance of Schwarzschild solution of one source fields could be written as

$$ds^2 = -(1-\frac{r^*}{r})(cdt)^2 + (1-\frac{r^*}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \quad (1)$$

In most publications, covariant metrics for the transformation were determined to be

$$g_{00} = -(1-\frac{r^*}{r}),\ g_{11} = (1-\frac{r^*}{r})^{-1},\ g_{22} = r^2,\ g_{33} = r^2 sin^2\theta \quad (2)$$

with coordinate differentials

$$dx^0 = cdt,\ dx^1 = dr,\ dx^2 = d\theta,\ dx^3 = d\varphi, \quad (3)$$

so that the invariant distance could be written brief form that

$$ds^2 = g_{ij} dx^i dx^j \quad (4)$$

It seems reasonable and natural in expressions and calculations. However, the negative sign in g_{00} does not come from transformation. It is the definition inherited from Minkowski space in which the distance is

$$d\varsigma^2 = -(cdt)^2 + dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \quad (5)$$

In general relativity, Pseudo Riemannian Space is just transformed from Minkowski space, thereby, the previous could be called covariant space and the latter contra variant space.

The negative sign in Minkowski space is physical determined setting. Pseudo Riemannian Space keeps the negative form of time item is of physical requirement rather than the consequence of space transformation especially of that of gravity. In fact, sometimes, the invariant distance could be expressed as

$$ds^2 = (1-\frac{r^*}{r})(cdt)^2 - (1-\frac{r^*}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\varphi^2 \quad (6)$$

so as well, the metrics g_{11}, g_{22} and g_{33} could neither be determined negative.

Nevertheless, the metrics r^2 and $r^2 sin^2\theta$ in Eq. (1-2) are exactly the distance factors of spherical coordinate systems. We will see that they could be drawn from a transformation from original spherical space $(dr, d\theta, d\varphi)$ to distance expressed spherical space$(r, rd\theta, rsin\theta d\varphi)$. Thus, those metrics are exactly of the composite metrics rather than pure gravitational metrics.

If the composite metrics were employed in analysis on space properties, such as derivatives, Christoffel symbols, curvatures, frequency shifts and accelerations etc., they would represent the properties of composite transformations. Once the composite metrics were presumingly employed for interpretations of pure gravitational effects, that will bring about mistakes inevitably.

In most of publications on traditional theory, both of these two geometrical problems have been involved in various of analysis and calculations. Some of them might have led to unbelievable wrong conclusions. It is valuable and very important to sponsor more investigations to deeply insight into the performances and subsequences of metric abuse in traditional frames.

It is suggested that the positive metric g_{00} and pure gravitational metrics for invariant distance Eq. (1) that

$$g_{00} = (1-\frac{r^*}{r}),\ g_{11} = (1-\frac{r^*}{r})^{-1},\ g_{22} = 1,\ g_{33} = 1 \quad (7)$$

could be employed instead of the composite metrics in Eq. (2) in most cases, with coordinate differentials

$$dx^0 = cdt,\ dx^1 = dr,\ dx^2 = rd\theta,\ dx^3 = rsin\theta d\varphi, \quad (8)$$

Thus, the invariant distance could be expressed as

$$ds^2 = -g_{00}(cdt)^2 + g_{11}dr^2 + g_{22}r^2d\theta^2 + g_{33}r^2sin^2\theta d\varphi^2$$
$$= -g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 \quad (9)$$

This suggestion will help to renovate those distorted equations and misleading conclusions in classical frames.

2.2. Two Physical Problems

2.2.1. Newtonian Gravitational Redshift

The equation of gravitational redshift can be drawn via Doppler redshift in a thought experiment that a light ray be emitted from the ceiling to the bottom or that is reversely performed from bottom to the ceiling, in a freely falling elevator cabin in a center source field [3]. It could be verified that any observer in the cabin would detect no frequency shift whatever the ways the light emitted. Suppose another observer outside the cabin keeping rest so that to have a relative velocity against the cabin, who will then detect a frequency shift other than the freefalling observer. What I want to say is that whoever freefalling will detect no frequency shift, no matter inside or outside the cabin. It is the relative motion that eliminates the gravitational frequency shift. Once the relative velocities catch up to a relativistic velocity, the gravitational frequency shift will then cannot be eliminated anymore.

As a light ray is down ward emitted from a cabin ceiling and at the same time the cabin is released to freely fall, the light front should spend a very short time interval to reach the bottom that

$$\Delta t = \frac{\Delta H}{c} \quad (10)$$

where, ΔH is the distance that the light ray travels from the start to the end which may approximately equal to the height of the cabin, c is light speed.

Thus, the cabin velocity increase is

$$\Delta v = a\Delta t \quad (11)$$

where a is gravitational acceleration, which approximately equals to gravity g case velocity does not reach relativistic level. We know it exactly has a minus value as its direction pointing to center source.

Doppler redshift is frequency shift between two observers that one has a relative motion to another, to detect frequency. Here the rest observer outside would have a relative velocity $-\Delta v$ or $+\Delta v$ to the cabin so that when they receive the light ray at the same time with the inner, the Doppler redshift could be calculated as

$$z_D = \frac{\Delta v}{c} = \frac{g\Delta H}{c^2} = \frac{\Delta\emptyset}{mc^2} \quad (12)$$

With the calculation of Doppler redshift, we know that the gravitational redshift has happened in the same value. Notwithstanding, I prefer to suggest a new methodology to get an equation for gravitational redshift, in which a proposal should be given that every photon at any position in a center source field could be assumed to have experienced a travel from a farthest point to the current position. This attempt may bring about more physical significances and comprehensive understandings.

A photon in one source field at position r with frequency $\nu_{0(r)}$ is set to have an imaginary primary frequency $\nu_{0(\infty)}$ at a farthest point

$$E_\infty = h\nu_{0(\infty)} \quad (13)$$

where h is Planck's constant. NB, we are not talking about quantum character of photons so that photonic energy momentum mentioned refers to statistic quantities.

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The corresponding dynamic mass comes from the mass-energy equation is

$$m = E_\infty c^{-2} \quad (14)$$

where c is light speed.

Then the gravitational potential at position r, especially as is in weak field with $r \gg r^*$, could be written as

$$\emptyset_r = \int_\infty^r \frac{GMm_r}{r^2} dr = -\frac{GM\bar{m}}{r} \approx -\frac{GMm}{r} = -\frac{r^*}{2r} mc^2 \quad (15)$$

where G is gravitational constant, M is the mass of the center source with $M \gg m_r$, m_r is photonic mass at position r, $\bar{m}$ is the mean mass for the integral and it could be approximately replaced by m case in weak field, and r^* is Schwarzschild radius written as $r^* = 2GM/c^2$.

The current dynamic energy is the summation of primary dynamic energy and released potential

$$E_r = h\nu_{0(r)} = E_\infty + [\emptyset_\infty - \emptyset_r] \approx mc^2 + \frac{r^*}{2r} mc^2 \approx h\nu_{0(\infty)}\left(1 + \frac{r^*}{2r}\right) \quad (16)$$

Case a light ray travels from positions r_1 to r_2, as is shown in Figure 1, gravitational redshift happens.

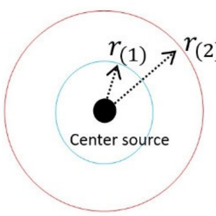

Figure 1. Center source field.

Gravitational redshift could be defined traditionally as

$$z_{g\lambda} = \frac{\lambda^1{}_{(2)} - \lambda^1{}_{(1)}}{\lambda^1{}_{(1)}} = \frac{\nu_{0(1)} - \nu_{0(2)}}{\nu_{0(2)}} \approx \frac{r^*}{r_{(1)}} \cdot \frac{r_{(2)} - r_{(1)}}{2r_{(2)} + r^*} \quad (17)$$

Considering weak field effects, the gravitational redshift is

$$z_{g\lambda} \approx r^* \cdot \frac{r_{(2)} - r_{(1)}}{2r_{(1)} r_{(2)}} = \frac{r^*}{2r_{(1)}} - \frac{r^*}{2r_{(2)}} = \frac{\emptyset_{(2)} - \emptyset_{(1)}}{mc^2} = \frac{\Delta\emptyset}{mc^2} \quad (18)$$

They are the forms of frequency shift of wave length based expression in weak fields, called redshift, where λ^1 and ν_0 are wave length and frequency tensors in contra variant space. Of course, a frequency shift could also be defined based on frequency. But we have been used to the forms previous in tradition. In the wave length based equation, redshift may go up to more than 1.0, while blueshift must have been limited in -1 to 0. Case in the form of frequency based equations, one could get blueshift greater than 1.0 but redshift limited in -1 to 0.

We could also carry out new forms of frequency shift for conveniences in special discussions, that a differential of wave length based redshift could be defined as

$$dz_\lambda = \frac{d\lambda^1{}_{(r)}}{\lambda^1{}_{(r)}} = dln\lambda^1{}_{(r)} \quad (19)$$

so that the integral form for $r_{(1)}$ to $r_{(2)}$

$$z_\lambda = ln\lambda^1{}_{(r)} \Big|_{r_{(1)}}^{r_{(2)}} = ln\frac{\lambda^1{}_{(2)}}{\lambda^1{}_{(1)}} = ln\frac{\nu_{0(1)}}{\nu_{0(2)}} \quad (20)$$

or a differential form of frequency based

$$dz_\nu = \frac{d\nu_{0(r)}}{\nu_{0(r)}} = dln\nu_{0(r)} \quad (21)$$

so that

$$z_\nu = ln\nu_{0(r)}\Big|_{r_{(1)}}^{r_{(2)}} = ln\frac{\nu_{0(2)}}{\nu_{0(1)}} = ln\frac{\lambda^1{}_{(1)}}{\lambda^1{}_{(2)}} = -ln\frac{\lambda^1{}_{(2)}}{\lambda^1{}_{(1)}} \quad (22)$$

Considering the energy expression of the Eq. (571) in the following sections into the Eq. (20), one can gain the redshift result as same as the Eq. (18). But they are based on different definitions of frequency shift so that they would perform physical implications a little different.

It is said that after the definitions of Eq. (20) and Eq. (22) there is

$$z_\lambda = -z_\nu \quad (23)$$

We have seen that they have shown difference from traditional equations. But for small frequency shift, both the two integral equations could be use instead of traditional equation.

Moreover, with Eq. (16), the gravitational frequency differential

$$d\nu_{0(r)} \approx d[\nu_{0(\infty)}\left(1+\frac{r^*}{2r}\right)] = -\frac{r^*}{2r^2}\nu_{0(\infty)}dr \quad (24)$$

Then the differential frequency shift goes

$$dz_{g\nu} = \frac{d\nu_{0(r)}}{\nu_{0(r)}} \approx \frac{-\frac{r^*}{2r^2}}{1+\frac{r^*}{2r}}dr = -\frac{r^*dr}{r(2r+r^*)} \quad (25)$$

This equation will be also taken into further discussions in the next sections. In following sections, the subscripts of frequency shift symbols will be neglected for convenience and in most cases we use the concept redshift to present frequency shifts.

2.2.2. Errors in the Equation of the So-Called Revisit Gravitational Redshift

A thought experiment has ever been employed to present the concept of the so-called revisit gravitational redshift [3,4], in which a pulse of light ray is supposed to be emitted from position 1, lasting for a time interval $\Delta t_{(1)}$, and be received at position 2 within a time interval $\Delta t_{(2)}$. The world lines of the of photons are shown in Figure 2. We then know that the two drawn lines are literally the world lines of the first photon and the final photon of the light pulse.

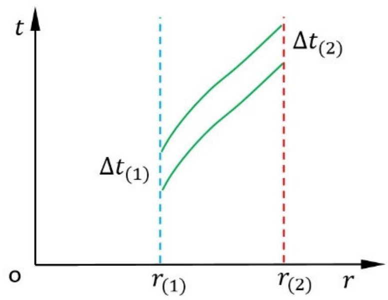

Figure 2. World lines for a pulse of photons.

Believing that the two world lines besides the $\Delta t_{(1)}$ and $\Delta t_{(2)}$ are parallel, it is known that the time interval $\Delta t_{(1)}$ equals to that of $\Delta t_{(2)}$. As the light frequency being inversely proportional to proper time interval $\Delta\tau$, the proper forms of frequency ratio were written in some textbooks as [3,4]

$$\frac{\nu_{(1)}}{\nu_{(2)}} = \frac{\Delta\tau_{(2)}}{\Delta\tau_{(1)}} = \frac{\sqrt{g_{00(2)}}\Delta t_{(2)}}{\sqrt{g_{00(1)}}\Delta t_{(1)}} = \frac{\sqrt{g_{00(2)}}}{\sqrt{g_{00(1)}}} \approx 1 + \frac{\phi_{(2)}-\phi_{(1)}}{mc^2} \quad (26)$$

where $\nu_{(1)}$ and $\nu_{(2)}$ are proper frequencies corresponding to position 1 and position 2, g_{00} is time metric of Schwarzschild, if the value is defined negative initially it should be modified to be positive. The right hand side of this equation is an approximate result with Schwarzschild solution of metric g_{00} in condition that $r_2 > r_1 \gg r^*$.

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Thus, the so-called revisit redshift is

$$z_{revisit} = \frac{\nu_{(1)}}{\nu_{(2)}} - 1 \approx \frac{\Delta\phi}{mc^2} \quad (27)$$

It is seemingly that the revisit form of equation for gravitational redshift was worked out.

But there are quite many errors in above equations. (i) The two world lines belong to the first photon and the final photon respectively, thus they might be controlled by emitter, so that the time intervals between the two lines do nothing with any light frequencies. (ii) Any time intervals between neighboring photons could be randomly assigned, so that these intervals also do nothing with any light frequencies. (iii) Frequency of a photon is the reciprocal of its photonic period and is the intrinsic property of a photon, so that it is independent to its positions relating to other photons and any variation of the frequency will not change its position in the pulse of photons. (iv) Detections of frequencies must involve with wave numbers and time intervals together, while this equation has made a mistake by comparing time intervals only. (v) It is said that the time intervals $\Delta t_{(1)}$ and $\Delta t_{(2)}$ do nothing with the physical events of frequency shift, they are just the records of the physical event of emission and receiving.

2.2.3. Further Investigation into the Revisit Gravitational Redshift

Following the rules in classical physics, the tensor of light wave period is a tensor with upper index

$$T^0 = \frac{dt}{dn} \quad (28)$$

where, T^0 is contra variant light period that is described by coordinate time, and dt is coordinate time lasting in which the number of dn waves may have traveled across a specific position. And the proper form of wave period

$$T = \frac{d\tau}{dn} \quad (29)$$

where, $d\tau$ is proper time lasting for dn number of waves to cross the position. So that there is

$$T = e_0 T^0 \; or \; T^0 = e^0 T \quad (30)$$

where, e_0 is the first component of covariant time base, and it should be noted that the base $\boldsymbol{e}_0$ is a vector but the component e_0 is not vector even though it is still a tensor. One can get more understandings for these expressions I would sponsor here and in followings.

In this way, we know that different values of the contra variant period and proper period correspond to a same physical issue of wave counting. Then it leads to a frequency tensor

$$\nu_0 = \frac{1}{T^0} = e_0 \nu = \frac{dn}{dt} \quad (31)$$

We have seen that, ν_0 is called covariant tensor traditionally, and ν is a proper tensor. This may have brought about confusions in that ν is actually covariant but ν_0 has been named the name yet. I am not going to change the naming methodology thoroughly right now because that may cause more difficulties and sounds more trivial.

Generally, some pure one-order tensors seem to be infinite small quantities such as dr, but for ν_0, it is of dn divided by dt, so that it is not an infinite small quantity. As for velocity tensors, they are really mixed tensors.

Theoretically, the covariant derivative of a frequency in a falling process into a center source can be written as

$$\frac{D\nu_0}{\partial r} = \frac{\partial \nu_0}{\partial r} - \Gamma^0_{10}\nu_0 \quad (32)$$

where, $\Gamma^\lambda_{\nu\mu}$ is Christoffel symbols.

We know that, as has been presented in Eq. (24), the contra variant derivative is

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$$\frac{\partial \nu_0}{\partial r} = -\frac{r^*}{2r^2}\nu_{0(\infty)} \quad (33)$$

Let us try to calculate the Christoffel symbols in Eq. (32) that

$$\Gamma_{10}^0 = \frac{1}{2} g^{0\lambda}\left(\frac{\partial g_{1\lambda}}{\partial x^0} + \frac{\partial g_{0\lambda}}{\partial x^1} - \frac{\partial g_{10}}{\partial x^\lambda}\right) \quad (34)$$

The Einstein summation convention has been and will be adopt unless additional declarations.

It is found that only in the condition of $\lambda = 0$ there is a nonvanishing item in the bracket of right hand side of Eq. (34), so that with Schwarzschild metrics $g_{00} = -(1 - \frac{r^*}{r})$ as is given traditionally, it turns to be

$$\Gamma_{10}^0 = \frac{1}{2} g^{00}\frac{\partial g_{00}}{\partial x^1} = \frac{1}{2}\left(-1 + \frac{r^*}{r}\right)^{-1}\left(-1 + \frac{r^*}{r}\right)' = \frac{r^*}{2r^2}\left(1 + \frac{r^*}{r - r^*}\right) \quad (35)$$

The covariant derivative will be calculated to be

$$\frac{D\nu_0}{\partial r} = -\frac{r^*}{2r^2}\nu_{0(\infty)} - \frac{r^*}{2r^2}\left(1 + \frac{r^*}{r - r^*}\right)\nu_0$$

$$= -\frac{r^*}{2r^2}\nu_{0(\infty)} - \frac{r^*}{2r^2}\left(1 + \frac{r^*}{r - r^*}\right)\left(1 + \frac{r^*}{2r}\right)\nu_{0(\infty)} \quad (36)$$

We could get an approximate solution for weak field that

$$\frac{D\nu_0}{\partial r} \approx 2\frac{\partial \nu_0}{\partial r} \quad (37)$$

As we have seen, it shows that the values of covariant redshift doubles that of the contra variant redshift.

2.2.4. Additional Discussions on the Thought Experiment of Freefalling Elevator Cabin

The thought experiment that observer in freefalling elevator cabin will detect no frequency shift is usually employed to discuss the equivalent principle for the support of general covariance. But in fact, that issue does nothing with covariance. It is just because that the gravitational redshift happens to be offset by Doppler redshift. In fact, this experiment is a comprehensive event that relates both to gravitational redshift and gravitational acceleration.

We know that in a freefalling cabin, freefalling observer will not detect any frequency shift no matter the light ray emitter is on the bottom or on the top to emit up or down to the receiver. Nevertheless, even if the light ray is not vertical, freefalling observer would also observe no frequency shift in the cabin.

For a case that a light ray is emitted to the top with an angle θ to the vertical line, the time interval for light ray from bottom to the top is

$$\Delta t = \frac{1}{c}\frac{\Delta H}{\cos\theta} \quad (38)$$

And the velocity increase of freefalling cabin is

$$\Delta v = a\Delta t = \frac{g}{c}\frac{\Delta H}{\cos\theta} \quad (39)$$

where, a is geometrical acceleration and g is gravity or the so called gravitational acceleration. We will see that it does not always equal to geometrical acceleration in some cases.

Then the velocity increase component at the direction of light ray

$$\Delta v_l = \Delta v \cos\theta = g\frac{\Delta H}{c} \quad (40)$$

So that we get the Doppler redshift again as

$$z_{\mathrm{D}} = \frac{\Delta v_l}{c} = \frac{g\Delta H}{c^2} = \frac{\Delta\emptyset}{mc^2} \quad (41)$$

In fact, the Doppler redshift for the detector in freefalling cabin could eliminate the gravitational frequency shift, even if the emitter is either outside of the cabin or with any low initial velocity. The real reason for non-detectable frequency shift in freefalling cabin is that the relative motion formed Doppler redshift just has a minus approximate value of gravitational frequency shift.

I prefer to put forward the case that the gravitational frequency shift cannot be eliminated by Doppler redshift. In the case that cabin has a relativistic initial velocity, the geometrical acceleration is not the total gravitational acceleration again as that in Eq. (380) as

$$a = \eta g, \eta < 1 \quad (42)$$

Thus, for light rays passing across the cabin there is the Doppler velocity of detector

$$\Delta v_l = a\Delta t = \eta g \frac{\Delta H}{c} \quad (43)$$

so that

$$z_{\mathrm{D}} = \frac{\Delta v_l}{c} = \eta \frac{g\Delta H}{c^2} = \eta \frac{\Delta\emptyset}{mc^2} \quad (44)$$

While the gravitational frequency is still the form

$$z_g = \frac{g\Delta H}{c^2} = \frac{\Delta\emptyset}{mc^2} \quad (45)$$

so that in this case

$$z_{\mathrm{D}} \neq z_g \quad (46)$$

We will see that in some situations in freefalling cabin one could detect frequency shift again, so that the thought experiment cannot support general covariance thoroughly. Of course, one can continue to argue that relativistic motion may bring about more sophisticated conditions on frequency shift.

2.2.5. An Investigation into Gravitational Acceleration

For a freefalling massive matter in gravitational field, the component of the velocity at the direction of radius of the center source is

$$V^1 = \frac{dx^1}{d\tau} \quad (47)$$

where, dx^1 is number one component of vector of moving distance, in this case it is dr.

There is the covariant derivative at one of coordinate direction

$$\frac{DV^1}{\partial\lambda} = \frac{\partial V^1}{\partial\lambda} + \Gamma^1_{\lambda\mu} V^\mu \quad (48)$$

Because $dx^0 = cdt$, and $dt = e^0 d\tau$, where e^0 is the nonvanishing component of the base $\boldsymbol{e}^0$. Thus, the covariant derivative by τ is

$$\frac{DV^1}{d\tau} = \frac{dV^1}{d\tau} + \frac{dx^0}{d\tau}\Gamma^1_{0\mu} V^\mu = a^1 + ce^0\Gamma^1_{0\mu} V^\mu \quad (49)$$

where, $a^1 = \frac{dV^1}{d\tau}$ is the contra variant acceleration of the matter, c is light speed.

NB, accelerations we have mentioned and will discuss later refer to geometrical accelerations, which will be something different from gravity g, in that the latter sometimes may also be called gravitational accelerations in some cases but in fact matters may not experience accelerating as that.

With the equation of Christoffel symbols

$$\Gamma^1_{0\lambda} = \frac{1}{2} g^{1\rho}\left(\frac{\partial g_{0\rho}}{\partial x^\lambda} + \frac{\partial g_{\lambda\rho}}{\partial x^0} - \frac{\partial g_{0\lambda}}{\partial x^\rho}\right) \quad (50)$$

case $\lambda = 1$ in this equation, there is $\Gamma^1_{01} = 0$. Then considering the condition of $\lambda = 0$, it is

$$\Gamma^1_{00} = \frac{1}{2} g^{1\rho}\left(\frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{0\rho}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\rho}\right) \quad (51)$$

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In this condition, only in the case of $\rho = 1$ there is the nonvanishing item in the bracket of right hand side, so that with Schwarzschild metrics $g_{00} = -(1 - \frac{r^*}{r})$ and $g^{11} = (1 - \frac{r^*}{r})$ given in tradition, there is

$$\Gamma^1_{00} = \frac{1}{2} g^{11}(-\frac{\partial g_{00}}{\partial x^1}) = \frac{1}{2}(1 - \frac{r^*}{r})(1 - \frac{r^*}{r})' = \frac{1}{2}(1 - \frac{r^*}{r})\frac{r^*}{r^2} \quad (52)$$

And with $V^0 = \frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = e^0 c$, Eq. (49) turns to be

$$\frac{DV^1}{d\tau} = a^1 + ce^0\Gamma^1_{00}V^0 = a^1 + e^0e^0c^2(1 - \frac{r^*}{r})\frac{r^*}{2r^2} \quad (53)$$

As a matter freefalls on to the earth, its acceleration could be calculated to be

$$a^1 = -e^0e^0\frac{GM}{r^2} = -e^0e^0c^2\frac{r^*}{2r^2} \quad (54)$$

Thus, there is the weak field solution

$$\frac{DV^1}{d\tau} \approx 0 \quad (55)$$

What deserving of more discussions is that the metric g_{00} designated to be negative form might have been involved with errors. In fact, metrics are defined to interpret transformations between contra variant space and covariant space for the geometrical and physical quantities, such as time and distance intervals, velocities, curvatures and covariant derivatives etc. As has discussed in previous, the minus sign in the equation of invariant distance is not of the results of transformation. The correct form of g_{00} must be of positive value. We could have more detailed discussions about this issue in next sections.

With the revised metrics $g_{00} = (1 - \frac{r^*}{r})$ and $g^{11} = (1 - \frac{r^*}{r})$, Γ^1_{00} would be drawn to be a negative value with respect to that in Eq. (52)

$$\Gamma^1_{00} = \frac{1}{2} g^{11}(-\frac{\partial g_{00}}{\partial x^1}) = \frac{1}{2}(1 - \frac{r^*}{r})(-1 + \frac{r^*}{r})' = -\frac{1}{2}(1 - \frac{r^*}{r})\frac{r^*}{r^2} \quad (56)$$

One can calculate the covariant derivative in Eq. (49) to be

$$\frac{DV^1}{d\tau} = a^1 + ce^0\Gamma^1_{00}V^0 = a^1 - e^0e^0c^2(1 - \frac{r^*}{r})\frac{r^*}{2r^2} \approx 2a^1 \quad (57)$$

This result is the real consequence of the equation of Christoffel symbols so that it sounds more reasonable with respect to the result in Eq. (37). But the calculations have revealed more problems in that the entire calculation shows irrelevant with velocity itself, in that at the start we expected a variation depending on velocity in the equation of covariant derivatives.

2.3. Discussions and Controversies

We have made more discussions on metric abuse and the mistaken calculations on the revisit gravitational redshift and the covariant acceleration of freefalls.

In the view of general covariance, the covariant derivatives of light frequency should go nonvanishing, so do the accelerations of freefalling matters. Someone may argue that the frequency is not a tensor, but that makes no sense because light frequency is the reciprocal of its wave period which involves with time coordinate.

What I urgently want to say is that these discussions are not enough. The most significant problem is that, the items in original covariant differentials in Eq. (49) show that matter's velocities expected to be multiplied with the base differentials, have been calculated to do nothing with the realistic velocity. We should know that the multiplied items in Eq. (48) originally indicate base variation ratio multiplied with the very tensors, but the final equation has skipped the influence of velocity that does lead to contradictions, and the time speed V^0 is a virtual velocity which might have been abused. Furthermore, it is so strange that the acceleration Eq. (53) and Eq. (57) have been treated to do nothing with Γ^1_{11}. There must be something wrong. These contradictions really bothered me until it is occasionally gone through one day, that the real problem is deeply hidden in the equations of Christoffel symbols.

Notwithstanding, a differential of velocity is the differential of that on the trajectory of matter's motion. Thus, the covariant time derivatives of something moving in rest fields cannot be treated directly as ordinary derivatives anymore as in Eq. (49) to Eq. (57). I am going to carry out detailed discussions on trajectory derivatives in next sections so that to interpret covariant time derivatives correctly.

3. Investigations on Christoffel Symbols and Bases

3.1. Classical Equations of Christoffel Symbols

Christoffel symbols have been defined as

$$\frac{\partial \boldsymbol{e}^{\mu}}{\partial x^{\nu}} = -\Gamma^{\mu}_{\nu\lambda}\boldsymbol{e}^{\lambda},\ \frac{\partial \boldsymbol{e}_{\mu}}{\partial x^{\nu}} = \Gamma^{\mu}_{\nu\lambda}\boldsymbol{e}_{\lambda} \quad (58)$$

There is nothing wrong with the definition in that the derivative of a base must have a direction and so that to be written as a linear combination of the total bases. The key problem is what the Christoffel symbols are.

In this and following sections, all symbols of vectors and matrix would be boldly written while their components and that of other quantities may be simply written, no matter they are tensors or not.

In most conditions, Christoffel symbols could be discussed by the derivation of metrics as

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \frac{\partial}{\partial x^{\lambda}}(\boldsymbol{e}_{\mu}\cdot\boldsymbol{e}_{\nu}) = \frac{\partial \boldsymbol{e}_{\mu}}{\partial x^{\lambda}}\cdot\boldsymbol{e}_{\nu} + \boldsymbol{e}_{\mu}\cdot\frac{\partial \boldsymbol{e}_{\nu}}{\partial x^{\lambda}} = \Gamma^{\rho}_{\lambda\mu}\boldsymbol{e}_{\rho}\cdot\boldsymbol{e}_{\nu} + \Gamma^{\rho}_{\lambda\nu}\boldsymbol{e}_{\rho}\cdot\boldsymbol{e}_{\mu} \quad (59)$$

For the derivative forms with alternative indexes mathematically, there will be

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \Gamma^{\rho}_{\lambda\mu}g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu}g_{\rho\mu} = 0 \quad (60)$$

$$\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \Gamma^{\rho}_{\mu\lambda}g_{\rho\nu} - \Gamma^{\rho}_{\mu\nu}g_{\rho\lambda} = 0 \quad (61)$$

$$\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} - \Gamma^{\rho}_{\nu\mu}g_{\rho\lambda} - \Gamma^{\rho}_{\nu\lambda}g_{\rho\mu} = 0 \quad (62)$$

In the case the so-called torsions $S^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\nu\mu}$ are set zero, the summation of the previous two equations minus the last one that

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} - 2\Gamma^{\rho}_{\lambda\mu}g_{\rho\nu} = 0 \quad (63)$$

Thereafter the equations of Christoffel symbols will be solved as [2,3,5]

$$\Gamma^{\rho}_{\lambda\mu} = \frac{1}{2}g^{\rho\nu}\left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\right) \quad (64)$$

Generally, Eq. (60) to Eq. (62) could also be rewritten as the original forms

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial \boldsymbol{e}_{\mu}}{\partial x^{\lambda}}\cdot\boldsymbol{e}_{\nu} - \frac{\partial \boldsymbol{e}_{\nu}}{\partial x^{\lambda}}\cdot\boldsymbol{e}_{\mu} = 0 \quad (65)$$

$$\frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial \boldsymbol{e}_{\lambda}}{\partial x^{\mu}}\cdot\boldsymbol{e}_{\nu} - \frac{\partial \boldsymbol{e}_{\nu}}{\partial x^{\mu}}\cdot\boldsymbol{e}_{\lambda} = 0 \quad (66)$$

$$\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \boldsymbol{e}_{\mu}}{\partial x^{\nu}}\cdot\boldsymbol{e}_{\lambda} - \frac{\partial \boldsymbol{e}_{\lambda}}{\partial x^{\nu}}\cdot\boldsymbol{e}_{\mu} = 0 \quad (67)$$

We will find that in some conditions the torsions do not always equal to zero. It is said that the mixed derivatives of bases $\frac{\partial \boldsymbol{e}_{\mu}}{\partial x^{\nu}}$ and $\frac{\partial \boldsymbol{e}_{\nu}}{\partial x^{\mu}}$ do not always equal, and then the Christoffel symbols with mixed subscripts $\Gamma^{\rho}_{\mu\nu}$ and $\Gamma^{\rho}_{\nu\mu}$ do not always equal, so that the Eq. (64) might be invalid in those conditions.

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3.2. Bases of Space Transformation

Any points in a Riemannian space of Riemannian manifold of dimension n has a neighborhood homeomorphic to a subset of Euclidean space of dimension n, so that there must be probable maps between the neighborhoods and the corresponding subsets. It is just to say that the coordinates of any points in Riemannian space could be expressed with the coordinates of corresponding points of Euclidean space, and reversely. If a part or the entire of a Riemannian space are continuous and differentiable, Euclidean coordinate lines could be drawn in the part or the entire of the Riemannian space. On the other side, coordinate lines of Riemannian space could also be drawn in the corresponding Euclidean space. For convenience, the Riemannian space could be called covariant space, and the corresponding Euclidean space could be called contra variant space. A contra variant space is curved in the view of its covariant space, and the covariant space is also curved in the view of contra variant space.

It is obvious that transformations of spaces are actual coordinate transformations. These transformations could happen between covariant space and contra variant space, as well as they could happen among homeomorphic Riemannian spaces. Coordinate transformations may perform in the way with unequal metrics as well as the way with equal metrics.

The more effective method for coordinate transformation is to define bases and distances for spaces. Two examples would be presented firstly for definitions and for following discussions.

Example 1. *Bases of Riemannian manifold of super surface*

The derivative vectors of 3-dimensional surfaces were usually employed to form bases in classical differential geometry. The curve space has an extra dimension than a plane space that could be called the super surface. The Riemannian manifold of the super surface in the 3-dimensional space (u, v, w) would have a homeomorphic Euclidean space (x, y) in the space (x, y, z). The coordinate lines x and y in contra variant space could be transformed to be $\xi(x)$ and $\eta(y)$ in covariant space, while u, v and w in covariant space be transformed to be $\alpha(u)$, $\beta(v)$ and $\gamma(w)$ in the space (x, y, z) as shown in Figures 3 and 4.

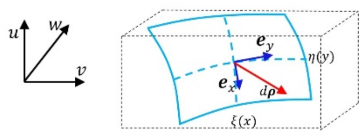

Figure 3. A super surface as covariant space in 3-dimensional space.

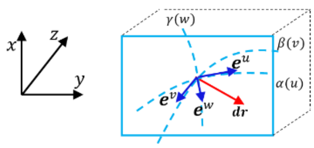

Figure 4. A coordinate plane (x, y) as contra variant space in 3-dimensional space.

The super surface could be determined by a vector function $\boldsymbol{\rho}$

$$\boldsymbol{\rho} = \boldsymbol{\rho}(u, v, w)|_{2\ of\ variables\ independent} \quad (68)$$

And the function could also be written as

$$\boldsymbol{\rho} = \boldsymbol{\rho}(\xi, \eta) \quad (69)$$

or

$$\boldsymbol{\rho} = \boldsymbol{\rho}(x, y) \quad (70)$$

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At the same time, the contra variant space could be defined by $\boldsymbol{r}$

$$\boldsymbol{r}=\boldsymbol{r}(x,y,z)|_{z=const.}=\boldsymbol{r}(\alpha,\beta,\gamma)|_{2\ of\ variables\ independent}=\boldsymbol{r}(x,y)=\boldsymbol{r}(\xi,\eta)\quad(71)$$

There will be varieties of available ways to develop the expressions of bases and distances. I prefer to put forward the followings might as well.

The way in super space:

In super space, the differential $d\boldsymbol{\rho}$ has 3 components

$$d\boldsymbol{\rho}=\begin{pmatrix}du\\dv\\dw\end{pmatrix}\quad(72)$$

That of differential $d\boldsymbol{r}$ could be simplified to be 2 dimensional because it just locates in the space (x,y)

$$d\boldsymbol{r}=\begin{pmatrix}dx\\dy\end{pmatrix}\quad(73)$$

To define a set of covariant bases for a position in covariant space by

$$\boldsymbol{e}_x=\frac{\partial\rho}{\partial x},\boldsymbol{e}_y=\frac{\partial\rho}{\partial y}\quad(74)$$

It should be pointed out that, in some publications, coordinate and vector symbols have been used reversely like $d\boldsymbol{\rho}$ and $d\boldsymbol{r}$, which would have brought about confusions.

In covariant space, the differential $d\boldsymbol{\rho}$ expressed by $d\boldsymbol{r}$ with covariant bases

$$d\boldsymbol{\rho}=dx\boldsymbol{e}_x+dy\boldsymbol{e}_y\quad(75)$$

Differential distance could be defined as

$$ds^2=du^2+dv^2+dw^2=d\boldsymbol{\rho}\cdot d\boldsymbol{\rho}=(dx\boldsymbol{e}_x+dy\boldsymbol{e}_y)\cdot(dx\boldsymbol{e}_x+dy\boldsymbol{e}_y)\quad(76)$$

If the bases are orthogonal, there is

$$ds^2=\boldsymbol{e}_x\cdot\boldsymbol{e}_xdx^2+\boldsymbol{e}_y\cdot\boldsymbol{e}_ydy^2\quad(77)$$

We have seen that the covariant bases are defined in covariant space to help contra variant coordinates to form covariant distances.

There are more complexities for a transformation between a super surface in 3-dimensional space and R^2 that the 2 covariant bases would have 3 components

$$\boldsymbol{e}_x=\begin{pmatrix}\frac{\partial u}{\partial x}\\\frac{\partial v}{\partial x}\\\frac{\partial w}{\partial x}\end{pmatrix},\boldsymbol{e}_y=\begin{pmatrix}\frac{\partial u}{\partial y}\\\frac{\partial v}{\partial y}\\\frac{\partial w}{\partial y}\end{pmatrix}\quad(78)$$

Define the contra variant bases for a point in the plane that

$$\boldsymbol{e}^u=\frac{\partial r}{\partial u},\boldsymbol{e}^v=\frac{\partial r}{\partial v},\boldsymbol{e}^w=\frac{\partial r}{\partial w}\quad(79)$$

The 3 contra variant bases all have 2 components as

$$\boldsymbol{e}^u=\begin{pmatrix}\frac{\partial x}{\partial u}\\\frac{\partial y}{\partial u}\end{pmatrix},\boldsymbol{e}^v=\begin{pmatrix}\frac{\partial x}{\partial v}\\\frac{\partial y}{\partial v}\end{pmatrix},\boldsymbol{e}^w=\begin{pmatrix}\frac{\partial x}{\partial w}\\\frac{\partial y}{\partial w}\end{pmatrix}\quad(80)$$

The differential $d\boldsymbol{r}$ expressed by $d\boldsymbol{\rho}$ in contra variant space

$$d\boldsymbol{r}=du\boldsymbol{e}^u+dv\boldsymbol{e}^v+dw\boldsymbol{e}^w\quad(81)$$

Of course, one can create transformation matrix to perform the relationship between $d\boldsymbol{r}$ and $d\boldsymbol{\rho}$ directly.

The distance could be defined as

$$d\varsigma^2 = dx^2 + dy^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = (du\boldsymbol{e}^u + dv\boldsymbol{e}^v + d\boldsymbol{e}^w) \cdot (du\boldsymbol{e}^u + dv\boldsymbol{e}^v + d\boldsymbol{e}^w) \quad (82)$$

If the bases are orthogonal

$$d\varsigma^2 = \boldsymbol{e}^u \cdot \boldsymbol{e}^u du^2 + \boldsymbol{e}^v \cdot \boldsymbol{e}^v dv^2 + \boldsymbol{e}^w \cdot \boldsymbol{e}^w dw^2 \quad (83)$$

One could imagine that in this condition you cannot give the relationship of metrics that g_{ii} equals to $1/g^{ii}$, in that the covariant bases have 3 components and contra variant bases have 2.

The way in tangent space:

Consequently, the issues could be simplified in tangent spaces. At a position $\boldsymbol{\rho}$ in the covariant space, there is a neighborhood which will be labeled with coordinate lines of $\xi(x)$ and $\eta(y)$, at the same time at the position $\boldsymbol{r}$, there is a corresponding neighborhood in contra variant space labeled with coordinate lines of x and y, as shown in Figures 5 and 6. Generally, coordinate lines could be set orthogonal. In most of publications, $\xi(x)$ and $\eta(y)$ were seen as x and y, but one should realize that the difference really matters.

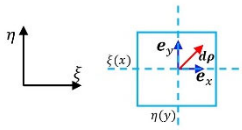

Figure 5. Covariant tangent space.

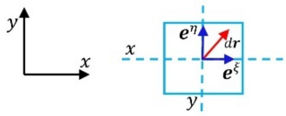

Figure 6. Contra variant tangent space.

One could define the differential vector in covariant space

$$d\boldsymbol{\rho} = \begin{pmatrix} d\xi \\ d\eta \end{pmatrix} \quad (84)$$

As a result, the bases

$$\boldsymbol{e}_x = \frac{\partial \boldsymbol{\rho}}{\partial x} = \begin{pmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \eta}{\partial x} \end{pmatrix}, \boldsymbol{e}_y = \frac{\partial \boldsymbol{\rho}}{\partial y} = \begin{pmatrix} \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial y} \end{pmatrix} \quad (85)$$

Again, there is the distance

$$ds^2 = d\xi^2 + d\eta^2 = d\boldsymbol{\rho} \cdot d\boldsymbol{\rho} = \boldsymbol{e}_x \cdot \boldsymbol{e}_x dx^2 + \boldsymbol{e}_y \cdot \boldsymbol{e}_y dy^2 \quad (86)$$

The differential $d\boldsymbol{r}$ keep the form as in Eq. (73), so that the contra variant bases could be defined as

$$\boldsymbol{e}^\xi = \frac{\partial \boldsymbol{r}}{\partial \xi} = \begin{pmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{pmatrix}, \boldsymbol{e}^\eta = \frac{\partial \boldsymbol{r}}{\partial \eta} = \begin{pmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \end{pmatrix} \quad (87)$$

Also, there is

$$d\varsigma^2 = dx^2 + dy^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = \boldsymbol{e}^\xi \cdot \boldsymbol{e}^\xi d\xi^2 + \boldsymbol{e}^\eta \cdot \boldsymbol{e}^\eta d\eta^2 \quad (88)$$

If the products of covariant bases and contra variant bases are concerned that

$$\boldsymbol{e}^\xi \cdot \boldsymbol{e}_x = \frac{\partial \xi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial y}{\partial \xi} \quad (89)$$

$$\boldsymbol{e}^\eta \cdot \boldsymbol{e}_y = \frac{\partial x}{\partial \eta}\frac{\partial \xi}{\partial y} + \frac{\partial y}{\partial \eta}\frac{\partial \eta}{\partial y} \quad (90)$$

Case covariant bases and contra variant bases are coaxial so that $\frac{\partial \eta}{\partial x} = \frac{\partial y}{\partial \xi} = 0$ on the line $\xi(x)$ and x as well as $\frac{\partial \xi}{\partial y} = \frac{\partial x}{\partial \eta} = 0$ on the line $\eta(y)$ and y as shown in Figure 5 and Figure 6, there will be

$$\boldsymbol{e}^{\xi} \cdot \boldsymbol{e}_x = 1 \quad (91)$$

and

$$\boldsymbol{e}^{\eta} \cdot \boldsymbol{e}_y = 1 \quad (92)$$

so that

$$g_{xx} = 1/g^{\xi\xi}, \; g_{yy} = 1/g^{\eta\eta} \quad (93)$$

Case covariant bases and contra variant bases are not coaxial. One can single out the line of that which is coaxial to the corresponding base. For example, for a base $\boldsymbol{e}_x$ which is uncoaxial to contra variant bases, one can build a direction base $\boldsymbol{e}^l$ on the direction of $\boldsymbol{e}_x$ with the direction cosine in contra variant space so that to draw the equation

$$\boldsymbol{e}^l = \boldsymbol{e}^{\xi} cos\alpha + \boldsymbol{e}^{\eta} sin\alpha \quad (94)$$

so that

$$\boldsymbol{e}^l \cdot \boldsymbol{e}_x = 1 \quad (95)$$

and

$$\boldsymbol{e}_m = \boldsymbol{e}_x cos\beta + \boldsymbol{e}_x sin\beta \quad (96)$$

so that

$$\boldsymbol{e}^{\xi} \cdot \boldsymbol{e}_m = 1 \quad (97)$$

Example 2. *Bases of Riemannian manifold of equal dimensions*

As a Riemanian manifold has equal dimensions with its contra variant space, it could be called equal dimension manifold. A plane space (u, v) maps to a plane space (x, y) could be taken for granted, as shown in Figures 7 and 8.

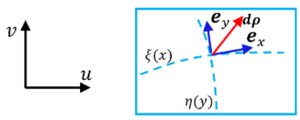

Figure 7. Covariant space.

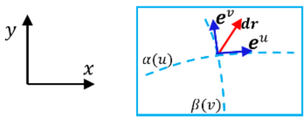

Figure 8. Contra variant space.

A differential vector in covariant space is

$$d\boldsymbol{r} = \begin{pmatrix} dx \\ dy \end{pmatrix} \quad (98)$$

The differential vector in contra variant space is

$$d\boldsymbol{\rho} = \begin{pmatrix} du \\ dv \end{pmatrix} \quad (99)$$

Thus, the definition of contra variant bases could be

$$\boldsymbol{e}^u = \frac{\partial \boldsymbol{r}}{\partial u}, \boldsymbol{e}^v = \frac{\partial \boldsymbol{r}}{\partial v} \quad (100)$$

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To express $d\boldsymbol{r}$ with $d\boldsymbol{\rho}$

$$d\boldsymbol{r} = du\boldsymbol{e}^u + dv\boldsymbol{e}^v \quad (101)$$

Also, there is the definition of covariant bases

$$\boldsymbol{e}_x = \frac{\partial \boldsymbol{\rho}}{\partial x}, \boldsymbol{e}_y = \frac{\partial \boldsymbol{\rho}}{\partial y} \quad (102)$$

So that the expression of $d\boldsymbol{\rho}$ by $d\boldsymbol{r}$ should be

$$d\boldsymbol{\rho} = dx\boldsymbol{e}_x + dy\boldsymbol{e}_y \quad (103)$$

Something different is that a covariant base has 2 components

$$\boldsymbol{e}_x = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix}, \boldsymbol{e}_y = \begin{pmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{pmatrix} \quad (104)$$

And a contra variant base also has 2 components

$$\boldsymbol{e}^u = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{pmatrix}, \boldsymbol{e}^v = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{pmatrix} \quad (105)$$

In the case that bases are orthogonal, the distance

$$ds^2 = du^2 + dv^2 = d\boldsymbol{\rho} \cdot d\boldsymbol{\rho} = \boldsymbol{e}_x \cdot \boldsymbol{e}_x dx^2 + \boldsymbol{e}_y \cdot \boldsymbol{e}_y dy^2 \quad (106)$$

and

$$d\varsigma^2 = dx^2 + dy^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = \boldsymbol{e}^u \cdot \boldsymbol{e}^u du^2 + \boldsymbol{e}^v \cdot \boldsymbol{e}^v dv^2 \quad (107)$$

One can investigate the performances of $\boldsymbol{e}^u \cdot \boldsymbol{e}_x$ and $\boldsymbol{e}^v \cdot \boldsymbol{e}_y$ case the bases coaxial or not.

3.3. Inequalities of Mixed Derivatives of Bases

In Riemannian differential geometry, the equality of Christoffel symbols of mixed subscripts is usually adopt the doctrine. But no forceful researches could provide reliable supports for it to be applied in general relativity. The truth is that the problem of mixed derivatives of bases in a pseudo Riemannian space are far different from the problem of normal mixed derivatives of a 3-dimensional surface in Euclidean geometry.

We could find out the truth that in the deduction of Γ_{00}^1 in Eq. (52), and Eq. (56), $\frac{\partial g_{00}}{\partial x^1}$ has been used instead of $\frac{\partial g_{11}}{\partial x^0}$ so that to gain $\Gamma_{00}^1 = \frac{1}{2} g^{11}(-\frac{\partial g_{00}}{\partial x^1})$. But we can easily calculate that $\frac{\partial \boldsymbol{e}_0}{\partial x^1}$ and $\frac{\partial \boldsymbol{e}_1}{\partial x^0}$ are not equal. It is said that the Eq. (52), and Eq. (56) have been calculated to be a nonvanishing value of Γ_{00}^1, which really relates to time derivatives of bases that must be determined to be zero in rest field originally. We might have found out the problems.

Now it is the time to carry out the first discussion on the inequality of mixed derivatives of bases. The mixed derivatives of bases are just special defined for bases alternative derivations. As transformation from contra variant space to covariant space is concerned, the covariant bases could be considered to be derived by the coordinate lines in chain rule

$$\boldsymbol{e}_x = \frac{\partial \boldsymbol{\rho}}{\partial x} = \frac{\partial \boldsymbol{\rho}}{\partial \xi}\frac{\partial \xi}{\partial x}, \boldsymbol{e}_y = \frac{\partial \boldsymbol{\rho}}{\partial y} = \frac{\partial r}{\partial \eta}\frac{\partial \eta}{\partial y} \quad (108)$$

where, $\frac{\partial \rho}{\partial \xi}$ and $\frac{\partial \rho}{\partial \eta}$ are the direction derivatives along the coordinate lines $\xi(x)$ and $\eta(y)$ in covariant space, and $d\xi$ and $d\eta$ are their differential lengths in covariant space, which could be called the covariant lengths. And there will be a setting that Einstein summation convention does not act on double $d\xi$ and double $d\eta$.

It should be pointed out that, in most mathematics and physics, mixed derivatives being confirmed to be equal is because in the Eq. (108) ∂x and ∂y is incorrectly understood to be the differential length in covariant space (u, v, w), but they are really the lengths in contra variant space (x, y, z). That is the reason we have carried out the concept of covariant length $d\xi$ and $d\eta$.

Thus, the mixed derivatives will be

$$\frac{\partial \boldsymbol{e}_x}{\partial y} = \frac{\partial^2 \boldsymbol{\rho}}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial \boldsymbol{\rho}}{\partial\xi}\frac{\partial^2\xi}{\partial x\partial y} \quad (109)$$

and

$$\frac{\partial \boldsymbol{e}_y}{\partial x} = \frac{\partial^2 \boldsymbol{\rho}}{\partial\eta\partial\xi}\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial x} + \frac{\partial \boldsymbol{\rho}}{\partial\eta}\frac{\partial^2\eta}{\partial y\partial x} \quad (110)$$

Consequently 3 conditions could be focused on:

Condition 1:

If there is an equality between the first items of the two equations that

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial\xi\partial\eta} = \frac{\partial^2 \boldsymbol{\rho}}{\partial\eta\partial\xi} \quad (111)$$

For example, in the super surface, the mixed derivatives of course have the equality just as the equality of normal mixed derivatives of a 3-dimensional surface in a Euclidean space.

In this case and if there is another equality for the last items of the two equations that

$$\frac{\partial \boldsymbol{\rho}}{\partial\xi}\frac{\partial^2\xi}{\partial x\partial y} = \frac{\partial \boldsymbol{\rho}}{\partial\eta}\frac{\partial^2\eta}{\partial y\partial x} \quad (112)$$

Then that must come to the conclusion

$$\frac{\partial \boldsymbol{e}_x}{\partial x} = \frac{\partial \boldsymbol{e}_y}{\partial y} \quad (113)$$

Otherwise, that depends.

It should be pointed out that $\frac{\partial \boldsymbol{\rho}}{\partial\xi}$ and $\frac{\partial \boldsymbol{\rho}}{\partial\eta}$ do not equal in general conditions because they have different directions and, in most cases, they are usually set orthogonal, so that if that equality of Eq. (112) happens, it asks for

$$\frac{\partial^2\xi}{\partial x\partial y} = \frac{\partial^2\eta}{\partial y\partial x} = 0 \quad (114)$$

We will see that in some cases it is really well satisfied.

Condition 2:

Most special if

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial\xi\partial\eta} \neq \frac{\partial^2 \boldsymbol{\rho}}{\partial\eta\partial\xi} \quad (115)$$

that indicate the first items of the two equations are not equal, but at the same time if the total equations are still equal that

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial \boldsymbol{\rho}}{\partial\xi}\frac{\partial^2\xi}{\partial x\partial y} = \frac{\partial^2 \boldsymbol{\rho}}{\partial\eta\partial\xi}\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial x} + \frac{\partial \boldsymbol{\rho}}{\partial\eta}\frac{\partial^2\eta}{\partial y\partial x} \quad (116)$$

We will still obtain the equality that

$$\frac{\partial \boldsymbol{e}_x}{\partial x} = \frac{\partial \boldsymbol{e}_y}{\partial y} \quad (117)$$

Condition 3:

This is the condition after the previous two conditions and else to them. Generally if

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} + \frac{\partial \boldsymbol{\rho}}{\partial\xi}\frac{\partial^2\xi}{\partial x\partial y} \neq \frac{\partial^2 \boldsymbol{\rho}}{\partial\eta\partial\xi}\frac{\partial\eta}{\partial y}\frac{\partial\xi}{\partial x} + \frac{\partial \boldsymbol{\rho}}{\partial\eta}\frac{\partial^2\eta}{\partial y\partial x} \quad (118)$$

No matter the first items of the two equations are equal or not, the mixed derivatives will perform inequality

$$\frac{\partial \boldsymbol{e}_x}{\partial x} \neq \frac{\partial \boldsymbol{e}_y}{\partial y} \quad (119)$$

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Then, turn to the issue of geometrical influence that the inequality of mixed derivatives will cause closure errors [6,7]. I prefer to give a brief presentation. Consider a differential in a curve line coordinate system expressed by bases along different coordinate paths as shown in Figure 9 that

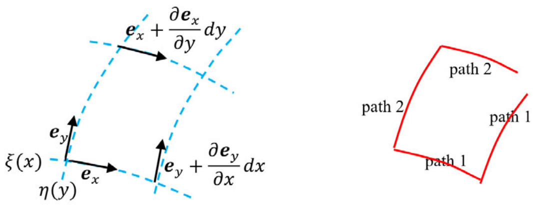

Figure 9. Bases vary in different paths.

in path 1,

$$(d\boldsymbol{\rho})_1 = \int_x^{x+dx} \boldsymbol{e}_x\, dx + \int_y^{y+dy} (\boldsymbol{e}_y + \frac{\partial e_y}{\partial x} dx)\, dy \quad (120)$$

The irregular expressions of same symbols of integral variables and integral range could be adopted in special cases.

By Taylor's approximation, it could be written as

$$(d\boldsymbol{\rho})_1 \approx \boldsymbol{e}_x dx + \frac{1}{2}\frac{\partial e_x}{\partial x} dx^2 + \boldsymbol{e}_y dy + \frac{\partial e_y}{\partial x} dxdy + \frac{1}{2}(\frac{\partial e_y}{\partial y} + \frac{\partial e_y}{\partial x \partial y} dx) dy^2 \quad (121)$$

We could also get the differential in path 2,

$$(d\boldsymbol{\rho})_2 \approx \boldsymbol{e}_y dy + \frac{1}{2}\frac{\partial e_y}{\partial y} dy^2 + \boldsymbol{e}_x dx + \frac{\partial e_x}{\partial y} dydx + \frac{1}{2}(\frac{\partial e_x}{\partial x} + \frac{\partial e_x}{\partial y \partial x} dy) dx^2 \quad (122)$$

Trimming off the 3-order infinite small quantities, the difference of $(d\boldsymbol{\rho})_1$ and $(d\boldsymbol{\rho})_2$ is

$$\Delta = (d\boldsymbol{\rho})_1 - (d\boldsymbol{\rho})_2 \approx (\frac{\partial e_y}{\partial x} - \frac{\partial e_x}{\partial y}) dydx \quad (123)$$

There will be a closure error in close path if the mixed derivatives of bases do not equal.

This result let me think about the probable closure error in the thought experiment of emission and receive of a light pulse in gravitational field that has been mentioned in section 2.2. It could be proposed that two closed lines of line ABD and line ACD present two physical events in space (t, r). In Figure 10, the line ABD named path 1 could be imagined of the event for a photon emitted from position A to B and waiting for $\Delta t_{(2)}$. Another line ACD named path 2 is of the event for another photon emitted from position C to D after a waiting for $\Delta t_{(1)}$. A question could be asked for who are imagined waiting, never mind, it could be the world. It has been mentioned that it is believed $\Delta t_{(1)} = \Delta t_{(2)}$. That is easily described that the two emission lines AC and BD are parallel in that the velocities of $\frac{dr}{dt}$ must be same for the two lines at same space positions A and C or B and D or whatever of corresponding positions. One can see one of the two lines is a copy of the other and experiences a vertical displacement.

It is natural to transform the path 1 of abd' and path 2 of acd from the space (t, r) to the space (τ, ρ), as shown in Figure 11. We know that light speed is invariant in covariant space, so that the emission lines in space (τ, ρ) must be parallel and straight lines ab and cd. On the other side, we have known that

$$\Delta\tau_{(1)} = e_{0(1)}\Delta t_{(1)} \text{ and } \Delta\tau_{(2)} = e_{0(2)}\Delta t_{(2)} \quad (124)$$

so that

$$\Delta\tau_{(1)} \neq \Delta\tau_{(2)} \quad (125)$$

If it is believed that $\Delta t_{(1)} = \Delta t_{(2)}$ as discussed previous, there must be closure error between path 1 and path 2 in covariant space that

$$\Delta = [e_{0(2)} - e_{0(1)}]\Delta t_{(1)} \quad (126)$$

This is really a paradox that the two path arrive at the different points d and d′ in covariant space while they are the same point in contra variant space. The inequalities of mixed derivatives of bases could provide the geometrical interpretations, but the problem is whether and how does it perform realities. Is that new surprise of light? The topics we are mainly focusing on have not been linked to this issue. I just carried out the problem in curiosity.

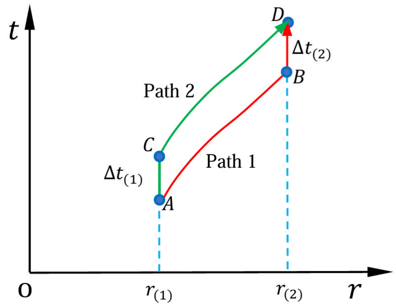

Figure 10. Closed paths in contra variant space.

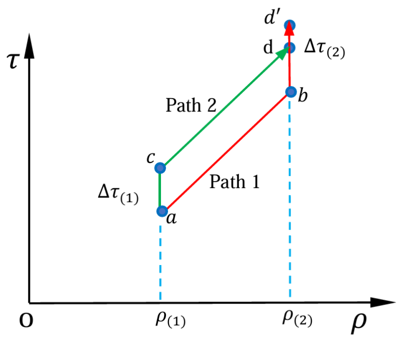

Figure 11. A presentation of closure error in the thought experiment of light pulse.

3.4. Verifications and Discussions

Example 1. *Original polar coordinate system to Cartesian coordinate system*

A polar coordinate system that we are familiar with is a transformation from its contra variant space of original polar system (r, θ), as shown in Figures 12 and 13.

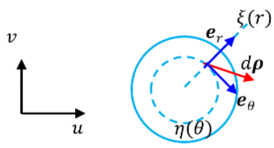

Figure 12. Covariant space.

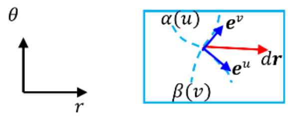

Figure 13. Contra variant space.

A position in contra variant space could be expressed by vector

$$\boldsymbol{r} = \boldsymbol{r}(r, \theta) \quad (127)$$

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and the differential is

$$\boldsymbol{dr} = \begin{pmatrix} dr \\ d\theta \end{pmatrix} \quad (128)$$

Corresponding position in covariant space, will be expressed by

$$\boldsymbol{\rho} = \boldsymbol{\rho}(u, v) \quad (129)$$

and the differential is

$$\boldsymbol{d\rho} = \begin{pmatrix} du \\ dv \end{pmatrix} \quad (130)$$

In the contra variant space, the differential distance between two positions could be defined as

$$d\varsigma^2 = \boldsymbol{dr} \cdot \boldsymbol{dr} = dr^2 + d\theta^2 \quad (131)$$

The system we have used to is the really the Cartesian coordinate system that has been transformed from original polar space (r, θ) with

$$u = rcos\theta, v = rsin\theta \quad (132)$$

The bases

$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{\rho}}{\partial r} = \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} \quad (133)$$

$$\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{\rho}}{\partial \theta} = \begin{pmatrix} -rsin\theta \\ rcos\theta \end{pmatrix} \quad (134)$$

so that

$$d\boldsymbol{\rho} = \boldsymbol{e}_r dr + \boldsymbol{e}_\theta d\theta \quad (135)$$

The contra variant bases could be defined that

$$\boldsymbol{e}^u = \frac{\partial \boldsymbol{r}}{\partial u}, \boldsymbol{e}^v = \frac{\partial \boldsymbol{r}}{\partial v} \quad (136)$$

It is sophisticated to give detailed expressions of contra variant bases. In fact, the contra variant bases and covariant bases are not coaxial so that it could be expected that

$$\boldsymbol{e}^u \cdot \boldsymbol{e}_r \neq 1 \quad (137)$$

and

$$\boldsymbol{e}^v \cdot \boldsymbol{e}_\theta \neq 1 \quad (138)$$

There is the invariant distance

$$ds^2 = du^2 + dv^2 = d\boldsymbol{\rho} \cdot d\boldsymbol{\rho} = \boldsymbol{e}_r \cdot \boldsymbol{e}_r dr^2 + \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta d\theta^2 = dr^2 + r^2 d\theta^2 \quad (139)$$

The mixed derivatives of bases

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \frac{\partial}{\partial \theta}\begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \quad (140)$$

$$\frac{\partial \boldsymbol{e}_\theta}{\partial r} = \frac{\partial}{\partial r}\begin{pmatrix} -rsin\theta \\ rcos\theta \end{pmatrix} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \quad (141)$$

We have seen the mixed derivatives got equal

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \frac{\partial \boldsymbol{e}_\theta}{\partial r} \quad (142)$$

It could also be verified in Eq. (109) and Eq. (110) that

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_{(r)} \partial \eta_{(\theta)}} \frac{\partial \xi_{(r)}}{\partial r} \frac{\partial \eta_{(\theta)}}{\partial \theta} + \frac{\partial \boldsymbol{\rho}}{\partial \xi_{(r)}} \frac{\partial^2 \xi_{(r)}}{\partial r \partial \theta} \quad (143)$$

and

$$\frac{\partial \boldsymbol{e}_\theta}{\partial r} = \frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)} \partial \xi_{(r)}} \frac{\partial \eta_{(\theta)}}{\partial \theta} \frac{\partial \xi_{(r)}}{\partial r} + \frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}} \frac{\partial^2 \eta_{(\theta)}}{\partial \theta \partial r} \quad (144)$$

The vector $\boldsymbol{\rho}$ is

$$\boldsymbol{\rho} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} r cos\theta \\ r sin\theta \end{pmatrix} \quad (145)$$

Because $d\xi_{(r)}$ is radius length dr, and $d\eta_{(\theta)}$ is arc length $rd\theta$, then

$$\frac{\partial \boldsymbol{\rho}}{\partial \xi_{(r)}} = \frac{\partial}{\partial r} \begin{pmatrix} r cos\theta \\ r sin\theta \end{pmatrix} = \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} \quad (146)$$

$$\frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}} = \frac{\partial}{r\partial \theta} \begin{pmatrix} r cos\theta \\ r sin\theta \end{pmatrix} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \quad (147)$$

Thus, the first item of Eq. (143) is

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_{(r)} \partial \eta_{(\theta)}} \frac{\partial \xi_{(r)}}{\partial r} \frac{\partial \eta_{(\theta)}}{\partial \theta} = \frac{\partial}{r\partial \theta} \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} \frac{\partial r}{\partial r} \frac{r\partial \theta}{\partial \theta} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \quad (148)$$

The second item is

$$\frac{\partial \boldsymbol{\rho}}{\partial \xi_{(r)}} \frac{\partial^2 \xi_{(r)}}{\partial r \partial \theta} = \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix} \frac{\partial}{r\partial \theta} \frac{\partial r}{\partial r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (149)$$

And the first item of Eq. (144)

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)} \partial \xi_{(r)}} \frac{\partial \eta_{(\theta)}}{\partial \theta} \frac{\partial \xi_{(r)}}{\partial r} = \frac{\partial}{\partial r} \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \frac{r\partial \theta}{\partial \theta} \frac{\partial r}{\partial r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (150)$$

The second item

$$\frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}} \frac{\partial^2 \eta_{(\theta)}}{\partial \theta \partial r} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \frac{\partial}{\partial r} \frac{r\partial \theta}{\partial \theta} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \quad (151)$$

so that

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \quad (152)$$

and

$$\frac{\partial \boldsymbol{e}_\theta}{\partial r} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} = \begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} \quad (153)$$

Obviously there is

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_{(r)} \partial \eta_{(\theta)}} \neq \frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)} \partial \xi_{(r)}} \quad (154)$$

and

$$\frac{\partial \boldsymbol{\rho}}{\partial \xi_{(r)}} \frac{\partial^2 \xi_{(r)}}{\partial r \partial \theta} \neq \frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}} \frac{\partial^2 \eta_{(\theta)}}{\partial \theta \partial r} \quad (155)$$

but the mixed derivatives are equal totally that

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \frac{\partial \boldsymbol{e}_\theta}{\partial r} \quad (156)$$

Again, we have got the equality of mixed derivatives. But to our surprise is that this solution really subject to condition 2. It is said that the first items of Eq. (143) and Eq. (144) do not equal. One of the reasons in this case, is that there is no super surface.

Example 2. *Original spherical surface coordinate system to Cartesian coordinate system*

A transformation original spherical system to Cartesian coordinate system are shown in Figures 14 and 15, in which

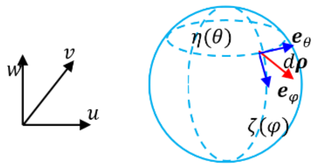

Figure 14. Covariant space.

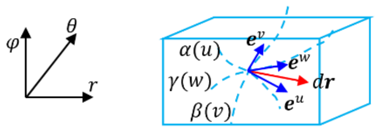

Figure 15. Contra variant space.

$$\boldsymbol{r} = \boldsymbol{r}(r, \theta, \varphi)|_{r=R} = \begin{pmatrix} \theta \\ \varphi \end{pmatrix}\Big|_{r=R} \quad (157)$$

Differential distance is

$$d\zeta^2 = d\boldsymbol{r} \cdot d\boldsymbol{r} = d\theta^2 + d\varphi^2 \quad (158)$$

And the coordinates of covariant space will be expressed with

$$\boldsymbol{\rho} = \boldsymbol{\rho}(u, v, w) = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad (159)$$

The original spherical coordinates could be transformed to Cartesian coordinates,

$$u = Rsin\theta cos\varphi, v = Rsin\theta sin\varphi, w = Rcos\theta \quad (160)$$

The bases could be defined as

$$\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{\rho}}{\partial \theta} = \begin{pmatrix} Rcos\theta cos\varphi \\ Rcos\theta sin\varphi \\ -Rsin\theta \end{pmatrix}, \boldsymbol{e}_\varphi = \frac{\partial \boldsymbol{\rho}}{\partial \varphi} = \begin{pmatrix} -Rsin\theta sin\varphi \\ Rsin\theta cos\varphi \\ 0 \end{pmatrix} \quad (161)$$

and

$$d\boldsymbol{\rho} = \boldsymbol{e}_\theta d\theta + \boldsymbol{e}_\varphi d\varphi \quad (162)$$

Thus, there is the covariant distance

$$ds^2 = d\boldsymbol{\rho} \cdot d\boldsymbol{\rho} = \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta dr^2 + \boldsymbol{e}_\varphi \cdot \boldsymbol{e}_\varphi dr^2 = R^2 d\theta^2 + R^2 sin^2\theta d\varphi^2 \quad (163)$$

The derivatives

$$\frac{\partial \boldsymbol{e}_\theta}{\partial \varphi} = \begin{pmatrix} -Rcos\theta sin\varphi \\ Rcos\theta cos\varphi \\ 0 \end{pmatrix}, \frac{\partial \boldsymbol{e}_\varphi}{\partial \theta} = \begin{pmatrix} -Rcos\theta sin\varphi \\ Rcos\theta cos\varphi \\ 0 \end{pmatrix} \quad (164)$$

so that

$$\frac{\partial \boldsymbol{e}_\theta}{\partial \varphi} = \frac{\partial \boldsymbol{e}_\varphi}{\partial \theta} \quad (165)$$

It could also be verified in Eq. (109) and Eq. (110) that

$$\frac{\partial \boldsymbol{e}_\theta}{\partial \varphi} = \frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)} \partial \zeta_{(\varphi)}} \frac{\partial \eta_{(\theta)}}{\partial \theta} \frac{\partial \zeta_{(\varphi)}}{\partial \varphi} + \frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}} \frac{\partial^2 \eta_{(\theta)}}{\partial \theta \partial \varphi} \quad (166)$$

and

$$\frac{\partial \boldsymbol{e}_\varphi}{\partial \theta} = \frac{\partial^2 \boldsymbol{\rho}}{\partial \zeta_{(\varphi)} \partial \eta_{(\theta)}} \frac{\partial \zeta_{(\varphi)}}{\partial \varphi} \frac{\partial \eta_{(\theta)}}{\partial \theta} + \frac{\partial \boldsymbol{\rho}}{\partial \zeta_{(\varphi)}} \frac{\partial^2 \zeta_{(\varphi)}}{\partial \varphi \partial \theta} \quad (167)$$

The vector $\boldsymbol{\rho}$ is

$$\boldsymbol{\rho} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} Rsin\theta cos\varphi \\ Rsin\theta sin\varphi \\ Rcos\theta \end{pmatrix} \quad (168)$$

Because $d\eta_{(\theta)}$ is arc length $Rd\theta$, and $d\zeta_{(\varphi)}$ is arc length $Rd\varphi$, then

$$\frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}} = \frac{\partial}{R\partial\theta}\begin{pmatrix} Rsin\theta cos\varphi \\ Rsin\theta sin\varphi \\ Rcos\theta \end{pmatrix} = \begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix} \quad (169)$$

$$\frac{\partial \boldsymbol{\rho}}{\partial \zeta_{(\varphi)}} = \frac{\partial}{R\partial\varphi}\begin{pmatrix} Rsin\theta cos\varphi \\ Rsin\theta sin\varphi \\ Rcos\theta \end{pmatrix} = \begin{pmatrix} -sin\theta sin\varphi \\ sin\theta cos\varphi \\ 0 \end{pmatrix} \quad (170)$$

Thus, the first item of Eq. (166) is

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)}\partial \zeta_{(\varphi)}}\frac{\partial \eta_{(\theta)}}{\partial\theta}\frac{\partial \zeta_{(\varphi)}}{\partial\varphi} = \frac{\partial}{R\partial\varphi}\begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix}\frac{R\partial\theta}{\partial\theta}\frac{R\partial\varphi}{\partial\varphi} = \begin{pmatrix} -Rcos\theta sin\varphi \\ Rcos\theta cos\varphi \\ 0 \end{pmatrix} \quad (171)$$

The second item is

$$\frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}}\frac{\partial^2 \eta_{(\theta)}}{\partial\theta\partial\varphi} = \begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix}\frac{\partial}{\partial\varphi}\frac{R\partial\theta}{\partial\theta} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (172)$$

And the first item of Eq. (167)

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \zeta_{(\varphi)}\partial \eta_{(\theta)}}\frac{\partial \zeta_{(\varphi)}}{\partial\varphi}\frac{\partial \eta_{(\theta)}}{\partial\theta} = \frac{\partial}{R\partial\theta}\begin{pmatrix} -sin\theta sin\varphi \\ sin\theta cos\varphi \\ 0 \end{pmatrix}\frac{R\partial\varphi}{\partial\varphi}\frac{R\partial\theta}{\partial\theta} = \begin{pmatrix} -Rcos\theta sin\varphi \\ Rcos\theta cos\varphi \\ 0 \end{pmatrix} \quad (173)$$

The second item

$$\frac{\partial \boldsymbol{\rho}}{\partial \zeta_{(\varphi)}}\frac{\partial^2 \zeta_{(\varphi)}}{\partial\varphi\partial\theta} = \begin{pmatrix} -sin\theta sin\varphi \\ sin\theta cos\varphi \\ 0 \end{pmatrix}\frac{\partial}{\partial\theta}\frac{R\partial\varphi}{\partial\varphi} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (174)$$

With Eq. (171) to Eq. (174) we found that

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)}\partial \zeta_{(\varphi)}} = \frac{\partial^2 \boldsymbol{\rho}}{\partial \zeta_{(\varphi)}\partial \eta_{(\theta)}} \quad (175)$$

and

$$\frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}}\frac{\partial^2 \eta_{(\theta)}}{\partial\theta\partial\varphi} = \frac{\partial \boldsymbol{\rho}}{\partial \zeta_{(\varphi)}}\frac{\partial^2 \zeta_{(\varphi)}}{\partial\varphi\partial\theta} \quad (176)$$

so that there is

$$\frac{\partial \boldsymbol{e}_\theta}{\partial\varphi} = \frac{\partial \boldsymbol{e}_\varphi}{\partial\theta} \quad (177)$$

One can see that this is of condition 1.

Example 3. *Original spherical coordinate system to Cartesian coordinate system*

A transformation from original spherical system to Cartesian coordinate system has been shown in Figures 16 and 17, in which

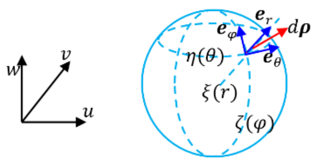

Figure 16. Covariant space.

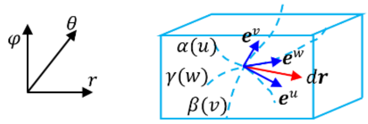

Figure 17. Contra variant space.

$$\boldsymbol{r} = \boldsymbol{r}(r,\theta,\varphi) = \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix} \quad (178)$$

Differential distance of original spherical space is

$$d\zeta^2 = d\boldsymbol{r}\cdot d\boldsymbol{r} = dr^2 + d\theta^2 + d\varphi^2 \quad (179)$$

And the coordinates of covariant space will be expressed with

$$\boldsymbol{\rho} = \boldsymbol{\rho}(u,v,w) = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad (180)$$

The original spherical coordinates could be transformed to Cartesian coordinates,

$$u = rsin\theta cos\varphi, v = rsin\theta sin\varphi, w = rcos\theta \quad (181)$$

Their differentials are

$$du = sin\theta cos\varphi dr + rcos\theta cos\varphi d\theta - rsin\theta sin\varphi d\varphi \quad (182)$$

$$dv = sin\theta sin\varphi dr + rcos\theta sin\varphi d\theta + rsin\theta cos\varphi d\varphi \quad (183)$$

$$dw = cos\theta dr - rsin\theta d\theta \quad (184)$$

and

$$d\boldsymbol{\rho} = \begin{pmatrix} du \\ dv \\ dw \end{pmatrix} \quad (185)$$

Thus, the invariant distance could be written as

$$ds^2 = d\boldsymbol{\rho}\cdot d\boldsymbol{\rho} = du^2 + dv^2 + dw^2$$
$$= sin^2\theta cos^2\varphi dr^2 + r^2cos^2\theta cos^2\varphi d\theta^2 + r^2sin^2\theta sin^2\varphi d\varphi^2$$
$$+2rsin\theta cos\theta cos^2\varphi drd\theta - 2r^2sin\theta cos\theta sin\varphi cos\varphi d\theta d\varphi - 2rsin\varphi cos\varphi sin^2\theta drd\varphi$$
$$+sin^2\theta sin^2\varphi dr^2 + r^2cos^2\theta sin^2\varphi d\theta^2 + r^2sin^2\theta cos^2\varphi d\varphi^2$$
$$+2rsin\theta cos\theta sin^2\varphi drd\theta + 2r^2sin\theta cos\theta sin\varphi cos\varphi d\theta d\varphi + 2rsin\varphi cos\varphi sin^2\theta drd\varphi$$
$$+cos^2\varphi dr^2 + r^2sin^2\theta d\theta^2 - 2rsin\theta cos\theta drd\theta$$
$$= dr^2 + r^2d\theta^2 + r^2sin^2\theta d\varphi^2 \quad (186)$$

The bases could be calculated as

$$\boldsymbol{e}_r = \frac{\partial\boldsymbol{\rho}}{\partial r} = \begin{pmatrix} sin\theta cos\varphi \\ sin\theta sin\varphi \\ cos\theta \end{pmatrix}, \boldsymbol{e}_\theta = \frac{\partial\boldsymbol{\rho}}{\partial\theta} = \begin{pmatrix} rcos\theta cos\varphi \\ rcos\theta sin\varphi \\ -rsin\theta \end{pmatrix}, \boldsymbol{e}_\varphi = \frac{\partial\boldsymbol{\rho}}{\partial\varphi} = \begin{pmatrix} -rsin\theta sin\varphi \\ rsin\theta cos\varphi \\ 0 \end{pmatrix} \quad (187)$$

and

$$d\boldsymbol{\rho} = \boldsymbol{e}_r dr + \boldsymbol{e}_\theta d\theta + \boldsymbol{e}_\varphi d\varphi \quad (188)$$

Thus, the invariant distance could also be calculated as

$$ds^2 = d\boldsymbol{\rho}\cdot d\boldsymbol{\rho} = \boldsymbol{e}_r\cdot\boldsymbol{e}_r dr^2 + \boldsymbol{e}_\theta\cdot\boldsymbol{e}_\theta d\theta^2 + \boldsymbol{e}_\varphi\cdot\boldsymbol{e}_\varphi d\varphi^2 = dr^2 + r^2d\theta^2 + r^2sin^2\theta d\varphi^2 \quad (189)$$

The derivatives

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$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix}, \frac{\partial \boldsymbol{e}_\theta}{\partial r} = \begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix}, \frac{\partial \boldsymbol{e}_\theta}{\partial \varphi} = \begin{pmatrix} -rcos\theta sin\varphi \\ rcos\theta cos\varphi \\ 0 \end{pmatrix}$$

$$\frac{\partial \boldsymbol{e}_\varphi}{\partial \theta} = \begin{pmatrix} -rcos\theta sin\varphi \\ rcos\theta cos\varphi \\ 0 \end{pmatrix}, \frac{\partial \boldsymbol{e}_\varphi}{\partial r} = \begin{pmatrix} -sin\theta sin\varphi \\ sin\theta cos\varphi \\ 0 \end{pmatrix}, \frac{\partial \boldsymbol{e}_r}{\partial \varphi} = \begin{pmatrix} -sin\theta sin\varphi \\ sin\theta cos\varphi \\ 0 \end{pmatrix} \quad (190)$$

so that

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \frac{\partial \boldsymbol{e}_\theta}{\partial r}, \frac{\partial \boldsymbol{e}_\theta}{\partial \varphi} = \frac{\partial \boldsymbol{e}_\varphi}{\partial \theta}, \frac{\partial \boldsymbol{e}_\varphi}{\partial r} = \frac{\partial \boldsymbol{e}_r}{\partial \varphi} \quad (191)$$

It could also be verified in Eq. (109) and Eq. (110) that

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_{(r)} \partial \eta_{(\theta)}} \frac{\partial \xi_{(r)}}{\partial r} \frac{\partial \eta_{(\theta)}}{\partial \theta} + \frac{\partial \boldsymbol{\rho}}{\partial \xi_{(r)}} \frac{\partial^2 \xi_{(r)}}{\partial r \partial \theta} \quad (192)$$

and

$$\frac{\partial \boldsymbol{e}_\theta}{\partial r} = \frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)} \partial \xi_{(r)}} \frac{\partial \eta_{(\theta)}}{\partial \theta} \frac{\partial \xi_{(r)}}{\partial r} + \frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}} \frac{\partial^2 \eta_{(\theta)}}{\partial \theta \partial r} \quad (193)$$

The vector $\boldsymbol{\rho}$ is

$$\boldsymbol{\rho} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} rsin\theta cos\varphi \\ rsin\theta sin\varphi \\ rcos\theta \end{pmatrix} \quad (194)$$

Because $d\xi_{(r)}$ is radius length dr, $d\eta_{(\theta)}$ is arc length $rd\theta$, and $d\zeta_{(\varphi)}$ is arc length $rd\varphi$, then

$$\frac{\partial \boldsymbol{\rho}}{\partial \xi_{(r)}} = \frac{\partial}{\partial r} \begin{pmatrix} rsin\theta cos\varphi \\ rsin\theta sin\varphi \\ rcos\theta \end{pmatrix} = \begin{pmatrix} sin\theta cos\varphi \\ sin\theta sin\varphi \\ cos\theta \end{pmatrix} \quad (195)$$

$$\frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}} = \frac{\partial}{r\partial \theta} \begin{pmatrix} rsin\theta cos\varphi \\ rsin\theta sin\varphi \\ rcos\theta \end{pmatrix} = \begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix} \quad (196)$$

$$\frac{\partial \boldsymbol{\rho}}{\partial \zeta_{(\varphi)}} = \frac{\partial}{r\partial \varphi} \begin{pmatrix} rsin\theta cos\varphi \\ rsin\theta sin\varphi \\ rcos\theta \end{pmatrix} = \begin{pmatrix} -sin\theta sin\varphi \\ sin\theta cos\varphi \\ 0 \end{pmatrix} \quad (197)$$

Thus, the first item of Eq. (192) is

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_{(r)} \partial \eta_{(\theta)}} \frac{\partial \xi_{(r)}}{\partial r} \frac{\partial \eta_{(\theta)}}{\partial \theta} = \frac{\partial}{r\partial \theta} \begin{pmatrix} sin\theta cos\varphi \\ sin\theta sin\varphi \\ cos\theta \end{pmatrix} \frac{\partial r}{\partial r} \frac{r\partial \theta}{\partial \theta} = \begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix} \quad (198)$$

The second item is

$$\frac{\partial \boldsymbol{\rho}}{\partial \xi_{(r)}} \frac{\partial^2 \xi_{(r)}}{\partial r \partial \theta} = \begin{pmatrix} sin\theta cos\varphi \\ sin\theta sin\varphi \\ cos\theta \end{pmatrix} \frac{\partial}{r\partial \theta} \frac{\partial r}{\partial r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (199)$$

And the first item of Eq. (193)

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)} \partial \xi_{(r)}} \frac{\partial \eta_{(\theta)}}{\partial \theta} \frac{\partial \xi_{(r)}}{\partial r} = \frac{\partial}{\partial r} \begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix} \frac{r\partial \theta}{\partial \theta} \frac{\partial r}{\partial r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (200)$$

The second item

$$\frac{\partial \boldsymbol{\rho}}{\partial \eta_{(\theta)}} \frac{\partial^2 \eta_{(\theta)}}{\partial \theta \partial r} = \begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix} \frac{\partial}{\partial r} \frac{r\partial \theta}{\partial \theta} = \begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix} \quad (201)$$

With Eq. (198) and Eq. (200) we found that

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_{(r)} \partial \eta_{(\theta)}} \neq \frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)} \partial \xi_{(r)}} \quad (202)$$

but there is

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \frac{\partial \boldsymbol{e}_\theta}{\partial r} \quad (203)$$

One can also calculate that

$$\frac{\partial^2 \boldsymbol{\rho}}{\partial \eta_{(\theta)} \partial \zeta_{(\varphi)}} = \frac{\partial^2 \boldsymbol{\rho}}{\partial \zeta_{(\varphi)} \partial \eta_{(\theta)}}, \frac{\partial^2 \boldsymbol{\rho}}{\partial \zeta_{(\varphi)} \partial \xi_{(r)}} \neq \frac{\partial^2 \boldsymbol{\rho}}{\partial \xi_{(r)} \partial \zeta_{(\varphi)}} \quad (204)$$

At the end we can obtain

$$\frac{\partial \boldsymbol{e}_\theta}{\partial \varphi} = \frac{\partial \boldsymbol{e}_\varphi}{\partial \theta}, \frac{\partial \boldsymbol{e}_\varphi}{\partial r} = \frac{\partial \boldsymbol{e}_r}{\partial \varphi} \quad (205)$$

One can see that one of them is of condition 1 and the others of them are of condition 2.

Additional discussion: deformed bases of example 3

If one of the bases in example 3 be set deformed as

$$\boldsymbol{e}_r = f(r)\begin{pmatrix} sin\theta cos\varphi \\ sin\theta sin\varphi \\ cos\theta \end{pmatrix} \quad (206)$$

where, $f(r) \neq const.$ is a function of coordinate r.

One will find that the derivatives

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = f(r)\begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix} + \frac{\partial f(r)}{\partial \theta}\begin{pmatrix} sin\theta cos\varphi \\ sin\theta sin\varphi \\ cos\theta \end{pmatrix} = f(r)\begin{pmatrix} cos\theta cos\varphi \\ cos\theta sin\varphi \\ -sin\theta \end{pmatrix} \quad (207)$$

and

$$\frac{\partial \boldsymbol{e}_r}{\partial \varphi} = f(r)\begin{pmatrix} -sin\theta sin\varphi \\ sin\theta cos\varphi \\ 0 \end{pmatrix} + \frac{\partial f(r)}{\partial \varphi}\begin{pmatrix} sin\theta cos\varphi \\ sin\theta sin\varphi \\ cos\theta \end{pmatrix} = f(r)\begin{pmatrix} -sin\theta sin\varphi \\ sin\theta cos\varphi \\ 0 \end{pmatrix} \quad (208)$$

while $\frac{\partial \boldsymbol{e}_\theta}{\partial r}$ and $\frac{\partial \boldsymbol{e}_\varphi}{\partial r}$ will still keep the results as in Eq. (190), that will cause

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} \neq \frac{\partial \boldsymbol{e}_\theta}{\partial r}, \frac{\partial \boldsymbol{e}_\varphi}{\partial r} \neq \frac{\partial \boldsymbol{e}_r}{\partial \varphi} \quad (209)$$

This result reminds us that the same performance would have happened in gravitational space time that will be put into discussions in next section.

4. Calculations on Christoffel Symbols and Covariant Derivatives

4.1. Metrics in Pseudo Riemannian Space

Pseudo Riemannian space is raised for the description of space time for general relativity after Minkowski space [8,9] and the distance of the latter is defined as

$$d\varsigma^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2 \quad (210)$$

The invariant distance in general relativity with Schwarzschild solution is

$$ds^2 = -(1 - \frac{r^*}{r})(cdt)^2 + (1 - \frac{r^*}{r})^{-1}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \quad (211)$$

As has been mentioned, in most publications [3,4,10–12], metrics were presented as

$$ds^2 = g_{00}(cdt)^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\varphi^2 \quad (212)$$

so that there could be a brief expression of invariant distance

$$ds^2 = g_{ij}dx^i dx^j \quad (213)$$

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in which $g_{00}=-(1-\frac{r^*}{r}),\ g_{11}=(1-\frac{r^*}{r})^{-1},\ g_{22}=r^2,\ g_{33}=r^2sin^2\theta$.

As we have discussed, they are not correct definition of metrics. In fact, it is one of the reasons that cause the wrong result of acceleration calculation in Eq. (53).

Former researchers have made efforts on the topics, for example, the concept of plural employed to reform the base $\boldsymbol{e}_0$ [6]. But plural bases for relativity is not a good idea. Another treatment is to define $x^0=ict$, which looks like more reasonable [7]. But we know that the negative sign of the first items of Eq. (211) does not come from transformations of spaces or coordinates. It is a kind of mathematical and physical setting.

The issues we are talking about are of transformations of tensors as time and distance differentials, covariant derivatives and curvatures that interpret the variations between covariant tensors and corresponding contra variant tensors. Those transformations do nothing with negative signs. The Christoffel symbols and metrics are employed to present the derivatives of bases and they do nothing with negative signs yet, because that a derivative of a base by a corresponding space time coordinate just relates to a transformation of time or distance differentials that do nothing with minus signs.

In fact, the distance factors r^2 and $r^2sin^2\theta$ in Eq. (211) involved in metrics in Eq. (212) and Eq. (213) is improper.

In order to clarify how does a contra variant space transformed to a covariant space that influence the definition of metrics, I want to present various of transformations from different forms of contra variant space to specific covariant space. The invariant distance under Schwarzschild solution could be taken for example.

Condition 1:

The transformation from contra variant space of original pseudo spherical space $(cdt,dr,d\theta,d\varphi)$ to covariant space of distance expressed pseudo spherical space $(cd\tau,d\rho,rd\theta,rsin\theta d\varphi)$ will be taken into investigations. The spaces have been labeled with their tangent spaces for convenience. We know that the metrics are of the multipliers that transform the contra variant distance to covariant distance.

The differentials of space time in contra variant space

$$dx^0=cdt,\ dx^1=dr,\ dx^2=d\theta,\ dx^3=d\varphi \quad (214)$$

For covariant space, the differentials will be

$$dx0=cd\tau,\ dx1=\mathrm{d}\rho,\ dx2=rd\theta,\ dx3=rsin\theta d\varphi \quad (215)$$

so that the invariant distance is

$$\begin{aligned} ds^2&=-(dx0)^2+(dx1)^2+(dx2)^2+(dx3)^2\\ &=-g_{00}(dx^0)^2+g_{11}(dx^1)^2+g_{22}(dx^2)^2+g_{33}(dx^3)^2\\ &=-(1-\frac{r^*}{r})(cdt)^2+(1-\frac{r^*}{r})^{-1}dr^2+r^2d\theta^2+r^2sin^2\theta d\varphi^2\\ &=-g_{00}(cdt)^2+g_{11}dr^2+g_{22}d\theta^2+g_{33}d\varphi^2 \quad (216)\end{aligned}$$

in which $g_{00}=(1-\frac{r^*}{r}),\ g_{11}=(1-\frac{r^*}{r})^{-1},\ g_{22}=r^2,\ g_{33}=r^2sin^2\theta$.

We have seen that the original pseudo spherical space $(cdt,dr,d\theta,d\varphi)$ is not of same unit coordinate system in that in Riemannian geometry the distance could be written as

$$\begin{aligned} d\varsigma^2&=-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2\\ &=-(cdt)^2+dr^2+d\theta^2+d\varphi^2 \quad (217)\end{aligned}$$

This is a definition of insufficient physical value.

Thus, the covariant bases

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$$\boldsymbol{e}_0=\frac{\partial\rho}{c\partial t}=\begin{pmatrix}(1-\frac{r^*}{r})^{1/2}\\0\\0\\0\end{pmatrix},\boldsymbol{e}_1=\frac{\partial\rho}{\partial r}=\begin{pmatrix}0\\(1-\frac{r^*}{r})^{-1/2}\\0\\0\end{pmatrix},\boldsymbol{e}_2=\frac{\partial\rho}{\partial\theta}=\begin{pmatrix}0\\0\\r\\0\end{pmatrix},\boldsymbol{e}_3=\frac{\partial\rho}{\partial\varphi}=\begin{pmatrix}0\\0\\0\\rsin\theta\end{pmatrix} \quad (218)$$

It is said that the r in $rd\theta$ and $rsin\theta$ in $rsin\theta d\varphi$ are exactly distance factors for angular coordinates. Obviously, we are not satisfied with the definitions in this kind of contra variant space. It will be interesting to step to the next condition for the consideration.

Condition 2:

There could be a transformation from contra variant space of distance expressed pseudo spherical space $(cdt, dr, rd\theta, rsin\theta d\varphi)$ to the covariant space of distance expressed pseudo spherical space $(cd\tau, d\rho, rd\theta, rsin\theta d\varphi)$ so that to have same unit coordinate system in contra variant space.

The differentials of space time in contra variant space

$$dx^0=cdt,\ dx^1=dr,\ dx^2=rd\theta,\ dx^3=rsin\theta d\varphi \quad (219)$$

For covariant space, the differentials are the same as the previous

$$dx0=cd\tau,\ dx1=\mathrm{d}\rho,\ dx2=rd\theta,\ dx3=rsin\theta d\varphi \quad (220)$$

so that the invariant distance is

$$ds^2=-(dx0)^2+(dx1)^2+(dx2)^2+(dx3)^2$$
$$=-g_{00}(dx^0)^2+g_{11}(dx^1)^2+g_{22}(dx^2)^2+g_{33}(dx^3)^2$$
$$=-(1-\frac{r^*}{r})(cdt)^2+(1-\frac{r^*}{r})^{-1}dr^2+r^2d\theta^2+r^2sin^2\theta d\varphi^2$$
$$=-g_{00}(cdt)^2+g_{11}dr^2+g_{22}r^2d\theta^2+g_{33}r^2sin^2\theta d\varphi^2 \quad (221)$$

in which $g_{00}=(1-\frac{r^*}{r}),\ g_{11}=(1-\frac{r^*}{r})^{-1},\ g_{22}=1,\ g_{33}=1.$

The contra variant space is of the same unit coordinate system in that the distance could be written as

$$d\varsigma^2=-(dx^0)^2+(dx^1)^2+(dx^2)^2+(dx^3)^2$$
$$=-(cdt)^2+dr^2+r^2d\theta^2+r^2sin^2\theta d\varphi^2 \quad (222)$$

The covariant bases could be presented as

$$\boldsymbol{e}_0=\frac{\partial\rho}{c\partial t}=\begin{pmatrix}(1-\frac{r^*}{r})^{1/2}\\0\\0\\0\end{pmatrix},\boldsymbol{e}_1=\frac{\partial\rho}{\partial r}=\begin{pmatrix}0\\(1-\frac{r^*}{r})^{-1/2}\\0\\0\end{pmatrix},\boldsymbol{e}_2=\frac{\partial\rho}{r\partial\theta}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\boldsymbol{e}_3=\frac{\partial\rho}{rsin\theta\partial\varphi}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \quad (223)$$

Condition 3:

There could be a transformation from contra variant space of original pseudo spherical space $(cdt, dr, d\theta, \mathrm{d}\varphi)$ to the covariant space of Cartesian space $(cd\tau, du, dv, dw)$, in which $r,\ \theta,\ \varphi$ are of spherical coordinates, $x^1,\ x^2,\ x^3$, the contra variant coordinates and $u,\ v,\ w$ are of Cartesian coordinates, $x1,\ x2,\ x3$, the proper coordinates.

The differentials of space time in contra variant space

$$dx^0=cdt,\ dx^1=dr,\ dx^2=d\theta,\ dx^3=d\varphi \quad (224)$$

For covariant space, the first coordinate differential

$$dx0=cd\tau=(1-\frac{r^*}{r})^{1/2}dt \quad (225)$$

The last three are of transformation as that in Eq. (182) to Eq. (184) as

$$dx1=du=(1-\frac{r^*}{r})^{-1/2}sin\theta cos\varphi dr+rcos\theta cos\varphi d\theta-rsin\theta sin\varphi d\varphi \quad (226)$$

$$dx2 = dv = (1-\frac{r^*}{r})^{-1/2} sin\theta sin\varphi dr + rcos\theta sin\varphi d\theta + rsin\theta cos\varphi d\varphi \quad (227)$$

$$dx3 = dw = (1-\frac{r^*}{r})^{-1/2} cos\theta dr - rsin\theta d\theta \quad (228)$$

The invariant distance could be calculated as

$$ds^2 = -g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$

$$= -(dx0)^2 + (dx1)^2 + (dx2)^2 + (dx3)^2$$

$$= -(cd\tau)^2 + du^2 + dv^2 + dw^2$$

$$= -(1-\frac{r^*}{r})(cdt)^2$$

$$+(1-\frac{r^*}{r})^{-1} sin^2\theta cos^2\varphi dr^2 + r^2cos^2\theta cos^2\varphi d\theta^2 + r^2 sin^2\theta sin^2\varphi d\varphi^2$$

$$+2r(1-\frac{r^*}{r})^{-1/2} sin\theta cos\theta cos^2\varphi dr d\theta - 2r^2 sin\theta cos\theta sin\varphi cos\varphi d\theta d\varphi$$

$$-2r(1-\frac{r^*}{r})^{-1/2} sin\varphi cos\varphi sin^2\theta dr d\varphi$$

$$+(1-\frac{r^*}{r})^{-1} sin^2\theta sin^2\varphi dr^2 + r^2cos^2\theta sin^2\varphi d\theta^2 + r^2 sin^2\theta cos^2\varphi d\varphi^2$$

$$+2r(1-\frac{r^*}{r})^{-1/2} sin\theta cos\theta sin^2\varphi dr d\theta + 2r^2 sin\theta cos\theta sin\varphi cos\varphi d\theta d\varphi$$

$$+2r(1-\frac{r^*}{r})^{-1/2} sin\varphi cos\varphi sin^2\theta dr d\varphi$$

$$+(1-\frac{r^*}{r})^{-1} cos^2\varphi dr^2 + r^2 sin^2\theta d\theta^2 - 2r(1-\frac{r^*}{r})^{-1/2} sin\theta cos\theta dr d\theta$$

$$= -(1-\frac{r^*}{r})(cdt)^2 + (1-\frac{r^*}{r})^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2$$

$$= -g_{00}(cdt)^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}d\varphi^2 \quad (229)$$

in which $g_{00} = (1-\frac{r^*}{r})$, $g_{11} = (1-\frac{r^*}{r})^{-1}$, $g_{22} = r^2$, $g_{33} = r^2 sin^2\theta$.

The distance of original pseudo spherical space is

$$d\varsigma^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$= -(cdt)^2 + dr^2 + d\theta^2 + d\varphi^2 \quad (230)$$

The covariant bases could be presented as

$$\boldsymbol{e}_0 = \frac{\partial\rho}{\partial(ct)} = \begin{pmatrix} \frac{\partial\tau}{\partial t} \\ \frac{\partial u}{c\partial t} \\ \frac{\partial v}{c\partial t} \\ \frac{\partial w}{c\partial t} \end{pmatrix}, \boldsymbol{e}_1 = \frac{\partial\rho}{\partial r} = \begin{pmatrix} c\frac{\partial\tau}{\partial r} \\ \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial r} \\ \frac{\partial w}{\partial r} \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} c\frac{\partial\tau}{\partial\theta} \\ \frac{\partial u}{\partial\theta} \\ \frac{\partial v}{\partial\theta} \\ \frac{\partial w}{\partial\theta} \end{pmatrix}, \boldsymbol{e}_3 = \begin{pmatrix} c\frac{\partial\tau}{\partial\varphi} \\ \frac{\partial u}{\partial\varphi} \\ \frac{\partial v}{\partial\varphi} \\ \frac{\partial w}{\partial\varphi} \end{pmatrix} \quad (231)$$

They could be calculated furtherly that

$$\boldsymbol{e}_0 = \begin{pmatrix} (1-\frac{r^*}{r})^{1/2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{e}_1 = \begin{pmatrix} 0 \\ sin\theta cos\varphi(1-\frac{r^*}{r})^{-1/2} \\ sin\theta sin\varphi(1-\frac{r^*}{r})^{-1/2} \\ cos\theta(1-\frac{r^*}{r})^{-1/2} \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ rcos\theta cos\varphi \\ rcos\theta sin\varphi \\ -rsin\theta \end{pmatrix}, \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ -rsin\theta sin\varphi \\ rsin\theta cos\varphi \\ 0 \end{pmatrix} \quad (232)$$

It could be verified that

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$$g_{00} = \boldsymbol{e}_0 \cdot \boldsymbol{e}_0 = (1 - \frac{r^*}{r}),\ g_{11} = \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = (1 - \frac{r^*}{r})^{-1},\ g_{22} = \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = r^2,\ g_{33} = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = r^2 sin^2\theta \quad (233)$$

It should be highlighted that the tangent spaces in contra variant space and covariant space are not coaxial so that their bases would have been in more complex relationships.

Again, we have seen the distance factors for angular coordinates were employed in metrics.

One can study one more condition for a transformation from contra variant space of distance expressed pseudo spherical space $(cdt, dr, rd\theta, rsin\theta d\varphi)$ to the covariant space of Cartesian space $(cd\tau, du, dv, dw)$, that could give briefer metrics but more complex bases. I prefer to cast further more discussions and seek for a few preliminary conclusions for the subsequent studies.

In the three conditions, we have seen the differences for metrics and bases from contra variant spaces to covariant space. In fact, the transformations are of that from a specific coordinate system transformed to another. Obviously, if a coordinate differential of contra variant space is directly expressed by angular quantity solely such as $d\theta$ or $d\varphi$, the velocity tensor must be an angular velocity rather than linear velocity. Consequently, if the metrics and bases would be involved with the distance factors of angular coordinates and employed to calculate descendant tensors such as Christoffel symbols, curvatures or exactly the linear velocities, that could lead to confusions and errors.

The negative time metric in traditional methodologies must be revised to be positive value to avoid false results as that in Eq. (53). We have realized the minus sign of the first item in the equation of invariant distance is of physical settings, that do nothing with gravitational transformation.

The coordinate setting of $(cd\tau, d\rho, rd\theta, rsin\theta d\varphi)$ and $(cd\tau, du, dv, dw)$ for covariant space would have equivalent metrics and invariant distances. But they would have different bases, in most cases, the Cartesian coordinate system for covariant space of one source fields is too trivial and not necessary.

In general, Christoffel symbols and metrics are just employed to present the calculations of bases, but they are not necessary. Bases could take the effect in every equation composed of Christoffel symbols and metrics. We will see the same answer while bases be employed instead of Christoffel symbols and metrics in the discussions in next sections. In fact, the calculations based on bases could be taken for the verifications to that based on Christoffel symbols and metrics. That is to say if any differences happen in the verifications, there must be something wrong in the calculations. This is another reason to argue about the negative sign of time metric.

In general relativity, it could be suggested to simplify the presentation that we only focus on the gravitational transformation. It is a transformation from contra variant space of distance expressed pseudo spherical space $(cdt, dr, rd\theta, rsin\theta d\varphi)$ to the covariant space of distance expressed pseudo spherical space $(cd\tau, d\rho, rd\theta, rsin\theta d\varphi)$ that just has been performed in condition 2 in previous. Thus, the transformations of any tensors only present gravitational effects. The invariant distance could be expressed as

$$ds^2 = -g_{00}(cdt)^2 + g_{11}dr^2 + g_{22}r^2 d\theta^2 + g_{33}r^2 sin^2\theta d\varphi^2 \quad (234)$$

In which, $g_{22} = g_{33} = 1$. The most particular, g_{00} must be a positive quantity.

These metrics could be called the gravitational metrics, while the metrics other to them could be called composite metrics and the metrics before gravitational transformation could be called original metrics or pseudo spherical transformation metrics.

For gravitational metrics of Schwarzschild solution, the bases are

$$\boldsymbol{e}_0 = \begin{pmatrix} (1 - \frac{r^*}{r})^{1/2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{e}_1 = \begin{pmatrix} 0 \\ (1 - \frac{r^*}{r})^{-1/2} \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (235)$$

It could be calculated that

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$$\frac{\partial e_0}{\partial r}=\begin{pmatrix}\frac{r^*}{2r^2}(1-\frac{r^*}{r})^{-1/2}\\0\\0\\0\end{pmatrix},\frac{\partial e_1}{\partial t}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\ (236)$$

so that

$$\frac{\partial e_0}{\partial r}\neq\frac{\partial e_1}{\partial t}\ (237)$$

This is of the condition 3 that we have discussed in Section 3.2.2. One can also calculate the inequality of mixed derivatives of bases of total metrics.

That reminds us that the inequality of mixed derivatives of bases will cause closure errors in space time, which would be left for more discussions elsewhere.

4.2. Bases, Tensors, and Their Derivatives

Tensors could be recognized as the quantities relating to coordinates in space time. Case a tensor varies in space time, the variation ratio could be inspected by derivation. The simplest tensor is position vector $\boldsymbol{\rho}(x0,x1,x2,x3,)$. You have seen that we are going to use middle subscriptions to express coordinates and tensors in covariant space time, though they are rarely mentioned in most of references. To study its variation in space time, one could define the distance variation ratio to form the bases

$$\boldsymbol{e}_i=\frac{\partial\boldsymbol{\rho}}{\partial x^i}\ (238)$$

For a tensor of coordinate dependent, such as a covariant tensor as

$$\boldsymbol{A}=A^i\boldsymbol{e}_i\ (239)$$

where, A^i is the number i component of the total contra variant tensor.

The derivatives of the tensor are

$$\frac{\partial\boldsymbol{A}}{\partial x^j}=\boldsymbol{e}_i\frac{\partial A^i}{\partial x^j}+A^i\frac{\partial\boldsymbol{e}_i}{\partial x^j}\ (240)$$

The differential of the $d\boldsymbol{A}$ could be defined to be covariant differential labeled as $D\boldsymbol{A}$, and then the derivative $\frac{\partial\boldsymbol{A}}{\partial x^j}$ will be defined the covariant derivative labeled as $\frac{D\boldsymbol{A}}{\partial x^j}$

$$\frac{D\boldsymbol{A}}{\partial x^j}=\frac{\partial\boldsymbol{A}}{\partial x^j}=\frac{\partial A^i\boldsymbol{e}_i}{\partial x^j}=\boldsymbol{e}_i\frac{\partial A^i}{\partial x^j}+A^i\frac{\partial\boldsymbol{e}_i}{\partial x^j}\ (241)$$

In these equations, the middle subscriptions of proper tensors maybe neglected conventionally so that it is expressed as $\boldsymbol{A}$. And the tensor $\boldsymbol{A}$ could be called proper tensor reluctantly because A_i has already been named covariant tensor conventionally. Because A^i or A_i is just a component, we could imagine that there must be the total quantity. That will be expressed to be $\boldsymbol{A}^{\uparrow}=(A^0,A^1,A^2,A^3)$ or $\boldsymbol{A}_{\downarrow}=(A_0,A_1,A_2,A_3)$ for convenience. Sometimes, row forms of expressions of vectors are employed to interpret components which are equivalent to the column forms.

Bases sometimes look like one-order tensors since they have a single index in expressions. But in fact, they really are two-order mixed tensors. For example, a component of contra variant base of collinear transformation, i.e., coordinate lines of contra variant space and covariant space coinciding, could be written as

$$e^{\mu}{}_{\nu}=\frac{\partial x^{\mu}}{\partial x\nu}\ (242)$$

where the contra variant base and proper coordinate differential are all labeled with middle index ν.

So that the base really is

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$$\boldsymbol{e}^{\mu} = \begin{pmatrix} e^{\mu}0 \\ e^{\mu}1 \\ e^{\mu}2 \\ e^{\mu}3 \end{pmatrix} \ (243)$$

If $e^{\mu}\nu$=0 case $\mu \neq \nu$, we could use e^{μ} instead of $e^{\mu}\mu$ for convenience, as it is the only nonvanishing component.

For any one order tensors there is a transformation

$$A^{\mu} = \boldsymbol{e}^{\mu} \cdot \boldsymbol{A} \ or \ A_{\mu} = \boldsymbol{e}_{\mu} \cdot \boldsymbol{A} \ \ (244)$$

But for two order tensors, there are some things different. For example, a component of contra variant velocity could be transformed from proper velocity

$$V^{\mu} = \frac{dx^{\mu}}{d\tau} = \boldsymbol{e}^{\mu} \cdot \frac{d\boldsymbol{\rho}}{d\tau} = \boldsymbol{e}^{\mu} \cdot \boldsymbol{V} \ \ (245)$$

We know that it may be seen as one-order tensor in practice, but it is really two order tensor.

As for whole contra variant velocity as $V_0^1 = \frac{dx^1}{dt}$, that would not be worked out by direct vector product. In fact, it could be composed independently by dx^1 and dt.

$$V_0^1 = \frac{dx^1}{dt} = \frac{\boldsymbol{e}^1 \cdot d\boldsymbol{\rho}}{\boldsymbol{e}^0 \cdot d\boldsymbol{\rho}} = \frac{e^1 dx1}{e^0 d\tau} = e^1 e_0 \frac{dx1}{d\tau} \ \ (246)$$

where, e^1 and e_0 are components of $\boldsymbol{e}^1$ and $\boldsymbol{e}_0$, which are of the simplified forms of $e^1 1$ and $e^0 0$ as we have discussed on Eq. (243) previously.

As has been shown in Eq. (246), it is impossible to get the value $e^1 e_0$ from $\boldsymbol{e}^1 \cdot \boldsymbol{e}_0$ and the latter is zero. Notwithstanding, a velocity is a derivative on matter's trajectory rather than an ordinary derivative. That will be further discussed in next sections.

4.3. On Definitions of Christoffel Symbols and Covariant Derivatives

Christoffel symbols were put forward to perform geometrical relationship that takes equivalent effects with derivatives of bases. I prefer to rewrite the equation [3,13]

$$\frac{\partial \boldsymbol{e}^{\mu}}{\partial x^{\lambda}} = -\Gamma^{\mu}_{\lambda\nu} \boldsymbol{e}^{\nu} \ \ (247)$$

The purpose of the equation is to consider the derivatives to be a function of bases, so that the right hand side item is really a kind of trivial types. In the summation items, $\boldsymbol{e}^{\nu}$ just act as direction indicators that would provide whole basic vectors of entire dimensions. And then, $\Gamma^{\mu}_{\lambda\nu}$ provide the coefficients of all directions. It is said, this definition has just provided an error-free frame for the functions of derivatives. That means there may be redundant designs for the coefficients.

Since there is the probability of inequality of mixed derivatives of bases, we should define a specific sequence for subscripts of Christoffel symbols. For the traditional reasons, $\Gamma^{\mu}_{\lambda\nu}$ will be defined as the coefficient of a derivative of $\boldsymbol{e}^{\mu}$ that is derived by x^{λ}, on a direction of $\boldsymbol{e}^{\nu}$, that requires unalterable subscripts of $\Gamma^{\mu}_{\lambda\nu}$.

In the case that contra variant bases and covariant bases are coaxial there would be the relationship $\boldsymbol{e}^{\mu} \cdot \boldsymbol{e}_{\mu} = 1$, where Einstein summation convention does not act on double μ. Thus,

$$\frac{\partial}{\partial x^{\lambda}} \left(\boldsymbol{e}^{\mu} \cdot \boldsymbol{e}_{\mu} \right) = 0 \ \ (248)$$

or

$$\frac{\partial \boldsymbol{e}^{\mu}}{\partial x^{\lambda}} \cdot \boldsymbol{e}_{\mu} + \frac{\partial \boldsymbol{e}_{\mu}}{\partial x^{\lambda}} \cdot \boldsymbol{e}^{\mu} = 0 \ \ (249)$$

Because the definition of Eq. (247), there is

$$-\Gamma^{\mu}_{\lambda\nu} \boldsymbol{e}^{\nu} \cdot \boldsymbol{e}_{\mu} + \frac{\partial \boldsymbol{e}_{\mu}}{\partial x^{\lambda}} \cdot \boldsymbol{e}^{\mu} = 0 \ \ (250)$$

Obviously, $\boldsymbol{e}^{\nu} \cdot \boldsymbol{e}_{\mu} = \boldsymbol{e}_{\nu} \cdot \boldsymbol{e}^{\mu}$, so that it is equivalent to that

$$-\Gamma^{\mu}_{\lambda\nu} \boldsymbol{e}_{\nu} \cdot \boldsymbol{e}^{\mu} + \frac{\partial \boldsymbol{e}_{\mu}}{\partial x^{\lambda}} \cdot \boldsymbol{e}^{\mu} = 0 \ \ (251)$$

As well as that

$$\frac{\partial \boldsymbol{e}_\mu}{\partial x^\lambda} = \Gamma^v_{\lambda\mu}\boldsymbol{e}_v \quad (252)$$

If the bases are not coaxial, the Eq. (252) will not take effect.

We now turn to the topic of covariant derivative which is exactly a derivative of a proper tensor

$$\frac{D\boldsymbol{A}}{\partial x^\lambda} = \frac{\partial \boldsymbol{A}}{\partial x^\lambda} \quad (253)$$

This highlightable concept is essentially carried out to perform general covariance. More statements should be casted on is that the tensor $\boldsymbol{A}$ is simplified expression of any probable middle labeled forms of $\boldsymbol{A}\uparrow$ or $\boldsymbol{A}\uparrow/\downarrow$ etc.

The contra variant component form also performs the same covariance as that

$$\frac{DA^\mu}{\partial x^\lambda} = \boldsymbol{e}^\mu \cdot \frac{D\boldsymbol{A}}{\partial x^\lambda} \quad (254)$$

You might have found that the tensor component that has been expressed by a total tensor is exactly partial expression rather than a whole expression. It is just of traditional operations. One can of course carry out whole form expression of $\boldsymbol{A}^\uparrow$ expressed by $\boldsymbol{A}$ with base matrix $[\boldsymbol{e}^\uparrow]$. But too more renovations in the performances will bring about more reading difficulties. I prefer to present equations in traditional forms as far as possible.

Comparing with the definition of $\frac{D\boldsymbol{A}}{\partial x^\lambda}$, $\frac{DA^\mu}{\partial x^\lambda}$ does not keep perfect physical senses. At most, it could be seen as a component of the tensor that the $\frac{D\boldsymbol{A}}{\partial x^\lambda}$ be seen in the view of contra variant space or a component of contra variant tensor $\boldsymbol{A}^\uparrow$ transformed from covariant one that

$$A^\mu = \boldsymbol{e}^\mu \cdot \boldsymbol{A} \quad (255)$$

Case it experiences a derivation as

$$\frac{\partial A^\mu}{\partial x^v} = \frac{\partial}{\partial x^v}(\boldsymbol{e}^\mu \cdot \boldsymbol{A}) = \frac{\partial \boldsymbol{e}^\mu}{\partial x^v} \cdot \boldsymbol{A} + \boldsymbol{e}^\mu \cdot \frac{\partial \boldsymbol{A}}{\partial x^v} = \frac{\partial \boldsymbol{e}^\mu}{\partial x^v} \cdot \boldsymbol{A} + \frac{DA^\mu}{\partial x^v} \quad (256)$$

There will be

$$\frac{DA^\mu}{\partial x^v} = \frac{\partial A^\mu}{\partial x^v} - \frac{\partial \boldsymbol{e}^\mu}{\partial x^v} \cdot \boldsymbol{A} = \frac{\partial A^\mu}{\partial x^v} + \Gamma^\mu_{v\lambda}A^\lambda \quad (257)$$

It is easy to study those covariant derivatives for covariant tensors

$$\frac{DA_\mu}{\partial x^v} = \frac{\partial A_\mu}{\partial x^v} - \frac{\partial \boldsymbol{e}_\mu}{\partial x^v} \cdot \boldsymbol{A} = \frac{\partial A_\mu}{\partial x^v} - \Gamma^\lambda_{v\mu}A_\lambda \quad (258)$$

We have seen that the methodologies of Christoffel symbols and the derivation directly from bases are something equivalent treatments that present the covariant derivatives. That of course may be use to inspect the problems of equations of Christoffel symbols. Since we have known that part of Christoffel symbols with mixed subscripts do not equal in space time, it is necessary to do more discussions.

4.4. Renovated Equation of Christoffel Symbols to Single out Mistakes in Traditional Calculations

4.4.1. Renovated Equation of Christoffel Symbols

As has been discussed that the traditional equation Eq. (64) was involved with errors in relativity, it is necessary to seek for renovated equations.

In practice, metrics are usually taken in to Christoffel connection analysis. It is proper to discuss the cases of orthogonal bases. For a series of bases $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, there is the metric defined as

$$g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j \quad (259)$$

For orthogonal bases, there is

$$\begin{cases} g_{ij} = 0, i \neq j \\ g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_i \neq 0, i = j \end{cases} \quad (260)$$

The derivatives

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$$\frac{\partial g_{ij}}{\partial x^\lambda} = \frac{\partial e_i}{\partial x^\lambda} \cdot \boldsymbol{e}_j + \frac{\partial e_j}{\partial x^\lambda} \cdot \boldsymbol{e}_i = 2\frac{\partial e_i}{\partial x^\lambda} \cdot \boldsymbol{e}_j, i = j \quad (261)$$

Then, the equation could have nonvanishing value, that

$$\frac{\partial g_{ij}}{\partial x^\lambda} = 2\Gamma^k_{\lambda i}\boldsymbol{e}_k \cdot \boldsymbol{e}_j = 2\Gamma^k_{\lambda i} g_{kj}, i = j \quad (262)$$

For spaces with orthogonal bases, only in the case $k = j$ there is $g_{kj} \neq 0$. Thus,

$$\Gamma^k_{\lambda i} = \frac{1}{2g_{kj}}\frac{\partial g_{ij}}{\partial x^\lambda}, k = i = j \quad (263)$$

Case contra variant space and covariant space are coaxial, $\boldsymbol{e}^k \cdot \boldsymbol{e}_j = 1$, and

$$\Gamma^k_{\lambda i} = \frac{1}{2}g^{kj}\frac{\partial g_{ij}}{\partial x^\lambda} = \boldsymbol{e}^k \cdot \frac{\partial e_i}{\partial x^\lambda}, k = i = j \quad (264)$$

The base expressed form could be calculated from the form of metrics. This equation could be called revised equation for Christoffel symbols in general relativity.

In fact, this equation could be drawn from Eq. (247) for the case of orthogonal bases.

For a result, this equation could be verified in a covariant derivative directly as that in Eq. (258) that

$$\Gamma^k_{\lambda i}A_k = \frac{1}{2}g^{kj}\frac{\partial g_{ij}}{\partial x^\lambda}A_k = \frac{1}{2}(\boldsymbol{e}^k \cdot \boldsymbol{e}^j)(2\frac{\partial e_i}{\partial x^\lambda} \cdot \boldsymbol{e}_j)A_k = A_k\boldsymbol{e}^k \cdot \frac{\partial e_i}{\partial x^\lambda} = \frac{\partial e_i}{\partial x^\lambda} \cdot \boldsymbol{A} \quad (265)$$

It should be pointed out that the Christoffel symbols are not necessary because that the issue only started from derivatives of bases, as consequences they surely might be taken the place by the operations of bases.

4.4.2. Christoffel symbols of One Source Fields

In one source fields, the invariant distance could be expressed as

$$ds^2 = -B(r)(cdt)^2 + A(r)dr^2 + r^2d\theta^2 + r^2sin^2\theta d\varphi^2$$
$$= -(1-\frac{r^*}{r})(cdt)^2 + (1-\frac{r^*}{r})^{-1}dr^2 + r^2d\theta^2 + r^2sin^2\theta d\varphi^2 \quad (266)$$

If the transformation is seen as that from contra variant space of original pseudo spherical space $(cdt, \mathrm{d}r, \mathrm{d}\theta, \mathrm{d}\varphi)$ to covariant space of distance expressed pseudo spherical space $(cd\tau, \mathrm{d}\rho, rd\theta, rsin\theta d\varphi)$, the metrics will be

$$g_{00} = (1-\frac{r^*}{r}),\ g_{11} = (1-\frac{r^*}{r})^{-1},\ g_{22} = r^2,\ g_{33} = r^2sin^2\theta \quad (267)$$

It is of course not appropriate ways to study gravitational fields, but this concern could discover the errors in traditional calculations. With the equation Eq. (264), Christoffel symbols could be calculated as following.

$$\begin{bmatrix} \Gamma^0_{00} & \Gamma^0_{01} & \Gamma^0_{02} & \Gamma^0_{03} \\ \Gamma^1_{00} & \Gamma^1_{01} & \Gamma^1_{02} & \Gamma^1_{03} \\ \Gamma^2_{00} & \Gamma^2_{01} & \Gamma^2_{02} & \Gamma^2_{03} \\ \Gamma^3_{00} & \Gamma^3_{01} & \Gamma^3_{02} & \Gamma^3_{03} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (268)$$

$$\begin{bmatrix} \Gamma^0_{10} & \Gamma^0_{11} & \Gamma^0_{12} & \Gamma^0_{13} \\ \Gamma^1_{10} & \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{13} \\ \Gamma^2_{10} & \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{13} \\ \Gamma^3_{10} & \Gamma^3_{11} & \Gamma^3_{12} & \Gamma^3_{13} \end{bmatrix} = \begin{bmatrix} \frac{r^*}{2r^2}(1-\frac{r^*}{r})^{-1} & 0 & 0 & 0 \\ 0 & -\frac{r^*}{2r^2}(1-\frac{r^*}{r})^{-1} & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r} \end{bmatrix} \quad (269)$$

$$\begin{bmatrix} \Gamma^0_{20} & \Gamma^0_{21} & \Gamma^0_{22} & \Gamma^0_{23} \\ \Gamma^1_{20} & \Gamma^1_{21} & \Gamma^1_{22} & \Gamma^1_{23} \\ \Gamma^2_{20} & \Gamma^2_{21} & \Gamma^2_{22} & \Gamma^2_{23} \\ \Gamma^3_{20} & \Gamma^3_{21} & \Gamma^3_{22} & \Gamma^3_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & cot\theta \end{bmatrix} \quad (270)$$

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$$\begin{bmatrix} \Gamma_{30}^{0} & \Gamma_{31}^{0} & \Gamma_{32}^{0} & \Gamma_{33}^{0} \\ \Gamma_{30}^{1} & \Gamma_{31}^{1} & \Gamma_{32}^{1} & \Gamma_{33}^{1} \\ \Gamma_{30}^{2} & \Gamma_{31}^{2} & \Gamma_{32}^{2} & \Gamma_{33}^{2} \\ \Gamma_{30}^{3} & \Gamma_{31}^{3} & \Gamma_{32}^{3} & \Gamma_{33}^{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (271)$$

On another hand, it is more interesting to investigate Christoffel symbols based on gravitational metrics that

$$g_{00} = (1 - \frac{r^*}{r}),\ g_{11} = (1 - \frac{r^*}{r})^{-1},\ g_{22} = 1,\ g_{33} = 1 \quad (272)$$

It is simpler than the previous so that the calculation could be presented in brief ways. In fact, only Γ_{10}^{0} and Γ_{11}^{1} are nonvanishing as shown as following

$$\begin{bmatrix} \Gamma_{00}^{0} & \Gamma_{01}^{0} & \Gamma_{02}^{0} & \Gamma_{03}^{0} \\ \Gamma_{00}^{1} & \Gamma_{01}^{1} & \Gamma_{02}^{1} & \Gamma_{03}^{1} \\ \Gamma_{00}^{2} & \Gamma_{01}^{2} & \Gamma_{02}^{2} & \Gamma_{03}^{2} \\ \Gamma_{00}^{3} & \Gamma_{01}^{3} & \Gamma_{02}^{3} & \Gamma_{03}^{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (273)$$

$$\begin{bmatrix} \Gamma_{10}^{0} & \Gamma_{11}^{0} & \Gamma_{12}^{0} & \Gamma_{13}^{0} \\ \Gamma_{10}^{1} & \Gamma_{11}^{1} & \Gamma_{12}^{1} & \Gamma_{13}^{1} \\ \Gamma_{10}^{2} & \Gamma_{11}^{2} & \Gamma_{12}^{2} & \Gamma_{13}^{2} \\ \Gamma_{10}^{3} & \Gamma_{11}^{3} & \Gamma_{12}^{3} & \Gamma_{13}^{3} \end{bmatrix} = \begin{bmatrix} \frac{r^*}{2r^2}(1 - \frac{r^*}{r})^{-1} & 0 & 0 & 0 \\ 0 & -\frac{r^*}{2r^2}(1 - \frac{r^*}{r})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (274)$$

$$\begin{bmatrix} \Gamma_{20}^{0} & \Gamma_{21}^{0} & \Gamma_{22}^{0} & \Gamma_{23}^{0} \\ \Gamma_{20}^{1} & \Gamma_{21}^{1} & \Gamma_{22}^{1} & \Gamma_{23}^{1} \\ \Gamma_{20}^{2} & \Gamma_{21}^{2} & \Gamma_{22}^{2} & \Gamma_{23}^{2} \\ \Gamma_{20}^{3} & \Gamma_{21}^{3} & \Gamma_{22}^{3} & \Gamma_{23}^{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (275)$$

$$\begin{bmatrix} \Gamma_{30}^{0} & \Gamma_{31}^{0} & \Gamma_{32}^{0} & \Gamma_{33}^{0} \\ \Gamma_{30}^{1} & \Gamma_{31}^{1} & \Gamma_{32}^{1} & \Gamma_{33}^{1} \\ \Gamma_{30}^{2} & \Gamma_{31}^{2} & \Gamma_{32}^{2} & \Gamma_{33}^{2} \\ \Gamma_{30}^{3} & \Gamma_{31}^{3} & \Gamma_{32}^{3} & \Gamma_{33}^{3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (276)$$

It has been investigated that in gravitational fields, the sub indexes of Christoffel symbols are not alternative. There are other two critical properties could be drawn for Christoffel symbols. The first one is that for orthogonal bases, it would have vanishing value that $\Gamma_{\lambda i}^{k} = 0$ case $k \neq i$. The second one is that for rest fields, once time derivation is concerned, there will be $\Gamma_{0i}^{k} = 0$.

It should be highlighted that Christoffel symbols are of field derivatives that they are far different from trajectory derivatives we will discuss in next section.

4.4.3. Mistakes in Traditional Calculations

Many publications have given the nonvanishing Christoffel symbols for one source fields, largely the same, with few minor distinctions. As the invariant distance

$$ds^2 = -B(r)(cdt)^2 + A(r)dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2$$

$$= -(1 - \frac{r^*}{r})(cdt)^2 + (1 - \frac{r^*}{r})^{-1} dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\varphi^2 \quad (277)$$

with metrics

$$g_{00} = -B(r) = -(1 - \frac{r^*}{r}),\ g_{11} = A(r) = (1 - \frac{r^*}{r})^{-1},\ g_{22} = r^2,\ g_{33} = r^2 sin^2\theta \quad (278)$$

They could be listed in Table 1. for more investigations.

Table 1. A list of nonvanishing Christoffel symbols in some publications.

$\Gamma_{00}^{1} = \frac{B'(r)}{2A(r)} = \frac{r^*}{2r^2}(1 - \frac{r^*}{r})$	$\Gamma_{11}^{1} = \frac{A'(r)}{2A(r)} = -\frac{r^*}{2r^2}(1 - \frac{r^*}{r})^{-1}$	$\Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{1}{r}$
$\Gamma_{01}^{0} = \Gamma_{10}^{0} = \frac{B'(r)}{2B(r)} = \frac{r^*}{2r^2}(1 - \frac{r^*}{r})^{-1}$	$\Gamma_{22}^{1} = -\frac{r}{A(r)} = -r(1 - \frac{r^*}{r})$	$\Gamma_{13}^{3} = \Gamma_{31}^{3} = \frac{1}{r}$
$\Gamma_{33}^{1} = -r\frac{sin^2\theta}{A(r)} = -r(1 - \frac{r^*}{r})sin^2\theta$	$\Gamma_{33}^{2} = -sin\theta cos\theta$	$\Gamma_{23}^{3} = \Gamma_{32}^{3} = cot\theta$

$\Gamma_{00}^{1} = \frac{B'(r)}{2A(r)} = \frac{r^*}{2r^2}(1 - \frac{r^*}{r})$	$\Gamma_{11}^{1} = \frac{A'(r)}{2A(r)} = -\frac{r^*}{2r^2}(1 - \frac{r^*}{r})^{-1}$	$\Gamma_{12}^{2} = \Gamma_{21}^{2} = \frac{1}{r}$
$\Gamma_{01}^{0} = \Gamma_{10}^{0} = \frac{B'(r)}{2B(r)} = \frac{r^*}{2r^2}(1 - \frac{r^*}{r})^{-1}$	$\Gamma_{22}^{1} = -\frac{r}{A(r)} = -r(1 - \frac{r^*}{r})$	$\Gamma_{13}^{3} = \Gamma_{31}^{3} = \frac{1}{r}$
$\Gamma_{33}^{1} = -r\frac{sin^2\theta}{A(r)} = -r(1 - \frac{r^*}{r})sin^2\theta$	$\Gamma_{33}^{2} = -sin\theta cos\theta$	$\Gamma_{23}^{3} = \Gamma_{32}^{3} = cot\theta$

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As have discussed in previous sections, in rest fields, Γ_{00}^{1} and Γ_{01}^{0} must be vanishing while they are wrong calculated in the table. Any $k \neq i$ will cause $\Gamma_{\lambda i}^{k} = 0$. Thus, Γ_{21}^{2}, Γ_{22}^{1}, Γ_{31}^{3}, Γ_{33}^{1}, Γ_{33}^{2} and Γ_{32}^{3} have been wrong calculated as well. Nevertheless, in nature, $\Gamma_{12}^{2} \neq \Gamma_{21}^{2}$, $\Gamma_{01}^{0} \neq \Gamma_{10}^{0}$, $\Gamma_{13}^{3} \neq \Gamma_{31}^{3}$, $\Gamma_{23}^{3} \neq \Gamma_{32}^{3}$ rather than they were mistakenly treated in the table.

One can furtherly calculate the base derivatives with the listed Christoffel symbols to find out more mistakes.

Table 2. Calculations on derivatives of bases of one source fields with given Christoffel symbols.

Derivatives of contra variant bases	T or F	Derivatives of covariant bases	T or F
$\frac{\partial \boldsymbol{e}^0}{\partial t} = -\Gamma_{00}^{0}\boldsymbol{e}^0 - \Gamma_{01}^{0}\boldsymbol{e}^1 - \Gamma_{02}^{0}\boldsymbol{e}^2 - \Gamma_{03}^{0}\boldsymbol{e}^3 = -\Gamma_{01}^{0}\boldsymbol{e}^1$	F	$\frac{\partial \boldsymbol{e}_0}{\partial t} = \Gamma_{00}^{0}\boldsymbol{e}_0 + \Gamma_{00}^{1}\boldsymbol{e}_1 + \Gamma_{00}^{2}\boldsymbol{e}_2 + \Gamma_{00}^{3}\boldsymbol{e}_3 = \Gamma_{00}^{1}\boldsymbol{e}_1$	F
$\frac{\partial \boldsymbol{e}^0}{\partial r} = -\Gamma_{10}^{0}\boldsymbol{e}^0 - \Gamma_{11}^{0}\boldsymbol{e}^1 - \Gamma_{12}^{0}\boldsymbol{e}^2 - \Gamma_{13}^{0}\boldsymbol{e}^3 = -\Gamma_{10}^{0}\boldsymbol{e}^0$	T	$\frac{\partial \boldsymbol{e}_0}{\partial r} = \Gamma_{10}^{0}\boldsymbol{e}_0 + \Gamma_{10}^{1}\boldsymbol{e}_1 + \Gamma_{10}^{2}\boldsymbol{e}_2 + \Gamma_{10}^{3}\boldsymbol{e}_3 = \Gamma_{10}^{0}\boldsymbol{e}_0$	T
$\frac{\partial \boldsymbol{e}^1}{\partial t} = -\Gamma_{00}^{1}\boldsymbol{e}^0 - \Gamma_{01}^{1}\boldsymbol{e}^1 - \Gamma_{02}^{1}\boldsymbol{e}^2 - \Gamma_{03}^{1}\boldsymbol{e}^3 = -\Gamma_{00}^{1}\boldsymbol{e}^0$	F	$\frac{\partial \boldsymbol{e}_1}{\partial t} = \Gamma_{01}^{0}\boldsymbol{e}_0 + \Gamma_{01}^{1}\boldsymbol{e}_1 + \Gamma_{01}^{2}\boldsymbol{e}_2 + \Gamma_{01}^{3}\boldsymbol{e}_3 = \Gamma_{01}^{0}\boldsymbol{e}_0$	F
$\frac{\partial \boldsymbol{e}^1}{\partial r} = -\Gamma_{10}^{1}\boldsymbol{e}^0 - \Gamma_{11}^{1}\boldsymbol{e}^1 - \Gamma_{12}^{1}\boldsymbol{e}^2 - \Gamma_{13}^{1}\boldsymbol{e}^3 = -\Gamma_{11}^{1}\boldsymbol{e}^1$	T	$\frac{\partial \boldsymbol{e}_1}{\partial r} = \Gamma_{11}^{0}\boldsymbol{e}_0 + \Gamma_{11}^{1}\boldsymbol{e}_1 + \Gamma_{11}^{2}\boldsymbol{e}_2 + \Gamma_{11}^{3}\boldsymbol{e}_3 = \Gamma_{11}^{1}\boldsymbol{e}_1$	T
$\frac{\partial \boldsymbol{e}^1}{\partial \theta} = -\Gamma_{20}^{1}\boldsymbol{e}^0 - \Gamma_{21}^{1}\boldsymbol{e}^1 - \Gamma_{22}^{1}\boldsymbol{e}^2 - \Gamma_{23}^{1}\boldsymbol{e}^3 = -\Gamma_{22}^{1}\boldsymbol{e}^2$	F	$\frac{\partial \boldsymbol{e}_1}{\partial \theta} = \Gamma_{21}^{0}\boldsymbol{e}_0 + \Gamma_{21}^{1}\boldsymbol{e}_1 + \Gamma_{21}^{2}\boldsymbol{e}_2 + \Gamma_{21}^{3}\boldsymbol{e}_3 = \Gamma_{21}^{2}\boldsymbol{e}_2$	F
$\frac{\partial \boldsymbol{e}^1}{\partial \varphi} = -\Gamma_{30}^{1}\boldsymbol{e}^0 - \Gamma_{31}^{1}\boldsymbol{e}^1 - \Gamma_{32}^{1}\boldsymbol{e}^2 - \Gamma_{33}^{1}\boldsymbol{e}^3 = -\Gamma_{33}^{1}\boldsymbol{e}^3$	F	$\frac{\partial \boldsymbol{e}_1}{\partial \varphi} = \Gamma_{31}^{0}\boldsymbol{e}_0 + \Gamma_{31}^{1}\boldsymbol{e}_1 + \Gamma_{31}^{2}\boldsymbol{e}_2 + \Gamma_{31}^{3}\boldsymbol{e}_3 = \Gamma_{31}^{3}\boldsymbol{e}_3$	F
$\frac{\partial \boldsymbol{e}^2}{\partial r} = -\Gamma_{10}^{2}\boldsymbol{e}^0 - \Gamma_{11}^{2}\boldsymbol{e}^1 - \Gamma_{12}^{2}\boldsymbol{e}^2 - \Gamma_{13}^{2}\boldsymbol{e}^3 = -\Gamma_{12}^{2}\boldsymbol{e}^2$	T	$\frac{\partial \boldsymbol{e}_2}{\partial r} = \Gamma_{12}^{0}\boldsymbol{e}_0 + \Gamma_{12}^{1}\boldsymbol{e}_1 + \Gamma_{12}^{2}\boldsymbol{e}_2 + \Gamma_{12}^{3}\boldsymbol{e}_3 = \Gamma_{12}^{2}\boldsymbol{e}_2$	T
$\frac{\partial \boldsymbol{e}^2}{\partial \theta} = -\Gamma_{20}^{2}\boldsymbol{e}^0 - \Gamma_{21}^{2}\boldsymbol{e}^1 - \Gamma_{22}^{2}\boldsymbol{e}^2 - \Gamma_{23}^{2}\boldsymbol{e}^3 = -\Gamma_{21}^{2}\boldsymbol{e}^1$	F	$\frac{\partial \boldsymbol{e}_2}{\partial \theta} = \Gamma_{22}^{0}\boldsymbol{e}_0 + \Gamma_{22}^{1}\boldsymbol{e}_1 + \Gamma_{22}^{2}\boldsymbol{e}_2 + \Gamma_{22}^{3}\boldsymbol{e}_3 = \Gamma_{22}^{1}\boldsymbol{e}_1$	F
$\frac{\partial \boldsymbol{e}^2}{\partial \varphi} = -\Gamma_{30}^{2}\boldsymbol{e}^0 - \Gamma_{31}^{2}\boldsymbol{e}^1 - \Gamma_{32}^{2}\boldsymbol{e}^2 - \Gamma_{33}^{2}\boldsymbol{e}^3 = -\Gamma_{33}^{2}\boldsymbol{e}^3$	F	$\frac{\partial \boldsymbol{e}_2}{\partial \varphi} = \Gamma_{32}^{0}\boldsymbol{e}_0 + \Gamma_{32}^{1}\boldsymbol{e}_1 + \Gamma_{32}^{2}\boldsymbol{e}_2 + \Gamma_{32}^{3}\boldsymbol{e}_3 = \Gamma_{32}^{3}\boldsymbol{e}_3$	F
$\frac{\partial \boldsymbol{e}^3}{\partial r} = -\Gamma_{10}^{3}\boldsymbol{e}^0 - \Gamma_{11}^{3}\boldsymbol{e}^1 - \Gamma_{12}^{3}\boldsymbol{e}^2 - \Gamma_{13}^{3}\boldsymbol{e}^3 = -\Gamma_{13}^{3}\boldsymbol{e}^3$	T	$\frac{\partial \boldsymbol{e}_3}{\partial r} = \Gamma_{13}^{0}\boldsymbol{e}_0 + \Gamma_{13}^{1}\boldsymbol{e}_1 + \Gamma_{13}^{2}\boldsymbol{e}_2 + \Gamma_{13}^{3}\boldsymbol{e}_3 = \Gamma_{13}^{3}\boldsymbol{e}_3$	T
$\frac{\partial \boldsymbol{e}^3}{\partial \theta} = -\Gamma_{20}^{3}\boldsymbol{e}^0 - \Gamma_{21}^{3}\boldsymbol{e}^1 - \Gamma_{22}^{3}\boldsymbol{e}^2 - \Gamma_{23}^{3}\boldsymbol{e}^3 = -\Gamma_{23}^{3}\boldsymbol{e}^3$	T	$\frac{\partial \boldsymbol{e}_3}{\partial \theta} = \Gamma_{23}^{0}\boldsymbol{e}_0 + \Gamma_{23}^{1}\boldsymbol{e}_1 + \Gamma_{23}^{2}\boldsymbol{e}_2 + \Gamma_{23}^{3}\boldsymbol{e}_3 = \Gamma_{23}^{3}\boldsymbol{e}_3$	T
$\frac{\partial \boldsymbol{e}^3}{\partial \varphi} = -\Gamma_{30}^{3}\boldsymbol{e}^0 - \Gamma_{31}^{3}\boldsymbol{e}^1 - \Gamma_{32}^{3}\boldsymbol{e}^2 - \Gamma_{33}^{3}\boldsymbol{e}^3 = -\Gamma_{31}^{3}\boldsymbol{e}^1 - \Gamma_{32}^{3}\boldsymbol{e}^2$	F	$\frac{\partial \boldsymbol{e}_3}{\partial \varphi} = \Gamma_{33}^{0}\boldsymbol{e}_0 + \Gamma_{33}^{1}\boldsymbol{e}_1 + \Gamma_{33}^{2}\boldsymbol{e}_2 + \Gamma_{33}^{3}\boldsymbol{e}_3 = \Gamma_{33}^{1}\boldsymbol{e}_1 + \Gamma_{33}^{2}\boldsymbol{e}_2$	F

It is very easy to be verified that some the derivatives in Table 2. were mistakenly calculated. As the results, there are 5 derivatives in every space were calculated correctly. Of course, these correct results are not the evidences for the reliability of traditional equation. The correct results should be calculated with the suggested equation Eq. (264).

Case gravitational metrics being taken for granted, there will be only $\frac{\partial \boldsymbol{e}^0}{\partial r}$, $\frac{\partial \boldsymbol{e}^1}{\partial r}$, $\frac{\partial \boldsymbol{e}_0}{\partial r}$ and $\frac{\partial \boldsymbol{e}_1}{\partial r}$ nonvanishing. These results indeed present the real properties of one source fields.

5. Trajectory Derivatives

5.1. The Differences between Trajectory Derivatives and Field Derivatives

The calculations on time derivatives may have caused mathematical abuse in classical theory. One source fields could be seen as rest fields. Thus, a time derivative of a field quantity should be zero. Case a matter moves in a space, it is the issue that the matter changes its position in a time interval and forms a motion trajectory. In this condition, to learn the acceleration is to study the position variation rather than the field variation. It is one of the reasons for that the errors occur in Eq. (53) and Eq. (57), in that the equation is to calculate in the way of field derivative rather than the way of moving matters.

It is valuable to reclassify tensors to be field tensors and motion tensors, thus field tensors may vary with field while motion tensors should vary both with field and matter's motion. For example, the bases only depend on gravitational field, while velocity of a matter may vary due to positions changing with time lasting. For example, case in one source field, a space derivative of a base may be nonvanishing, but a time derivative of a base must be zero, nevertheless, the base relating to the matter's motions in space time would vary because the coordinates vary on trajectory.

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A trajectory of motions of a matter should be a directed curve line in a space, from the start to the end. It is said that a trajectory must be a single parametrical curve line. Theoretically, the parameter maybe naturally the line it passes through, and as well it could be time that the motion experienced. What worthy of highlight is that these parameters are simultaneous and linearly dependent. It is said that a record of a parameter corresponds to a sole record of another. The parameters indicate the sequences.

It is no harm to discuss the trajectory vector $\boldsymbol{\lambda}$ as a directed curve line in space time, in that the trajectory is a function of single variable. There is

$$\boldsymbol{\lambda}(x^0,x^1,x^2,x^3)=\boldsymbol{\lambda}(\lambda)=\boldsymbol{\lambda}(t) \quad (279)$$

where, λ is the length of trajectory.

It is said that the trajectory could be of parametrical equations as

$$x^0=x^0(t),x^1=x^1(t),x^2=x^2(t),x^3=x^3(t) \quad (280)$$

We have seen that the time coordinate and the space coordinates on trajectory are correlative. Time coordinate is different from space coordinates in physics.

A tensor variation ratio during a time interval on trajectory could be defined to be trajectory derivative that

$$(\frac{D\boldsymbol{A}}{dx^\mu})_{tr}=\frac{D\boldsymbol{A}}{d\lambda}\frac{d\lambda}{dx^\mu} \quad (281)$$

For example

$$(\frac{D\boldsymbol{A}}{dt})_{tr}=\frac{D\boldsymbol{A}}{d\lambda}\frac{d\lambda}{dt} \quad (282)$$

where, Einstein summation convention does not act on double λ because trajectory is just a single line. The directed differential length $d\lambda$ is the differential of matter's trajectory, so that $\frac{D\boldsymbol{A}}{d\lambda}d\lambda$ is the covariant differential of tensor $\boldsymbol{A}$ between two neighbourhood positions on the trajectory. Thus, the so-called trajectory derivative is really a kind of line derivative that is derived by a parameter of trajectory.

It should be noted that a trajectory of matter's motion indicates time sequences corresponding to positions on the trajectory. There is the substantial difference between trajectory derivatives and primary derivatives. A differential on trajectory is the distance interval that a matter has past across, so that the velocity is a trajectory derivative

$$\boldsymbol{V}_0^{\uparrow}=(\frac{d\boldsymbol{\lambda}}{dt})_{tr}=\frac{d\boldsymbol{\lambda}}{dt}$$

$$=(\frac{dx^0(t)}{dt},\frac{dx^1(t)}{dt},\frac{dx^2(t)}{dt},\frac{dx^3(t)}{dt})$$

$$=(V_0^0,V_0^2,V_0^3,V_0^4) \quad (283)$$

We have seen that trajectory differentials are of basic differentials so that velocities do not have to be expressed in forms of covariant derivatives.

As mentioned above, bases are defined by field derivatives such as

$$\boldsymbol{e}^\mu=\frac{\partial\boldsymbol{r}}{\partial x\mu}=(\frac{\partial x^0}{\partial x\mu},\frac{\partial x^1}{\partial x\mu},\frac{\partial x^2}{\partial x\mu},\frac{\partial x^3}{\partial x\mu}) \quad (284)$$

where, $dx\mu$ is a proper coordinate differential, so that it is middle labeled.

For the velocity, the differential dx^μ is exactly defined on a trajectory of a matter, so that there is the probability that

$$\boldsymbol{V}_0^{\uparrow}\neq 0 \quad (285)$$

It is said that velocity tensor itself is literally trajectory derivatives. Trajectory derivatives of general tensor should be labeled for discrepancy. For example, the bases along a trajectory in rest field

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$$(\frac{d\boldsymbol{e}_i}{dt})_{tr} = \frac{d\boldsymbol{e}_i}{d\lambda}\frac{d\lambda}{dt} \neq 0 \quad (286)$$

while field derivatives

$$\frac{\partial \boldsymbol{e}_i}{\partial t} = 0 \quad (287)$$

Eq. (282) could be calculated as

$$(\frac{D\boldsymbol{A}}{dt})_{tr} = \frac{D(A^i\boldsymbol{e}_i)}{d\lambda}\frac{d\lambda}{dt} = \boldsymbol{e}_i\frac{dA^i}{d\lambda}\frac{d\lambda}{dt} + A^i\frac{d\boldsymbol{e}_i}{d\lambda}\frac{d\lambda}{dt} \quad (288)$$

And $\frac{dA^i}{d\lambda}\frac{d\lambda}{dt}$ is also time derivative, and there is

$$(\frac{dA^i}{dt})_{tr} = \frac{dA^i}{d\lambda}\frac{d\lambda}{dt} \quad (289)$$

where, Einstein summation convention does not act on double λ.

Thus, we have seen the differences between trajectory derivatives and field derivatives.

5.2. Calculations on Trajectory Derivatives

On the other hand, the value of matter's velocities are really trajectory derivatives although they would be written in simple forms in most cases,

$$V_0^\mu = (V_0^\mu)_{tr} = \frac{dx^\mu}{d\lambda}\frac{d\lambda}{dt} = \alpha_\lambda^\mu V_0^\uparrow \quad (290)$$

where, $\alpha_\lambda^\mu = \frac{\partial x^\mu}{\partial \lambda}$ is one of the direction cosines of the trajectory line $\boldsymbol{\lambda}$ on the direction μ, so that V_0^μ is one of component of $\boldsymbol{V}_0^\uparrow$ on that direction and $V_0^\uparrow$ is the value of $\boldsymbol{V}_0^\uparrow$. $\boldsymbol{V}_0^\uparrow$ is vector form of V_0^λ so that it could also be written as $\boldsymbol{V}_0^\lambda$ equivalently, anyway the symbol ↑ employed instead of λ is because the latter may cause confused with the labels of components.

One can sponsor more discussions on the topics of α_λ^0 case in variable fields. I think that time coordinate is independent with space coordinates by nature.

Covariant derivatives on matter's trajectories should be

$$(\frac{D\boldsymbol{A}}{dt})_{tr} = [\frac{D(\boldsymbol{e}_iA^i)}{dt}]_{tr}$$

$$= \boldsymbol{e}_i\frac{dA^i}{d\lambda}\frac{d\lambda}{dt} + A^i\frac{d\boldsymbol{e}_i}{d\lambda}\frac{d\lambda}{dt}$$

$$= \boldsymbol{e}_i(\frac{dA^i}{dt})_{tr} + A^i\frac{\partial\boldsymbol{e}_i}{\partial x^\mu}\frac{dx^\mu}{d\lambda}\frac{d\lambda}{dt}$$

$$= \boldsymbol{e}_i(\frac{dA^i}{dt})_{tr} + A^i\frac{\partial\boldsymbol{e}_i}{\partial x^\mu}\alpha_\lambda^\mu V_0^\uparrow$$

$$= \boldsymbol{e}_i(\frac{dA^i}{dt})_{tr} + A^i\frac{\partial\boldsymbol{e}_i}{\partial x^\mu}V_0^\mu \quad (291)$$

where, Einstein summation convention does not act on double λ.

We have seen that the covariant forms of basic derivatives could be their original forms, such as the velocities which are composed of original coordinate differentials. That is because the original coordinate differentials have been defined directly as that in Eq. (246) and Eq. (283). Only the tensors composed of higher order derivatives could have covariant forms with geometrical items as in Eq. (291). Therefore, any tensor $\boldsymbol{A}$ mentioned in covariant equation refers to higher order derivatives, at least one order derivatives. For examples, a velocity is of higher order tensors, in that it is of derivatives of original coordinates so that it could have higher derivation with Eq. (291) to get higher order covariant derivatives, but it is of basic derivatives so that a velocity itself cannot be drawn from Eq. (291). A little interesting, the frequency is of higher order derivatives as has defined in Eq. (31).

Mathematically, the concept of higher order of derivatives may be involved with complexities, that will not be furtherly discussed here trivially.

Eq. (291) could be expanded to be

$$(\frac{D\boldsymbol{A}}{dt})_{tr} = \boldsymbol{e}_i(\frac{dA^i}{dt})_{tr} + A^i(\frac{\partial \boldsymbol{e}_i}{\partial x^0}V_0^0 + \frac{\partial \boldsymbol{e}_i}{\partial x^1}V_0^1 + \frac{\partial \boldsymbol{e}_i}{\partial x^2}V_0^2 + \frac{\partial \boldsymbol{e}_i}{\partial x^3}V_0^3) \quad (292)$$

The expression in component forms could also be worked out as

$$(\frac{DA^\mu}{dt})_{tr} = (\frac{dA^\mu}{dt})_{tr} + A^i(\frac{\partial \boldsymbol{e}_i}{\partial x^0}\cdot \boldsymbol{e}^\mu V_0^0 + \frac{\partial \boldsymbol{e}_i}{\partial x^1}\cdot \boldsymbol{e}^\mu V_0^1 + \frac{\partial \boldsymbol{e}_i}{\partial x^2}\cdot \boldsymbol{e}^\mu V_0^2 + \frac{\partial \boldsymbol{e}_i}{\partial x^3}\cdot \boldsymbol{e}^\mu V_0^3) \quad (293)$$

For the spaces with orthogonal bases, the expression in the way of Christoffel symbols is

$$(\frac{DA^\mu}{dt})_{tr} = (\frac{dA^\mu}{dt})_{tr} + \Gamma_{0i}^\mu A^i V_0^0 + \Gamma_{1i}^\mu A^i V_0^1 + \Gamma_{2i}^\mu A^i V_0^2 + \Gamma_{3i}^\mu A^i V_0^3 \quad (294)$$

Christoffel symbols are of the property of gravitational fields rather than the property of motions as has been defined. In rest fields, there will be $\frac{\partial \boldsymbol{e}_i}{\partial x^0} = 0$, so that $\Gamma_{0i}^\mu = 0$. It is said that for one source field, a trajectory derivative could be based on a space trajectory that

$$(\frac{D\boldsymbol{A}}{dt})_{tr} = \boldsymbol{e}_i(\frac{dA^i}{dt})_{tr} + A^i(\frac{\partial \boldsymbol{e}_i}{\partial x^1}V_0^1 + \frac{\partial \boldsymbol{e}_i}{\partial x^2}V_0^2 + \frac{\partial \boldsymbol{e}_i}{\partial x^3}V_0^3) \quad (295)$$

and

$$(\frac{DA^\mu}{dt})_{tr} = (\frac{dA^\mu}{dt})_{tr} + \Gamma_{1i}^\mu A^i V_0^1 + \Gamma_{2i}^\mu A^i V_0^2 + \Gamma_{3i}^\mu A^i V_0^3 \quad (296)$$

On trajectories pointing to source center, it is

$$(\frac{DA^\mu}{dt})_{tr} = (\frac{dA^\mu}{dt})_{tr} + \Gamma_{1i}^\mu A^i V_0^1 \quad (297)$$

It should be highlighted that the trajectory derivatives could also be defined in distance derivatives as

$$(\frac{D\boldsymbol{A}}{d\lambda})_{tr} = [\frac{D(\boldsymbol{e}_i A^i)}{d\lambda}]_{tr}$$

$$= \boldsymbol{e}_i\frac{dA^i}{d\lambda} + A^i\frac{d\boldsymbol{e}_i}{d\lambda}$$

$$= \boldsymbol{e}_i\frac{dA^i}{d\lambda} + A^i\frac{d\boldsymbol{e}_i}{dx^\mu}\frac{dx^\mu}{d\lambda}$$

$$= \boldsymbol{e}_i\frac{dA^i}{d\lambda} + A^i\frac{d\boldsymbol{e}_i}{dx^\mu}\alpha_\lambda^\mu \quad (298)$$

or an expanded presentation as

$$(\frac{D\boldsymbol{A}}{d\lambda})_{tr} = \boldsymbol{e}_i\frac{dA^i}{d\lambda} + A^i(\frac{d\boldsymbol{e}_i}{dx^0}\alpha_\lambda^0 + \frac{d\boldsymbol{e}_i}{dx^1}\alpha_\lambda^1 + \frac{d\boldsymbol{e}_i}{dx^2}\alpha_\lambda^2 + \frac{d\boldsymbol{e}_i}{dx^3}\alpha_\lambda^3) \quad (299)$$

or the component forms

$$(\frac{DA^\mu}{d\lambda})_{tr} = (\frac{dA^\mu}{d\lambda})_{tr} + A^i(\frac{\partial \boldsymbol{e}_i}{\partial x^0}\cdot \boldsymbol{e}^\mu \alpha_\lambda^0 + \frac{\partial \boldsymbol{e}_i}{\partial x^1}\cdot \boldsymbol{e}^\mu \alpha_\lambda^1 + \frac{\partial \boldsymbol{e}_i}{\partial x^2}\cdot \boldsymbol{e}^\mu \alpha_\lambda^2 + \frac{\partial \boldsymbol{e}_i}{\partial x^3}\cdot \boldsymbol{e}^\mu \alpha_\lambda^3) \quad (300)$$

The expression in the way of Christoffel symbols as

$$(\frac{DA^\mu}{d\lambda})_{tr} = (\frac{dA^\mu}{d\lambda})_{tr} + \Gamma_{0i}^\mu A^i \alpha_\lambda^0 + \Gamma_{1i}^\mu A^i \alpha_\lambda^1 + \Gamma_{2i}^\mu A^i \alpha_\lambda^2 + \Gamma_{3i}^\mu A^i \alpha_\lambda^3 \quad (301)$$

In rest fields, there is

$$(\frac{D\boldsymbol{A}}{d\lambda})_{tr} = \boldsymbol{e}_i(\frac{dA^i}{d\lambda})_{tr} + A^i(\frac{\partial \boldsymbol{e}_i}{\partial x^1}\cdot \boldsymbol{e}^\mu \alpha_\lambda^1 + \frac{\partial \boldsymbol{e}_i}{\partial x^2}\cdot \boldsymbol{e}^\mu \alpha_\lambda^2 + \frac{\partial \boldsymbol{e}_i}{\partial x^3}\cdot \boldsymbol{e}^\mu \alpha_\lambda^3) \quad (302)$$

and

$$(\frac{DA^\mu}{d\lambda})_{tr} = (\frac{dA^\mu}{d\lambda})_{tr} + \Gamma_{1i}^\mu A^i \alpha_\lambda^1 + \Gamma_{2i}^\mu A^i \alpha_\lambda^2 + \Gamma_{3i}^\mu A^i \alpha_\lambda^3 \quad (303)$$

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which just performs a special appearance of trajectory derivatives.

For trajectory of freefalling perpendicular to source center, $\lambda = r$, so that

$$(\frac{DA^{\mu}}{dr})_{tr} = (\frac{dA^{\mu}}{dr})_{tr} + \Gamma^{\mu}_{1i}A^{i} \quad (304)$$

where, $\alpha^{1}_{1} = 1$ has been hidden in and Γ^{μ}_{1i} will get nonvanishing value only in the case $i = 1$. These equations could also be expressed in covariant tensors that

$$(\frac{DA_{\mu}}{dt})_{tr} = (\frac{dA_{\mu}}{dt})_{tr} - \Gamma^{i}_{0\mu}A_{i}V^{0}_{0} - \Gamma^{i}_{1\mu}A_{i}V^{1}_{0} - \Gamma^{i}_{2\mu}A_{i}V^{2}_{0} - \Gamma^{i}_{3\mu}A_{i}V^{3}_{0} \quad (305)$$

and

$$(\frac{DA_{\mu}}{d\lambda})_{tr} = (\frac{dA_{\mu}}{d\lambda})_{tr} - \Gamma^{i}_{0i}A_{i}\alpha^{0}_{\lambda} - \Gamma^{i}_{1i}A_{i}\alpha^{1}_{\lambda} - \Gamma^{i}_{2i}A_{i}\alpha^{2}_{\lambda} - \Gamma^{i}_{3i}A_{i}\alpha^{3}_{\lambda} \quad (306)$$

For one source fields, they are

$$(\frac{DA_{i}}{dt})_{tr} = (\frac{dA_{i}}{dt})_{tr} - \Gamma^{i}_{1\mu}A_{i}V^{1}_{0} - \Gamma^{i}_{2\mu}A_{i}V^{2}_{0} - \Gamma^{i}_{3\mu}A_{i}V^{3}_{0} \quad (307)$$

and

$$(\frac{DA_{\mu}}{d\lambda})_{tr} = (\frac{dA_{\mu}}{d\lambda})_{tr} - \Gamma^{i}_{1i}A_{i}\alpha^{1}_{\lambda} - \Gamma^{i}_{2i}A_{i}\alpha^{2}_{\lambda} - \Gamma^{i}_{3i}A_{i}\alpha^{3}_{\lambda} \quad (308)$$

On trajectories pointing to source center, they are

$$(\frac{DA_{i}}{dt})_{tr} = (\frac{dA_{i}}{dt})_{tr} - \Gamma^{i}_{1\mu}A_{i}V^{1}_{0} \quad (309)$$

and

$$(\frac{DA_{\mu}}{dr})_{tr} = (\frac{dA_{\mu}}{dr})_{tr} - \Gamma^{i}_{1\mu}A_{i} \quad (310)$$

As for mixed tensors were concerned, their trajectory covariant derivatives could be written as

$$(\frac{DA^{\nu}_{\mu}}{dt})_{tr} = (\frac{dA^{\nu}_{\mu}}{dt})_{tr} + \Gamma^{\nu}_{0i}A^{i}_{\mu}V^{0}_{0} + \Gamma^{\nu}_{1i}A^{i}_{\mu}V^{1}_{0} + \Gamma^{\nu}_{2i}A^{i}_{\mu}V^{2}_{0} + \Gamma^{\nu}_{3i}A^{i}_{\mu}V^{3}_{0}$$
$$-\Gamma^{i}_{0\mu}A^{\nu}_{i}V^{0}_{0} - \Gamma^{i}_{1\mu}A^{\nu}_{i}V^{1}_{0} - \Gamma^{i}_{2\mu}A^{\nu}_{i}V^{2}_{0} - \Gamma^{i}_{3\mu}A^{\nu}_{i}V^{3}_{0} \quad (311)$$

and

$$(\frac{DA^{\nu}_{\mu}}{d\lambda})_{tr} = (\frac{dA^{\nu}_{\mu}}{d\lambda})_{tr} + \Gamma^{\nu}_{0i}A^{i}_{\mu}\alpha^{0}_{\lambda} + \Gamma^{\nu}_{1i}A^{i}_{\mu}\alpha^{1}_{\lambda} + \Gamma^{\nu}_{2i}A^{i}_{\mu}\alpha^{2}_{\lambda} + \Gamma^{\nu}_{3i}A^{i}_{\mu}\alpha^{3}_{\lambda}$$
$$-\Gamma^{i}_{0\mu}A^{\nu}_{i}\alpha^{0}_{\lambda} - \Gamma^{i}_{1\mu}A^{\nu}_{i}\alpha^{1}_{\lambda} - \Gamma^{i}_{2\mu}A^{\nu}_{i}\alpha^{2}_{\lambda} - \Gamma^{i}_{3\mu}A^{\nu}_{i}\alpha^{3}_{\lambda} \quad (312)$$

For one source fields, they are

$$(\frac{DA^{\nu}_{\mu}}{dt})_{tr} = (\frac{dA^{\nu}_{\mu}}{dt})_{tr} + \Gamma^{\nu}_{1i}A^{i}_{\mu}V^{1}_{0} + \Gamma^{\nu}_{2i}A^{i}_{\mu}V^{2}_{0} + \Gamma^{\nu}_{3i}A^{i}_{\mu}V^{3}_{0}$$
$$-\Gamma^{i}_{1\mu}A^{\nu}_{i}V^{1}_{0} - \Gamma^{i}_{2\mu}A^{\nu}_{i}V^{2}_{0} - \Gamma^{i}_{3\mu}A^{\nu}_{i}V^{3}_{0} \quad (313)$$

and

$$(\frac{DA^{\nu}_{\mu}}{d\lambda})_{tr} = (\frac{dA^{\nu}_{\mu}}{d\lambda})_{tr} + \Gamma^{\nu}_{1i}A^{i}_{\mu}\alpha^{1}_{\lambda} + \Gamma^{\nu}_{2i}A^{i}_{\mu}\alpha^{2}_{\lambda} + \Gamma^{\nu}_{3i}A^{i}_{\mu}\alpha^{3}_{\lambda}$$
$$-\Gamma^{i}_{1\mu}A^{\nu}_{i}\alpha^{1}_{\lambda} - \Gamma^{i}_{2\mu}A^{\nu}_{i}\alpha^{2}_{\lambda} - \Gamma^{i}_{3\mu}A^{\nu}_{i}\alpha^{3}_{\lambda} \quad (314)$$

On trajectory pointing to source center, they are

$$(\frac{DA^{\nu}_{\mu}}{dt})_{tr} = (\frac{dA^{\nu}_{\mu}}{dt})_{tr} + \Gamma^{\nu}_{1i}A^{i}_{\mu}V^{1}_{0} - \Gamma^{i}_{1\mu}A^{\nu}_{i}V^{1}_{0} \quad (315)$$

and

$$(\frac{DA^{\nu}_{\mu}}{dr})_{tr} = (\frac{dA^{\nu}_{\mu}}{dr})_{tr} + \Gamma^{\nu}_{1i}A^{i}_{\mu} - \Gamma^{i}_{1\mu}A^{\nu}_{i} \quad (316)$$

5.3. Derivatives of Condensed Matters

We have learned about the concepts of ordinary derivative, substance derivative and covariant derivative in the past, and now trajectory derivative was carried out for motion matters. It is necessary to make more discussions to clarify their differences and physical applications.

Ordinary derivative is defined to describe the variations of physical quantities themselves. An ordinary time derivative or ordinary space derivative of a tensor is the variational ratio of the tensor by nature no matter the matter moving or not. For example, field derivatives of bases are of ordinary derivatives in that they could vary with positions and time but they will not vary with motions.

Covariant derivatives are specially defined for the variations of covariant tensors. Sometimes a covariant derivative may be written in contra variant forms is just a projection of the corresponding covariant forms.

Trajectory derivatives have been carried out to describe the variation of tensors about matter's motions. Owing to any motion trajectory must be a sole parametrical curve, a trajectory derivative is a derivative that is derived by that curve. A position on trajectory would have only a single direction, so that the trajectory derivative at that position is derivative at that direction. That is the reason the trajectory derivative has been defined to prevent from abuses. A trajectory derivative may be of covariant derivatives or not. A covariant derivative may be of trajectory derivatives or not.

Substance derivatives are derivatives defined for condensed matters to describe geometrical variations in motions. In fact, substance derivative is the definition of quantities of macro statistics. For example, for condensed matters, stress, strain, density and temperature etc. are statistical quantities that should be estimated by dimensions and volumes. As motions and flows happens, the quantities will vary, coordinate dependently. Substance derivative is defined to describe the variations owing to deformations and convections. It is said that substance derivative is a kind of description to specially present inner interactions of matters. More exceptional, for the motions of mass point, it is not necessary.

Now there will be two topics for derivatives that should be furtherly discussed. One is that the most of covariant derivatives we talk about are of trajectory derivatives but trajectory derivatives may be contra variant derivatives. The other is that trajectory derivatives could be seen as the development of substance derivatives, because they both are trajectories on motions, nothing but the trajectory derivatives should be considered once the space time is variable.

As has discussed, most of covariant derivatives we have discussed are of trajectory derivatives, such as velocity, acceleration, as well as the light frequency shift on trajectory. That is because most of the problems we concerned are of that on matter's motions so that the covariant derivatives must be trajectory derivatives. On the other hand, the contra variant derivatives are also trajectory derivatives. But for some special cases, some tensors have no covariant derivatives or it could be said that there are not great differences between covariant and contra variant derivatives. For example, velocity is that kind of tensors, which could be space distance derived by time interval, but because the differentials of space and time are of one order quantities, they will not be expanded to the forms of covariant derivatives. As for the further derivation of velocity, the acceleration, it really have covariant derivatives, because the acceleration is the second order quantity. The order mentioned is numbers of derivations that the differentials involved has experienced.

In fact, the substance derivative is really of another kind of the trajectory derivatives, because they are both on matter's motions. In most case, we discuss on ordinary dynamics in weak fields in which metrics do not vary greatly at different positions, so that the property of trajectory covariant derivative will not perform, so far as that has not been realized in practices. If the dynamics for condensed matters in strong fields were discussed, more comprehensive conditions should be taken for granted such as coordinate dependent metrics and internal interactions. In these cases, substance derivatives and the trajectory derivatives will be combined to form more complicated dynamics.

5.4. Covariant Trajectory Derivatives and General Covariance

Trajectory derivatives are of derivatives on trajectories of matter motions in space time, which perform the covariant properties of motions. Christoffel symbols are of the functions of field derivatives of bases, which perform the properties of gravitational fields. In the equation of covariant trajectory derivatives, Christoffel symbols are employed to present the variations of bases on the direction of motions.

The equation of covariant trajectory derivatives is the pure geometrical relationship of trajectory derivatives in different spaces to present covariance. The covariant trajectory time derivatives are of the most basic and respective problems in general relativity. Accelerations are of trajectory time derivatives. I would like to sponsor more discussions on accelerations respectively for massive matters and for light rays. Moreover, one can study some other trajectory covariant derivatives such as the problems of gravitational redshift. That will be put into discussions in next sections.

Firstly, the accelerations for massive matters: In contra variant space, gravity is defined the external force. We know that $(\frac{dV_0^\mu}{dt})_{tr}$ is the result of whole external forces, while $(\frac{DV_0^\mu}{dt})_{tr}$ shows more complexities. The covariant derivative equation just presents the geometrical transformation. However, we know that in contra variant space of one source fields, metrics are defined invariant so that one will find the results of $(\frac{dV_0^\mu}{dt})_{tr}$ do nothing with coordinates and motions. But for $(\frac{DV_0^\mu}{dt})_{tr}$, because of variable metrics, it is coordinate dependent or trajectory dependent.

Secondly, the accelerations for light rays: The invariant light speed is defined the invariance of light speed in covariant space, so that it is obvious that there will be $(\frac{Dc_0^\mu}{dt})_{tr} \equiv 0$ or $(\frac{Dc}{dt})_{tr} \equiv 0$. As for that in contra variant space, the apparent light speed c_0^μ is variable so that the contra variant derivative $(\frac{dc_0^\mu}{dt})_{tr} \neq 0$. It is said that for light rays, the covariant derivatives are coordinate independent, while the contra variant derivatives are coordinate dependent.

In comparison, we have seen that light rays perform thoroughly inversed with respect to massive matters. The covariant trajectory derivative equation, being a transformation equation, just acts as a bridge to present the trajectory derivatives in different spaces. That shows that massive matters and light rays perform different inertial motions in every specific space. That reveals the probabilities that the covariant trajectory derivative equation does not determine the covariant derivatives to be definitive zero. What do the covariant derivatives be at the end? I think that, in scientific recognitions, they must depend on the physical realities. I prefer to sponsor theoretical and experimental verifications on these topics in following sections.

6. Theoretical Verifications

Because of the inequality of Christoffel symbols of mixed subscripts, the classical Christoffel symbol equations could not be used any more in the theory of general relativity. The covariant derivatives in gravitational field should be considered in their correct forms.

6.1. On Gravitational Redshifts

Taking centripetal light propagations for granted, a distance derivative of contra variant frequency is exactly trajectory derivative as in Eq. (310)

$$(\frac{D\nu_0}{dr})_{tr} = (\frac{d\nu_0}{dr})_{tr} - \Gamma_{10}^0 \nu_0 \quad (317)$$

where, the first item of right side of the equation is contra variant derivative that has been drawn in Eq. (24).

We have seen that the Eq. (317) has the same form with the Eq. (32), but they have been based on different perceptions on the derivatives on matter's motions.

It is sure to consider the tensor of frequency and its derivative to be vectors, but in traditions it is not of a rare necessity. It has been mentioned that Christoffel symbol Γ_{10}^0 were employed correctly with the form $\Gamma_{10}^0 = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial x^1}$ as has been shown in Eq. (35). As a result, it will lead to a true answer

$$\Gamma_{10}^{0}=\frac{1}{2}g^{00}\frac{\partial g_{00}}{\partial r}=\frac{1}{2}(1-\frac{r^{*}}{r})^{-1}(1-\frac{r^{*}}{r})'=\frac{r^{*}}{2r^{2}}(1+\frac{r^{*}}{r-r^{*}}) \quad (318)$$

The covariant derivative will be calculated to be

$$(\frac{Dv_0}{\partial r})_{tr}=-\frac{r^{*}}{2r^{2}}v_{0(\infty)}-\frac{r^{*}}{2r^{2}}(1+\frac{r^{*}}{r-r^{*}})(1+\frac{r^{*}}{2r})v_{0(\infty)} \quad (319)$$

The approximate solution for weak field is that

$$(\frac{Dv_0}{\partial r})_{tr}\approx 2\frac{dv_0}{dr} \quad (320)$$

It is said that the solution Eq. (37) is confirmed again, and it reveals that the covariant redshift is approximately double of the contra variant one in weak field, which departs from general covariance.

Notwithstanding, one can also make a time derivative for centripetal propagations

$$(\frac{Dv_0}{dt})_{tr}=\frac{dv_0}{dr}\frac{dr}{dt}-\Gamma_{10}^{0}v_0\frac{dr}{dt} \quad (321)$$

For light rays at that trajectory, $\frac{dr}{dt}=c_0^1$. It becomes

$$(\frac{Dv_0}{dt})_{tr}=(\frac{dv_0}{dt})_{tr}-c_0^1\Gamma_{10}^{0}v_0 \quad (322)$$

The first item of right hand side can also be transformed to be

$$(\frac{dv_0}{dt})_{tr}=\frac{dv_0}{dr}\frac{dr}{dt}=c_0^1\frac{dv_0}{dr} \quad (323)$$

so that

$$(\frac{Dv_0}{dt})_{tr}=c_0^1(\frac{dv_0}{dr}-\Gamma_{10}^{0}v_0)=c_0^1(\frac{Dv_0}{dr})_{tr}\approx 2c_0^1\frac{dv_0}{dr} \quad (324)$$

6.2. On Accelerations

One can take matter freefalls for example to study the acceleration in gravitational fields. We have pointed out that accelerations we discussed refer to geometrical accelerations. The entire contra variant acceleration is the derivative of whole contra variant velocity as that

$$a_{00}^{1}=(\frac{dV_0^1}{dt})_{tr}=\frac{d^2r}{dt^2} \quad (325)$$

where, $V_0^1=\frac{dr}{dt}$.

And entire covariant acceleration is a pure covariant derivative

$$a1/00=(\frac{DV1/0}{d\tau})_{tr}=\frac{d^2\rho}{d\tau^2} \quad (326)$$

where, $V1/0=\frac{d\rho}{d\tau}$.

The most important is that some tensors have been labeled with detailed middle indexes, such as $a1/00$, hence they may help to provide explicit expressions, in which the symbol / is employed to divide the middle upper and middle lower indexes. In fact, the middle-labeled indexes are to label the coordinates of covariant space, while the indexes labeled in the so-called contra variant tensors and covariant tensors are both the labels of contra variant coordinates. Thus, the confusions on naming rules have been well solved that it will be more explicit to name a tensor based on the relative coordinates of corresponding spaces.

With the relationship between covariant derivatives, it could be drawn that

$$e^1e_0e_0a1/00=(\frac{DV_0^1}{dt})_{tr} \quad (327)$$

To study the covariant derivative expression with Christoffel symbols, with Eq. (315) there is

$$(\frac{DV_0^1}{dt})_{tr} = (\frac{dV_0^1}{dt})_{tr} + \Gamma_{11}^1 (V_0^1)^2 - \Gamma_{10}^0 (V_0^1)^2 \quad (328)$$

With $\Gamma_{11}^1 = e^1 \frac{\partial e_1}{\partial r}$, $\Gamma_{10}^0 = e^0 \frac{\partial e_0}{\partial r}$, $e^0 \partial e_0 = -e_0 \partial e^0$ and $(\frac{dV_0^1}{dt})_{tr} = a_{00}^1$, it is

$$(\frac{DV_0^1}{dt})_{tr} = a_{00}^1 + e^1 \frac{\partial e_1}{\partial r} (V_0^1)^2 - e^0 \frac{\partial e_0}{\partial r} (V_0^1)^2$$

$$= a_{00}^1 + e^1 \frac{\partial e_1}{\partial r} (V_0^1)^2 + e_0 \frac{\partial e^0}{\partial r} (V_0^1)^2$$

$$= a_{00}^1 + e^1 e_0 \frac{\partial}{\partial r} (e_1 e^0) (V_0^1)^2 \quad (329)$$

With Eq. (327), we obtain

$$a1/00 = e^0 e^0 e_1 (\frac{DV_0^1}{dt})_{tr} = e^0 e^0 e_1 a_{00}^1 + e^0 \frac{\partial}{\partial r} (e_1 e^0) (V_0^1)^2 \quad (330)$$

One can find that we have use Γ_{11}^1 and Γ_{10}^0 in the calculation of $a1/00$, rather than Γ_{00}^1 that has been used in Eq. (53) and Eq. (57). As we have discussed, the value of Γ_{00}^1 is really of zero.

As has been said that covariant derivatives could also be developed in a direct way without Christoffel symbols

$$a1/00 = (\frac{DV1/0}{d\tau})_{tr} = (\frac{d^2\rho}{d\tau^2})_{tr} = [\frac{d}{\partial \tau} (\frac{e_1 dr}{e_0 dt})]_{tr} = [\frac{d}{e_0 dt} (\frac{e_1 dr}{e_0 dt})]_{tr}$$

$$= e^0 \frac{e_1}{e_0} [\frac{d}{dt} (\frac{dr}{dt})]_{tr} + e^0 \frac{dr}{dt} [\frac{d}{dt} (\frac{e_1}{e_0})]_{tr}$$

$$= e^0 e^0 e_1 a_{00}^1 + e^0 V_0^1 [\frac{d}{dt} (e^0 e_1)]_{tr}$$

$$= e^0 e^0 e_1 a_{00}^1 + e^0 V_0^1 \frac{\partial}{\partial r} (e^0 e_1) \frac{dr}{dt}$$

$$= e^0 e^0 e_1 a_{00}^1 + e^0 \frac{\partial}{\partial r} (e^0 e_1) (V_0^1)^2 \quad (331)$$

It could be found that this equation has been far different from the Eq. (53) and Eq. (57), because errors in Christoffel symbol equation have been eliminated and at the same time the concept of trajectory derivatives will help to calculate an acceleration in right way. These discussions have presented further verifications for the revised equation of Christoffel symbols of Eq. (264).

By the way, it is interesting to take some discussions on some trivial concepts such as a_{00}^0 and the covariant form $a0/00$ of massive matters. Since V_0^0 and $V0/0$ are the velocities of contra variant time and proper time but not the real velocity of light

$$V_0^0 = \frac{dx^0}{dt} = \frac{d(ct)}{dt} = c \quad (332)$$

and

$$V0/0 = \frac{e_0 dx^0}{d\tau} = \frac{e_0 d(ct)}{d\tau} = c \quad (333)$$

Then their derivatives are just the accelerations of time coordinates that

$$a_{00}^0 = (\frac{dV_0^0}{dt})_{tr} = 0 \quad (334)$$

while

$$(\frac{DV_0^0}{dt})_{tr} = (\frac{dV_0^0}{dt})_{tr} + \Gamma_{10}^0 V_0^0 V_0^1 - \Gamma_{10}^0 V_0^0 V_0^1 = 0 \quad (335)$$

so that

$$a0/00 = 0 \quad (336)$$

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As light propagation at a direction of a radius is concerned, we know that light speed c keeps invariant in covariant space, so that there is

$$\frac{Dc}{d\tau} = 0 \quad (337)$$

Case discussing the performance of contra variant light speed in perpendicular propagations, with invariant distance, there is

$$ds^2 = -g_{00}c^2dt^2 + g_{11}dr^2 = 0 \quad (338)$$

and then the light speed in contra variant space will be

$$c_0^1 = \frac{dr}{dt} = \sqrt{\frac{g_{00}}{g_{11}}}c = e_0e^1c \quad (339)$$

where, positive g_{00} is set instead of a negative g_{00} as has suggested previous.

Then the acceleration

$$a_{00}^1 = (\frac{dc_0^1}{dt})_{tr} = [\frac{d}{dt}(e_0e^1c)]_{tr} = c\frac{\partial}{\partial r}(e_0e^1)\frac{\partial r}{\partial t} = cc_0^1\frac{\partial}{\partial r}(e_0e^1) = c^2e_0e^1\frac{\partial}{\partial r}(e_0e^1) \quad (340)$$

With Eq. (329) the covariant derivative is

$$(\frac{Dc_0^1}{dt})_{tr} = (\frac{dc_0^1}{dt})_{tr} - e_1e^0\frac{\partial}{\partial r}(e^1e_0)(c_0^1)^2$$

$$= a_{00}^1 - e_1e^0\frac{\partial}{\partial r}(e^1e_0)(c_0^1)^2$$

$$= c^2e_0e^1\frac{\partial}{\partial r}(e_0e^1) - c^2e_0e^1\frac{\partial}{\partial r}(e^1e_0) = 0 \quad (341)$$

Of course, we could obtain the result only by a judgment that

$$(\frac{Dc_0^1}{dt})_{tr} = e^1e_0e_0\frac{Dc}{d\tau} = 0 \quad (342)$$

7. Experimental Verifications

Every tensor involved with measurable quantities could have probabilities to be performed in practice with measured quantities to verify their theoretical expressions. In space time, space intervals and time intervals are all measurable quantities so that they surely could be employed to perform the space and time dependent tensors.

The methodology of the so-called revisit gravitational redshift encourages me to sponsor a realistic analysis method to further verify the general covariance, which will present solutions all based on physical events of realities. Physical events always have substantial existences so that they can help to create irrefutable conclusions. We know that physical events may be record both in contra variant space and covariant space that might provide different values for physical quantities, but both of them actually represent the same physical realities.

7.1. On Measurable Experiments

Measurable quantities could be used to describe physical events, which may be coordinate independent or not. Coordinate independent quantities of course show invariance in physical events in different spaces, such as wave numbers, which could be record as images or texts at specific times and positions. However, coordinate dependent quantities measured in site may really depend. For examples, distance measurements not only depend on in-site space coordinates but also depend on the in-site rulers, so as well, time measurements also depend on both in-site time coordinates and the in-site clocks. We may imagine that the space rulers and clocks their selves maybe also vary. Logically, records of these quantities are recognizable even if they are in farthest distance to the bystanders.

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Case a measurement equipment varies with time space, whether the measurement quantity measured is in contra variant space quantity or covariant space quantity? With general covariance, it has been believed that rulers will shrink when they go closer to the center source corresponding to the space interval to become shorter. And also, it has been expected that clocks will go variant corresponding to their dynamic conditions.

However, after those inspections in previous sections, we know that general covariance does not work in some circumstances. Energy and momentum of a matter may not keep covariant in covariant space, while light speed may keep covariant spectacularly. On another side, our discussions may have led to a theoretical inference that matters may experience relativistic emission when they go to a center source and then shrink because of the variation of energy structure. That could be called covariant deformations.

Once we measure space and time intervals at a position, maybe they are not committed to be contra variant quantities or covariant quantities, because our clocks and rulers may vary uncommitted. Anyway, we deem we can measure. In another word, we could indeed measure something so that they might relate to the corresponding ones. Thus, it is not harmful to suppose one of the series of measurable quantities could be measured in following discussions, for example, the contra variant distances or contra variant time intervals. And then they will be valid to be transformed from one to another. That will help us to do more analysis for comparisons and discussions.

7.2. Measurable Verifications for Gravitational Redshift

For the issue of redshift, we are going to sponsor the physical events of wave number counting. It is known that light frequency investigation should be accomplished by indirect techniques and sometimes it may come out with deviations. But it is supposed here that the wave number of the light is countable, or it is believed that light wave could be seen and record. This assumption actually may not do harm to our understanding to the realities, because that indeed will not change the realities and the events of wave counting in that the measurements themselves are also physical processes.

The event of wave number counting could be specified as the record of a number of waves to past a position in a time interval, and it could also be simplified to be one wave corresponding to a time lasting of the light ray propagating a wave length distance. On another side indirectly, one can get wave number by measuring wave length, based on the assumption of invariant light speed. But the apparent light speed might be variable so that the indirect method is not a good idea.

If there is a photon propagating from position 1 to position 2 in a one source field as shown in Figure 18, which correspond to coordinates $r_{(1)}$and $r_{(2)}$,

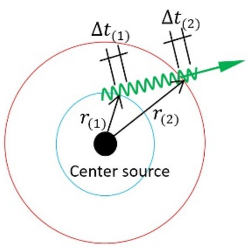

Figure 18. A photon traveling in center source field.

The wave number counting events should be carried out at the time that the photon passes the position 1 and position 2. In very short time $\Delta t_{(1)}$ and $\Delta t_{(2)}$, we will count the corresponding wave number $\Delta n_{(1)}$ and $\Delta n_{(2)}$. It should be pointed out that the time intervals here do nothing with the time intervals in Figure 2 in Section 2.2, in that they belong to different physical events that the ones we are talking about are of the records of wave counting events and the ones in Figure 2 are of the records of pulse emission and receiving events.

Since the frequencies should be calculated as

$$\nu_{0(1)} = \frac{\Delta n_{(1)}}{\Delta t_{(1)}}\ and\ \nu_{0(2)} = \frac{\Delta n_{(2)}}{\Delta t_{(2)}}\ (343)$$

Redshift has been defined as

$$z_{\text{contra}} = \frac{\nu_{0(1)} - \nu_{0(2)}}{\nu_{0(2)}}\ (344)$$

With the measurable records, it could be rewritten as

$$z_{\text{contra}} = \frac{\Delta n_{(1)}\Delta t_{(2)}}{\Delta n_{(2)}\Delta t_{(1)}} - 1\ \ (345)$$

In the event of wave counting, wave numbers are invariant in both contra variant space and covariant space, but the time intervals $\Delta t_{(1)}$ and $\Delta t_{(2)}$ will vary to be $\Delta \tau_{(1)}$ and $\Delta \tau_{(2)}$. In fact, every physical event keeps the only one event, whereas the different describing metrics lead to different results in the different spaces.

Naturally, gravitational redshift in covariant space is

$$z_{\text{revisit}} = \frac{\nu_{(1)} - \nu_{(2)}}{\nu_{(2)}} = \frac{\Delta n_{(1)}\Delta \tau_{(2)}}{\Delta n_{(2)}\Delta \tau_{(1)}} - 1\ \ (346)$$

where, this redshift symbol labeled with revisit is because it corresponds to that one named in classical equations.

As we know,

$$\Delta \tau = e_0 \Delta t\ \ (347)$$

It turns to be

$$z_{\text{revisit}} = \frac{e_{0(2)}}{e_{0(1)}}\frac{\Delta n_{(1)}\Delta t_{(2)}}{\Delta n_{(2)}\Delta t_{(1)}} - 1\ \ (348)$$

As in a field of a center source, the metrics take the forms of Schwarzschild solution, the equation will be drawn as

$$z_{\text{revisit}} = \left(1 + \frac{\emptyset_2 - \emptyset_1}{c^2} + o\right)\frac{\Delta n_{(1)}\Delta t_{(2)}}{\Delta n_{(2)}\Delta t_{(1)}} - 1\ \ (349)$$

We know that the z_{contra} in Eq. (345) could have been measured in the physical event of wave number counting that of course equals to that in Eq. (18), so that

$$\frac{\Delta n_{(1)}\Delta t_{(2)}}{\Delta n_{(2)}\Delta t_{(1)}} = \left(1 + \frac{\emptyset_2 - \emptyset_1}{c^2} + o\right)\ \ (350)$$

Thus, the covariant redshift in weak field is obtained

$$z_{\text{revisit}} = \left(1 + \frac{\emptyset_2 - \emptyset_1}{c^2} + o\right)^2 - 1 \approx 2\frac{\Delta\emptyset}{mc^2} \approx 2z_{\text{contra}}\ \ (351)$$

It is said that, the revisit gravitational redshift is double of that of contra variant one.

As the equation of contra variant redshift is concerned, we know that it could be of course drawn by counting two wave numbers in two equivalently specified time intervals. For example, set $\Delta t_{(1)} = \Delta t_{(2)} = \Delta t$ which are measured at positions of $r_{(1)}$ and $r_{(2)}$, then $\Delta n_{(1)}$ and $\Delta n_{(2)}$ should represent the difference of frequency without time intervals. So that

$$z_{\text{contra}} = \frac{\Delta n_{(1)}\Delta t}{\Delta n_{(2)}\Delta t} - 1 = \frac{\Delta n_{(1)}}{\Delta n_{(2)}} - 1\ \ (352)$$

We know that $\Delta n_{(1)}$ and $\Delta n_{(2)}$ present the wave numbers with respect to $\Delta t_{(1)} = \Delta t_{(2)} = \Delta t$.

As for revisit redshift, one will still get different covariant time intervals because the metrics go varied. Thus, it is always doubled of the previous.

$$z_{\text{revisit}} = \frac{\Delta n_{(1)}}{\Delta n_{(2)}}\frac{e_{0(2)}}{e_{0(1)}}\frac{\Delta t}{\Delta t} - 1 \approx 2z_{\text{contra}}\ \ (353)$$

It should be pointed out that in some experiments on gravitational redshift, only one timing clock was designed for time interval measurement. In this case, a wrong setting of proper time intervals may be taken into consideration, for example some sole clock timing experiments, so that the experimental redshift may be presented as

$$z_{\text{experimental}} = \frac{\Delta n_{(1)}\Delta\tau_{(2)}}{\Delta n_{(2)}\Delta\tau_{(1)}} - 1 = \frac{\Delta n_{(2)}}{\Delta n_{(1)}}\frac{e_{0(\text{x})}}{e_{0(\text{x})}}\frac{\Delta t_{(2)}}{\Delta t_{(1)}} - 1 = z_{\text{contra}} \quad (354)$$

where, $e_{0(\text{x})}$ is base component at clock position of $r_{(1)}$ or $r_{(2)}$ or any position others to them.

We can find out those completed experiments observations [14–16] on gravitational redshift will be easy to be verified to have only worked out the results of contra variant frequency shift.

Of course, one can calculate the real proper time intervals by time interval transformation between sole timing position and frequency shift positions. That will finally help to work out revisit gravitational redshift as have discussed.

7.3. Measurable Verifications for Acceleration

7.3.1. Measurable Quantities and Measurable Acceleration

Firstly, I prefer to rise a controversy of a freefalling on the Earth that if a matter freefalls from rest with velocity $V_0^1{}_{(1)} = 0$ as well as $V1/0_{(1)} = 0$ by nature, we do know that it will move quite faster with velocity $V_0^1{}_{(2)}$ after traveling a distance in a time interval because of gravity. In traditional theory, we know that the proper velocity $V1/0_{(2)} = e_0 e^1 V_0^1{}_{(2)}$. Considering the weak field effect, there is $e_0 e^1 \approx 1$. Hence comes the controversy that the covariant acceleration must be great than zero because the matter has started from rest to a quite apparent motion. That is really contradicted with the principle of general covariance with a requirement of zero covariant acceleration. Nevertheless, considering that $V_0^1{}_{(2)}$ and $V1/0_{(2)}$ are still non-relativistic velocities, it is easy to estimated that the accelerations are also approximately equal that $a1/00 \approx a_{00}^1$. The following works of so-called realistic verifications in this section are exactly to be sponsored to solve these controversies thoroughly.

A freefalling test with initial velocity is going to be put forward, in which a matter freely falls to source center from a position $r_{(1)}$ to a position $r_{(2)}$ as shown in Figure 19. Once the velocities at the two positions are measured, the average values could be estimated with the velocities difference and the interval distance. Considering the condition on the surface of the Earth, a freefall with a rarely big velocity and a rarely small traveling would be performed. For example, a velocity of more than 10000 m/s, could be seen as a constant accelerating motion even in covariant space, in that a covariant derivative is expected to be linear with velocity.

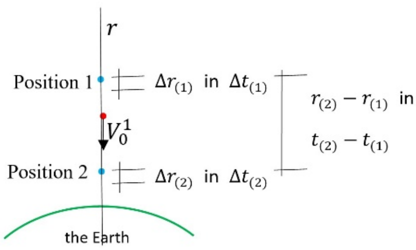

Figure 19. Freefalling measurement.

It is supposed that contra variant distances and time intervals are measurable as has been discussed previous. Based on the measurements, velocity at position 1 could be written as

$$V_0^1{}_{(1)} = \frac{\Delta r_{(1)}}{\Delta t_{(1)}} \quad (355)$$

where the $\Delta r_{(1)}$ and $\Delta t_{(1)}$ are measured distance and time intervals when the matter goes by the position $r_{(1)}$, so that they are both tensors of contra variant space. For reasons of convenience their bracketed sub indexes here are only employed to represent positions.

As well as that at position 2

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$$V^1_{0\ (2)} = \frac{\Delta r_{(2)}}{\Delta t_{(2)}} \ (356)$$

Because the velocity at position 2 is the result of acceleration, it could be written in integral form

$$V^1_{0\ (2)} = V^1_{0\ (1)} + \int_{t_{(1)}}^{t_{(2)}} a^1_{00} dt$$

$$= V^1_{0\ (1)} + \overline{a^1_{00}}[t_{(2)} - t_{(1)}] \ (357)$$

where, the mean acceleration $\overline{a^1_{00}}$ is the integral point value, and $t_{(2)} - t_{(1)}$ is time interval for the matter traveling from $r_{(1)}$ to $r_{(2)}$.

So that the mean acceleration is

$$\overline{a^1_{00}} = \frac{V^1_{0\ (2)} - V^1_{0\ (1)}}{t_{(2)} - t_{(1)}} \ (358)$$

On the other hand, with covariant bases there are the relationships of contra variant quantities and proper ones

$$\Delta\rho_{(1)} = e_{1_{(1)}}\Delta r_{(1)}, \Delta\tau_{(1)} = e_{0_{(1)}}\Delta t_{(1)}, \Delta\rho_{(2)} = e_{1_{(2)}}\Delta r_{(2)}, \Delta\tau_{(2)} = e_{0_{(2)}}\Delta t_{(2)} \ (359)$$

One can use the mean metric to calculate the proper time intervals from position 1 to positon 2

$$\tau_{(2)} - \tau_{(1)} = \int_{t_{(1)}}^{t_{(2)}} e_0 dt = \overline{e_0}(t_{(2)} - t_{(1)}) \ (360)$$

where the $\overline{e_0}$ is the value at integral point, and it is suggested to be evaluated approximately as following in weak field

$$\overline{e_0} \approx 0.5(e_{0_{(1)}} + e_{0_{(2)}}) \ (361)$$

The proper velocities

$$V1/0_{(1)} = \frac{\Delta\rho_{(1)}}{\Delta\tau_{(1)}} \ (362)$$

$$V1/0_{(2)} = \frac{\Delta\rho_{(2)}}{\Delta\tau_{(2)}} \ (363)$$

And the integral relationship in covariant space that

$$V1/0_{(2)} = V1/0_{(1)} + \int_{\tau_{(1)}}^{\tau_{(2)}} a1/00 d\tau$$

$$= V1/0_{(1)} + \overline{a1/00}(\tau_{(2)} - \tau_{(1)}) \ (364)$$

And also, we get the mean covariant acceleration with Lagranian mean value theorem of integration that

$$\overline{a1/00} = \frac{V1/0_{(2)} - V1/0_{(1)}}{\tau_{(2)} - \tau_{(1)}} \ (365)$$

It could be rewritten as

$$\overline{a1/00} = \frac{\frac{e_{1(2)}\Delta r_{(2)}}{e_{0(2)}\Delta t_{(2)}} - \frac{e_{1(1)}\Delta r_{(1)}}{e_{0(1)}\Delta t_{(1)}}}{\overline{e_0}(t_{(2)} - t_{(1)})} = \frac{\frac{e_{1(2)}}{e_{0(2)}}V^1_{0\ (2)} - \frac{e_{1(1)}}{e_{0(1)}}V^1_{0\ (1)}}{\overline{e_0}(t_{(2)} - t_{(1)})} \ (366)$$

It is of course the measuring forms of covariant acceleration of a freefall. And then it could be compared to that of contra variant one.

We would like substitute the equation of contra variant velocity 2 of Eq. (357) into this equation. That is

$$\overline{a1/00} = \overline{e^0}\frac{\frac{e_{1(2)}}{e_{0(2)}}[(t_{(2)} - t_{(1)})\overline{a^1_{00}} + V^1_{0\ (1)}] - \frac{e_{1(1)}}{e_{0(1)}}V^1_{0\ (1)}}{t_{(2)} - t_{(1)}}$$

$$= \overline{e^0}\frac{e_{1(2)}}{e_{0(2)}}\overline{a_{00}^1} + \overline{e^0}\frac{\frac{e_{1(2)}}{e_{0(2)}} - \frac{e_{1(1)}}{e_{0(1)}}}{t_{(2)} - t_{(1)}}V_0^1{}_{(1)}$$

$$= \overline{e^0}e_{1(2)}e^0{}_{(2)}\overline{a_{00}^1} + \overline{e^0}\frac{e_{1(2)}e^0{}_{(2)} - e_{1(1)}e^0{}_{(1)}}{t_{(2)} - t_{(1)}}V_0^1{}_{(1)} \quad (367)$$

It could be transformed to be

$$\overline{a1/00} = \overline{e^0}e_{1(2)}e^0{}_{(2)}\overline{a_{00}^1} + \overline{e^0}\frac{e_{1(2)}e^0{}_{(2)} - e_{1(1)}e^0{}_{(1)}}{t_{(2)} - t_{(1)}}V_0^1{}_{(1)}\frac{r_{(2)} - r_{(1)}}{r_{(2)} - r_{(1)}}$$

$$= \overline{e^0}e_{1(2)}e^0{}_{(2)}\overline{a_{00}^1} + \overline{e^0}\frac{e_{1(2)}e^0{}_{(2)} - e_{1(1)}e^0{}_{(1)}}{r_{(2)} - r_{(1)}}V_0^1{}_{(1)}\overline{V_0^1} \quad (368)$$

Here we have got the transformed form of covariant acceleration of freefalls.

Nevertheless, with Lagrangian differential mean value theorem, we can write down the differential form as

$$\overline{a1/00} = \overline{e^0}e_{1(2)}e^0{}_{(2)}\overline{a_{00}^1} + \overline{e^0}\overline{\frac{\partial}{\partial r}(e_1e^0)}V_0^1{}_{(1)}\overline{V_0^1} \quad (369)$$

Or the form of reverse bases

$$\overline{a1/00} = \overline{e^0}e_{1(2)}e^0{}_{(2)}\overline{a_{00}^1} - \overline{e^0}e_{1(2)}e_{1(1)}e^0{}_{(2)}e^0{}_{(1)}\overline{\frac{\partial}{\partial r}(e^1e_0)}V_0^1{}_{(1)}\overline{V_0^1} \quad (370)$$

Thus, by the way, another kind of proof of differential analysis of the Eq. (330) or Eq. (331) has been completed, in the way of measurable experiment. It should be highlighted that the accelerations we have discussed refer to geometrical accelerations.

7.3.2. Examples

Some terrestrial experiments are going to be put forward, that matters with initial velocity freefall in vacuum circumstance with in 1000m height to the ground. Both at the start point position 1 and end point position 2, the matter's velocities will be measured. And of course, the space and time intervals between position 1 and 2 that depend on the so-called geodesic line will be measured together so that to calculate the mean accelerations.

Some basic data of the Earth have already been tested certainly, so that we can take the standard value for our experiments, such as the total mass of the Earth $M = 5.97237 \times 10^{24} kg$, and the position on the ground could be assigned to have a radial coordinate $R_{\oplus} = 6.371393 \times 10^6 m$. We could also take the gravitational constant $G = 6.67259 \times 10^{-11} Nm^2/kg^2$, with the light speed c = 299792458 m/s thus the gravitational radius will be calculated as

$$r^* = \frac{2GM}{c^2} = 8.8680827 \times 10^{-3} m \quad (371)$$

With Newtonian equation and Schwarzschild's solution, some positional data could be listed in Table 3.

Table 3. Base components, their derivatives and gravity at experimental positions.

r	e_1	e_0	GM/r^2	$\partial(e_1e^0)/\partial r$
6372.393	1.00000000069582045	0.99999999930417955	9.81377376	2.183859184×10-16
6371.493	1.00000000069591874	0.99999999930408126	9.81654643	2.184476187×10-16
6371.393	1.00000000069592966	0.99999999930407034	9.81685457	2.184544758×10-16

So far as we have discussed, the accelerations a_{00}^1 and $a1/00$ are really geological quantities, and now it is necessary to make an extending study.

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For convenience, we are going to talk about the acceleration of freefalls which are determined on radius to the center source. We know that all kinds of interactions could be seen as momentum exchanges between matters, as that

$$dP = d(mv) = vdm + mdv \quad (372)$$

For the convenience, some quantities discussed in this section will not be marked with tensor index anymore.

In conditions of low velocity motions, the theory of special relativity indicates small mass variations, thus

$$dP \approx mdv \quad (373)$$

For the cases of high velocity motions, one should take a total analysis. Now the total action could be defined as

$$\Lambda = \frac{d(mv)}{mdt} = \frac{vdm}{mdt} + \frac{dv}{dt} \quad (374)$$

It is said, the total action includes mass variant action and velocity variant acceleration, and the latter also could be called geometrical acceleration.

With a momentum variation, kinetic energy will vary a difference

$$dE_k = vdP = v^2 dm + mvdv \quad (375)$$

At the same time, the mass energy equation of differential form is

$$dE_k = c^2 dm \quad (376)$$

Thus, there will be

$$c^2 dm = v^2 dm + mvdv \quad (377)$$

To be divided by time differential, that will lead to the expression of geometrical acceleration

$$a_{00}^1 = \frac{dv}{dt} = \frac{c^2 - v^2}{mv}\frac{dm}{dt} \quad (378)$$

Now one can define a coefficient of geometrical acceleration

$$\eta = \frac{\frac{dv}{dt}}{\Lambda} = \frac{\frac{c^2-v^2}{mv}\frac{dm}{dt}}{\frac{vdm}{mdt} + \frac{c^2-v^2}{mv}\frac{dm}{dt}} = 1 - \frac{v^2}{c^2} \quad (379)$$

In one source field, gravitational geometrical acceleration for a freefall is

$$a_{00}^1 = \eta \frac{GM}{r^2} = \eta g \quad (380)$$

where g is the total action of gravity.

We will see that geometrical acceleration declines as velocity goes up to a relativistic level, and it goes to zero as velocity closely catches up to light speed.

If a matter is accelerated from rest, the total energy includes rest part and kinetic part

$$mc^2 = m_0 c^2 + \xi m v^2 \quad (381)$$

where m is relativistic mass and m_0 is rest mass.

We know that in special relativity there is

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (382)$$

Back to Eq. (381), there is

$$\xi = (1 - \sqrt{1 - \frac{v^2}{c^2}})\frac{c^2}{v^2} \quad (383)$$

Then the total kinetic energy

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$$E_k = \xi m v^2 = (1 - \sqrt{1 - \frac{v^2}{c^2}})\frac{c^2}{v^2} m v^2 \quad (384)$$

Case v/c is a small value, ξ would be close to 0.5, so that

$$E_k \approx \frac{1}{2} m v^2 \quad (385)$$

On the occasion of freefall, the variation of kinetic energy

$$\xi_2 m_2 v^2 - \xi_1 m_1 v_1^2 = \int_{r_{(1)}}^{r_{(2)}} m g ds = \bar{m}\bar{g}s \quad (386)$$

Again with Eq. (377) the energy difference in an experiment

$$\Delta m c^2 \approx \Delta m v^2 + m_1 v \Delta v \quad (387)$$

Thus, there is

$$\Delta m = m_1 \frac{v \Delta v}{c^2 - v^2} \quad (388)$$

Considering $\xi_1 \approx \xi_2$ in experiments and v_1 is quite big, the difference of kinetic energy could be written as

$$\begin{aligned} \Delta E_k = \xi_2 m_2 v^2 - \xi_1 m_1 v_1^2 &= \xi_2 (m_1 + \Delta m)(v_1 + \Delta v)^2 - \xi_1 m_1 v_1^2 \\ &\approx \xi_1 (m_1 + \Delta m)(v_1^2 + 2 v_1 \Delta v) - \xi_1 m_1 v_1^2 \\ &= \xi_1 (\Delta m v_1^2 + 2 m_1 v_1 \Delta v) \\ &= \xi_1 m_1 (\frac{v_1 \Delta v}{c^2 - v_1^2} v_1^2 + 2 v_1 \Delta v) \\ &\approx m_1 g s \quad (389) \end{aligned}$$

so that

$$\Delta v \approx \frac{gs}{\xi_1 v_1 (\frac{v_1^2}{c^2 - v_1^2} + 2)} \quad (390)$$

Thus,

$$t_{(2)} - t_{(1)} = \Delta t = \frac{\Delta v}{a_{00}^1} \quad (391)$$

Unfortunately, this solution cannot come up with a higher accuracy than that

$$\Delta t = \frac{s}{v_1 + 0.5 \Delta v} \quad (392)$$

After then, we are going to sponsor series of freefalling experiments. Contra variant accelerations and covariant accelerations for every position are easy to calculate. While measurable covariant acceleration $\overline{a1/00}$ could be obtained with measured distances and time intervals via Eq. (366). But it is convenient to calculate with Eq. (367), in that the latter is just a transformation of the previous. And in this equation, space intervals would be gained with Newtonian equations and time intervals $t_{(2)} - t_{(1)}$ with Eq. (392) for convenience. One may argue that the measured quantities might come from calculation. That doesn't matter, because the equation has been verified for hundreds of years, therefore it is sure that the calculated quantities have equal value with that by measuring. And then the mean covariant acceleration will be taken to compare with the contra variant acceleration a_{00}^1 calculated with Eq. (380) and covariant acceleration $a1/00$ calculated with Eq. (330) or Eq. (331). Calculation results have been listed in Table 4. and 5.

Table 4. Terrestrial freefalling experiments from position 6371.493 to 6371.393.

Analysis method	Theoretical analysis				Measurable analysis	
Positions	$r_{(1)}$=6371.493		$r_{(2)}$=6371.393		From $r_{(1)}$ to $r_{(2)}$	
Initial velocity	Contra variant acceleration	Covariant acceleration	Contra variant acceleration	Covariant acceleration	measured time intervals	Covariant acceleration
$V_0^1{}_{(1)}$ (m/s)	$a_{00}^1{}_{(1)}$ (m/s^2)	$a_{1/00}{}_{(1)}$ (m/s^2)	$a_{00}^1{}_{(2)}$ (m/s^2)	$a_{1/00}{}_{(2)}$ (m/s^2)	$t_{(2)}-t_{(1)}$ (s)	$\overline{a_{1/00}}$ (m/s^2)
0	9.8165464	9.8165464	9.8168546	9.8168546	4.5137	9.8167
100000	9.8165453	9.8165453	9.8168535	9.8168535	1.0×10^{-3}	9.8167
10000000	9.8056240	9.8274692	9.8059318	9.8277770	1.0×10^{-5}	9.8276
100000000	8.7243083	10.908784	8.7245822	10.909127	1.0×10^{-6}	10.909
200000000	5.4475940	14.185499	5.4477651	14.185944	5.0×10^{-7}	14.186
250000000	2.9900583	16.643034	2.9901522	16.643557	4.0×10^{-7}	16.643
299792458	0	19.632910	0	19.633526	3.3356×10^{-7}	19.633

Table 5. Terrestrial freefalling experiments from position 6372.393 to 6371.393.

Analysis method	Theoretical analysis				Measurable analysis	
Positions	$r_{(1)}$=6372.393		$r_{(2)}$=6371.393		From $r_{(1)}$ to $r_{(2)}$	
Initial velocity	Contra variant acceleration	Covariant acceleration	Contra variant acceleration	Covariant acceleration	measured time intervals	Covariant acceleration
$V_0^1{}_{(1)}$ (m/s)	$a_{00}^1{}_{(1)}$ (m/s^2)	$a_{1/00}{}_{(1)}$ (m/s^2)	$a_{00}^1{}_{(2)}$ (m/s^2)	$a_{1/00}{}_{(2)}$ (m/s^2)	Δt (s)	$\overline{a_{1/00}}$ (m/s^2)
0	9.8137738	9.8137738	9.8168546	9.8168546	14.276	9.8153
100000	9.8137730	9.8137751	9.8168535	9.8168535	1.0×10^{-2}	9.8153
10000000	9.8028548	9.8246934	9.8059318	9.8277770	1.0×10^{-4}	9.8262
100000000	8.7218446	10.905704	8.7245822	10.909127	1.0×10^{-5}	10.909
200000000	5.4460559	14.181493	5.4477651	14.185944	5.0×10^{-6}	14.186
250000000	2.9892144	16.383343	2.9901522	16.643557	4.0×10^{-6}	16.644
299792458	0	19.627364	0	19.633526	3.3356×10^{-6}	19.635

8. Conclusions and Inferences and Their Applications

8.1. Conclusions

Previous discussions will lead to two conclusions for matters' motions in gravitational fields:

1) For light: Light speed keeps general covariance in covariant space, but light frequency keeps conservation in contra variant space.

2) For massive matters: Massive matter's velocity does not perform general covariance, but it will keep conservation in contra variant space.

These conclusions indicate the partial breaking of general covariance and the failure of gravity geometrization. In fact, gravity geometrization has succeeded in motions of light rays but failed in that of massive matters. One of the reasons is that covariant derivatives of massive matter's velocities depend on the values of velocities corresponding, while velocities are variable even in covariant space, so that it is impossible to geometrize the massive matter's gravity only with source-depending metrics.

8.2. Inferences

The conclusions are really different from classical theory of general relativity and they will then lead to natural inferences. I prefer to focus on the inferences on kinematics and relativistic release:

1) For kinematics: General covariance goes break by a large range. During the motions in gravitational field, all matters, including light rays, will keep energy and momentum conservation in

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contra variant space rather than that in covariant space. Only for light rays they may keep velocity invariant in covariant space, but their energy and momentum will still keep conservation in contra variant space. Energy momentum conservations are the conservations under the condition of gravitational potential conversions. It is said that there is only one exception in realities, the light speed invariance, which will lead to the validity Lagrangian of light ray propagation. While for massive matters, Lagrangian goes invalid. At any positions in gravity fields, massive matters always have opportunities to be accelerated up to and keep velocities close to absolute-light-speed. Consequently, there will be a question that what is the reason for light rays run in covariance and reveal the metrics of Schwarzschild? The logical reason is the momentum conversion together with Lorentz covariance that determines the performance of light ray propagations.

2) For relativistic release: Since apparent light speed may vary in gravitational field, that will bring changes to interaction efficiency between and within particles of massive matters so as to influence fine structures. For electromagnet forces, there will be of variations of momentum exchanges. It is also reasonable to predict that the speed of gluons relating to the strong interactions is general covariant like that of photons. On the other side, these interactions keep energy momentum conservations at the same time. Therefore, case massive matters inflow enough distance in gravitational fields, they might get to excited state and release energy, which could be called relativistic release, just as excited electrons in atoms might do. The difference is that relativistic releases might be releases experiencing thoroughly exciting of the matters both in intrinsic structures and exterior structures, including exciting of electrons. Matters may also experience covariant deformation after relativistic release because of new equivalent state. Moreover, relativistic emissions and absorptions may reveal the mystery of intrinsic structures just as the exciting electrons in atoms might do.

8.3. Applications

Detailed discussions on some applications will be sponsored consequently that will greatly support the conclusions and inferences.

1) On kinematics: General covariance and physical quantity conservativeness are of the handles to rectify the classical equations, especially the principle of moment and mass energy conservations. It would be seen that those efforts to employ the geodesic equation or covariant derivatives to build kinetic equations have already gone failed, in that covariant derivatives might be nonvanishing. Renovated solutions on light ray propagations and massive matter's motion as applications might provide forceful verifications on conclusions and inferences.

2) On relativistic release: The concept of equivalent state would be carried out to estimate the energy exceeding for inflow matters so that to discuss energy release, which will then lead to relativistic redshift of emission and absorption. Equivalent state also relates to relativistic deformation that might perform another kind of covariance. Researches on relativistic release will bring about fantastic interpretations on tremendous observations on quasars and active galactic nuclei.

8.4. On Conservativeness

If a physical quantity being conservative in covariant space or contra variant space is to say the quantity keep invariant at any positions. Consequently, the forms of conservative quantities must be expressed in subsidiary conservatives. It should be highlighted that a position in gravitational fields refers to a definite physical reality no matter how does it be described in covariant space or contra variant space. Thus, a quantity at a position always refers to a definite physical quantity whatever described in any coordinate space. A physical quantity in a gravitational field could be conservative or not in one of those descriptions. For example, light speed is a conservative quantity at any positions in covariant space but it is variable in contra variant space. If a physics expressed by conservative quantities, one could give their values directly, else if not, the variations should be performed by coordinates of space time. In a word, the physical quantities defined should perform physical realities.

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Momentum and energy keep conservations in contra variant space is something special that they are keeping conservations after considerations of the momentum and energy exchanges with gravitational fields.

A quantity conservative in a space may be formed with subsidiary quantities in another space or not. We will see in next section that the angular moment of light rays keep conservation in contra variant space but the expression would be written in form of covariant light speed.

9. Kinematics and Dynamics

9.1. The Most Important

The second Newtonian law interprets the mechanism of accelerative motions of massive matters so that to form the dynamics. Case in the conditions that matters have relativistic velocities, forces acting on matters will cause not only the variations of velocities but also the variations of matter's mass. It should be highlighted that a force really is of a statistic quantity rather than an essential physical quantity. In fact, a force is just a performance of exchange of momentum, as well as mass energy at the mean time. Thus, that physics could be called the relativistic dynamics.

But for light propagations, the second Newtonian law will not take effects anymore. Even in the case that a force is vertical to a light ray, we will see that the second Newtonian law remains invalid. That is the reason we suggest the concept of kinematics that others to the concept of dynamics. If we persistently employ the concept of dynamics, it should be a new one.

No matter the space time been determined by what kind of metrics and labeled by what kind of coordinates, it is just a methodology for descriptions for physical events. None of them would have priorities. Physics is on earth depends on its nature rather on spaces defined. The most important is the conservative principles in realities.

It is easy to imagine that geodesic line could be employed for the solution of kinematic trajectories of matters, because general relativity expects conservations in curve space. But we will find out that geodesic equation or covariant derivatives have not really taken effect in the solving of the kinematics in the past century. We know the reason is that covariant derivatives may be nonvanishing so that those imposed settings of vanishing covariant derivatives might cause discrepancies with respect to realities.

Most of methodologies published for kinematic trajectories were based on the so-called Lagrangian. Besides these conditions, contra variant angular momentum conservation has been used in all those solutions. One can imagine that this condition is apparently contradicted with general covariance. In fact, it is always the greatest reason for me to persist in this issue with more efforts.

Finally, the Lorentz covariance is also a kind of constraint condition, since it has been involved in the settings of Minkowski space and pseudo Riemannian space.

9.2. Falsification of the Employment of Geodesic Line for Kinematic Equation

We have drawn the conclusion that covariant accelerations of matters in gravitational fields may be nonvanishing so that the geodesic lines which ask for zero value of the covariant derivatives cannot be employed to be the kinematic equation for matter's motions. We will find great many faults in those equations with which they declare the kinematic equations coming from geodesic lines.

The geodesic line equations presented by Weinberg [17] with metrics given by

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\varphi^2 \quad (393)$$

is that

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \quad (394)$$

That has been calculated to perform as components as [17]

$$0 = \frac{d^2r}{d\tau^2} + \frac{A'(r)}{2A(r)}\left(\frac{dr}{d\tau}\right)^2 - \frac{r}{A(r)}\left(\frac{d\theta}{d\tau}\right)^2 - r\frac{sin^2\theta}{A(r)}\left(\frac{d\varphi}{d\tau}\right)^2 + \frac{B'(r)}{2A(r)}\left(\frac{dt}{d\tau}\right)^2 \quad (395)$$

$$0=\frac{d^2\theta}{d\tau^2}+\frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau}-sin\theta cos\theta(\frac{d\varphi}{d\tau})^2 \quad (396)$$

$$0=\frac{d^2\varphi}{d\tau^2}+\frac{2}{r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau}+2cot\theta\frac{d\varphi}{d\tau}\frac{d\theta}{d\tau} \quad (397)$$

$$0=\frac{d^2t}{d\tau^2}+\frac{B'(r)}{B(r)}\frac{dt}{d\tau}\frac{dr}{d\tau} \quad (398)$$

And then, with $\theta=\pi/2$, the so-called kinematic equation, were finally drawn as

$$A(r)(\frac{dr}{d\tau})^2+\frac{L^2}{r^2}-\frac{1}{B(r)}=-E \quad (399)$$

where, $L=r^2\frac{d\varphi}{d\tau}$ and E are set constants.

Christoffel symbols in equations above could be listed in Table 6. as following.

Table 6. Probable calculations of Christoffel symbols in equations above.

$\Gamma^1_{00}=\frac{B'(r)}{2A(r)}$	$\Gamma^1_{11}=\frac{A'(r)}{2A(r)}$	$\Gamma^0_{01}=\Gamma^0_{10}=\frac{B'(r)}{B(r)}$
$\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{r}$	$\Gamma^1_{22}=-\frac{r}{A(r)}$	$\Gamma^3_{13}=\Gamma^3_{31}=\frac{1}{r}$
$\Gamma^1_{33}=-r\frac{sin^2\theta}{A(r)}$	$\Gamma^2_{33}=-sin\theta cos\theta$	$\Gamma^3_{23}=\Gamma^3_{32}=cot\theta$

It seems that the kinematic equation has been created by covariant derivatives. But it should be pointed out that Eq. (395) to Eq. (399) have gone wrong. Because we have discussed that the covariant derivatives could be nonvanishing in some occasions, so that there must be something wrong involved.

We have been told that the Eq. (394) is the equation of geodesic lines. In this equation, What kinds of differentials dx^μ will be defined that determines the final forms of calculation deserve of more attentions. One can find out that in calculations above, they are defined as

$$dx^0=cdt\ ,\ dx^1=dr,\ dx^2=d\theta,\ dx^1=d\varphi \quad (400)$$

We are going to sponsor investigations in two ways for a comparison. Firstly, might as well, a transformation from the original pseudo spherical space defined as $(cdt,dr,d\theta,d\varphi)$ to the distance expressed pseudo spherical space defined as $(cd\tau,d\rho,rd\theta,rsin\theta d\varphi)$, just as the same as previous discussions, could be put into considerations. For convenience, the invariant distance should be rewritten as

$$ds^2=-B(r)(cdt)^2+A(r)dr^2+r^2d\theta^2+r^2sin^2\theta d\varphi^2 \quad (401)$$

where, signs in the equation will be suggested to be modifiedand the metrics would rather be defined as

$$g_{00}=B(r),\ g_{11}=A(r),\ g_{22}=r^2,\ g_{33}=r^2sin^2\theta \quad (402)$$

in which all metrics have been set positive as has been discussed so that to lead to correct results.

In fact, this kind of transformation is to make analysis from original spherical space with unequal units of $cdt,\ dr,\ d\theta\ d\varphi$ in demensions to the distance expressed spherical space with equal units of $cdt,\ dr,\ rd\theta\ rsin\theta d\varphi$ in demensions. That might cause a little confused, but this treatment is to define dx^μ as same as that in Eq. (395) to Eq. (398) so that to make comparison. I would like to make more comparisons to show the correct methodologies for the calculations on covariant derivatives. The very important is that the covariant derivatives we are talking about are the derivatives of velocities on the trajectories of matter's motions.

The Eq. (294) could be employed for the solution, so that the derivative components could be performed with and then

$$(\frac{DV^1}{\partial\tau})_{tr}=(\frac{D}{\partial\tau}\frac{dr}{d\tau})_{tr}=(\frac{dV^1}{d\tau})_{tr}+\Gamma^1_{0i}V^iV^0+\Gamma^1_{1i}V^iV^1+\Gamma^1_{2i}V^iV^2+\Gamma^1_{3i}V^iV^3$$

$$=(\frac{dV^1}{d\tau})_{tr}+\Gamma^1_{01}V^1V^0+\Gamma^1_{11}V^1V^1+\Gamma^1_{21}V^1V^2+\Gamma^1_{31}V^1V^3$$

$$= (\frac{dV^1}{d\tau})_{tr} + \Gamma^1_{11}V^1V^1$$

$$= \frac{d^2r}{d\tau^2} + \frac{A'(r)}{2A(r)}(\frac{dr}{d\tau})^2 \quad (403)$$

where, $V^1 = \frac{dr}{\partial\tau}$, $\Gamma^1_{01} = 0$,$\Gamma^1_{21} = 0$ and $\Gamma^1_{31} = 0$, so that there is an only nonvanishing item of $\Gamma^1_{11} = \frac{A'(r)}{2A(r)} \neq 0$.

$$(\frac{DV^2}{\partial\tau})_{tr} = (\frac{D}{d\tau}\frac{d\theta}{d\tau})_{tr} = (\frac{dV^2}{d\tau})_{tr} + \Gamma^2_{0i}V^iV^0 + \Gamma^2_{1i}V^iV^1 + \Gamma^2_{2i}V^iV^2 + \Gamma^2_{3i}V^iV^3$$

$$= (\frac{dV^2}{d\tau})_{tr} + \Gamma^2_{02}V^2V^0 + \Gamma^2_{12}V^2V^1 + \Gamma^2_{22}V^2V^2 + \Gamma^2_{32}V^2V^3$$

$$= (\frac{dV^2}{d\tau})_{tr} + \Gamma^2_{12}V^2V^1$$

$$= \frac{d^2\theta}{d\tau^2} + \frac{1}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau} \quad (404)$$

where, $V^2 = \frac{d\theta}{\partial\tau}$, and $\Gamma^2_{02} = 0$, $\Gamma^2_{22} = 0$ and $\Gamma^2_{32} = 0$, so that there will be an only nonvanishing item $\Gamma^2_{12} = \frac{1}{r} \neq 0$.

$$(\frac{DV^3}{\partial\tau})_{tr} = (\frac{D}{d\tau}\frac{d\varphi}{d\tau})_{tr} = (\frac{dV^3}{d\tau})_{tr} + \Gamma^3_{0i}V^iV^0 + \Gamma^3_{1i}V^iV^1 + \Gamma^3_{2i}V^iV^2 + \Gamma^3_{3i}V^iV^3$$

$$= (\frac{dV^3}{d\tau})_{tr} + \Gamma^3_{03}V^3V^0 + \Gamma^3_{13}V^3V^1 + \Gamma^3_{23}V^3V^2 + \Gamma^3_{33}V^3V^3$$

$$= (\frac{dV^3}{d\tau})_{tr} + \Gamma^3_{13}V^3V^1 + \Gamma^3_{23}V^3V^2$$

$$= \frac{d^2\varphi}{d\tau^2} + \frac{1}{r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau} + cot\theta\frac{d\varphi}{d\tau}\frac{d\theta}{d\tau} \quad (405)$$

where, $V^3 = \frac{d\varphi}{\partial\tau}$, and $\Gamma^3_{03} = 0$ and $\Gamma^3_{33} = 0$, so that there will be two nonvanishing items that $\Gamma^3_{13} = \frac{1}{r} \neq 0$ and $\Gamma^3_{23} = cot\theta \neq 0$.

$$(\frac{DV^0}{\partial\tau})_{tr} = (\frac{D}{d\tau}\frac{cdt}{d\tau})_{tr} = (\frac{dV^0}{d\tau})_{tr} + \Gamma^0_{0i}V^iV^0 + \Gamma^0_{1i}V^iV^1 + \Gamma^0_{2i}V^iV^2 + \Gamma^0_{3i}V^iV^3$$

$$= (\frac{dV^0}{d\tau})_{tr} + \Gamma^0_{00}V^0V^0 + \Gamma^0_{10}V^0V^1 + \Gamma^0_{20}V^0V^2 + \Gamma^0_{30}V^0V^3$$

$$= (\frac{dV^0}{d\tau})_{tr} + \Gamma^0_{10}V^0V^1$$

$$= c\frac{d^2t}{d\tau^2} + \frac{B'(r)}{2B(r)}\frac{cdt}{d\tau}\frac{dr}{d\tau} \quad (406)$$

where, $V^0 = \frac{dt}{\partial\tau}$, and $\Gamma^0_{00} = 0$, $\Gamma^0_{20} = 0$ and $\Gamma^0_{30} = 0$, so that there will be an only nonvanishing item $\Gamma^0_{10} = \frac{B'(r)}{2B(r)} \neq 0$.

We have seen that some errors in the Eq. (395) to Eq. (398) have been rectified. In fact, it is easy to find out calculation errors. If any $i \neq j$ the Christoffel symbol of $\Gamma^j_{\mu i}$=0, in that the bases we discussed are orthogonal.

Secondly, we are going to study another condition that the physical quantities could be transformed from the contra variant space of distance expressed coordinates $(cdt, dr, rd\theta, rsin\theta d\varphi)$ to the covariant space of distance expressed coordinates $(cd\tau, d\rho, rd\theta, rsin\theta d\varphi)$. The invariant distance could be written as

$$ds^2 = -B(r)(cdt)^2 + A(r)dr^2 + (rd\theta)^2 + (rsin\theta d\varphi)^2$$

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$$= -g_{00}(cdt)^2 + g_{11}dr^2 + g_{22}(rd\theta)^2 + g_{33}(rsin\theta d\varphi)^2 \quad (407)$$

Thus, $g_{00} = B(r),\ g_{11} = A(r),\ g_{22} = 1,\ g_{33} = 1$. As has mentioned previous, in this condition, only gravitational metrics has been considered in the expressions, and as result, these metrics will really make sense in physics. More surprising, this kind of settings would easily help us to verify the calculation previous. One may argue that this transformation has overcome Riemannian manifold definition because the contra variant space is not a R^4. But on earth in mathematics, that doesn't matter because we know that the space could map to a R^4 if you will, and anyway, it is not important to make a R^4 or something different and it is not important to discuss these topics in Riemannian geometry or not. The derivation regulars are still available. Then the derivatives should be performed with Eq. (294) that

$$(\frac{DV^1}{\partial\tau})_{tr} = (\frac{D}{\partial\tau}\frac{dr}{d\tau})_{tr} = (\frac{dV^1}{d\tau})_{tr} + \Gamma^1_{0i}V^iV^0 + \Gamma^1_{1i}V^iV^1 + \Gamma^1_{2i}V^iV^2 + \Gamma^1_{3i}V^iV^3$$

$$= (\frac{dV^1}{d\tau})_{tr} + \Gamma^1_{01}V^1V^0 + \Gamma^1_{11}V^1V^1 + \Gamma^1_{21}V^1V^2 + \Gamma^1_{31}V^1V^3$$

$$= (\frac{dV^1}{d\tau})_{tr} + \Gamma^1_{11}V^1V^1$$

$$= \frac{d^2r}{d\tau^2} + \frac{A'(r)}{2A(r)}(\frac{dr}{d\tau})^2 \quad (408)$$

where, $V^1 = \frac{dr}{\partial\tau}$, and $\Gamma^1_{01} = 0,\ \Gamma^1_{21} = 0$ and $\Gamma^1_{31} = 0$, so that there will be an only nonvanishing item $\Gamma^1_{11} = \frac{A'(r)}{2A(r)} \neq 0$.

$$(\frac{DV^2}{\partial\tau})_{tr} = (\frac{D}{d\tau}\frac{rd\theta}{d\tau})_{tr} = (\frac{dV^2}{d\tau})_{tr} + \Gamma^2_{0i}V^iV^0 + \Gamma^2_{1i}V^iV^1 + \Gamma^2_{2i}V^iV^2 + \Gamma^2_{3i}V^iV^3$$

$$= (\frac{dV^2}{d\tau})_{tr} + \Gamma^2_{02}V^2V^0 + \Gamma^2_{12}V^2V^1 + \Gamma^2_{22}V^2V^2 + \Gamma^2_{32}V^2V^3$$

$$= (\frac{dV^2}{d\tau})_{tr} + 0$$

$$= r\frac{d^2\theta}{d\tau^2} + \frac{d\theta}{d\tau}\frac{dr}{d\tau} \quad (409)$$

where, $V^2 = \frac{rd\theta}{\partial\tau}$, and $\Gamma^2_{02} = 0,\ \Gamma^2_{12} = 0,\ \Gamma^2_{22} = 0,\ \Gamma^2_{32} = 0$ because that $g_{22} = 1$.

$$(\frac{DV^3}{\partial\tau})_{tr} = (\frac{D}{d\tau}\frac{rsin\theta d\varphi}{d\tau})_{tr} = (\frac{dV^3}{d\tau})_{tr} + \Gamma^3_{0i}V^iV^0 + \Gamma^3_{1i}V^iV^1 + \Gamma^3_{2i}V^iV^2 + \Gamma^3_{3i}V^iV^3$$

$$= (\frac{dV^3}{d\tau})_{tr} + \Gamma^3_{03}V^3V^0 + \Gamma^3_{13}V^3V^1 + \Gamma^3_{23}V^3V^2 + \Gamma^3_{33}V^3V^3$$

$$= (\frac{dV^3}{d\tau})_{tr} + 0$$

$$= rsin\theta\frac{d^2\varphi}{d\tau^2} + sin\theta\frac{d\varphi}{d\tau}\frac{dr}{d\tau} + rcos\theta\frac{d\varphi}{d\tau}\frac{d\theta}{d\tau} \quad (410)$$

where, $V^3 = \frac{rsin\theta d\varphi}{\partial\tau}$, and $\Gamma^3_{03} = 0,\ \Gamma^3_{13} = 0,\ \Gamma^3_{23} = 0,\ \Gamma^3_{33} = 0$ because that $g_{33} = 1$. The $(\frac{dV^3}{d\tau})_{tr}$ has been calculated by chain rule is because the trajectory derivative is a derivative on the parametric motion line.

$$(\frac{DV^0}{\partial\tau})_{tr} = (\frac{D}{d\tau}\frac{cdt}{d\tau})_{tr} = (\frac{dV^0}{d\tau})_{tr} + \Gamma^0_{0i}V^iV^0 + \Gamma^0_{1i}V^iV^1 + \Gamma^0_{2i}V^iV^2 + \Gamma^0_{3i}V^iV^3$$

$$= (\frac{dV^0}{d\tau})_{tr} + \Gamma^0_{10}V^0V^1$$

$$= c\frac{d^2t}{d\tau^2} + \frac{B'(r)}{2B(r)}\frac{cdt}{d\tau}\frac{dr}{d\tau} \quad (411)$$

where, $V^0 = \frac{dt}{\partial\tau}$, and $\Gamma^0_{00} = 0$, $\Gamma^0_{20} = 0$ and $\Gamma^0_{30} = 0$, so that there will be an only nonvanishing item $\Gamma^0_{10} = \frac{B'(r)}{2B(r)} \neq 0$.

In comparisons of the previous two calculations, we could find subtle nuance in that they are settled by different x^μ and V^μ. But they have really given the equivalent results, in that both of them could be transformed to uniform covariant derivatives $(\frac{DV}{\partial\tau})_{tr}$. The latter calculation is very easy to be done. The most important is that the comparison of calculations have verified the conclusions on inequality of mixed subscript Christoffel symbols. One could easily make a checking computation for the latter so that to use the simplified expression of the latter to verified the solution of the previous. That will finally indicate the errors in classical theory, as well as that in Weinberg's calculations on geodesic equations.

Many efforts [11,12,17] have been made to attempt to prove the conservative principles after the geodesic equations. They attempt to show that

$$r^2\frac{d\varphi}{d\tau} = const. \quad (412)$$

$$(1 - \frac{r^*}{r})\frac{dt}{d\tau} = const. \quad (413)$$

are of the results of the Eq. (397), and the Eq. (398).

It is easy to find that these works involve with errors. In comparisons on the results of $(\frac{DV^2}{\partial\tau})_{tr}$ and $(\frac{DV^3}{\partial\tau})_{tr}$ in previous two kinds of strategies, we will find that these two derivatives do nothing with gravity influence, and they are just come from transformation of spherical coordinates so that any doctrines after that to form angular momentum conservative principle would be lack of supports. We will make further verifications in next sections that these two equations are all of false ones. In fact, the Eq. (412) is not a correct form of angular momentum, and the Eq. (413) does nothing with energy expressions. We will see that, motion trajectories could be calculated based on neither zero covariant derivatives, nor the so-called the geodesic lines.

9.3. Classical Equations of Light Ray Deflection

It is indicated in some books that the Lagrangian relates to Euler-Lagrangian equation and geodesic equation [3,4]. It is trivial to continue the discussions on whatever of the origins. I will say that the Lagrangian for light rays is absolutely correct, because we will see that it is just the expression of composition of light speed components in covariant space. I think that many people may have realized the problems in the calculations of geodesic equation. Perhaps, it is the reason that the Lagrangian is employed for the equation of matter trajectories in more and more publications. In fact, the methodology of velocity composition could also be employed to solve the trajectory of massive matter's motions. But it should be highlighted that the Lagrangian equation is not proper for massive matters, which will be presented in following discussions.

Incomprehensibly, traditional solving process for Lagrangian equations seem like to do nothing with geodesic line equation and covariant differentials. On the other side, we could find that those solved trajectories all involved with contra variant angular momentum conservation, which also indicates that the solved trajectories may be not real geodesic lines.

9.3.1. Classical Equations After Traditional Settings

Take the problem of the light rays passing across the Sun for granted, as shown in Figure 20.

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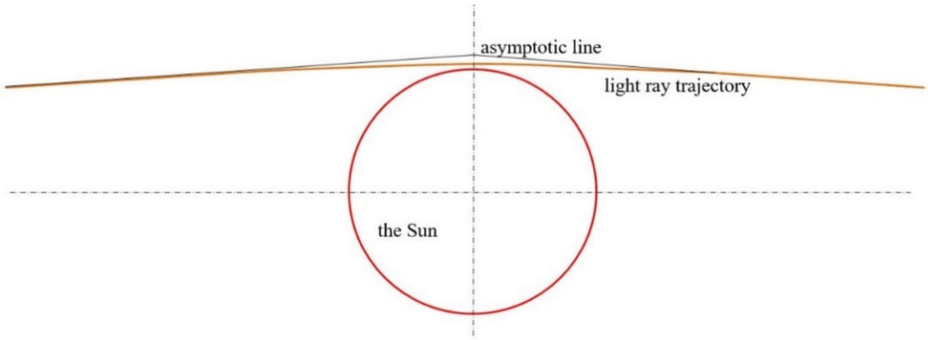

Figure 20. Light rays passing across the Sun.

It is expected in classical theories that, for the propagations of light rays, the Lagrangian is zero

$$\mathcal{L} = -(1-\frac{r^*}{r})c^2\dot{t}^2 + (1-\frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = 0 \quad (414)$$

where, Schwarzschild metrics has been concerned. Thereby, trajectory derivatives by proper time have been specially expressed for conveniences as $\dot{t} = \frac{dt}{d\tau}$, $\dot{r} = \frac{dr}{d\tau}$ and $\dot{\varphi} = \frac{d\varphi}{d\tau}$, which will also be employed in following expressions.

Two items were always set to be constant in most publications [3,4] as

$$\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = r^2\dot{\varphi} = L \quad (415)$$

$$\frac{1}{2}\frac{\partial\mathcal{L}}{\partial\dot{t}} = (1-\frac{r^*}{r})c^2\dot{t} = c^2E \quad (416)$$

Because of the multiplier r^2 used, L is really an equivalent contra variant angular momentum rather than a covariant one, so that this setting seems like to insert the contra variant conservation in. It is said that the setting of L does not actually coincide with general covariance. The setting of constant E make sense neither because of the variables of r involved in. The mathematical reason of the wrong settings is that any motions of matters in space time are of the problems on their trajectories. As we all know, every trajectory is of monotropic function while the equations Eq. (415) and Eq. (416) perform wrong results in derivations. And anyway, variational methods often propose $\frac{\partial\mathcal{L}}{\partial\lambda} = 0$ rather than that being set a constant. Even so, we will find out that the wrong setting of angular moment together with the setting of constant E have led to a not bad result, which may have performed two negatives making a positive.

With the two settings, the Lagrangian would be reformed to be

$$c^2E^2 - \dot{r}^2 - (1-\frac{r^*}{r})\frac{L^2}{r^2} = 0 \quad (417)$$

Setting $r' = \frac{dr}{d\varphi} = \frac{\dot{r}}{\dot{\varphi}}$, $u = \frac{1}{r}$, and then $r' = \frac{dr}{d\varphi} = -\frac{u'}{u^2}$, there is

$$\frac{c^2E^2}{L^2} - u'^2 - (1-r^*u)u^2 = 0 \quad (418)$$

Derived by φ, it is

$$u'' + u = \frac{3}{2}r^*u^2 \quad (419)$$

In this equation, the right hand side item is a smaller quantity than every left side item so that it could be set zero for an approximate solution $u = sin\varphi/b$, where b is a constant, which could be proved to be the distance from solar center to the asymptotic line in this case, so that it could also be called asymptotic distance. Replacing the very small right hand item with the solution, there is

$$u'' + u = \frac{3r^*}{2b^2}sin^2\varphi \quad (420)$$

There is a particular solution

$$u_1 = \frac{3r^*}{4b^2}(1 + \frac{1}{3}cos2\varphi) \quad (421)$$

Then the approximate solution of Eq. (419) is obtained

$$u = \frac{sin\varphi}{b} + \frac{3r^*}{4b^2}(1 + \frac{1}{3}cos2\varphi) \quad (422)$$

For an infinite large r, the u is infinitesimal one, there is

$$\varphi_\infty = -\frac{r^*}{b} \quad (423)$$

Observational deflection of light rays passing by the edge of the Sun is

$$\Delta = 2|\varphi_\infty| \approx 1.75'' \quad (424)$$

At the position $\varphi = \frac{\pi}{2}$ and $r = R$, there is

$$\frac{1}{R} = \frac{1}{b} + \frac{3r^*}{4b^2}(1 - \frac{1}{3}) \quad (425)$$

or

$$b \approx R + \frac{r^*}{2} \quad (426)$$

That will bring about contradictions for the setting of Eq. (415) that at peak point there is

$$L_R = r^2\dot{\varphi} = Rc \quad (427)$$

while at a very far point, there is

$$L_\infty = r_\infty^2\dot{\varphi} = bc = (R + \frac{r^*}{2})c \neq L_R \quad (428)$$

It is said that the equations have been solved incorrectly. Might as well, we would find out an inevitable solution after the inappropriate setting of Eq. (415) in next section.

9.3.2. Errors Hidden in the Solving Process

It is obvious that a mistake has been involved in Eq. (416), because in gravitational field, $(1 - \frac{r^*}{r})c^2\dot{t}$ really varies with r. In fact, the Eq. (414) could be solved directly only based on the setting of Eq. (415) as following.

Considering $\dot{t} = \frac{dt}{d\tau} = (1 - \frac{r^*}{r})^{-1/2}$, the Lagrangian really is

$$-c^2 + (1 - \frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = 0 \quad (429)$$

or

$$-c^2 + \dot{\rho}^2 + r^2\dot{\varphi}^2 = 0 \quad (430)$$

It is really the composition of light speed in covariant space. Any equation that leads to $(1 - \frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 \neq c^2$ is unacceptable, as that has been done in Eq. (417).

Transform the equation by multiplying a $(1 - \frac{r^*}{r})$

$$(1 - \frac{r^*}{r})c^2 - \dot{r}^2 - (1 - \frac{r^*}{r})r^2\dot{\varphi}^2 = 0 \quad (431)$$

If setting peak point radius coordinate to be R and setting the angular moment at that point to be $L = Rc = r^2\dot{\varphi} = const$, with $u = 1/r$, it becomes

$$(1 - r^*u)c^2 - u'^2R^2c^2 - (1 - r^*u)u^2R^2c^2 = 0 \quad (432)$$

To be derived by φ, it is

$$u'' + u = \frac{3}{2}r^*u^2 - \frac{r^*}{2R^2} \quad (433)$$

With $\frac{3r^*}{2b^2}sin^2\varphi$ instead of $\frac{3}{2}r^*u^2$, it becomes

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$$u'' + u = \frac{3r^*}{2b^2} sin^2\varphi - \frac{r^*}{2R^2} \quad (434)$$

Because $R \approx b$, it could be solved as

$$u = \frac{sin\varphi}{b} + \frac{r^*}{4b^2}(1 + cos2\varphi) \quad (435)$$

One will obtain the solution of the equation as

$$\varphi_\infty = -\frac{r^*}{2b} \quad (436)$$

At the peak point as $\varphi = \frac{\pi}{2}$, we will get the constant b that

$$\frac{1}{R} = \frac{1}{b} + \frac{r^*}{4b^2}(1 - 1) \quad (437)$$

so that

$$b = R \quad (438)$$

And the setting of Eq. (415) has been well kept that

$$L_R = r^2\dot{\varphi} = Rc \quad (439)$$

and

$$L_b = r^2\dot{\varphi} = bc = Rc = L_R \quad (440)$$

This is really the inevitable solution for classical equations but it is not a true result for realities. We have seen that the classical equations to have been solved to an answer Eq. (422) accurately up to the observation results is just caused by the wrong settings of the declared energy momentum conservations of Eq. (415) and Eq. (416). We will see in next sections that the angular momentum in Eq. (432) is not correct as well.

9.4. Momentum, Energy, and Angular Momentum Conservativeness

We have drawn the conclusion that light momentum keeps conservation in contra variant space rather than covariant space, and then of course, so does the light mass energy. In fact, apparent light speed or so-called contra variant light speed may varies in contra variant space, but light momentum and energy will not be affected by apparent speeds. In fact, neither geodesic equation nor the derivations of Lagrangian could help proving the Eq. (415) and Eq. (416), in that Eq. (415) and Eq. (416) are substantially not correct.

Considering a light ray goes a vertical distance on the Earth, one could gain the mass variation as

$$m_r = \frac{h\nu_{0(\infty)}}{c^2}(1 + \frac{r^*}{2r}) = m_\infty(1 + \frac{r^*}{2r}) \quad (441)$$

It could be called simplified equation of mass in gravitational field.

Light momentum could be expressed as

$$P = \frac{h\nu_0}{c} = m_r c \quad (442)$$

or the momentum square

$$P^2 = m_r^2 c^2 \quad (443)$$

Case in contra variant space, apparent light speed varies with position so that that speed cannot be used in expressions of light momentum directly. The invariant light speed $c =$ const. could be employed to present invariance of momentum. Eq. (442) performs full variation with gravity by m_r, that is the performance of momentum conservation. In fact, light momentum depends on frequency, just as m_r does. If we ask more for a deep reason, that should be mass energy equation.

Case in covariant space, if light momentum will also be expressed by frequency, that will vary with bases additionally.

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We know the Lagrangian in one source fields is

$$c^2 = (1 - \frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 \quad (444)$$

With Lagrangian substituted in conservative momentum square, that turns to be

$$P^2 = m_r^2[(1 - \frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2] \quad (445)$$

or

$$P^2 = m_r^2[(1 - \frac{r^*}{r})^{-2}(\frac{dr}{dt})^2 + (1 - \frac{r^*}{r})^{-1}r^2(\frac{d\varphi}{dt})^2] \quad (446)$$

As we have discussed, light speed cannot be directly composed in contra variant space but can be done in covariant space. This is a reason the Lagrangian is employed in conservative momentum square.

In one source fields, the momentum vector could be discomposed to be components of centripetal and tangent

$$\boldsymbol{P} = \boldsymbol{P}_c + \boldsymbol{P}_t \quad (447)$$

or

$$P^2 = P_c^2 + P_t^2 \quad (448)$$

Obviously, the tangent momentum relates to tangent velocity and centripetal momentum relates to centripetal velocity. It could also be inferred that the tangent component varies with the corresponding velocity, and so does the centripetal one.

So that there must be

$$P_c^2 = m_r^2(1 - \frac{r^*}{r})^{-1}\dot{r}^2 = m_r^2(1 - \frac{r^*}{r})^{-2}(\frac{dr}{dt})^2 \quad (449)$$

and

$$P_t^2 = m_r^2 r^2\dot{\varphi}^2 = m_r^2(1 - \frac{r^*}{r})^{-1}r^2(\frac{d\varphi}{dt})^2 \quad (450)$$

In one source field, the angular moment conservation could be expressed as

$$L^2 = r^2P_t^2 = m_r^2 r^4\dot{\varphi}^2 = m_r^2(1 - \frac{r^*}{r})^{-1}r^4(\frac{d\varphi}{dt})^2 = \text{const.} \quad (451)$$

It should be highlighted that we are talking about the moment conservation in contra variant space. It is amazing that the angular momentum should be expressed in the form of $r^2m_r\dot{\varphi}$ or $(1 - \frac{r^*}{r})^{-1/2}r^2m_r\frac{d\varphi}{d\tau}$ rather than the form of $r^2m_r\frac{d\varphi}{dt}$, or we have seen that contra variant light momentum could be only directly and partially composed in covariant form. The real reason is that the invariant light speed c is just employed for the expressions by invariance. By the way, m_r is of contra variant forms as defined in the Eq. (441). It will be discussed in next sections that the light momentum would be discomposed in covariant space rather in contra variant space, but they keep conservations in the latter. That is really surprising.

The issues of light momentum have always been one of the controversies in physics for more than a hundred years [18]. The problem is the difficulty of assessing the light momentum in transparent materials between Minkowski's equation [19] and Abraham's equation [20]. To one's surprise, we would have made the conclusion different from both of them, after the discussions in previous sections, because of Lorentz covariance.

Nevertheless, angular momentum equations in gravitational fields would bring about new surprises on. We will find that the surprises not only to have been picked up from the expressions, but also to be hidden in the kinematics of light propagations in gravitational fields. These efforts might bring about tiny contributions for the attempt to answer the question of Einstein about 'What are light quanta?' [21] I appreciate what Leonhardt has said that light continues to surprise [22].

9.5. Revisit Equations for Light Ray Trajectory

9.5.1. Renovation and Resolution

We have recognized that it is the conservative principles that really controls the solutions. In fact, light rays in gravitational field may undergo mass energy variation.

As has mentioned previous, the light mass at the peak point is

$$m_R = \frac{h\nu_{0(\infty)}}{c^2}(1+\frac{r^*}{2R}) \quad (452)$$

And it varies at position r

$$m_r = \frac{h\nu_{0(\infty)}}{c^2}(1+\frac{r^*}{2r}) = \frac{1+\frac{r^*}{2r}}{1+\frac{r^*}{2R}} m_R \quad (453)$$

These two equations are involved with the energy conservation in contra variant space rather than that in covariant space.

For a light ray passing by a one source field, there is the angular momentum conservation as

$$L = r^2 m_r \dot{\varphi} = R m_R c = const. \quad (454)$$

It should be highlighted again that the contra variant light momentum has been expressed by c and $\dot{\varphi}$ which are of covariant space quantities rather than c_0^μ and $\frac{d\varphi}{dt}$ of contra variant ones. In fact, it could be proved that c_0^μ and $\frac{d\varphi}{dt}$ cannot be taken to form momentum conservation, if one takes efforts to have a try. That is because momentum variation depends on m_r that perform the effect of gravity, or in another words, the gravity input energy into the m_r. The invariant light speed c employed reveals that light momentum variation really depends on frequency rather than real velocity, just as that light propagates in transparent materials. That perhaps is of real surprise.

Considering the Eq. (452) and Eq. (453), Eq. (454) will lead to

$$\dot{\varphi} = \frac{R m_R c}{m_r r^2} = \frac{Rc}{r^2}\frac{1+\frac{r^*}{2R}}{1+\frac{r^*}{2r}} \quad (455)$$

Setting $B = (R+\frac{r^*}{2})$, and in weak field, the item $1/(1+\frac{r^*}{2r}) \approx (1-\frac{r^*}{2r})$, thus

$$\dot{\varphi} = \frac{\mathrm{Bc}}{r^2}(1-\frac{r^*}{2r}) \quad (456)$$

I prefer to present the Lagrangian again

$$-c^2 + (1-\frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = 0 \quad (457)$$

Define $r' = \frac{dr}{d\varphi}$, so that $\dot{r} = \frac{dr}{d\varphi}\frac{d\varphi}{d\tau} = r'\dot{\varphi}$. There is

$$-c^2 + (1-\frac{r^*}{r})^{-1} r'^2\dot{\varphi}^2 + r^2\dot{\varphi}^2 = 0 \quad (458)$$

Insert the Eq. (456) into Eq. (458), the latter becomes

$$-c^2 + (1-\frac{r^*}{r})^{-1} r'^2[\frac{1}{r^2}(1-\frac{r^*}{2r})]^2 B^2 c^2 + r^2[\frac{1}{r^2}(1-\frac{r^*}{2r})]^2 B^2 c^2 = 0 \quad (459)$$

With $(1-\frac{r^*}{2r})^2 \approx 1-\frac{r^*}{r}$, it is

$$-1 + r'^2\frac{B^2}{r^4} + (1-\frac{r^*}{r})\frac{B^2}{r^2} = 0 \quad (460)$$

With $u = 1/\mathrm{r}$, and $r' = \frac{dr}{d\varphi} = -\frac{u'}{u^2}$ it turns to be

$$-\frac{1}{B^2} + u'^2 + (1-ur^*)u^2 = 0 \quad (461)$$

We have seen the similar form of Eq. (418), but this equation comes from the settings of the real conservative principles.

Make a secondary order derivation for the equation,

$$u'' + u = \frac{3}{2}r^* u^2 \quad (462)$$

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Considering r is very big value so that the last item is very small so that the solution will be close to horizontal line at near R positions, the following equation could be presented

$$u'' + u = 0 \quad (463)$$

It could be solved to be

$$u = \frac{sin\varphi}{b} \quad (464)$$

It is a horizontal line with a perpendicular distance b to the center of the Sun.

The right item of Eq. (462) could be replaced with the simple solution for approximation, because the deviation is also very small. So that the equation could be reformed to be

$$u'' + u = \frac{3r^*}{2b^2} sin^2\varphi \quad (465)$$

Case at positions far from R, the right item could be an approximation of the right item of Eq. (462) to close to zero.

Once again, we could obtain the solution for the differential equation that

$$u = \frac{sin\varphi}{b} + \frac{3r^*}{4b^2}\left(1 + \frac{1}{3}cos2\varphi\right) \quad (466)$$

And then the deflection angle

$$\varphi_\infty = -\frac{r^*}{b} \quad (467)$$

Case $\varphi = \frac{\pi}{2}$, there is

$$\frac{1}{R} = \frac{1}{b} + \frac{r^*}{2b^2} \quad (468)$$

Because $b \gg r^*$ it becomes

$$R \approx b - \frac{r^*}{2} \quad (469)$$

or

$$b = R + \frac{r^*}{2} \quad (470)$$

To verify the momentum conservativeness that

$$L_R = m_R Rc \quad (471)$$

At a position $r \gg R$

$$L_b = m_b bc = \frac{1 + \frac{r^*}{2r_\infty}}{1 + \frac{r^*}{2R}} m_R bc = \frac{1}{R + \frac{r^*}{2}} R m_R bc = m_R Rc = L_R \quad (472)$$

It seems that we have got the same results as that of classical equations. But the truth is that the solution is the results after the conclusions of momentum and mass conservations which completely others to that of classical theory and at the same time the assumptions of Eq. (415) and Eq. (416) are thoroughly given up. It is said that the angular momentum keep conservation in the way as in Eq. (454) rather than that in Eq. (415). This solution is real solution.

The most important is that the real kinematics of light propagation has been worked out.

9.5.2. Detailed Discussions on the Coordinates of the Light Ray Trajectory

The trajectories of light ray in gravitational fields have more details behind the previous solution. More discussions may help to discover more realities. Further discussions are going to be sponsored to make more detailed analysis for understanding of two issues, positions and angles.

The first one is to recognize the various kinds of lines. The most important line is the real light ray that is emitted from a farthest star to the observers on the Earth. This line should be curved as it goes closely to the Sun. Prolonging the straight parts of the light ray, one will gain the crossed straight lines, the asymptotic lines of light trajectory. They will be parallel to the two radial coordinate vectors r_∞ left and r_∞ right. There is another important line is the straight line from farthest star to the observer, that will present the real star direction from observer to the star.

The second one is to recognize those angles. φ_∞ is the second coordinate of farthest point on the light ray. Because the Earth is far enough to the Sun, the elevation angle of observer view line could be seen as φ_∞. And because the star is very very far from the Sun, the angle between straight line to the star and horizontal line could be also seen as φ_∞. Thus, the total deflect angle of $\Delta_{Observe}$ is approximate $2\varphi_\infty$. They could be shown in Figure 21.

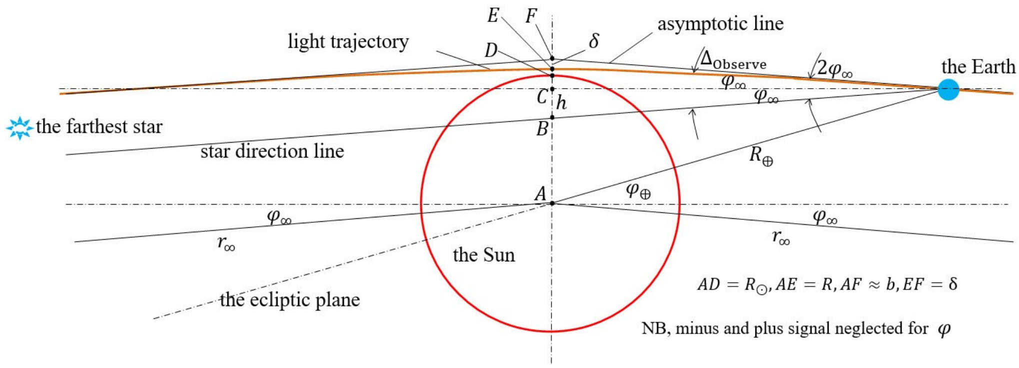

Figure 21. Detailed relationships for angles of light rays and view lines and coordinates.

For the coordinate of the Earth, $u = 1/A_u$, where A_u is astronomical unit, the simple solution could be used

$$\frac{1}{A_u} \approx \frac{\varphi_\oplus}{b} \quad (473)$$

where, $\varphi_\oplus$ is second coordinate of the Earth.

Thus, there is

$$\varphi_\oplus \approx \frac{b}{A_u} \quad (474)$$

Because we know that $b \approx R_\odot$, where $R_\odot$ is solar radius, so that

$$\varphi_\oplus \approx 0.004654478 \text{ rad} \quad (475)$$

The height of the horizon line to light ray

$$CE = R - A_u sin\varphi_\oplus \quad (476)$$

It is difficult to get a not bad accurate solution. However, we can turn to discuss the value of $h = BC$ instead. For the approximate of $\varphi_\infty \approx \frac{1}{2}\Delta_{\text{observer}}$, it could be gain

$$h \approx A_u sin\varphi_\infty \approx 635 \text{ km} \quad (477)$$

in the upper height $h = CF$.

There is a very small difference between real ray trajectory and asymptotic line, the $\delta = EF$. For the condition that light rays run from farthest position to the position they pass by the Sun, there is the equation after angular momentum conservativeness

$$(1 + \frac{r^*}{2r_\infty})b = (1 + \frac{r^*}{2R})R \quad (478)$$

NB, b is perpendicular length to asymptotic line, which is the moment distance for farthest positions, and only in approximate cases, it could be seen as $R + \delta$. R is peak point radius, and it need not be determined to be $R_\odot$ in these discussions theoretically.

Then it is easy to obtain

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$$b = R + \frac{r^*}{2} \quad (479)$$

Peak difference between R and b is

$$\delta = \frac{r^*}{2} = 1.477 \text{ km} \quad (480)$$

It is more difficult to investigate such a fine distance in practice that not only because of the observational accuracy but also due to the coordinating of the peak point of light ray. The development of very-long-baseline interferometry have the capability of measuring angular separations and changes in angles as small as 10^{-4} seconds of arc [23]. That shows probabilities for the quite good accuracy for fine angle measuring. This issue perhaps cannot be solved easily.

At the farther positions of a light ray, it could be proposed that $cos\varphi \approx 1$ and $cos2\varphi \approx 1$, so that the Eq. (466) could be rewritten in approximate forms as

$$\frac{1}{r} = \frac{1}{b} sin\varphi + \frac{r^*}{b^2} cos\varphi \quad (481)$$

or

$$b = r(sin\varphi + \frac{r^*}{b} cos\varphi) \quad (482)$$

Because $b \gg r^*$, furtherly write an approximate equation for Eq. (482) that

$$b = rsin(\varphi + \frac{r^*}{b}) \approx rsin(\varphi + \frac{r^*}{R}) \quad (483)$$

That is the asymptotic equation and we have invented the method to create asymptotic equations. It is a straight line with a distance b to the source center.

We can prolong the straight line to $\varphi = \frac{\pi}{2}$, where we will see that the top point is just a little bit higher than horizontal line, as shown in Figure 22, that the top point coordinate is

$$r = b/sin(\frac{\pi}{2} + \frac{r^*}{b}) \quad (484)$$

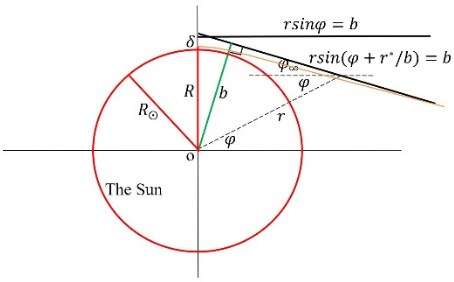

Figure 22. Asymptotic line with an overplayed φ_∞.

It is easy to verify in Figure 22 that the Eq. (483) is the function of a straight line.

In fact, the Eq. (481) and Eq. (482) both are of the equation of asymptotic line. They could also be written as

$$r = \frac{b}{sin\varphi + r^* cos\varphi / b} \quad (485)$$

It should be mentioned that the equation $u = sin\varphi/b$ or $rsin\varphi = b$ is a horizontal line with a distance b to the Sun center.

For the case that a light passes closely by the edge of the sun, there might be some influences from solar corona. It is a good idea to left a distance from the edge, for example, the position of $1.5R_\odot$ or even further.

9.5.3. A Wrong Treatment for Light Propagations

If the angular momentum conservativeness would be insisted for light propagations as

$$L = Rm_{\rm R}c = r^2 m_r \frac{d\varphi}{dt} \quad (486)$$

It is seemingly that $\frac{d\varphi}{dt}$ presents the contra variant angular velocity of photons which is expected to correspond to contra variant angular momentum. And moreover, it could be argued to take $Rm_{\rm R}c_0^{\mu}$ as expressions of contra variant angular momentum, that will exactly lead to conservation break as gravitational redshift is concerned.

Now for the equation Eq. (486), there is

$$\frac{d\varphi}{dt} = \frac{Rm_{\rm R}c}{m_r r^2} = \frac{Rc}{r^2}\frac{1+\frac{r^*}{2R}}{1+\frac{r^*}{2r}} \quad (487)$$

Set $B = (R + \frac{r^*}{2})$. In weak field, the item $1/(1 + \frac{r^*}{2r}) \approx (1 - \frac{r^*}{2r})$, and $(1 - \frac{r^*}{2r})^2 \approx 1 - \frac{r^*}{r}$, thus

$$\frac{d\varphi}{dt} = \frac{(1-\frac{r^*}{2r})Bc}{r^2} \quad (488)$$

Because the Lagrangian with coordinate time is

$$-c^2 + (1 - \frac{r^*}{r})^{-2}(\frac{dr}{dt})^2 + (1 - \frac{r^*}{r})^{-1} r^2 (\frac{d\varphi}{dt})^2 = 0 \quad (489)$$

It could be transformed to be

$$c^2 - (1 - \frac{r^*}{r})^{-1} r'^2 \frac{B^2c^2}{r^4} - \frac{B^2c^2}{r^2} = 0 \quad (490)$$

or

$$(1 - \frac{r^*}{r}) - r'^2 \frac{B^2}{r^4} - (1 - \frac{r^*}{r})\frac{B^2}{r^2} = 0 \quad (491)$$

or

$$(1 - r^*u) - u'^2B^2 - (1 - r^*u)u^2B^2 = 0 \quad (492)$$

To be derived to be

$$u'' + u = \frac{3}{2}r^*u^2 - \frac{r^*}{2B^2} \quad (493)$$

Considering $b \approx B$, it could be solved as

$$u = \frac{sin\varphi}{b} + \frac{r^*}{4b^2} + \frac{r^*}{4b^2}cos2\varphi \quad (494)$$

One will obtain that

$$\varphi_\infty = -\frac{r^*}{2b} \quad (495)$$

One can verify the peak point on the case of $\varphi = \pi/2$, and $r = R$, and then will gain

$$b = R \quad (496)$$

It is obvious that this solution of b cannot keep the conservation of angular momentum. The main reason is that the angular velocity defined in the angular momentum is not the direct component of light speed in Eq. (489). As a comparison, that in Eq. (457) is. Naturally, the solution Eq. (494) is a wrong answer for light ray propagations after a wrong setting.

9.6. Trajectories of Massive Matters

9.6.1. Discussions on Lagrangian

It is believed in classical theory that Lagrangian for massive matters have a special form [3,10] that others to that of light rays that

$$\mathcal{L} = -(1 - \frac{r^*}{r})\dot{t}^2c^2 + (1 - \frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = -c^2 \quad (497)$$

As has discussed that for the first item in Eq. (497) $\dot{t}^2=(dt/d\tau)^2=g^{00}=(1-\frac{r^*}{r})^{-1}$, it will naturally lead to

$$-c^2+(1-\frac{r^*}{r})^{-1}\dot{r}^2+r^2\dot{\varphi}^2=-c^2 \quad (498)$$

so that

$$(1-\frac{r^*}{r})^{-1}\dot{r}^2+r^2\dot{\varphi}^2\equiv 0 \quad (499)$$

It is completely unacceptable that the covariant velocity results in absolute zero. But we know that the two settings of Eq. (415) and Eq. (416) seem to have been employed to solved the problem. Consequently, the Eq. (497) will be then reformed with them that

$$(1-\frac{r^*}{r})^{-1}c^2E^2-(1-\frac{r^*}{r})^{-1}\dot{r}^2-\frac{L^2}{r^2}=c^2 \quad (500)$$

It is the traditional form of dynamic equation for massive matters. Of course, it is of fantastic expressions of no sense.

On another side, even the original Lagrangian cannot work for massive matters

$$\mathcal{L}=-c^2+(1-\frac{r^*}{r})^{-1}\dot{r}^2+r^2\dot{\varphi}^2=0 \quad (501)$$

in that for massive matters we know that the last two items are definitely unequal to c^2 in most cases.

In fact, you know there is no more appropriate dynamic equation for massive matters better than that

$$(V_0^s)^2-(\frac{dr}{dt})^2-r^2(\frac{d\varphi}{dt})^2=0 \quad (502)$$

where, V_0^s is modulus of space velocity of massive matters.

The Eq. (502) really performs velocity composition rather than the Eq. (500) and Eq. (501). The covariant velocity composition could be shown as

$$(Vs/0)^2=\dot{\rho}^2+r^2\dot{\varphi}^2 \quad (503)$$

But it is difficult to be employed for dynamic equations, in that the velocity $\boldsymbol{V}s/0$ is not conservative quantity so that it cannot be employed in expressions of kinetic energy.

As has discussed above, it is easily to carried out the momentum equation for massive matters that

$$P^2={m_r}^2(\frac{dr}{dt})^2+{m_r}^2(\frac{rd\varphi}{dt})^2 \quad (504)$$

and the angular momentum equations for massive matters that

$$L=Rm_{\mathrm{R}}V_0^s=r^2m_r\frac{d\varphi}{dt} \quad (505)$$

where, $V_0^s=V_0^\varphi$ at the peak point.

9.6.2. Traditional Methodology for Planet Orbits

As the two settings of Eq. (415) and Eq. (416) adopt for the Lagrangian Eq. (497), there will be

$$c^2E^2-\dot{r}^2-(1-\frac{r^*}{r})\frac{L^2}{r^2}=(1-\frac{r^*}{r})c^2 \quad (506)$$

One could find that the main item of c^2 might sounds literally unreasonable, although the problem will not bring about great obstacles in process of calculation, because first items in Eq. (506) will be eliminated by a further derivation. Never mind, we will promptly pick up the problem in the issues of time delay in next section.

Set $u=1/r$ and with $\dot{r}=\frac{dr}{d\varphi}\frac{d\varphi}{d\tau}$ and $r'=\frac{dr}{d\varphi}=-\frac{u\prime}{u^2}$ it could be reformed as

$$\frac{c^2E^2}{L^2}-{u'}^2-(1-r^*u)u^2=(1-r^*u)\frac{c^2}{L^2} \quad (507)$$

To make further derivation, there is

$$u''+u=\frac{3}{2}r^*u^2+\frac{r^*c^2}{2L^2} \quad (508)$$

For a planet orbit, the so-called angular momentum $L=RV\varphi/0\ll Rc$, so that in right side of the equation the item $\frac{3}{2}r^*u^2$ is a very small quantity than that of $\frac{r^*c^2}{2L^2}$, where, $V\varphi/0=r\frac{d\varphi}{d\tau}$ in traditional settings. Approximation of the equation could be

$$u''+u=\frac{r^*c^2}{2L^2} \quad (509)$$

It is the famous Binet equation. That has a solution

$$u=A(1-esin\varphi) \quad (510)$$

where, $A = \frac{r^*c^2}{2L^2}$ and e is orbit eccentricity.

The item $\frac{3}{2}r^*u^2$ could be replaced by the solution approximately that

$$\frac{3}{2}r^*u^2 = \frac{3}{2}r^*A^2(1-esin\varphi)^2 = \frac{3}{2}r^*A^2(1-2esin\varphi+e^2sin^2\varphi) \quad (511)$$

Then, the Eq. (508) becomes

$$u''+u = A+\frac{3}{2}r^*A^2-3r^*A^2esin\varphi+\frac{3}{2}r^*A^2e^2sin^2\varphi \quad (512)$$

Considering the conditions of very small eccentricity, in right side, the last item as a variable in absolute value is far less than the third item, as well as the second item as constant is far less than the first item. There will be an approximate form of the equation

$$u''+u \approx A-3r^*A^2esin\varphi \quad (513)$$

We could try to solve the split equations of Eq. (513) of small eccentricity that

$$u''+u = A \quad (514)$$

with

$$u_1 = A(1-esin\varphi) \quad (515)$$

and the other equation

$$u''+u = -3r^*A^2esin\varphi \quad (516)$$

with

$$u_2 = \frac{3}{2}r^*A^2e\varphi cos\varphi \quad (517)$$

There is the general solution

$$u = u_1+u_2 = A(1-esin\varphi+\frac{3}{2}r^*Ae\varphi cos\varphi) \quad (518)$$

where $\Delta\varphi = -\frac{3}{2}r^*Ae\varphi$ as absolute value is far less than φ. By Taylor equation or sine equation of two angles, we have

$$\sin(\varphi+\Delta\varphi) \approx sin\varphi+\Delta\varphi cos\varphi \quad (519)$$

so that

$$u \approx A-Aesin[(1-\frac{3}{2}r^*A)\varphi] \quad (520)$$

This equation presents the perihelion precession that

$$\Delta\varphi = \frac{3}{2}r^*A\varphi \quad (521)$$

Thus, the precession in a revolution

$$\Delta\varphi_{2\pi} = 3r^*A \quad (522)$$

as shown in Figure 23

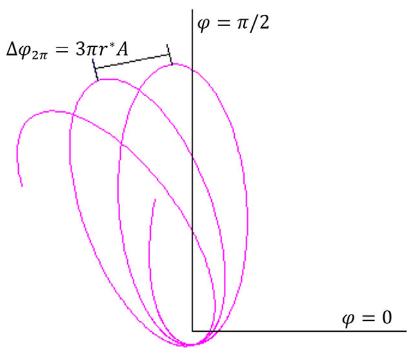

Figure 23. The trajectory of planet.

It should be noted, that there will be more conditions in which the Eq. (513) might be invalid in that the orbits may have greater eccentricities, such as some of asteroids and comets in solar system would be. In the case of greater eccentricities, one must entirely solve the Eq. (512) to gain more precise solutions. In fact, for the orbit of Mercury, it has an eccentricity of about 0.206 which implies a considerable deviation from the Eq. (513).

Neglecting the very small constant item in Eq. (512), it could be rewritten as

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$$u'' + u \approx A - 3r^*A^2 e sin\varphi + \frac{3}{2}r^*A^2e^2sin^2\varphi \quad (523)$$

We could try to solve the split equations of Eq. (523) that the first one

$$u'' + u = A \quad (524)$$

with a solution

$$u_1 = A(1 - esin\varphi) \quad (525)$$

and the second one

$$u'' + u = -3r^*A^2esin\varphi \quad (526)$$

with

$$u_2 = \frac{3}{2}r^*A^2e\varphi cos\varphi \quad (527)$$

and the third one

$$u'' + u = \frac{3}{2}r^*A^2e^2sin^2\varphi \quad (528)$$

As the same form of the Eq. (465) with the solution Eq. (466), a particular solution of Eq. (528) could be drawn as

$$u_3 = \frac{3}{4}r^*A^2e^2[1 + \frac{1}{3}\cos(2\varphi)] = \frac{1}{2}r^*A^2e^2 + \frac{1}{2}r^*A^2e^2cos^2\varphi \quad (529)$$

Thus, we could give the general solution of Eq. (523) as

$$u = A(1 - esin\varphi) + \frac{3}{2}r^*A^2e\varphi cos\varphi + \frac{1}{2}r^*A^2e^2 + \frac{1}{2}r^*A^2e^2cos^2\varphi \quad (530)$$

Because the last three items in right hand side are all small items than the first, neglecting the small constant item of third one, it is easy to gain an approximate solution as

$$u \approx A - Aesin(\varphi - \frac{3}{2}r^*A\varphi + \frac{1}{2}r^*Ae\cos\varphi) \quad (531)$$

This equation presents the orbit precession that

$$\Delta\varphi = \frac{3}{2}r^*A\varphi - \frac{1}{2}r^*Ae\cos\varphi \quad (532)$$

We have gained more precise solution with an additional orbit precession, which could be called in-revolution-vibrating precession.

The orbit precession could be rewritten as

$$\Delta\varphi = \Delta\varphi_1 + \Delta\varphi_2 \quad (533)$$

in which the main precession is

$$\Delta\varphi_1 = \frac{3}{2}r^*A\varphi \quad (534)$$

and the in-revolution-vibrating precession is

$$\Delta\varphi_2 = -\frac{1}{2}r^*Ae\cos\varphi \quad (535)$$

Thus, the orbit precessions in a number n revolution at specific coordinate φ could be calculated as

$$\Delta\varphi_{(2n+1/2)\pi} = (3n + 0.75)\pi r^*A \quad (536)$$

$$\Delta\varphi_{(2n+1)\pi} = (3n + 1.5)\pi r^*A + \frac{1}{2}r^*Ae \quad (537)$$

$$\Delta\varphi_{(2n+3/2)\pi} = (3n + 2.25)\pi r^*A \quad (538)$$

$$\Delta\varphi_{(2n+2)\pi} = (3n + 3)\pi r^*A + \frac{1}{2}r^*Ae \quad (539)$$

We have seen that the in-revolution-vibrating precessions are fixed on specific directions, so that the maximum precession may not happen at perihelion or aphelion. As aphelion of revolutions go from $\frac{\pi}{2}$ to π, their aphelion in-revolution-vibrating precessions increase from zero to maximum, as shown in Figure 24.

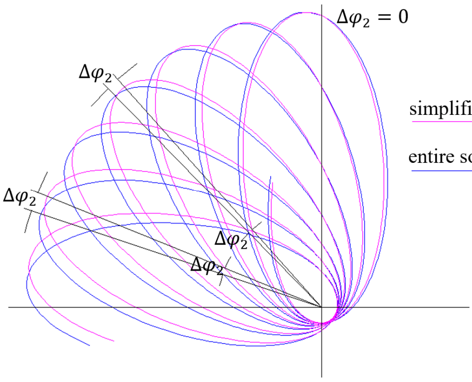

Figure 24. Comparison of simplified solution and entire solution in the second quadrant.

Surely, in-revolution-vibrating precessions are very small for planet orbits in solar system. However, they could be more obvious for motions in strong fields.

The most interesting is that the solution of in-revolution-vibrating precessions may have caused symmetry break. We can figure out the giant circulation as aphelion advances up to an entire circle, shown as Figure 25. One can find out that the aphelion of revolutions goes densified at positions near to $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, and goes loosened at positions near to π and 2π. It is a break of geometrical symmetry, because physical realities of one source field do not provide inhomogeneous conditions. Maybe the geometrical symmetry break caused by classical methodology is a self-falsification. If the methodology is reliable and its solution is correct, one can observe in-revolution-vibrating precessions in observations on strong field motions.

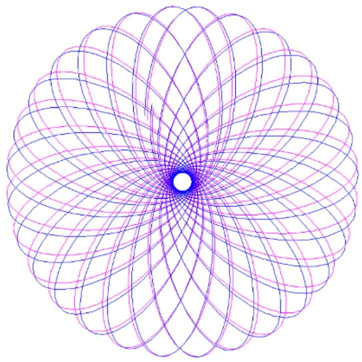

Figure 25. Comparison of precessions in a giant circulation.

9.6.3. Traditional Treatment for Close-to-Light-Speed Particles

Studies on the motion of close-to-light-speed particles in gravitational fields rarely have been seen in any publications probably due to sophisticated situations. I decide to study the issue firstly on traditional methodology is because that more faults would be found out in the process that may forcefully support the conclusions in section 8.

It is believed in traditional methodologies that the Lagrangian Eq. (497) also could be applied in the dynamics of the motion of close-to-light-speed particles that

$$\mathcal{L} = -(1-\frac{r^*}{r})\dot{t}^2c^2 + (1-\frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = -c^2 \quad (540)$$

So as well, the differential equation will be gained as same as Eq. (508) that

$$u'' + u = \frac{3}{2}r^*u^2 + \frac{r^*c^2}{2L^2} \quad (541)$$

But it should be highlighted that the Eq. (541) will be far different from Eq. (508) in that the angular momentum in Eq. (541) is $L = Rc$ which is far greater than that in Eq. (508). Thus, one will see that the two items in Eq. (541) is in same order of magnitudes while the first item in Eq. (508) is a very small quantity with respect to the second one.

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Using light speed c to perform close-to-light-speed is an approximate treatment in numerical analysis.

We know that the trajectory is very close to the horizontal line $u = sin\varphi/b$ at positions that r is not very farther than R. Otherwise at positions that r is very farther than R, the item $\frac{3}{2}r^*u^2$ will go to zero. Thus, the equation could be reformed to be

$$u'' + u = \frac{3r^*}{2b^2}sin^2\varphi + \frac{r^*c^2}{2L^2} \quad (542)$$

The reform will not greatly influence the result of the original equation in that in a little farther area $\frac{3r^*}{2b^2}sin^2\varphi$ together with $\frac{3}{2}r^*u^2$ both are very small with respect to $\frac{r^*c^2}{2L^2} = \frac{r^*}{2R^2}$.

There is the solution

$$u = \frac{sin\varphi}{b} + \frac{5r^*}{4b^2}(1 + \frac{1}{5}cos2\varphi) \quad (543)$$

And then the deflection angle

$$\varphi_\infty = -\frac{3r^*}{2b} \quad (544)$$

Case $\varphi = \frac{\pi}{2}$, there is

$$\frac{1}{R} = \frac{1}{b} + \frac{r^*}{b^2} \quad (545)$$

Because $b \approx R \gg r^*$ it becomes

$$R \approx b - r^* \quad (546)$$

or

$$b = R + r^* \quad (547)$$

The asymptotic line will be

$$r = \frac{b}{sin\varphi + \frac{3r^*}{2b}cos\varphi} \quad (548)$$

To verify the issue of momentum conservativeness that

$$L_R = Rc \quad (549)$$

At a position $r \gg R$

$$L_b = bc = (R + r^*)c \neq L_R = Rc \quad (550)$$

The deflection angle seems something strange. The peak difference $\delta = r^*$ shows un-conservative of angular momentum. This result is not far beyond expectation in that this traditional solution is involved with errors. We will see more detailed problems on the trajectory in the study of corresponding time spending in next sub-sections.

9.6.4. Renovated Equations for Close-to-Light-Speed Massive Particles

We have got the conclusion that massive particles will not run with general covariance. They run with Newtonian laws. The velocity composition for close-to-light-speed massive particles really is

$$-c^2 + (\frac{dr}{dt})^2 + r^2(\frac{d\varphi}{dt})^2 = 0 \quad (551)$$

The conservativeness of angular momentum is as discussed previous

$$L = Rm_R c = r^2 m_r \frac{d\varphi}{dt} \quad (552)$$

One will find that the momentum defined is rare different from that of light rays. The constant c is not the speed of light while it really is the approximate speed of close-to-light-speed massive matter. So that the c in Eq. (551) is just a velocity composition of close-to-light-speed motion. The Eq. (551) and Eq. (552) will perform the differences of conservations of motions of massive matter from that of light fly. We will see these differences in the equations of low velocity motions of massive matters in next sub-sections.

The angular velocity

$$\frac{d\varphi}{dt}=\frac{Rm_{R}c}{m_{r}r^{2}}=\frac{Rc}{r^{2}}\frac{1+\frac{r^{*}}{2R}}{1+\frac{r^{*}}{2r}} \quad (553)$$

Set $B=R+\frac{r^{*}}{2}$ and in weak field, the item $\frac{1}{1+\frac{r^{*}}{2r}}\approx 1-\frac{r^{*}}{2r}$, and $(1-\frac{r^{*}}{2r})^{2}\approx 1-\frac{r^{*}}{r}$. Thus,

$$\frac{d\varphi}{dt}=\frac{1}{r^{2}}(1-\frac{r^{*}}{2r})Bc \quad (554)$$

Together with $\frac{dr}{dt}=\frac{dr}{d\varphi}\frac{d\varphi}{dt}=r'\frac{d\varphi}{dt}$, the Eq. (551) turns to be

$$c^{2}-(1-\frac{r^{*}}{r})r'^{2}\frac{B^{2}c^{2}}{r^{4}}-(1-\frac{r^{*}}{r})\frac{B^{2}c^{2}}{r^{2}}=0 \quad (555)$$

or

$$(1+\frac{r^{*}}{r})-r'^{2}\frac{B^{2}}{r^{4}}-\frac{B^{2}}{r^{2}}=0 \quad (556)$$

With $u=1/r$, and $r'=dr/d\varphi=-u'/u^{2}$, there is

$$(1+r^{*}u)-u'^{2}B^{2}-u^{2}B^{2}=0 \quad (557)$$

To take further derivation on, it turns to

$$u''+u=\frac{r^{*}}{2B^{2}} \quad (558)$$

It has a same form with Binet equation for planet orbit, the Eq. (509), for which the most special is that this equation is created for the matters with close-to-light-speed, while the latter is for lower speed matters.

Considering $b\approx B$, it could be solved as

$$u=\frac{sin\varphi}{b}+\frac{r^{*}}{2b^{2}} \quad (559)$$

Obviously, there is

$$\varphi_{\infty}=-\frac{r^{*}}{2b} \quad (560)$$

It is easy to calculate in Eq. (559) that at the peak point, $\varphi=\pi/2$, and $r=R$, so that

$$b=R+\frac{r^{*}}{2} \quad (561)$$

or with angular momentum conservation there is

$$(1+\frac{r^{*}}{2r_{\infty}})b=(1+\frac{r^{*}}{2R})R \quad (562)$$

Again, we obtain

$$b=R+\frac{r^{*}}{2} \quad (563)$$

The asymptotic line

$$r=\frac{b}{sin\varphi+r^{*}cos\varphi/b/2} \quad (564)$$

It is said that the trajectories of light rays and close-to-light-speed particles present different φ_{∞} but they present the same peak difference δ. The same peak difference is because of the result of momentum conservation.

In fact, these studies have help to create the dynamics of close-to-light-speed particles in gravitational field. By the way, one can compare the solution Eq. (559) with the Eq. (543) in section

9.6.3 that they have given far different result of φ_∞ and different value of b, that indicates errors involved in the Eq. (541).

9.6.5. General Equations for Massive Matters

Let us investigate a motion of a massive matter from any specific position R to another position r in one source fields. A matter moving in a gravitational field will be simplified to be a mass point. Considering a variable dynamic mass m_r, the gravity

$$F = -\frac{GMm_r}{r^2} = -m_r c^2 \frac{r^*}{2r^2} \quad (565)$$

Dynamic energy converted from gravitational potential by a free motion from R to r is

$$E_{conv} = -\int_R^r m_\rho c^2 \frac{r^*}{2\rho^2} d\rho \quad (566)$$

and the total energy

$$E = m_r c^2 = m_R c^2 + E_{conv} \quad (567)$$

where, dynamic mass m_r and m_R correspond to possible velocities.

The function of total energy could be written as

$$m_r c^2 = m_R c^2 - \int_R^r m_\rho c^2 \frac{r^*}{2\rho^2} d\rho \quad (568)$$

Setting energy at position R as a known quantity, this equation could be derived to be

$$m_r' = -m_r \frac{r^*}{2r^2} \quad (569)$$

It could be integrated to be

$$ln\, m_r - ln\, m_R = \frac{r^*}{2r} - \frac{r^*}{2R} \quad (570)$$

Then we gain the expression of variable mass that

$$m_r = m_R e^{(\frac{r^*}{2r} - \frac{r^*}{2R})} = m_\infty e^{\frac{r^*}{2r}} \quad (571)$$

where, m_r and m_R are dynamic mass corresponding to specific velocities, and m_∞ is defined as an imaginary concept corresponding to a reduction mass at a farthest position. This equation is general expression of mass variation with respect to the simplified expression in Eq. (441). Particularly for massive matters in some cases the mass m_∞ could have imaginary velocity. This equation could be called general mass equation in gravitational fields.

And we would like to carry out the expanded expression of gravity equation that

$$F = -m_r c^2 \frac{r^*}{2r^2} = -m_R c^2 \frac{r^*}{2r^2} e^{(\frac{r^*}{2r} - \frac{r^*}{2R})} = -m_\infty c^2 \frac{r^*}{2r^2} e^{\frac{r^*}{2r}} \quad (572)$$

One could find that the relativistic mass may not keep mass conservation any more that may surprise us, but it really has been revealed in realities. This is another kind of comprehensive physics which I will not make further discussions in this section. Mass of matter does matter [24]. The truths stay in realities.

Dynamic energy will be converted during a moving of a matter that

$$E_{conv} = m_r c^2 - m_R c^2 = m_R c^2 [e^{(\frac{r^*}{2r} - \frac{r^*}{2R})} - 1] \quad (573)$$

It should be highlighted that the expression of exponential function of mass variation of course could be used for light ray propagations and close-to-light-speed massive particles for higher accuracy analysis case in strong fields.

For massive matters, with special relativity, the equation of dynamic energy at a position r is

$$E_k = \xi m_r V_r^2 \quad (574)$$

where, $\xi = (1-\sqrt{1-\frac{V_r^2}{c^2}})\frac{c^2}{V_r^2}$, see Eq. (383).

In gravitational field the dynamic energy varies with positions that

$$E_k = \xi_r m_r V_r^2 = \xi_R m_R V_R^2 + E_{conv} = \xi_R m_R V_R^2 + m_R c^2[e^{(\frac{r^*}{2r}-\frac{r^*}{2R})} - 1] \quad (575)$$

Conversion energy E_{conv} could be positive case potential release or negative case potential withdrawn. However, dynamic energy E_k will be always greater than zero, so that the variable r will be limited in some specific cases that depends on initial conditions.

NB, for the convenience of expressions, the tensors of velocities, frequencies or the components maybe not written in tensor format anymore case they may not bring about confusions for understandings, for examples, V_r and V_R refer to the velocity at position r and R.

It should be further discussed here that we have seen dynamic mass energy may come from the release of gravitational potential. That will then perform as inertial mass and gravitational mass. If in two source system, they move closer or farther will cause mass increase or lose in that we incline to realize that potential does not act as mass. We are not sure that in this condition mass conservativeness is available or not. I am inclining to say no. Maybe this discussion involves with new physics. In this section, this controversy does not really matter. This discussion just presents the issue for more concerns.

With Eq. (571) and Eq. (575), velocity square is obtained that

$$V_r^2 = \frac{\xi_R}{\xi_r} e^{(\frac{r^*}{2R}-\frac{r^*}{2r})} V_R^2 + \frac{1}{\xi_r} e^{(\frac{r^*}{2R}-\frac{r^*}{2r})}[e^{(\frac{r^*}{2r}-\frac{r^*}{2R})} - 1]c^2$$
$$= \frac{\xi_R}{\xi_r} e^{(\frac{r^*}{2R}-\frac{r^*}{2r})} V_R^2 + \frac{1}{\xi_r}[1 - e^{(\frac{r^*}{2R}-\frac{r^*}{2r})}]c^2 \quad (576)$$

Case the velocity V_r in a section of the trajectory is close to light speed so that $\xi_r \approx \xi_R \approx 1.0$, there will be

$$V_r^2 \approx e^{(\frac{r^*}{2R}-\frac{r^*}{2r})} c^2 + [1 - e^{(\frac{r^*}{2R}-\frac{r^*}{2r})}]c^2 = c^2 \quad (577)$$

where, R is a known position in the section.

Obviously, this condition will lead to the dynamics of close-to-light-speed particles discussed in section 9.6.4, but we will find that there could be a little different that the variable mass could be written in more general forms than previous.

With angular momentum conservation, there is

$$L = r^2 m_r \frac{d\varphi}{dt} = Rcm_R = const. \quad (578)$$

where, R is radius of the peak point and c is the corresponding velocity.

With mass energy conservation, there is

$$\frac{d\varphi}{dt} = \frac{\mathrm{Rc}}{r^2}\frac{m_R}{m_r} = \frac{\mathrm{Rc}}{r^2} e^{(\frac{r^*}{2R}-\frac{r^*}{2r})} \quad (579)$$

We have seen that the conservativeness is far different from that of light rays.

The velocity composition for close-to-light-speed particles could be expressed as

$$-c^2 + (\frac{dr}{dt})^2 + r^2(\frac{d\varphi}{dt})^2 = 0 \quad (580)$$

With $\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = r'\frac{d\varphi}{dt}$, it becomes

$$-c^2 + (r')^2 \frac{1}{r^4} e^{(\frac{r^*}{R}-\frac{r^*}{r})} R^2 c^2 + \frac{1}{r^2} e^{(\frac{r^*}{R}-\frac{r^*}{r})} R^2 c^2 = 0 \quad (581)$$

or

$$-e^{(\frac{r^*}{r}-\frac{r^*}{R})} + (r')^2 \frac{1}{r^4} R^2 + \frac{1}{r^2} R^2 = 0 \quad (582)$$

Setting $u = 1/r$, it is

$$-e^{(ur^*-\frac{r^*}{R})} + (u')^2 R^2 + u^2 R^2 = 0 \quad (583)$$

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After a derivation there is

$$-\frac{r^*}{2R^2}e^{(ur^*-\frac{r^*}{R})}+u''+u=0 \quad (584)$$

It could be solved for close-to-light-speed motions in strong fields. For the condition that $R \gg r^*$ and $r \gg r^*$, the exponent item could be treated as

$$e^{ur^*-\frac{r^*}{R}} \approx 1+(ur^*-\frac{r^*}{R})e^{ur^*-\frac{r^*}{R}}+\frac{1}{2}(ur^*-\frac{r^*}{R})^2 e^{ur^*-\frac{r^*}{R}}$$
$$\approx 1+(ur^*-\frac{r^*}{R})(1+ur^*-\frac{r^*}{R})+\frac{1}{2}(ur^*-\frac{r^*}{R})^2(1+ur^*-\frac{r^*}{R})$$
$$=1+ur^*-\frac{r^*}{R}+(ur^*-\frac{r^*}{R})^2+\frac{1}{2}(ur^*-\frac{r^*}{R})^2(1+ur^*-\frac{r^*}{R})$$
$$\approx 1+ur^*-\frac{r^*}{R}+\frac{3}{2}(ur^*-\frac{r^*}{R})^2$$
$$\approx 1+ur^*-\frac{r^*}{R}+\frac{3}{2}u^2r^{*2}+\frac{3}{2}\frac{r^{*2}}{R^2}-3u\frac{r^{*2}}{R} \quad (585)$$

Thus, the Eq. (584) could be simplified to be

$$u''+u(1-\frac{1}{2}\frac{r^{*2}}{R^2}+\frac{3}{2}\frac{r^{*3}}{R^3})=\frac{3}{4}\frac{r^{*3}}{R^2}u^2+\frac{r^*}{2R^2}-\frac{1}{2}\frac{r^{*2}}{R^3}+\frac{3}{4}\frac{r^{*3}}{R^4} \quad (586)$$

If the higher small items being deleted, it is

$$u''+u(1-\frac{1}{2}\frac{r^{*2}}{R^2})=\frac{r^*}{2R^2}-\frac{1}{2}\frac{r^{*2}}{R^3} \quad (587)$$

Set

$$\omega^2=1-\frac{1}{2}\frac{r^{*2}}{R^2} \quad (588)$$

Case $R \gg r^*$,

$$\omega=1-\frac{1}{4}\frac{r^{*2}}{R^2} \quad (589)$$

It could be solved approximately as

$$u=\frac{sin\omega\varphi}{b}+\frac{r^*}{2R^2}-\frac{1}{2}\frac{r^{*2}}{R^3} \quad (590)$$

As r is very big and φ is very small there is $sin\omega\varphi \approx \omega\varphi$, so that

$$0=\frac{\omega\varphi_\infty}{b}+\frac{r^*}{2R^2}-\frac{1}{2}\frac{r^{*2}}{R^3} \quad (591)$$

Thus,

$$\varphi_\infty=-\frac{1}{\omega}(\frac{r^*}{2R}-\frac{1}{2}\frac{r^{*2}}{R^2}) \quad (592)$$

It should be highlighted that the peak point is at the coordinate angle $\frac{\pi}{2\omega}$ rather than $\frac{\pi}{2}$, as shown in Figure 26, so that the half branch deflect angle is not the angle φ_∞. The real half branch deflection could be calculated as

$$\frac{1}{2}\Delta=\varphi_\infty-\frac{r^{*2}}{4R^2}\frac{\pi}{2}$$
$$=-\frac{1}{\omega}(\frac{r^*}{2R}-\frac{1}{2}\frac{r^{*2}}{R^2})-\frac{\pi r^{*2}}{8R^2}$$
$$\approx -\frac{r^*}{2R}+(\frac{1}{2}-\frac{\pi}{8})\frac{r^{*2}}{R^2} \quad (593)$$

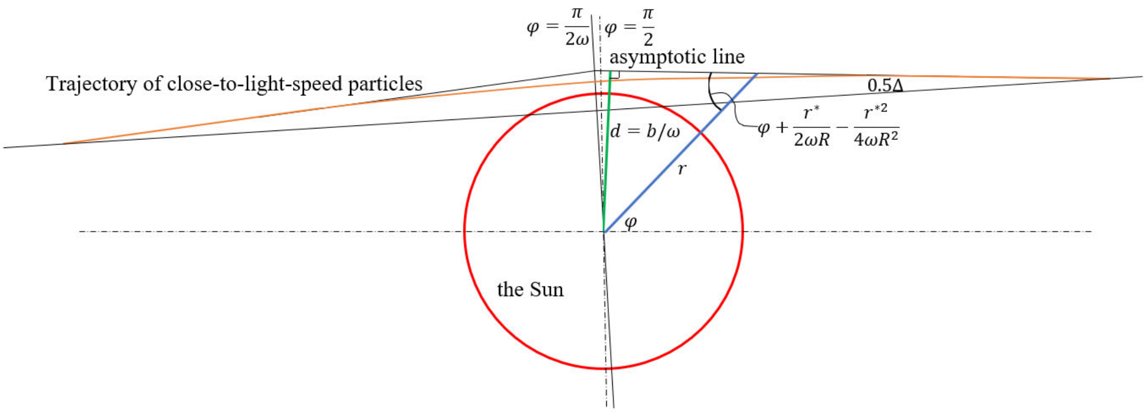

Figure 26. Close-to-light-speed particles passing across the Sun.

As $\omega\varphi = \frac{\pi}{2}$, there is

$$\frac{1}{R} = \frac{1}{b} + \frac{r^*}{2R^2} - \frac{1}{2}\frac{r^{*2}}{R^3} \quad (594)$$

Case $R \gg r^*$, there is

$$b = \frac{1}{\frac{1}{R} - \frac{r^*}{2R^2} + \frac{1}{2}\frac{r^{*2}}{R^3}}$$

$$= \frac{R(1 + \frac{r^*}{2R} - \frac{1}{2}\frac{r^{*2}}{R^2})}{1 - (\frac{r^*}{2R} - \frac{1}{2}\frac{r^{*2}}{R^2})^2}$$

$$= \frac{R(1 + \frac{r^*}{2R} - \frac{1}{2}\frac{r^{*2}}{R^2})}{1 - \frac{r^{*2}}{4R^2} - \frac{r^{*4}}{4R^4} + \frac{1}{2}\frac{r^{*3}}{R^3}}$$

$$\approx R(1 + \frac{r^*}{2R} - \frac{1}{2}\frac{r^{*2}}{R^2})(1 + \frac{r^{*2}}{4R^2})$$

$$\approx R(1 + \frac{r^*}{2R} - \frac{1}{4}\frac{r^{*2}}{R^2}) \quad (595)$$

We have seen that this solution is a little different from that of Eq. (561) in last subsection. That is because the general mass equation has been applied so that the small items were involved in to give a more general solution. We will see that this equation could also been drawn in general ways in following discussions.

For the study of asymptotic line, the rare part of the trajectory could be considered. The Eq. (590) could be rewritten as

$$\frac{1}{r} = \frac{1}{b} sin\omega\varphi + \frac{r^*}{2R^2} - \frac{r^{*2}}{2R^3} \quad (596)$$

or

$$b = r sin\omega\varphi + r\frac{br^*}{2R^2} - r\frac{br^{*2}}{2R^3} \quad (597)$$

With $R + \frac{r^*}{2}$ instead of the constant b in right hand side, it becomes

$$b \approx r[sin\omega\varphi + (R + \frac{r^*}{2})\frac{r^*}{2R^2} - (R + \frac{r^*}{2})\frac{r^{*2}}{2R^3}]$$

$$\approx r(sin\omega\varphi + \frac{r^*}{2R} + \frac{r^{*2}}{4R^2} - \frac{r^{*2}}{2R^2})$$

$$\approx r(sin\omega\varphi + \frac{r^*}{2R} - \frac{r^{*2}}{4R^2}) \quad (598)$$

Now we are going to consider the conditions that r is very big in very far area. The equation could be performed in an approximate form as asymptotic line that

$$b = rsin(\omega\varphi + \frac{r^*}{2R} - \frac{r^{*2}}{4R^2}) \quad (599)$$

But it is not an equation of a straight line. For the farther positions, ω close to 1.0 and φ is very small, it could be rewritten as

$$b = r\omega\sin(\varphi + \frac{r^*}{2\omega R} - \frac{r^{*2}}{4\omega R^2}) \quad (600)$$

or

$$\frac{b}{\omega} = rsin(\varphi + \frac{r^*}{2\omega R} - \frac{r^{*2}}{4\omega R^2}) \quad (601)$$

This is the straight asymptotic line that could be prolonged to the overtop of the peak point. It is found that the distance b is not the asymptotic distance any more as that in Figure 26. The real asymptotic distance is

$$d = \frac{b}{\omega}$$
$$\approx R(1 + \frac{1}{4}\frac{r^{*2}}{R^2})(1 + \frac{r^*}{2R} - \frac{1}{4}\frac{r^{*2}}{R^2})$$
$$\approx R(1 + \frac{r^*}{2R}) \quad (602)$$

It seems that the angular moment conservations would be as same as has discussed previous. But it should be mentioned that in this condition the general mass equation should be considered so that the angular moment conservative equation should be presented as

$$de^{\frac{r^*}{2r\infty}} = Re^{\frac{r^*}{2R}} \quad (603)$$

In higher precisions, the perpendicular distance should be

$$d = R(1 + \frac{r^*}{2R} + \frac{1}{8}\frac{r^{*2}}{R^2}) \quad (604)$$

There is a little difference between the Eq. (602) and the Eq. (604). It could be seen as that the approximate solution has caused the difference. Of course, the latter one is of higher precisions. One can seek for even higher-precision solution for the dynamic equation to compare the perpendicular distance.

Now we will be focusing on the motions more generalized than that of close-to-light-speed, the motions including them of close-to-light-speed or not.

The velocity composition for them could be expressed as

$$-V_r^2 + (\frac{dr}{dt})^2 + r^2(\frac{d\varphi}{dt})^2 = 0 \quad (605)$$

With angular momentum conservativeness, there is

$$L = r^2 m_r \frac{d\varphi}{dt} = Rm_R V_R = const. \quad (606)$$

where, R is radius of the perihelion or peak point and V_R is the corresponding velocity and that is vertical to the radius.

Of course, the angular momentum conservation for lower velocity motion will be the same as that of close-to-light-speed massive matters.

So that

$$\frac{d\varphi}{dt} = \frac{\mathrm{R}}{r^2}\frac{m_R}{m_r}V_R = \frac{\mathrm{R}}{r^2}V_R e^{(\frac{r^*}{2R} - \frac{r^*}{2r})} \quad (607)$$

Together with the Eq. (576), the Eq. (607) and the setting of $\frac{dr}{dt} = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = r'\frac{d\varphi}{dt}$, the Eq. (605) becomes

$$-\frac{\xi_R}{\xi_r}e^{(\frac{r^*}{2R} - \frac{r^*}{2r})}V_R^2 - \frac{1}{\xi_r}[1 - e^{(\frac{r^*}{2R} - \frac{r^*}{2r})}]c^2 + (r')^2\frac{1}{r^4}e^{(\frac{r^*}{R} - \frac{r^*}{r})}R^2V_R^2 + \frac{1}{r^2}e^{(\frac{r^*}{R} - \frac{r^*}{r})}R^2V_R^2 = 0 \quad (608)$$

or

$$-\frac{\xi_R}{\xi_r}e^{(\frac{r^*}{2r}-\frac{r^*}{2R})}V_R^2-\frac{1}{\xi_r}[e^{(\frac{r^*}{r}-\frac{r^*}{R})}-e^{(\frac{r^*}{2r}-\frac{r^*}{2R})}]c^2+(r')^2\frac{1}{r^4}R^2V_R^2+\frac{1}{r^2}R^2V_R^2=0 \quad (609)$$

Setting $u=1/r$, it is

$$-\frac{\xi_R}{\xi_r}e^{(\frac{r^*}{2}u-\frac{r^*}{2R})}V_R^2-\frac{1}{\xi_r}[e^{(ur^*-\frac{r^*}{R})}-e^{(\frac{r^*}{2}u-\frac{r^*}{2R})}]c^2+(u')^2R^2V_R^2+u^2R^2V_R^2=0 \quad (610)$$

To be derived once more, there will be

$$-\frac{r^*\xi_R}{4\xi_r}e^{(\frac{r^*}{2}u-\frac{r^*}{2R})}V_R^2-\frac{r^*}{2\xi_r}[e^{(ur^*-\frac{r^*}{R})}-\frac{1}{2}e^{(\frac{r^*}{2}u-\frac{r^*}{2R})}]c^2+u''R^2V_R^2+uR^2V_R^2=0 \quad (611)$$

This is a general equation for the dynamics of any massive matters moving in gravitational fields, no matter the velocity is close-to-light-speed or un-close-to-light-speed. One can solve the equation by numerical method or linearize the exponent items with nonlinear coefficients and solve it by segments, for example, the methodology of Newtonian tangent lines.

We can yet simplify the equations in some special conditions.

Case in the conditions that $R\gg r^*$ and $r\gg r^*$, in which we know that $\frac{r^*}{R}-\frac{r^*}{r}$ and $\frac{r^*}{2R}-\frac{r^*}{2r}$ are both close to zero, the Eq. (611) could be rewritten in an approximate form that

$$-\frac{r^*\xi_R}{4\xi_r}(1+\frac{r^*}{2}u-\frac{r^*}{2R})V_R^2-\frac{r^*}{2\xi_r}[(1+r^*u-\frac{r^*}{R})-\frac{1}{2}(1+\frac{1}{2}r^*u-\frac{r^*}{2R})]c^2+u''R^2V_R^2+uR^2V_R^2=0 \quad (612)$$

or

$$-\frac{r^*\xi_R}{4\xi_r}(1+\frac{r^*}{2}u-\frac{r^*}{2R})V_R^2-\frac{r^*}{2\xi_r}(\frac{1}{2}+\frac{3}{4}r^*u-\frac{3r^*}{4R})c^2+u''R^2V_R^2+uR^2V_R^2=0 \quad (613)$$

Neglecting small items, it becomes

$$u''+(1-\frac{3}{8\xi_r}\frac{r^{*2}c^2}{R^2V_R^2}-\frac{\xi_R}{8\xi_r}\frac{r^{*2}}{R^2})u=\frac{1}{4\xi_r}\frac{r^*c^2}{R^2V_R^2}+\frac{\xi_R}{4\xi_r}\frac{r^*}{R^2} \quad (614)$$

If more smaller quantities kept in Eq. (614), it could be verified that this equation could be degraded to the Eq. (587) for close-to-light-speed matters, case $\xi_r\approx\xi_R\approx 1.0$ and $V_R=c$.

For periodic motions, setting

$$\omega^2=1-\frac{3}{8\xi_r}\frac{r^{*2}c^2}{R^2V_R^2}-\frac{\xi_R}{8\xi_r}\frac{r^{*2}}{R^2} \quad (615)$$

The Eq. (614) could be solved [25] to be

$$u=A[1-esin(\omega\varphi)] \quad (616)$$

where, e is orbit eccentricity.

With this solution being substituted back into Eq. (614), it is obtained that

$$\mathrm{A}=(\frac{1}{4\xi_r}\frac{r^*c^2}{R^2V_R^2}+\frac{\xi_R}{4\xi_r}\frac{r^*}{R^2})\omega^{-2} \quad (617)$$

Approximation of Eq. (610) in the condition of $R\gg r^*$ and $r\gg r^*$ could be presented as

$$-\frac{\xi_R}{\xi_r}(1+\frac{r^*}{2}u-\frac{r^*}{2R})V_R^2-\frac{1}{\xi_r}[(1+ur^*-\frac{r^*}{R})-(1+\frac{r^*}{2}u-\frac{r^*}{2R})]c^2+(u')^2R^2V_R^2+u^2R^2V_R^2=0 \quad (618)$$

Neglecting very small items, it is

$$-\frac{\xi_R}{\xi_r}V_R^2+\frac{1}{2\xi_r}\frac{r^*}{R}c^2-\frac{1}{2\xi_r}r^*uc^2+(u')^2R^2V_R^2+u^2R^2V_R^2=0 \quad (619)$$

With Eq. (616), it becomes

$$-\frac{\xi_R}{\xi_r}V_R^2+\frac{c^2}{2\xi_r}[\frac{r^*}{R}-r^*A+r^*Aesin(\omega\varphi)]+A^2\omega^2e^2cos^2(\omega\varphi)R^2V_R^2+[A-Aesin(\omega\varphi)]^2R^2V_R^2=0 \quad (620)$$

where, e is orbit eccentricity.

Case $\varphi=0$ there is

$$-\frac{\xi_R}{\xi_r}V_R^2+\frac{1}{2\xi_r}\frac{r^*}{R}c^2-\frac{1}{2\xi_r}r^*Ac^2+A^2\omega^2e^2R^2V_R^2+A^2R^2V_R^2=0 \quad (621)$$

where, e is orbit eccentricity.

This equation is the velocity composition equation at the position of $\varphi = 0$. The first three items correspond to the component of minus total systematic dynamic energy, in which, the previous one is of the initial condition and the other two comes from potential conversion. The last item is of the component from angular dynamic energy. The fourth item is of the component from radial dynamic energy.

So that the eccentricity could be written as

$$e \approx [\frac{(\frac{\xi_R}{\xi_r} - A^2R^2)V_R^2 - \frac{1}{2\xi_r}(1-AR)\frac{r^*}{R}c^2}{A^2\omega^2R^2V_R^2}]^{1/2} \quad (622)$$

It should be highlighted that, in the motion with $\xi_r \neq \xi_R$, one cannot apply one solution of the Eq. (616) into the entire trajectory because ξ_r may vary in a long run. A solution could only be presented for a limited segment of the trajectory.

In conditions that the variation from V_r^2 to V_R^2 is considerable not very big, we will see that $\xi_r \approx \xi_R$. The Eq. (613) turns to be

$$-\frac{r^*}{4}(1 + \frac{r^*}{2}u - \frac{r^*}{2R})V_R^2 - \frac{r^*}{2\xi_r}(\frac{1}{2} + \frac{3}{4}r^*u - \frac{3r^*}{4R})c^2 + u''R^2V_R^2 + uR^2V_R^2 = 0 \quad (623)$$

With small items trimmed, there is

$$u'' + (1 - \frac{3}{8\xi_r}\frac{r^{*2}c^2}{R^2V_R^2} - \frac{1}{8}\frac{r^{*2}}{R^2})u = \frac{1}{4\xi_r}\frac{r^*c^2}{R^2V_R^2} + \frac{1}{4}\frac{r^*}{R^2} \quad (624)$$

Setting

$$\omega^2 = 1 - \frac{3}{8\xi_r}\frac{r^{*2}c^2}{R^2V_R^2} - \frac{1}{8}\frac{r^{*2}}{R^2} \quad (625)$$

Case ω^2 is very close to 1.0, there is an approximation that

$$\omega \approx 1 - \frac{3}{16\xi_r}\frac{r^{*2}c^2}{R^2V_R^2} - \frac{1}{16}\frac{r^{*2}}{R^2} \quad (626)$$

The equation would have a solution

$$u = A[1 - esin(\omega\varphi)] \quad (627)$$

where, e is orbit eccentricity.

With this solution being substituted back into Eq. (624), it is obtained that

$$A = (\frac{1}{4\xi_r}\frac{r^*c^2}{R^2V_R^2} + \frac{1}{4}\frac{r^*}{R^2})\omega^{-2} \quad (628)$$

The eccentricity becomes

$$e \approx [\frac{(1 - A^2R^2)V_R^2 - \frac{1}{2\xi_r}(1-AR)\frac{r^*}{R}c^2}{A^2\omega^2R^2V_R^2}]^{1/2} \quad (629)$$

Sometimes, we prefer to pay more attentions on the performances on irrelativistic velocity motions in which V_r^2 is far less than c^2 and at the same time $\xi_r \approx \xi_R \approx 0.5$, as that of the motions of planets in solar system. The dynamic equation Eq. (624) becomes

$$u'' + (1 - \frac{3}{4}\frac{r^{*2}c^2}{R^2V_R^2})u = \frac{1}{2}\frac{r^*c^2}{R^2V_R^2} \quad (630)$$

and

$$\omega^2 = 1 - \frac{3}{4}\frac{r^{*2}c^2}{R^2V_R^2} \quad (631)$$

For planets in solar system, ω^2 is close to 1.0, so that

$$\omega \approx 1 - \frac{3}{8}\frac{r^{*2}c^2}{R^2V_R^2} \quad (632)$$

The solution for the Eq. (630) is

$$u = A[1 - esin(\omega\varphi)] \quad (633)$$

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where, e is orbit eccentricity.
It could be calculated that

$$A = \frac{1}{2}\frac{r^* c^2}{R^2 V_R^2}\omega^{-2} \quad (634)$$

One will see that A in this equation is a little greater than that in Eq. (510).
The eccentricity is

$$e \approx [\frac{(1-A^2R^2)V_R^2-(1-AR)\frac{r^*}{R}c^2}{A^2\omega^2R^2V_R^2}]^{1/2} \quad (635)$$

Some publications have given the following expression for eccentricity

$$L^2 = a(1-e^2)GM \quad (636)$$

where, $L^2 = R^2V_R^2$, a is the longer semi-major axis distance, G is gravitational constant and M is the mass of central source. But this is not a real solution for eccentricity, because that the a solved in Eq. (633) is $a = A^{-1}(1-e^2)^{-1}$ which is a reformation of Eq. (636). The eccentricity e and the semi-major axis distance cannot be worked out independently.

The perihelion precession of solar planets could be presented as

$$\Delta\varphi \approx \frac{3}{8}\frac{r^{*2}c^2}{R^2V_R^2}\varphi \approx \frac{3}{4}r^*A\varphi \quad (637)$$

or the precession per revolution as

$$\Delta\varphi_{2\pi} \approx \frac{3\pi}{4}\frac{r^{*2}c^2}{R^2V_R^2} = \frac{3\pi}{2}r^*A\varphi \quad (638)$$

That could be shown as in Figure 27.

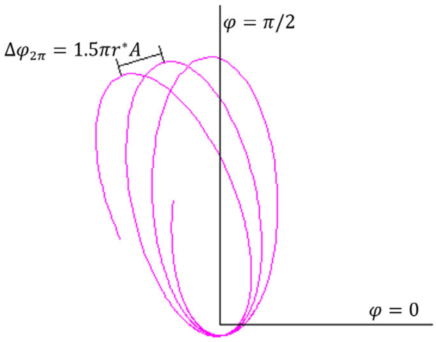

Figure 27. The renovated trajectory of solar planet.

The most surprising is that this solution of perihelion precession is non-general-relativistic. It is a half of the value of the classical solution. Obviously, the classical solution is involved with errors in that once the relativistic mass being considered will absolutely lead to another result. It is practicable to carry out more experiments of motions closely around the Sun to verified the conclusions of Eq. (633).

One can focus on more sophisticated conditions for massive matters traveling in gravitational fields, especially for those motions in strong fields and with relativistic velocities.

Perhaps this result is the only one in which we have focused on that is in contradictions with the observations that have declared perihelion precessions of planets in solar system [17]. Some observations on PSR J0737-3039A/B [26,27] have shown quite big deviations from classical predictions, which had been expected to have perihelion precession together with geodetic precession. Nevertheless, the experiment of Gravity Probe B [28] shows that geodetic item has got accurate results while frame-dragging item has not, in which it could be recognized that the geodetic item is really the effect of special relativity. Observations on PSR 1913+16 [29] have provided more evidences on energy conversions rather than perihelion precessions. The observations have not been well done for the verifications of perihelion precessions of S2 and S62 [30,31] in our galaxy. It could be expected that more observations on them and so on would be sponsored to furtherly verify the results of perihelion precessions.

9.6.6. Dynamics within the Event Horizons

We know that we know less about the physics with in the event horizons. But it is interesting to imagine the possibilities that might happen. It could be predicted that a massive matter with initial velocity high enough will have opportunities to escape from a black hole.

The massive matters would have velocity limit of light speed. That indicates a minimum radius for circular motions in one source fields. It has been discussed in previous that vertical component of gravity would lead to Newtonian geometrical acceleration at that direction. This will then support the equation of centrifugal motion case the matter catches up to the velocity close to light speed. The equilibrium equation could be written as

$$m\frac{c^2}{r} = \frac{GMm}{r^2} \quad (639)$$

The minimum radius

$$r_{min} = \frac{GM}{c^2} = \frac{r^*}{2} \quad (640)$$

It indicates that there will be three conditions for the close-to-light-speed motion on close to $r^*/2$ circle. Condition 1 is that matters move along a circle orbit under which the matter keep circular motion. Condition 2 is that matters move at inner inclined direction at a start time so that they cannot keep periodic motion and will fall to source center finally. Condition 3 is that at the initial state, the motion direction is outer inclined, which could lead to complex results dependently.

It could be imagined that the dynamics in all the regions within event horizon and close outer space will be greatly sophisticated. I would like to make more discussions on radial escape. With the Eq. (571), the total mass for a matter escape to event horizon from inner area is

$$m_r = m_0 e^{(\frac{r^*}{2r} - \frac{r^*}{2r^*})} = m_0 e^{(\frac{r^*}{2r} - \frac{1}{2})} \quad (641)$$

and the mass to escape to free is

$$m_r = m_0 e^{(\frac{r^*}{2r} - \frac{r^*}{2r\infty})} = m_0 e^{\frac{r^*}{2r}} \quad (642)$$

where, m_0 is defined due to velocity after escape that has been set zero.

One can calculate the minimum escape energy for matters at different positions. A brief calculation is shown in Table 7.

Table 7. Energy for matters to escape from within event horizon.

from Initial positions	for escape to event horizon		or escape to free	
of times of r^*	dynamic mass	velocity/c	dynamic mass	velocity/c
$0.1r^*$	$90.017m_0$	0.9999383	$148.413m_0$	0.9999773
$0.2r^*$	$7.389m_0$	0.99080	$12.182m_0$	0.99663
$0.5r^*$	$1.649m_0$	0.795	$2.718m_0$	0.930
$0.9r^*$	$1.057m_0$	0.324	$1.743m_0$	0.819

9.7. More Discussions on Light Propagation

General mass equation certainly could be employed in the equations of light ray propagations. It deserves more efforts to make a try.

9.7.1. General Mass Equation Applied for Light Rays

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For a light ray pass across the peak point in one source fields, as has mentioned previous, the light mass at the peak point could be expressed

$$m_R = \frac{h\nu_{0(\infty)}}{c^2} e^{\frac{r^*}{2R}} \quad (643)$$

And it varies at position r

$$m_r = \frac{h\nu_{0(\infty)}}{c^2} e^{\frac{r^*}{2r}} \quad (644)$$

There is the angular momentum conservation as

$$L = r^2 m_r \dot{\varphi} = R m_R c = const. \quad (645)$$

Then, the angular velocity

$$\dot{\varphi} = \frac{R m_R c}{m_r r^2} = \frac{Rc}{r^2} e^{\frac{r^*}{2R} - \frac{r^*}{2r}} \quad (646)$$

The Lagrangian for light rays is

$$-c^2 + (1 - \frac{r^*}{r})^{-1} \dot{r}^2 + r^2 \dot{\varphi}^2 = 0 \quad (647)$$

With the setting of $r' = \frac{dr}{d\varphi}$, so that $\dot{r} = \frac{dr}{d\varphi}\frac{d\varphi}{d\tau} = r'\dot{\varphi}$, There is

$$-c^2 + (1 - \frac{r^*}{r})^{-1} r'^2 \dot{\varphi}^2 + r^2 \dot{\varphi}^2 = 0 \quad (648)$$

Take the Eq. (646) into Eq. (648), the latter becomes

$$-c^2 + (1 - \frac{r^*}{r})^{-1} r'^2 [\frac{1}{r^2} e^{\frac{r^*}{2R} - \frac{r^*}{2r}}]^2 R^2 c^2 + r^2 [\frac{1}{r^2} e^{\frac{r^*}{2R} - \frac{r^*}{2r}}]^2 R^2 c^2 = 0 \quad (649)$$

In weak fields, $e^{-\frac{r^*}{r}} \approx 1 - \frac{r^*}{r}$, so that it is natural to consider whether the metrics may have general forms. That will be taken into detailed discussions in next sections.

Anyway, the second item in left hand side of Eq. (649) could be rewritten approximately to be

$$(1 - \frac{r^*}{r})^{-1} r'^2 [\frac{1}{r^2} e^{\frac{r^*}{2R} - \frac{r^*}{2r}}]^2 R^2 c^2 \approx \frac{r'^2}{r^4} e^{\frac{r^*}{2R}} R^2 c^2 \quad (650)$$

The Eq. (649) turns to be

$$-1 + r'^2 \frac{R^2}{r^4} e^{\frac{r^*}{R}} + \frac{R^2}{r^2} e^{\frac{r^*}{R} - \frac{r^*}{r}} = 0 \quad (651)$$

With $u = 1/\mathrm{r}$, and $r' = \frac{dr}{d\varphi} = -\frac{u'}{u^2}$ it turns to be

$$-\frac{1}{R^2} e^{-\frac{r^*}{R}} + u'^2 + e^{-r^* u} u^2 = 0 \quad (652)$$

To make a further derivation for it, there is

$$u'' + u e^{-r^* u} - \frac{r^*}{2} u^2 e^{-r^* u} = 0 \quad (653)$$

We know that in the conditions that $r \gg r^*$ the exponent item could be simplified as

$$e^{-r^* u} \approx 1 - r^* u e^{-r^* u} + \frac{1}{2}(-r^* u)^2 e^{-r^* u}$$

$$= 1 - r^* u (1 - r^* u) + \frac{1}{2}(r^* u)^2 (1 - r^* u)$$

$$= 1 - r^* u + \frac{3}{2}(r^* u)^2 - \frac{1}{2}(r^* u)^3 \quad (654)$$

The Eq. (653) could be simplified to be

$$u'' + u[1 - r^* u + \frac{3}{2}(r^* u)^2 - \frac{1}{2}(r^* u)^3] - u^2 \frac{r^*}{2}[1 - r^* u + \frac{3}{2}(r^* u)^2 - \frac{1}{2}(r^* u)^3] = 0 \quad (655)$$

Neglecting the higher small items, it is

$$u'' + u = \frac{3r^*}{2}u^2 - 2r^{*2}u^3 \quad (656)$$

The cubic items in the right hand side may be not easily solved. I prefer to suggest a not bad methodology of segmental solution. That is to solve the equation in every small segment of the trajectory that could also be seen as the approximate solution.

In a segment, the coordinate r could be defined to be R_x, thus the Eq. (656) could be rewritten as

$$u'' + u = (\frac{3r^*}{2} - \frac{2r^{*2}}{R_x})u^2 \quad (657)$$

A solution for the differential equation is that

$$u = \frac{sin\varphi}{b} + \frac{3r^*}{4b^2}(1 - \frac{4r^*}{3R_x})(1 + \frac{1}{3}cos2\varphi) \quad (658)$$

This equation will cause different b in different segment.

It is easy to calculate the half branch deflection with respect to the coordinate of farther position that $R_x \to \infty$

$$0 = \frac{\varphi_\infty}{b} + \frac{3r^*}{4b^2}(1 + \frac{1}{3}) \quad (659)$$

There is

$$\varphi_\infty = -\frac{r^*}{b} \quad (660)$$

At the position of $\varphi = \frac{\pi}{2}$, the first constant b could be worked out as

$$\frac{1}{R} = \frac{1}{b} + \frac{3r^*}{4b^2}(1 - \frac{4r^*}{3R})(1 - \frac{1}{3}) \quad (661)$$

There is the approximate solution

$$b_0 = R + \frac{r^*}{2R}(1 - \frac{7r^*}{6R}) \quad (662)$$

We can study the asymptotic line as $R_x \to \infty$ that the solution Eq. (658) becomes

$$\frac{1}{r} = \frac{sin\varphi}{b_\infty} + \frac{3r^*}{4b_\infty^2}(1 + \frac{1}{3}cos2\varphi) \quad (663)$$

Because $r \to \infty$ and φ is very small, it could be rewritten as

$$b_\infty = r(sin\varphi + \frac{r^*}{b_\infty}) = r\sin(\varphi + \frac{r^*}{b_\infty}) \quad (664)$$

This is the equation of asymptotic line of light ray trajectory based on the farthest segment.

With the conservation of angular moment, there is

$$b_\infty e^{\frac{r^*}{2r_\infty}} = Re^{\frac{r^*}{2R}} \quad (665)$$

In higher precisions, the perpendicular distance should be

$$b_\infty = R(1 + \frac{r^*}{2R} + \frac{1}{8}\frac{r^{*2}}{R^2}) \quad (666)$$

9.7.2. Falsification on General Metrics

The discussions previous indicates general metrics in gravitational fields. In fact, the gravity geometrization is to offset the influence of gravity by metrics. We have seen that gravity of matters may experience exponent variation in the way of general mass equation that indicates the probability for metrics to be renovated. Suppose the metrics of Schwarzschild solution could be renovated to be

$$g_{00} = e^{-\frac{r^*}{r}} \quad (667)$$

and

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$$g_{11} = e^{\frac{r^*}{r}} \quad (668)$$

As the performance of general equations of light ray propagations have been discussed in previous, we are going to make a try of general metrics employed for revisit gravitational redshift with respect to the simplified equation and metrics employed in Eq. (318). The Christoffel symbols could be calculated as

$$\Gamma_{10}^{0} = \frac{1}{2} g^{00} \frac{\partial g_{00}}{\partial r} = \frac{1}{2} e^{\frac{r^*}{r}} (e^{-\frac{r^*}{r}})' = \frac{r^*}{2r^2} \quad (669)$$

With Eq. (571), general frequency equation could be written as

$$\nu_0 = e^{\frac{r^*}{2r}} \nu_{0(\infty)} \quad (670)$$

The deriviative is

$$(\frac{d\nu_0}{\partial r})_{tr} = -\frac{r^*}{2r^2} e^{\frac{r^*}{2r}} \nu_{0(\infty)} \quad (671)$$

Thus, the covariant derivative

$$(\frac{D\nu_0}{dr})_{tr} = (\frac{d\nu_0}{dr})_{tr} - \Gamma_{10}^{0} \nu_0$$

$$= -\frac{r^*}{2r^2} e^{\frac{r^*}{2r}} \nu_{0(\infty)} - \frac{r^*}{2r^2} e^{\frac{r^*}{2r}} \nu_{0(\infty)}$$

$$\equiv 2(\frac{d\nu_0}{dr})_{tr} \quad (672)$$

It seems reasonable and probable. Unfortunately, this attempt has being run in the wrong way. I tackled this problem in such trivial cases is to highlight the concept of general mass equation that is not the concept of rest fields but the concept of moving matters. It is of accumulation of mass energy from potential release or withdraw. The gravity of Eq. (572) is not of real gravity equation but of the gravity variation of free motion in gravitational fields. On another side, metrics are of the nature of the rest fields that are fixed to describe the space time for any matters rest or moving with rest mass or accumulated mass. Metrics will not vary with the motions of matters and do nothing with the variations of dynamic mass energy. We will see that the general mass equation could be employed in detectable frequency shift in Eq. (850) in next sections because frequency shift is the issue of motions, but the metrics relating to variable apparent light speed keep invariant in rest fields.

It is very important to sponsor the discussions that the general mass equation is only available for independently moving matters. The accumulations of mass energy would bring no change to the properties of rest fields their selves.

9.8. Time Spending Problems

For a light ray passing by the Sun, the apparent light speed varies with the positions and directions on the trajectory. There should be a difference between the real time spending and that time spending calculated based imaginary light speed instead of apparent light speed so that to seem like a time delay for a light ray arrival. Shapiro proposed the tests of time delay of radar signals which transmitted from the Earth to pass by the edge of the Sun to another planet or satellite and then they would be reflected back to the Earth [32,33]. The observations on time delay of radar echoes would forcefully support the theory of general relativity as well as that of light ray deflect.

However, the solutions of time delay must have involved in the problems with light trajectory, that the solution process has inherited the errors in classical equations of light trajectory, so that it is necessary to make a detailed discussion to investigate.

9.8.1. Classical Solution for Radar Echoes

In classical procedure, with the assumptions of $\dot{t}(1-\frac{r^*}{r}) = E$ and $r^2\dot{\varphi} = L$, the Lagrangian $\mathcal{L} = -(1-\frac{r^*}{r})c^2\dot{t}^2 + (1-\frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = 0$ could be transformed to be

$$\dot{r}^2 = c^2E^2 - (1-\frac{r^*}{r})\frac{L^2}{r^2} \quad (673)$$

as that in Eq. (417).

And again with $\dot{t}(1-\frac{r^*}{r}) = E$, the $\dot{r}$ could be deformed to be

$$\dot{r} = \frac{dr}{dt}\dot{t} = \frac{dr}{dt}E(1-\frac{r^*}{r})^{-1} \quad (674)$$

Of course, this treatment is unreasonable as well as that acted on $\dot{t}$ in Eq. (673). Obviously, this will bring about additional influences than that in light deflect problem. Now, we are just reviewing the classical methodology for further investigations.

Thus, the Lagrangian becomes

$$(1-\frac{r^*}{r})^{-3}(\frac{dr}{dt})^2 = c^2(1-\frac{r^*}{r})^{-1} - \frac{1}{r^2}\frac{L^2}{E^2} \quad (675)$$

At the peak point, $\frac{dr}{dt} = 0$ so that there is

$$\frac{L^2}{E^2} = c^2R^2(1-\frac{r^*}{R})^{-1} \quad (676)$$

where, R is the coordinate of the peak point.

Taking it back into the Eq. (675), there is

$$(1-\frac{r^*}{r})^{-3}(\frac{dr}{dt})^2 - c^2(1-\frac{r^*}{r})^{-1} + c^2\frac{R^2}{r^2}(1-\frac{r^*}{R})^{-1} = 0 \quad (677)$$

or

$$(\frac{dr}{dt})^2 = c^2(1-\frac{r^*}{r})^2 - c^2\frac{R^2}{r^2}(1-\frac{r^*}{r})^3(1-\frac{r^*}{R})^{-1} \quad (678)$$

Then, the time spending from coordinate R to coordinate r could be integrated as

$$t|_R^r = \frac{1}{c}\int_R^r(1-\frac{r^*}{r})^{-1}[1-(1-\frac{r^*}{r})(1-\frac{r^*}{R})^{-1}\frac{R^2}{r^2}]^{-1/2}dr \quad (679)$$

Because $\frac{r^*}{r}$ and $\frac{r^*}{R}$ are very small, it could be written as

$$t|_R^r = \frac{1}{c}\int_R^r(1-\frac{r^*}{r})^{-1}[1-(1-\frac{r^*}{r}+\frac{r^*}{R})\frac{R^2}{r^2}]^{-1/2}dr \quad (680)$$

It has an approximate solution [4] as

$$t|_R^r \approx \frac{1}{c}\int_R^r(1-\frac{R^2}{r^2})^{-1/2}[1+\frac{r^*}{r}+\frac{r^*R}{2r(r+R)}]dr$$
$$= \frac{1}{c}[\sqrt{r^2-R^2} + r^*\,ln(\frac{r+\sqrt{r^2-R^2}}{R}) + \frac{r^*}{2}\sqrt{\frac{r-R}{r+R}}] \quad (681)$$

9.8.2. Errors in Classical Methodology

We have known that the assumption Eq. (416) in classical equations is incorrect. In fact, the equation could be solved without this assumption. Such as the Lagrangian

$$-c^2 + (1-\frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = 0 \quad (682)$$

With $L = Rc = r^2\dot{\varphi} = const.$ it is

$$-c^2 + (1-\frac{r^*}{r})^{-1}\dot{r}^2 + \frac{c^2R^2}{r^2} = 0 \quad (683)$$

And we know that

$$\dot{r} = \frac{dr}{dt}\dot{t} = \frac{dr}{dt}\frac{dt}{d\tau} = \frac{dr}{dt}(1-\frac{r^*}{r})^{-1/2} \quad (684)$$

so that

$$(\frac{dr}{dt})^2 = c^2(1-\frac{r^*}{r})^2 - c^2(1-\frac{r^*}{r})^2\frac{R^2}{r^2} \quad (685)$$

In this equation the postulation Eq. (416) has been given up. That will cause a natural solution after the setting momentum conservation, Eq. (415).

The differential relationship could be integrated that

$$t|_R^r = \frac{1}{c}\int_R^r(1-\frac{r^*}{r})^{-1}(1-\frac{R^2}{r^2})^{-1/2}dr \quad (686)$$

Approximate solution is

$$t|_R^r = \frac{1}{c}[\sqrt{r^2-R^2} + r^* \, ln(\frac{r+\sqrt{r^2-R^2}}{R})] \quad (687)$$

One can find that this treatment has just brought about a little deviation from the previous. That is because the trajectory is only a little different from that one at deflect angle but goes nearly same distance with that one. This investigation has revealed the different treatments on conservativeness between Eq. (673) and Eq. (683). Of course, this equation is incorrect either, because the wrong setting of light angular momentum. We will present the result under correct expressions of light momentum conservation and mass conservation in next subsections.

9.8.3. Renovated Equations for Radar Echoes

The revisit equation of trajectory may help to get new performances of the issue.

For the Lagrangian

$$-c^2 + (1-\frac{r^*}{r})^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = 0 \quad (688)$$

In previous section, we have got the angular velocity expression based on energy momentum conservativeness as discussed in previous sections that $L = Rm_R c = r^2 m_r \dot{\varphi}$, so that

$$\dot{\varphi} = \frac{Rm_R c}{m_r r^2} = \frac{Rc}{r^2}\frac{1+\frac{r^*}{2R}}{1+\frac{r^*}{2r}} \quad (689)$$

Setting $B = (R + \frac{r^*}{2})$ and in weak field the item $1/(1+\frac{r^*}{2r}) \approx (1-\frac{r^*}{2r})$, there is

$$\dot{\varphi} = \frac{1}{r^2}(1-\frac{r^*}{2r})Bc \quad (690)$$

Then the Lagrangian becomes

$$-c^2 + (1-\frac{r^*}{r})^{-1}\dot{r}^2 + \frac{1}{r^2}(1-\frac{r^*}{r})B^2c^2 = 0 \quad (691)$$

With $\dot{r} = \frac{dr}{dt}\frac{dt}{d\tau} = \frac{dr}{dt}(1-\frac{r^*}{r})^{-1/2}$, it could be rewritten as

$$-c^2 + (1-\frac{r^*}{r})^{-2}(\frac{dr}{dt})^2 + (1-\frac{r^*}{r})\frac{B^2c^2}{r^2} = 0 \quad (692)$$

With $B = (R + \frac{r^*}{2})$ it becomes

$$-c^2 + (1-\frac{r^*}{r})^{-2}(\frac{dr}{dt})^2 + (1-\frac{r^*}{r})(1+\frac{r^*}{R})\frac{R^2c^2}{r^2} = 0 \quad (693)$$

so that there is

$$(\frac{dr}{dt})^2 = c^2[(1-\frac{r^*}{r})^2 - (1-\frac{r^*}{r})^3(1+\frac{r^*}{R})\frac{R^2}{r^2}] \quad (694)$$

or

$$(dt)^2 = \frac{1}{c^2}[(1-\frac{r^*}{r})^2 - (1-\frac{r^*}{r})^3(1+\frac{r^*}{R})\frac{R^2}{r^2}]^{-1}(dr)^2 \quad (695)$$

or

$$dt = \frac{1}{c}[(1-\frac{r^*}{r})^2 - (1-\frac{r^*}{r})^3(1+\frac{r^*}{R})\frac{R^2}{r^2}]^{-1/2}dr \quad (696)$$

The time spending from coordinate R to coordinate r could be written as an integration as

$$t|_R^r = \frac{1}{c}\int_R^r (1-\frac{r^*}{r})^{-1}(1-\frac{R^2}{r^2}+\frac{r^*}{r}\frac{R^2}{r^2}-\frac{r^*}{R}\frac{R^2}{r^2})^{-1/2}dr \quad (697)$$

or

$$t|_R^r = \frac{1}{c}\int_R^r (1+\frac{r^*}{r})[1-\frac{R^2}{r^2}+(\frac{r^*}{r}-\frac{r^*}{R})\frac{R^2}{r^2}]^{-1/2}dr \quad (698)$$

For a function

$$f(x) = x^{-1/2} \quad (699)$$

where, $x = 1-\frac{R^2}{r^2}$, the derivative

$$f'(x) = -\frac{1}{2}x^{-3/2} \quad (700)$$

Considering $\frac{r^*}{r}$ is very small, the rear part in the integral Eq. (698) could be simplified as

$$[1-\frac{R^2}{r^2}+(\frac{r^*}{r}-\frac{r^*}{R})\frac{R^2}{r^2}]^{-1/2} \approx (1-\frac{R^2}{r^2})^{-\frac{1}{2}} - \frac{1}{2}(1-\frac{R^2}{r^2})^{-3/2}(\frac{r^*R^2}{r^3}-\frac{r^*R}{r^2}) \quad (701)$$

Then the time integration will be reformed as

$$t|_R^r \approx \frac{1}{c}\int_R^r [(1-\frac{R^2}{r^2})^{-1/2}(1+\frac{r^*}{r}) - \frac{1}{2}(1-\frac{R^2}{r^2})^{-3/2}(\frac{r^*R^2}{r^3}-\frac{r^*R}{r^2})]dr \quad (702)$$

where in the last item, a multiplier $(1+\frac{r^*}{r})$ has been simplified to be 1.

The first part of the integration could be calculated to be

$$\text{part1} = \frac{1}{c}\int_R^r (1-\frac{R^2}{r^2})^{-1/2}(1+\frac{r^*}{r})dr = \frac{1}{c}[\sqrt{r^2-R^2} + r^*\, ln(\frac{r+\sqrt{r^2-R^2}}{R})] \quad (703)$$

and the last part is done as

$$\text{part2} = -\frac{1}{c}\int_R^r \frac{1}{2}(1-\frac{R^2}{r^2})^{-\frac{3}{2}}(\frac{r^*R^2}{r^3}-\frac{r^*R}{r^2})dr = \frac{1}{c}(\frac{rr^*}{2\sqrt{r^2-R^2}} - \frac{r^*R}{2\sqrt{r^2-R^2}}) \quad (704)$$

so that the total integration is

$$t|_R^r \approx \frac{1}{c}[\sqrt{r^2-R^2} + r^*\, ln(\frac{r+\sqrt{r^2-R^2}}{R}) + \frac{r^*}{2}\sqrt{\frac{r-R}{r+R}}] \quad (705)$$

9.8.4. Falsification of Traditional Methodology on Time Spending of Close-to-Light-Speed Particles

Time delay of close-to-light-speed particles is the same issue as the trajectory. Nevertheless, we could see more problems revealed in this issue.

As Lagrangian is employed in traditional methodologies as Eq. (506) with the two settings as

$$\dot{r}^2 = -(1-\frac{r^*}{r})c^2 + c^2E^2 - (1-\frac{r^*}{r})\frac{L^2}{r^2} \quad (706)$$

it could be rewritten as

$$\dot{r}^2 = \frac{r^*}{r}c^2 + c^2(E^2-1) - (1-\frac{r^*}{r})\frac{L^2}{r^2} \quad (707)$$

And with the setting in traditional method as that has been applied in the Eq. (674) for issue of radar echoes

$$\dot{r} = \frac{dr}{dt}\dot{t} = \frac{dr}{dt}E(1-\frac{r^*}{r})^{-1} \quad (708)$$

it is

$$(1-\frac{r^*}{r})^{-2}(\frac{dr}{dt})^2E^2 = \frac{r^*}{r}c^2 + (E^2-1)c^2 - (1-\frac{r^*}{r})\frac{L^2}{r^2} \quad (709)$$

At the peak point, $\frac{dr}{dt} = 0$, and $L = Rc$ so that there is

$$E^2 = 2(1-\frac{r^*}{R}) \quad (710)$$

where, R is the coordinate of the peak point.

It could be imagined that in the traditional solution of light ray, constant E could be calculate to be an approximate of 1.0 as that in Eq. (676), but in this condition it should go to about 2.0.

Therefore, the equation is

$$2(1-\frac{r^*}{r})^{-2}(\frac{dr}{dt})^2(1-\frac{r^*}{R})=\frac{r^*}{r}c^2+(1-\frac{2r^*}{R})c^2-(1-\frac{r^*}{r})\frac{R^2c^2}{r^2} \quad (711)$$

It could be reformed as

$$(\frac{dr}{dt})^2=\frac{1}{2}[(1-\frac{r^*}{r})^2\frac{r^*}{r}(1+\frac{r^*}{R})c^2+(1-\frac{r^*}{r})^2(1-\frac{r^*}{R})c^2-(1-\frac{r^*}{r})^3(1+\frac{r^*}{R})\frac{R^2c^2}{r^2}] \quad (712)$$

The time spending could be drawn from the integration

$$t|_R^r=\frac{\sqrt{2}}{c}\int_R^r(1-\frac{r^*}{r})^{-1}[\frac{r^*}{r}(1+\frac{r^*}{R})+(1-\frac{r^*}{R})-(1-\frac{r^*}{r})(1+\frac{r^*}{R})\frac{R^2}{r^2}]^{-1/2}dr \quad (713)$$

Some small items could be neglected that

$$t|_R^r=\frac{\sqrt{2}}{c}\int_R^r(1-\frac{r^*}{r})^{-1}[1+\frac{r^*}{r}-\frac{r^*}{R}-(1-\frac{r^*}{r})(1+\frac{r^*}{R})\frac{R^2}{r^2}]^{-1/2}dr \quad (714)$$

or

$$t|_R^r=\frac{\sqrt{2}}{c}\int_R^r(1-\frac{r^*}{r})^{-1}[1+\frac{r^*}{r}-\frac{r^*}{R}-(1-\frac{r^*}{r}+\frac{r^*}{R})\frac{R^2}{r^2}]^{-1/2}dr \quad (715)$$

or

$$t|_R^r=\frac{\sqrt{2}}{c}\int_R^r(1-\frac{r^*}{r})^{-1}[1-\frac{R^2}{r^2}+(\frac{r^*}{r}-\frac{r^*}{R})(1+\frac{R^2}{r^2})]^{-1/2}dr \quad (716)$$

It is a good idea to reform the items in root function above that for the function

$$f(x)=x^{-1/2} \quad (717)$$

where, $x=1-\frac{R^2}{r^2}$ which could present main quantity of the function with respect to the rear part which just performs a higher order small item. The derivative

$$f'(x)=-\frac{1}{2}x^{-3/2} \quad (718)$$

Considering $\frac{r^*}{r}$ and $\frac{r^*}{R}$ are very small, the rear part in the integral Eq. (716) could be simplified as

$$[1-\frac{R^2}{r^2}+(\frac{r^*}{r}-\frac{r^*}{R})(1+\frac{R^2}{r^2})]^{-1/2}\approx(1-\frac{R^2}{r^2})^{-\frac{1}{2}}-\frac{1}{2}(1-\frac{R^2}{r^2})^{-3/2}(\frac{r^*}{r}-\frac{r^*}{R})(1+\frac{R^2}{r^2}) \quad (719)$$

Then the integration could be gain as

$$t|_R^r\approx\frac{\sqrt{2}}{c}\int_R^r[(1-\frac{R^2}{r^2})^{-\frac{1}{2}}(1+\frac{r^*}{r})-\frac{1}{2}(1-\frac{R^2}{r^2})^{-\frac{3}{2}}(\frac{r^*}{r}-\frac{r^*}{R})+\frac{1}{2}(1-\frac{R^2}{r^2})^{-\frac{3}{2}}(\frac{r^*R^2}{r^3}-\frac{r^*R}{r^2})]dr \quad (720)$$

where in the last item, a multiplier $(1+\frac{r^*}{r})$ has been simplified to be 1.

The first part of the integration could be calculated to be

$$\text{part1}=\frac{\sqrt{2}}{c}\int_R^r(1-\frac{R^2}{r^2})^{-1/2}(1+\frac{r^*}{r})dr=\frac{\sqrt{2}}{c}\int_R^r[r(r^2-R^2)^{-1/2}+r^*(r^2-R^2)^{-1/2})dr$$

$$=\frac{\sqrt{2}}{c}[\sqrt{r^2-R^2}+r^*\,ln(\frac{r+\sqrt{r^2-R^2}}{R})] \quad (721)$$

while the third part is

$$\text{part3}=-\frac{\sqrt{2}}{2c}\int_R^r(1-\frac{R^2}{r^2})^{-\frac{3}{2}}(\frac{r^*R^2}{r^3}-\frac{r^*R}{r^2})dr=\frac{\sqrt{2}}{4c}\frac{rr^*}{\sqrt{r^2-R^2}}-\frac{\sqrt{2}}{4c}\frac{r^*R}{\sqrt{r^2-R^2}}=\frac{\sqrt{2}}{4c}\sqrt{\frac{r-R}{r+R}} \quad (722)$$

Then, the second part is

$$\text{part2}=-\frac{\sqrt{2}}{2c}\int_R^r(1-\frac{R^2}{r^2})^{-3/2}(\frac{r^*}{r}-\frac{r^*}{R})dr \quad (723)$$

It is better to be split to two parts for integrations

$$\text{part21}=-\frac{\sqrt{2}}{2c}\int_R^r(1-\frac{R^2}{r^2})^{-3/2}\frac{r^*}{r}dr=-\frac{\sqrt{2}}{2c}\int_R^r(r^2-R^2)^{-3/2}r^2r^*dr$$

$$=-\frac{\sqrt{2}}{2c}\int_R^r(r^2-R^2)^{-3/2}(r^2-R^2+R^2)r^*dr$$

$$= -\frac{\sqrt{2}}{2c}\int_R^r [r^*(r^2-R^2)^{-1/2} + (r^2-R^2)^{-3/2}R^2r^*]dr$$

$$= -\frac{\sqrt{2}}{2c}[r^*\, ln(\frac{r+\sqrt{r^2-R^2}}{R}) - \frac{r^*r}{\sqrt{r^2-R^2}}\Big|_R^r] \quad (724)$$

The last item would be involved with an infinite problem at lower bound, but it could be solved by an offset with another infinite problem in next equation.

$$\text{part22} = \frac{\sqrt{2}}{2c}\int_R^r (1-\frac{R^2}{r^2})^{-3/2}\frac{r^*}{R}dr = \frac{\sqrt{2}}{2c}\int_R^r (r^2-R^2)^{-3/2}r^3\frac{r^*}{R}dr$$

$$= \frac{\sqrt{2}}{4c}\int_R^r (r^2-R^2)^{-3/2}(r^2-R^2+R^2)\frac{r^*}{R}]dr^2$$

$$= \frac{\sqrt{2}}{4c}\int_R^r [\frac{r^*}{R}(r^2-R^2)^{-1/2} + (r^2-R^2)^{-3/2}Rr^*]d(r^2-R^2)$$

$$= \frac{\sqrt{2}}{2c}(\frac{r^*}{R}\sqrt{r^2-R^2} - \frac{r^*R}{\sqrt{r^2-R^2}}\Big|_R^r) \quad (725)$$

The final solution could be drawn as

$$t|_R^r \approx \frac{\sqrt{2}}{c}[(1+\frac{1}{2}\frac{r^*}{R})\sqrt{r^2-R^2} + \frac{1}{2c}r^*\, ln(\frac{r+\sqrt{r^2-R^2}}{R}) + \frac{3}{4c}\sqrt{\frac{r-R}{r+R}}] \quad (726)$$

It is an unacceptable solution with coefficient $\sqrt{2}$ in those items, especially in the first item the approximate $\frac{\sqrt{2}}{c}\sqrt{r^2-R^2}$ will cause a giant time delay with respect to the expected time interval $\frac{1}{c}\sqrt{r^2-R^2}$. That is because of the treatment in Eq. (708) that the left item has got a double value mistakenly. This result has revealed a very strong clue that the setting treatment of Eq. (708) so as well the treatment in Eq. (417) involves with errors certainly.

The treatment in Eq. (709) is unreasonable. With $\dot{t} = \sqrt{g^{00}} = (1-\frac{r^*}{r})^{-1/2}$ instead of $\dot{t} = E(1-\frac{r^*}{r})^{-1} = \sqrt{2}(1-\frac{r^*}{r})^{-1/2}$, we can retrieve the calculation on the Eq. (707) to prevent Eq. (709). Thus, the Eq. (707) could be written as

$$(1-\frac{r^*}{r})^{-1}(\frac{dr}{dt})^2 = \frac{r^*}{r}c^2 + c^2(E^2-1) - (1-\frac{r^*}{r})\frac{L^2}{r^2} \quad (727)$$

Case $\varphi = \frac{\pi}{2}$ then $\frac{dr}{dt} = 0$, we get the same $E^2 = 2(1-\frac{r^*}{R})$.

It could be reformed as

$$(\frac{dr}{dt})^2 = (1-\frac{r^*}{r})\frac{r^*}{r}c^2 + (1-\frac{r^*}{r})(1-\frac{2r^*}{R})c^2 - (1-\frac{r^*}{r})^2\frac{R^2c^2}{r^2} \quad (728)$$

Time spending could be solved by integration

$$t|_R^r = \frac{1}{c}\int_R^r (1-\frac{r^*}{r})^{-1}[\frac{r^*}{r}(1+\frac{r^*}{r}) + (1+\frac{r^*}{r})(1-\frac{2r^*}{R}) - \frac{R^2}{r^2}]^{-1/2}dr \quad (729)$$

It could be reformed as

$$t|_R^r = \frac{1}{c}\int_R^r (1-\frac{r^*}{r})^{-1}[1-\frac{R^2}{r^2} + (\frac{2r^*}{r} - \frac{2r^*}{R})]^{-1/2}dr \quad (730)$$

Because

$$[1-\frac{R^2}{r^2} + (\frac{2r^*}{r} - \frac{2r^*}{R})]^{-1/2} \approx (1-\frac{R^2}{r^2})^{-\frac{1}{2}} - (1-\frac{R^2}{r^2})^{-3/2}(\frac{r^*}{r} - \frac{r^*}{R}) \quad (731)$$

The time integration could be rewritten as

$$t|_R^r \approx \frac{1}{c}\int_R^r [(1-\frac{R^2}{r^2})^{-1/2}(1+\frac{r^*}{r}) - (1-\frac{R^2}{r^2})^{-\frac{3}{2}}(\frac{r^*}{r} - \frac{r^*}{R})]dr \quad (732)$$

in which,

$$\text{part1} = \frac{1}{c}\int_R^r (1-\frac{R^2}{r^2})^{-1/2}(1+\frac{r^*}{r})dr = \frac{1}{c}\int_R^r [r(r^2-R^2)^{-1/2} + r^*(r^2-R^2)^{-1/2})dr$$

$$= \frac{1}{c}[\sqrt{r^2-R^2} + r^*\, ln(\frac{r+\sqrt{r^2-R^2}}{R})] \quad (733)$$

$$\text{part2} = -\frac{1}{c}\int_R^r (1-\frac{R^2}{r^2})^{-3/2}(\frac{r^*}{r}-\frac{r^*}{R})dr \quad (734)$$

To split part 2 to two parts

$$\text{part21} = -\frac{1}{c}\int_R^r (1-\frac{R^2}{r^2})^{-3/2}\frac{r^*}{r}dr = -\frac{1}{c}\int_R^r (r^2-R^2)^{-3/2} r^2 r^* dr$$

$$= -\frac{1}{c}\int_R^r (r^2-R^2)^{-3/2}(r^2-R^2+R^2)r^* dr$$

$$= -\frac{1}{c}\int_R^r [r^*(r^2-R^2)^{-1/2} + (r^2-R^2)^{-3/2}R^2 r^*]dr$$

$$= -\frac{1}{c}[r^*\, ln\left(\frac{r+\sqrt{r^2-R^2}}{R}\right) - \frac{r^* r}{\sqrt{r^2-R^2}}\Big|_R^r] \quad (735)$$

and

$$\text{part22} = \frac{1}{c}\int_R^r (1-\frac{R^2}{r^2})^{-3/2}\frac{r^*}{R}dr = \frac{1}{c}\int_R^r (r^2-R^2)^{-3/2} r^3 \frac{r^*}{R}dr$$

$$= \frac{1}{2c}\int_R^r (r^2-R^2)^{-3/2}(r^2-R^2+R^2)\frac{r^*}{R}]dr^2$$

$$= \frac{1}{2c}\int_R^r [\frac{r^*}{R}(r^2-R^2)^{-1/2} + (r^2-R^2)^{-3/2}Rr^*]d(r^2-R^2)$$

$$= \frac{1}{c}(\frac{r^*}{R}\sqrt{r^2-R^2} - \frac{r^* R}{\sqrt{r^2-R^2}}\Big|_R^r) \quad (736)$$

The final solution could be drawn as

$$t|_R^r \approx \frac{1}{c}[(1+\frac{r^*}{R})\sqrt{r^2-R^2} + \sqrt{\frac{r-R}{r+R}}] \quad (737)$$

This solution sounds more acceptable than the previous but it is still involved with errors.

9.7.5. Renovated Solution for Close-to-Light-Speed Particles

Now, let us discuss another kind of no-delay time spending, the time spending of close-to-light-speed particles. The velocity composition of close-to-light-speed particles could be expressed as

$$-c^2 + (\frac{dr}{dt})^2 + r^2(\frac{d\varphi}{dt})^2 = 0 \quad (738)$$

With angular momentum conservativeness as that in Eq. (554), there is

$$-c^2 + (\frac{dr}{dt})^2 + (1-\frac{r^*}{r})\frac{B^2c^2}{r^2} = 0 \quad (739)$$

or

$$-c^2 + (\frac{dr}{dt})^2 + (1-\frac{r^*}{r})(1+\frac{r^*}{R})\frac{R^2c^2}{r^2} = 0 \quad (740)$$

so that the time spending integration could be built as

$$t|_R^r = \frac{1}{c}\int_R^r [1-(1-\frac{r^*}{r})(1+\frac{r^*}{R})\frac{R^2}{r^2}]^{-1/2}dr$$

$$= \frac{1}{c}\int_R^r [1-\frac{R^2}{r^2}-(\frac{r^*}{r}-\frac{r^*}{R})\frac{R^2}{r^2}]^{-1/2}dr$$

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$$\approx \frac{1}{c}\int_R^r [(1-\frac{R^2}{r^2})^{-1/2} - \frac{1}{2}(1-\frac{R^2}{r^2})^{-3/2}(\frac{r^* R^2}{r^3} - \frac{r^* R}{r^2})]dr$$

$$= \frac{1}{c}[\sqrt{r^2-R^2} + \frac{r^*}{2}\sqrt{\frac{r-R}{r+R}}] \quad (741)$$

9.8.6. Equations of Time Delay

We have found that the revisit solution of time spending of radar echoes is the same with classical. The reality is that the classical treatment has got the same trajectory by additional settings, so that it is undoubtful to gain a same time spend with. In fact, the problem of time delay of radar echoes is just a kind of performance of light deflection. The results in these discussions are also just extensions of that in the problem of light deflection. The classical equation for time delay of radar echoes is to defined the difference between time interval of light rays and that of an imaginary motion of absolute light speed, and the half length of the trajectory has been coarsely set to be

$$l = \sqrt{r^2-R^2} \quad (742)$$

So that the classical equation of half branch time delay could be calculated with Eq. (705) as

$$\Delta t = \frac{1}{c}[r^* \ln(\frac{r+\sqrt{r^2-R^2}}{R}) + \frac{r^*}{2}\sqrt{\frac{r-R}{r+R}}] \quad (743)$$

But the half branch time delay of light rays, with respect to the time spending of close-to-light-speed particles Eq. (741), will be a little different that

$$\Delta t \approx \frac{r^*}{c}\ln(\frac{r+\sqrt{r^2-R^2}}{R}) \quad (744)$$

One may argue that the trajectory of close-to-light-speed particles will be different to light rays. In fact, the real length of the light trajectory also does not equal to $\sqrt{r^2-R^2}$. One can easily take measures to work out the real length of that trajectory. I prefer to give a more accurate value than Eq. (710) that could be estimated by geometrical relationship as shown in Figure 22 while $r \gg R \approx b$ that

$$l_{accurate} = \sqrt{r^2-R^2} + Rsin(-\varphi_\infty) \approx \sqrt{r^2-R^2} + r^* \quad (745)$$

where, $\varphi_\infty = -r^*/b$ is deflect angle of light ray.

So that a real half branch time delay of light rays with respect to an absolute motion on the very trajectory is

$$\Delta t = \frac{1}{c}[r^*\ln(\frac{r+\sqrt{r^2-R^2}}{R}) + \frac{r^*}{2}\sqrt{\frac{r-R}{r+R}} - r^*] \approx \frac{1}{c}[r^* \ln(\frac{r+\sqrt{r^2-R^2}}{R}) - \frac{r^*}{2}] \quad (746)$$

We know that the time delay has been verified to be very high accurate value with respect to the classical equation, that is because the trajectory length has always been set to be $\sqrt{r^2-R^2}$, which is indeed not accurate length of trajectory line.

If consider the close-to-light-speed particles, the real length of its trajectory line could be estimated to be

$$l_p = \sqrt{r^2-R^2} + Rsin(-\varphi_\infty) = \sqrt{r^2-R^2} + \frac{r^*}{2} \quad (747)$$

where, $\varphi_\infty = -r^*/(2b)$ is deflect angle of particles.

With Eq. (741), the half branch time delay of close-to-light-speed particles is

$$\Delta t_p = 0 \quad (748)$$

The zero time delay is because the particles keep approximate velocity during flying on the trajectory. This result is a certainty of velocity conservation for massive particles.

Something different from trajectory investigation, the test of time delay of light and close-to-light-speed particle propagation might allow a quite big separation to the Sun edge thereby to provide not bad accuracy. Furthermore, it could be expected to sponsor experiments to emit light

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rays and massive particles on a straight line from a point not very close to the Sun at the same time for time delay verification.

9.9. Comparative Researches on Numerical Solutions and Algebraic Solutions

9.9.1. The Invalidity of Newtonian Second Law in Light Propagation

It is interesting that perhaps there is the probability to carry out a new numerical method to calculate the light trajectories or close-to-light-speed motions in gravitational fields, which could be called ballistic method. Considering that the gravity component parallel to the motion trajectory will not bring about any changes for close-to-light-speed motions, we could only consider the calculation on motion variation due to the vertical component of gravity so that to determine a differential coordinate on the trajectory. Thus, in the way incremental, the trajectory could be solved at last.

Firstly, it is proficient to take a motion of close-to-light-speed massive particle in one source field into discussion, as shown in Figure 28.

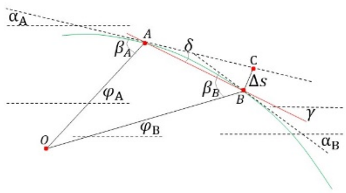

Figure 28. A diagram of ballistic method for a close-to-light-speed motion.

It is believed that for a massive particle in weak field, the vertical deviation could be solved by Newtonian second law

$$\Delta s = \frac{1}{2} g_v \Delta t^2 \quad (749)$$

where, g_v is the component of gravity vertical to the velocity of the particle.

The coordinates at position A are known to be $(r_{\mathrm{A}}, \varphi_{\mathrm{A}})$ or $(x_{\mathrm{A}}, y_{\mathrm{A}})$. As well, the trajectory will have a direction with angle α_{A} at that position. Then, the distance in a time interval Δt the particle travelling is

$$AB = c\Delta t \quad (750)$$

The deviation angle of AB to AC is

$$\delta \approx \Delta s / AB \quad (751)$$

and

$$\gamma = \alpha_{\mathrm{A}} + \delta \quad (752)$$

Thus, the coordinates at position B could be calculated as

$$x_{\mathrm{B}} = x_{\mathrm{A}} + AB cos\gamma \quad (753)$$

$$y_{\mathrm{B}} = y_{\mathrm{A}} + AB sin\gamma \quad (754)$$

or

$$r_{\mathrm{B}} = \sqrt{x_B^2 + y_B^2} \quad (755)$$

$$\varphi_{\mathrm{B}} = \arccos\frac{x_{\mathrm{B}}}{r_{\mathrm{B}}} \quad (756)$$

It should be highlighted that the angle γ is not the direction angle α_{B}. The α_{B} could be calculated by angular momentum conservation that

$$(1 + \frac{r^*}{2R})R = (1 + \frac{r^*}{2r_{\mathrm{B}}})r_{\mathrm{B}} sin\beta_B \quad (757)$$

where, $\beta_B = \alpha_B + \varphi_B$.

Now back to the fly of light ray from position A to position B. It might be imagined to calculate the deviation Δs also by Newtonian second law. That sounds naturally to see the photons as light speed particles with dynamic mass so that to deviate in the same way as massive particles do. Unfortunately, it is impossible in that that kind of calculation will lead to the result very close to that of massive particles. It seems like that the Newtonian second law sounds invalid in light propagation even at the vertical direction. Perhaps one can propose a double gravity for a simulation on light propagation with respect to that of massive particles and it could be imagined that the results would be of not bad precisions. But the truth is that they are really not of a same issue in that they would have different asymptotic distance b. An interesting clue is that light rays are un-accelerable even at the direction vertical to the motion. Maybe that could be employed to interpret the invalidity of Newtonian second law.

Anyway, there is still an opportunity to explore the ballistic trajectory method for them. That is to perform by trial methodologies. We can imagine that after point A there will be a wave front after Huygens' postulation, then, some points on it may be selected for further considerations. With angular momentum conservativeness and Lagrangian, one can calculate the velocity and angle α_B, as well as coordinates as shown in Figure 29. Thus, the deviations of the points could be estimated to help for further trials. I have made more efforts to try but that have not been done, although I still believe the probabilities.

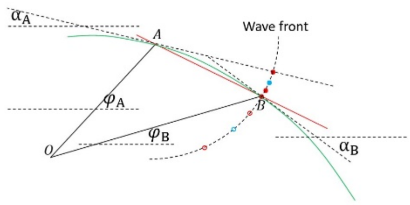

Figure 29. Probability of ballistic method for light deviation.

9.9.2. Comparisons of Numerical Solutions and Algebraic Solutions

To make comparisons with numerical and algebraic solutions is not only a kind of further verification but also a further support to the conclusion of energy moment conservativeness, and at last, a surprise of light propagation. As has discussed previous, ballistic method could be developed to calculate the trajectories of close-to-light massive particles. It could be applied by dividing trajectory to finite segments and calculating coordinates with Eq. (749) to Eq. (757) step by step. The invalidity of Newtonian second law for light propagation is actually another kind of support to the inferences of kinematics.

Notwithstanding, difference method for all differential equations could also be employed for more comparisons. Difference method has shown great advantages in scientific calculations and many excellent schemes have been developed to serve for more complex requirements.

For the equation of light ray propagation

$$u'' + u = \frac{3}{2} r^* u^2 \quad (758)$$

The simple central difference scheme could be suggested that

$$u'' = \frac{u(\varphi+\Delta\varphi) - 2u(\varphi) + u(\varphi-\Delta\varphi)}{(\Delta\varphi)^2} \quad (759)$$

Thus, the difference equation could be built as

$$\frac{u(\varphi+\Delta\varphi) - 2u(\varphi) + u(\varphi-\Delta\varphi)}{(\Delta\varphi)^2} + u(\varphi) = \frac{3}{2} r^* u^2(\varphi) \quad (760)$$

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Pre-exercises show that if step intervals defined by $\Delta\varphi < 0.1°$, the calculations would get quite good accuracy.

As for the dynamic equation of close-to-light massive particles

$$u'' + u = \frac{r^*}{2b^2} \quad (761)$$

could also be built simply as

$$\frac{u(\varphi+\Delta\varphi) - 2u(\varphi) + u(\varphi-\Delta\varphi)}{(\Delta\varphi)^2} + u(\varphi) = \frac{r^*}{2b^2} \quad (762)$$

Thus, we would sponsor a comparison with the analytical solutions of Eq. (466) for light rays and Eq. (559) for close-light-speed matters, difference method solutions of Eq. (760) for light rays and Eq. (762) for close-light-speed matters, the ballistic trajectory method solutions only for close-light-speed matters mentioned in section 9.9.1, together with the asymptotic lines and horizontal line for references.

Numerical methods of difference and ballistic were carried in office computer and others were completed by the calculator of my mobile phone, in that the phone calculator could provide 8 floating point precisions more than that of the Fortran software in the computer.

It is shown in Table 8. that difference method and ballistic method still perform not bad precisions in the results of upper parts of the trajectories. Of course, both of the two methods are of step by step arithmetic so as to accumulate quite amount of deviations at the rear parts of the trajectories. Anyway, the two numerical methods are reliable, that greatly supports the inferences of kinematics.

It is easy to develop more optimal schemes for difference method and the way of optimal secant seeking instead of the tangent seeking for ballistic method or any other effective technologies to improve the precisions of numerical analyses. That will confirm the conclusions and inferences that have been drawn previous.

Table 8. Comparison of the analytical, difference method and the ballistic trajectory method solutions.

coordinate	coordinate radius (m) of solution of light ray by method of			coordinate radius (m) of solution of close-to-light-speed massive particle by method of				coordinate radius (m) of
(degree)	analytical	difference	asymptotic line	analytical	difference	ballistic	asymptotic line	horizontal line
90	695,500,000	695,500,000	695,501,477	695,500,000	695,500,000	695,500,000	695,501,477	695,501,477
80	706,229,140	706,229,166	706,230,180	706,229,186	706,229,216	706,229,226	706,230,444	706,230,708
70	740,135,345	740,135,367	740,136,069	740,135,540	740,135,570	740,135,590	740,136,641	740,137,212
60	803,093,469	803,093,487	803,093,961	803,093,961	803,093,991	803,094,020	803,094,945	803,095,929
50	907,909,142	907,909,153	907,909,463	907,910,181	907,910,213	907,910,251	907,911,080	907,912,699
40	1,082,002,548	1,082,002,549	1,082,002,744	1,082,004,645	1,082,004,679	1,082,004,738	1,082,005,482	1,082,008,219
30	1,390,992,617	1,390,992,592	1,390,992,723	1,390,997,047	1,390,997,085	1,390,997,170	1,390,997,839	1,391,002,954
20	2,033,486,508	2,033,486,403	2,033,486,554	2,033,497,655	2,033,497,694	2,033,497,838	2,033,498,416	2,033,510,279
10	4,005,136,911	4,005,136,314	4,005,136,922	4,005,184,405	4,005,184,395	4,005,184,750	4,005,185,149	4,005,233,377
0.26	153,123,772,603	153,122,660,623	153,123,772,604	153,195,382,906	153,195,143,112	153,195,397,701	153,195,383,644	153,267,061,697
0.01	3,890,278,441,952	3,889,557,122,200	3,890,278,441,977	3,937,035,389,754	3,936,873,242,381	3,937,033,549,851	3,937,035,390,479	3,984,929,947,953
0	163,784,893,532,900	162,515,766,173,612	163,784,893,516,163	327,569,787,065,801	326,450,054,739,264	327,616,498,942,582	327,569,787,031,587	∞

9.10. Additional Discussions

9.10.1. On Geodesic Line, Inertial Motion, and General Covariance

We have discussed those derivatives theoretically that in covariant space the acceleration of a massive matter could not be really calculate to be zero and even revisit light frequency shift will perform double of that of contra variant space. There is only the light speed that always serve the general covariance as the result the acceleration of a light ray may perform nonvanishing in covariant space in any conditions. These conclusions would have then challenged the theory of geodesic lines, because geodesic lines ask for thoroughly nonvanishing solutions for covariant derivatives. We also have proved that geodesic equation cannot be really employed for the solutions of massive matter's motions as well as the motions of light rays.

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Now it is the time to make additional discussions on the tips between matter's trajectories, inertial motions, and geodesic lines.

General covariance asks for matters in covariant space to perform inertial motions just as they perform in no gravity fields. That requires trajectories of matters stay on the lines of geodesic lines in gravitational fields.

Geodesic lines are of the so-called straight lines of a covariant space that present the geometrical properties of the space. But inertial motions must ask for more. For example, an inertial trajectory must serve for any matters to run on, either the massive matters or light rays. The more important is that inertial trajectory must be run on by any matters with any velocities with same direction. It could be said that inertial motions share trajectories.

Back to the massive matter's trajectories, it is no doubt that any two matters with different velocities in gravitational field with same initial positions and directions, will fly apart, so that to fly on two trajectories, just as matter's motions in solar system. These trajectories of course correspond to two lines in covariant space. It is said that the matters with different velocities may not share trajectory in covariant space so that the motions are not inertial motions in covariant space and the trajectories are not geodesic lines.

As for light rays, due to the consequence of invariant light speed, they will come up to zero accelerations in covariant space and we might have seen that light rays could share trajectories in different frequencies, that really performs general covariance. But the trajectories of light rays are not that of real inertial motions or geodesic lines yet, in that light rays do not share trajectories with massive matters.

The motions of close-to-light-speed particles are something special, in that they would have zero acceleration approximately just as light rays, but they do not share trajectories with light rays and un-close-to-light-speed particles as have discussed.

9.10.2. On 4-Dimensional and 3-Dimensional Velocities

We all know that vector composition could be expressed in vector summation as well as in way of component square, by the Pythagorean theorem.

For example, 4-dimensional velocity in contra variant space as

$$\boldsymbol{V}_0^{\uparrow} = (V_0^0, V_0^1, V_0^2, V_0^3) \quad (763)$$

could have a composition of

$$\boldsymbol{V}_0^{\uparrow} = \boldsymbol{V}_0^0 + \boldsymbol{V}_0^1 + \boldsymbol{V}_0^2 + \boldsymbol{V}_0^3 \quad (764)$$

where $\boldsymbol{V}_0^{\mu}$ is defined the component vector at the direction μ which has a value V_0^{μ}.
or the form of

$$-(V_0^{\uparrow})^2 = -(V_0^0)^2 + (V_0^1)^2 + (V_0^2)^2 + (V_0^3)^2 \quad (765)$$

where $V_0^0 = \frac{cdt}{dt} = c$, and $V_0^{\uparrow}$ is module of $\boldsymbol{V}_0^{\uparrow}$. Only in the condition that matter's velocity catches up close to light speed c, the modules may be close to zero.

As in covariant space, there is

$$\boldsymbol{V}\uparrow/0 = (V0/0, V1/0, V2/0, V3/0) \quad (766)$$

That would have a composition of

$$\boldsymbol{V}\uparrow/0 = \boldsymbol{V}0/0 + \boldsymbol{V}1/0 + \boldsymbol{V}2/0 + \boldsymbol{V}3/0 \quad (767)$$

or the form of

$$-(V\uparrow/0)^2 = -(V0/0)^2 + (V1/0)^2 + (V2/0)^2 + (V3/0)^2 \quad (768)$$

where $V0/0 = \frac{cd\tau}{d\tau} = c$, and $(V\uparrow/0)^2$ could have a value uncertain to compare with c^2 so that the setting of $\mathcal{L} = -c^2$ is really a fault treatment.

For light rays, the contra variant velocity will be composed to be nonvanishing

$$\boldsymbol{c}_0^{\uparrow} = \boldsymbol{c}_0^0 + \boldsymbol{c}_0^1 + \boldsymbol{c}_0^2 + \boldsymbol{c}_0^3 \quad (769)$$

or

$$-(c_0^{\uparrow})^2 = -(c_0^0)^2 + (c_0^1)^2 + (c_0^2)^2 + (c_0^3)^2 \neq 0 \quad (770)$$

where $c_0^0 = \frac{cdt}{dt} = c$.

But their velocity composition in covariant space there is

$$\boldsymbol{c} \uparrow/0 = \boldsymbol{c}0/0 + \boldsymbol{c}1/0 + \boldsymbol{c}2/0 + \boldsymbol{c}3/0 \quad (771)$$

or

$$-(c \uparrow/0)^2 = -(c0/0)^2 + (c1/0)^2 + (c2/0)^2 + (c3/0)^2 = 0 \quad (772)$$

where $c0/0 = \frac{cd\tau}{d\tau} = c$, and because the summation of the last 3 items is c^2 we will then gain zero value of $c \uparrow/0$ just as the Lagrangian has been defined.

We have seen that 4-dimensional velocity does not always make sense. It is not a bad idea to return to the expressions of 3-dimensional velocity in some conditions that we call them true velocities.

$$\boldsymbol{V}_0^s = \boldsymbol{V}_0^1 + \boldsymbol{V}_0^2 + \boldsymbol{V}_0^3 \quad (773)$$

where, $\boldsymbol{V}_0^s$ is total quantity of the true velocity.

or the form of

$$(V_0^s)^2 = (V_0^1)^2 + (V_0^2)^2 + (V_0^3)^2 < c^2 \quad (774)$$

In some cases, $(V_0^s)^2$ could be greater than $(c_0^s)^2$.

As in covariant space, there is

$$\boldsymbol{V}s/0 = \boldsymbol{V}1/0 + \boldsymbol{V}2/0 + \boldsymbol{V}3/0 \quad (775)$$

or

$$(Vs/0)^2 = (V1/0)^2 + (V2/0)^2 + (V3/0)^2 \quad (776)$$

where, $(Vs/0)^2$ could have a value less than or equal to or great than c^2.

For light rays, the contra variant velocity will be composed to be

$$\boldsymbol{c}_0^s = \boldsymbol{c}_0^1 + \boldsymbol{c}_0^2 + \boldsymbol{c}_0^3 \quad (777)$$

or

$$(c_0^s)^2 = (c_0^1)^2 + (c_0^2)^2 + (c_0^3)^2 \leq c^2 \quad (778)$$

And in covariant space there is

$$\boldsymbol{c}s/0 = \boldsymbol{c}1/0 + \boldsymbol{c}2/0 + \boldsymbol{c}3/0 \quad (779)$$

or

$$(cs/0)^2 = (c1/0)^2 + (c2/0)^2 + (c3/0)^2 = c^2 \quad (780)$$

It could be found that in some cases, matter's velocity may be greater than light velocity that

$$(V_0^s)^2 \geq (c_0^s)^2 \quad (781)$$

or

$$(Vs/0)^2 \geq (cs/0)^2 \quad (782)$$

9.10.3. On Light Momentum and Massive Matter's Momentum

As have discussed, momentum and energy refer to that issues which will keep conservation during whole movement, so much as we cannot confirm that they are only valid in contra variant space. We have seen that for light rays, frequencies could be employed to perform momentum and energy, but for massive matters, there will be something different. It is based on physical realities that the conservative quantities be defined. Therefore, I will not discuss any transformed forms of momentum and energy, so far as to deal them as tensors. Owing to mass energy equation, matter's total mass could be seen to be equivalent to total energy. These treatments will not do any harms to our understandings on realities. One of the reasons is that conservative quantities are of something invariant. For example, as light speed c is used as invariant value in some cases, it could be treated as a scalar. Momentum and energy keep conservation except inputs or outputs.

As we talk about light momentum of square forms, the modulus of 4-dimensional velocities $(c_0^{\uparrow})^2$ in Eq. (770) and $(c \uparrow/0)^2$ in Eq. (772) cannot be selected for momentum equation, in that $(c \uparrow/0)^2$ is zero and $(c_0^{\uparrow})^2$ is close to zero in weak fields. As for the modulus of 3-dimensional

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velocities $(c_0^S)^2$ in Eq. (778), it varies in gravitational fields. So that $(cs/0)^2 \equiv c^2$ in Eq. (780) is the only choice.

For square momentum of massive matters, the modulus of 4-dimensional velocities $(V_0^{\uparrow})^2$ in Eq. (765) and $(V\uparrow/0)^2$ in Eq. (768) also cannot be employed because of similar reasons. The $(Vs/0)^2$ in Eq. (776) is not a conservative quantity, as has been proved previous, while $(V_0^s)^2$ in Eq. (774) is so that the latter could be employed for momentum expression.

As has mentioned, light momentum will be performed in special forms that

$$P^2 = m^2[(c1/0)^2 + (c2/0)^2 + (c3/0)^2] \quad (783)$$

or

$$\boldsymbol{P} = \boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3 = m\boldsymbol{c}1/0 + m\boldsymbol{c}2/0 + m\boldsymbol{c}3/0 \quad (784)$$

The most surprising is that the vector of momentum would have direction others to the tangent line of trajectory, as shown in Figure 30. For a light ray propagating in one source fields, the momentum direction will coincide with trajectory at peak point or be asymptotically close to the trajectory in very farther area.

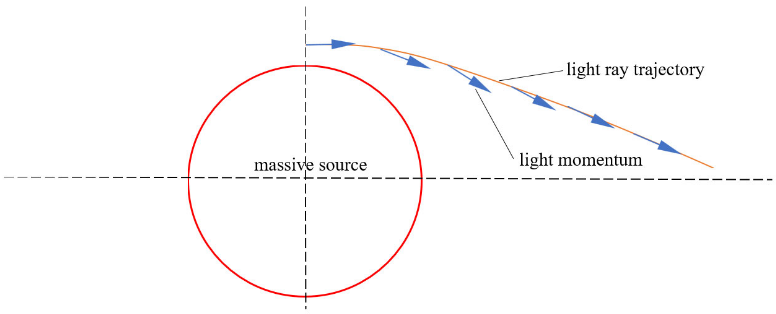

Figure 30. Light momentum vectors on the trajectory.

The surprising forms of light momentum encourages me to make a further conjecture that a light ray refracted from a media to another would have equivalent light momentums. That deserves more verifications.

Momentum of massive matters could be expressed as

$$P^2 = m^2[(V_0^1)^2 + (V_0^2)^2 + (V_0^3)^2] \quad (785)$$

or

$$\boldsymbol{P} = \boldsymbol{P}_1 + \boldsymbol{P}_2 + \boldsymbol{P}_3 = m\boldsymbol{V}_0^1 + m\boldsymbol{V}_0^2 + m\boldsymbol{V}_0^3 \quad (786)$$

The vector of momentum of massive matters will always coincide with trajectory in gravitational fields. Obviously, if the covariant velocity $\boldsymbol{V}s/0$ of massive matters be selected for momentum equation, that momentum could also show deviation with trajectory.

9.11. Summarizations on Conservativeness, Kinematics and Dynamics

In previous sections, the trajectory covariant derivatives of massive matter, the covariant accelerations, have been proved to be nonvanishing. As a while, the trajectory covariant accelerations of the light rays could be proved to be vanishing, exceptionally, the so called revisit gravitational redshift is nonvanishing. The more important is that these proofs are basically based on realities and so that to present realities. These proofs have provided forceful structure that determines the kinematics and dynamics of light rays and massive matters, in the way of conservative principle.

After the discussions in previous sections, it is necessary to make a comprehensive summarization on the equations of kinematics and dynamics in gravitational fields. In fact, the most important problems on the issues are of the expressions of velocity composition, momentum and mass energy. What we are focusing on is to seek for the conservative momentum equation, the conservative velocity equation and the conservative mass energy.

It should be highlighted that the conservative velocity equation refers to the expression of inertial motion except additional forces. I will emphasize that, as talking about inertial motions in contra variant space, gravity will be seen as additional force. As for light ray propagations in covariant space, we know that the gravity was declared to have experienced geometrization so that they keep inertial motions. Of cause, we have discussed that light momentum and energy keep conservation in contra variant space rather than covariant space, that reveals more sophistications.

As has discussed, massive matters in contra variant space, they keep inertial motions except exterior or interior interactions happening, including the gravity coming from massive sources, so that the contra variant velocity was found for conservative quantity. As for light rays, they keep invariant light speed in covariant space while the corresponding contra variant velocity might vary in gravitational fields. Not far beyond imagine, the invariant light speed was determined for conservative quantity.

In fact, the conservative velocities are of space velocity rather than 4-dimension velocity. We have seen that contra variant light speed $\boldsymbol{c}_0^{\uparrow}$ and $\boldsymbol{c}_0^{s}$ are variable quantities, while the 4-dimension velocity $\boldsymbol{c}\uparrow/0$ is zero. Theoretically, one cannot refuse $\boldsymbol{c}_0^{s}$ for kinematic equation instead of the Lagrangian, but it is difficult to link the velocity with mass energy because mass energy is conservative quantities in contra variant space while $\boldsymbol{c}_0^{s}$ might not keep conservations. The space velocity $\boldsymbol{c}s/0$ is the only choice of conservative velocity of light rays. That is the reason that the Lagrangian could be employed as kinematic equation for light rays. As for massive matters, the covariant velocities $\boldsymbol{V}\uparrow/0$ and $\boldsymbol{V}s/0$ do not present inertial motions in covariant space, while $\boldsymbol{V}_0^{\uparrow}$ may be up to invariant light speed case in rest or decrease to zero case in close-to-light-speed motions, so that it is not proper to be determined. If the velocity composition of $\boldsymbol{V}s/0$ is selected for dynamic equation, it is also difficult to link $\boldsymbol{V}s/0$ with mass energy because $\boldsymbol{V}s/0$ does not keep conservations. Finally, we have seen the only choice for massive matters, the composition of $\boldsymbol{V}_0^{s}$ instead of Lagrangian in traditional methodologies.

The most surprising is what space the conservative quantities keep conservation in and the way of their expressions. Normally, massive matters have contra variant conservative quantities which keep conservations in contra variant space. But for light rays, it is more sophisticated. The invariant light speed keeps conservations in covariant space and has covariant expressions, while light momentum keeps conservations in contra variant space but has partially covariant expressions. We know the reason is of the light speed. It is said that expressions of conservative quantities may not keep corresponding expressions, nothing but the conservative quantities usually being selected. One may ague how does the light momentum perform conservations in that space. The gravitational redshift is of a perfect demonstration. In contra variant space, as exterior forces, gravity could transport momentum that equals to momentum variation from frequency shift. That reveals light momentum and mass energy keep conservations in contra variant space.

As have said, mass energy keeps conservations in contra variant space. The experiments on the electric field causing deflections of accelerated charged particles [34] and the experiments on gravitational redshift of light rays have provided forceful evidences of mass energy equation and their conservativeness. The simplified mass equation could present most details of kinematics as has been shown in previous sections. But in strong fields or in conditions of more accurate discussions, the general mass equation is necessary.

We have seen that the general equations for massive matters, the Eq. (611), could be simplified to the Eq. (614) in the fields non-extreme. And for close-to-light-speed massive matters, the general dynamic equation Eq. (614) could be changed to Eq. (587) by combining likely items and furtherly simplified to the Binet equation, the Eq. (558), and for lower speed matters Eq. (614) could regress to Eq. (630) and furtherly to Binet equation Eq. (509). One would see that the two kinds of variations base on different conditions that previous has velocities up to light speed so that ξ will be close to 1.0, while the latter has low velocities so that ξ is about 0.5. Nevertheless, the two Binet equations Eq. (558) and Eq. (509) perform far different conditions in that the right side item in previous is far smaller than that of the latter because of the difference of angular momentum, so that the previous corresponds to the answer close to a straight line while the latter corresponds to the answer of a ellipse. The relationship between these equations indicates the exponent mass equation is the final

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form of mass equation, which could perform more precise physics. The general equations of both lower velocity motions and close-to-light-speed motions present trajectory perihelion. That reveals same origination in mass equation and energy equation.

Light mass could also be expressed in general forms, which brings about high accuracies and the applications in strong fields. Undoubtedly, the general equation Eq. (653) could be simplified to the Eq. (656), and regress to simplified equation, Eq. (462). Something different for light rays is that in weak fields the simplified solution has not bad precision yet. But for massive matters, general equation is necessary to present trajectory perihelion. Anyway, the regressibility of general equations has shown their universality on mass conservativeness.

It is said that all massive matters in different gravitational fields with different velocities could have the uniform dynamic equation. Light rays have another. Why do they, the light rays and massive matters, have different equations? It is not because of different Lagrangian. It is because of the conservativity that we are discussing on the general covariance, the inertial motion, the conservation of mass energy and the conservation of momentum.

10. Relativistic Release and Relativistic Frequency Shift

It is predicted that in one source field, the inflow of matters may cause relativistic release. Sole matters moving into the source center for a separation is of course exactly a kind of inflow. But I want to point out that the inflows of accretions of quasars and active galactic nuclei are of normal performances in celestial evolutions. As the matter of facts, they have not been well investigated, so that it is necessary to cast a few concentrations on.

10.1. Dynamics of Accretions of Quasars and Active Galactic Nuclei

10.1.1. Galactic Accretions and Planet Rings

The discovery of quasars and active galactic nuclei is of the greatest advancing of the astronomy in the 1960s [35,36]. Quasars and active galactic nuclei are of the most spectacular extragalactic objects not only because of their extraordinary giant radiations [37] but also due to the complicated spectra and very high redshifts. The emission spectra of quasars and active galactic nuclei are really non-thermal continuum [38]. Those issued models for quasars and active galactic nuclei in the past decades, such as the so-called standard model [39] or unified model [40], have not really interpreted the mechanism of accretion inflow and energy release. This is the reason that I want to sponsor a brief discussion on the dynamics of accretions of quasars and active galactic nuclei.

Galactic accretion disks exactly experience resemblant kinematics with planet ring systems in that both of them are of surrounding dusts revolving the center source, loosely inside the Roche limit [41]. I am going to sponsor a study firstly with respect to the planet ring systems so that to renovate those recognitions upon the evolution and fueling mechanism of galactic accretions.

Planet rings are not static structures [42,43]. Their evolutions involve with numerous internal and external processes in which Keplerian shear acts the key role. It causes rings to spread [44] in all process. As we have seen, Saturn rings usually have that striking refined structures.

In the condition that ring particles are small enough, the electromagnet interactions will then overcome the inertial forces so that to control the motion within the rings. Such particles or together with gases getting to form colloid, will be as a whole to perform fluid behaviors so as to bear pressure. Under the driving effects from outer boundaries, a fluid ring may go bolder cubically because pressure increasing helps particles to move deviate vertically to the ring plane, such as the Halo ring of Jupiter [45]. Spectrum observations predict that the particles in Halo ring are less than submicron. Optical observations also show that Halo ring seems blue and gray with respect to that the main rings show red color [45]. It is because that the fine particles scatter more blue rays, just like our sky does, while coarse particles scatter more red.

Firstly, I want to make a discussion based on a fluid ring in the condition that the center velocity equals to Keplerian velocity.

10.1.2. The Dynamic Models of Fluid Rings

For Keplerian motions of matters around center object, there is

$$mr\omega_k^2 = \frac{GMm}{r^2} \quad (787)$$

where, m is mass of a particle, G is gravitational constant, M is the mass of center source, r is distance to the center and $\omega_k = \frac{v_k}{r}$ is angular velocity while v_k is tangent velocity.

The tangent velocity and angular velocity will increase as radius getting smaller that

$$v_k = \frac{(GM)^{0.5}}{r^{0.5}} \ and \ \omega_k = \frac{(GM)^{0.5}}{r^{1.5}} \quad (788)$$

Keplerian kinetic energy could be expressed as

$$E_k = \frac{1}{2}mv_k^2 = \frac{1}{2}m\frac{GM}{r} \quad (789)$$

Keplerian motion determines constant angular momentum for a particle at specific position.

$$L_k = rmv_k = mr^{0.5}(GM)^{0.5} \quad (790)$$

Namely, there is Keplerian shear at the specific position, and the shearing rate can be derived to be

$$\frac{dv_k}{dr} = -\frac{(GM)^{0.5}}{2r^{1.5}} \ and \ \frac{d\omega_k}{dr} = -\frac{3(GM)^{0.5}}{2r^{2.5}} \quad (791)$$

I do not think that the dynamics of planet rings is very easy to be built in that the evolutions of the rings must be rarely sophisticated. I prefer to suggest three simplified models that may reveal a little bit of probabilities.

Model 1: Central Keplerian motion with driving front and resistant front

It is imagined that a ring keeps Keplerian velocity at its center position while outer part has velocity lower than Keplerian velocity so that to form driving force, and inner part has velocity higher than Keplerian velocity so that to form resistant force. In this way, the driving force and resistant force could help to form fluid pressure to bolder the ring at vertical dimension as shown in Figure 31. That really others to the flat rings as that of Saturn.

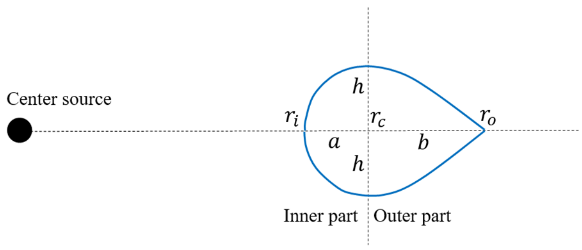

Figure 31. Fluid ring in one source field.

For example, in the outer part of a fluid ring, if the velocity is lower to form driving force, the pressure could be calculated as

$$p = \int_{r_o}^{r}(\rho\frac{GM}{x^2} - \rho x\omega^2)dx \quad (792)$$

where, ρ is density of fluid, r_o is radius from source center to the outer edge of fluid ring, and the angular velocity ω is less than Keplerian angular velocity ω_k of the very point so that to give driving force. If a linear function is assumed for an element of outer part

$$\omega^2 = (\omega_k^2 - \frac{r-r_c}{r_o-r_c}\Delta\omega_o^2) \quad (793)$$

where, r_c refers to radius of pressure center and $\Delta\omega_o^2$ represents the angular velocity square difference with respect to ω_k^2, which is the Keplerian angular velocity square at the very position r.

This equation presents an active falling of the outer part particles because of losing of energy and velocity.

Concern that at every position there is the relationship

$$\rho \frac{GM}{r^2} = \rho r \omega_k^2 \quad (794)$$

The Eq. (792) becomes

$$p = \int_{r_o}^{r} \rho x \frac{x - r_c}{r_o - r_c} \Delta\omega_o^2 \, dx \quad (795)$$

Giving the pressure of inner edge and outer edge equal to zero, and constant density, and $r_o - r_c \ll r_c$, there will be an approximate solution of center pressure

$$p_c \approx \frac{1}{2} \rho r_c (r_o - r_c) \Delta\omega_o^2 = \frac{1}{2} \rho r_c b \Delta\omega_o^2 \quad (796)$$

where, $b = r_o - r_c$ represents the outer radius of the fluid ring.

On another side, the resistant force determined by the higher velocity of inner parts, with velocity as

$$\omega^2 = (\omega_k^2 + \frac{r_c - r}{r_c - r_i} \Delta\omega_i^2) \quad (797)$$

In which ω^2 is greater than ω_k^2 so that to form a passive resistant force.

There will be the center pressure

$$p_c \approx \frac{1}{2} \rho r_c a \Delta\omega_i^2 \quad (798)$$

where, $a = r_c - r_i$ represents inner radius of fluid ring.

On the direction vertical to the plane, it is easily to calculate [36] that

$$p_c \approx \int_0^h \rho r \omega_{ck}^2 \frac{y}{r} dy = \frac{\rho h^2}{2} \omega_{ck}^2 \quad (799)$$

where, h represents vertical height from top or the bottom to the center of the ring, y is the distance to pressure center and ω_{ck} is Keplerian angular velocity at pressure center.

Thus, the relationships between semi axes are

$$a \approx \frac{b \Delta\omega_i^2}{\Delta\omega_o^2} \quad (800)$$

$$a \approx \frac{h^2 \omega_{ck}^2}{r_c \Delta\omega_i^2} \quad (801)$$

$$b \approx \frac{h^2 \omega_{ck}^2}{r_c \Delta\omega_o^2} \quad (802)$$

The outer edge of the ring may be called driving front, depending on a random falling of outer particles, that forms an ambiguous boundary. The inner edge of the ring comes from passive driving, may be called driven front, so that to form a sharp edge as seen in Halo ring [45], just like curved water surface in an accelerated container. Of course, fluid rings also can be flattened and split to be refined structures so that to perform more complicated than Halo ring.

Model 2: Driving condition with no shear motion in inner part

It is assumed that the driving force will form a pressure gradient in inner part that just fit the gravity difference coming from Keplerian velocity difference so that to form no shear motion as shown in Figure 32.

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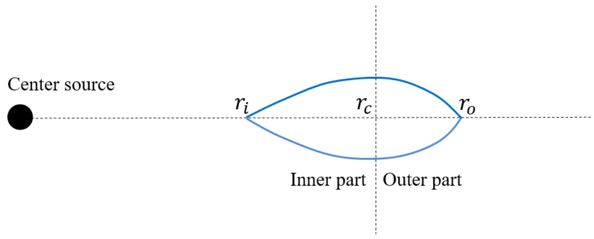

Figure 32. Driving condition with no shear motion in inner part. (Note, the curved line of cross section of the ring is an imaginary result probable, concerning variable density).

There will be

$$\omega_{ik} = \omega_i = \omega_c \quad (803)$$

It is said that any differential slice in the inner part may be accelerated by outer squeezing to catch up the velocity ω_{ik}.

That indicates an increase of the angular velocity square to meet the force equilibrium

$$\Delta\omega^2 = \omega_{ik}^2 - \omega_k^2 = \frac{GM}{{r_i}^3} - \frac{GM}{r^3} \quad (804)$$

Thus the ring pressure could be written as

$$p = \int_{r_i}^{r} \rho x \Delta\omega^2 \, dx = \int_{r_i}^{r} \rho x (\frac{GM}{{r_i}^3} - \frac{GM}{x^3}) \, dx \quad (805)$$

Assume a constant density, the central pressure is about

$$p_c \approx \rho \frac{GM}{{r_i}^2}(r_c - r_i) - \rho \frac{GM}{r_i r_c}(r_c - r_i)$$

$$\approx \rho \frac{GM}{{r_i}^2}(1 - \frac{r_i}{r_c})^2 \quad (806)$$

That will lead to a special state that inner part of the ring has no Keplerian shear. But this state may need strict conditions and may be destroyed by further evolution in a long term.

Model 3: Momentum conversion and ring split

After then, I want to discuss the split of a ring under the condition of developed Keplerian shear. If a fluid ring is well Keplerian sheared, it could be predicted that the ring will go thinner. We know that there may be Keplerian shear at any position in the ring. But we can make an analysis on the angular momentum conversion in the middle point which could then at last split the ring to two parts. For convenience, it could be defined as two rings originally, the ring i and the ring $i+1$ as shown in Figure 33. They will have their own Keplerian velocity points at r_i and r_{i+1}. In a time interval, the two rings will complete quite amount of angular momentum conversion because of the Keplerian shear, which is just like viscous friction acts in the interface that makes the conversion.

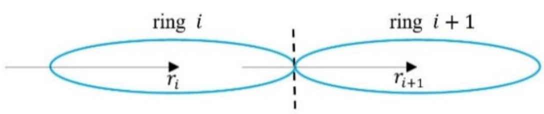

Figure 33. Angular momentum conversion between neighboring rings.

Setting the mass of rings $m_i = m_{i+1} = m$, the relative angular momentum difference between two neighboring rings could be estimated as

$$rm\Delta v = -m\frac{(GM)^{0.5}}{2r^{0.5}}\Delta r \quad (807)$$

where $\Delta r = r_{i+1} - r_i$ and $r \approx 0.5(r_{i+1} + r_i)$

The exchange of angular momentum may be expressed as

$$\Delta L = -\beta m \frac{(GM)^{0.5}}{2r^{0.5}} \Delta r \quad (808)$$

where, momentum exchange ratio $0 \le \beta \le 1$.

It is said that after a Keplerian shear, the outer one should get increase of angular momentum and the inner one should get that decline. For the reasons of kinetic mechanism, both the inner ring and outer ring will be driven apart from original position. If inner ring goes a difference δr, the difference of Keplerian angular momentum could be calculated as

$$\delta L_k = m \frac{(GM)^{0.5}}{2r^{0.5}} \delta r \quad (809)$$

Let δL_k equals to ΔL, it is obtained that

$$\delta r = \beta \Delta r \quad (810)$$

For the inner ring, it goes to shrink by falling a minus δr separation so that to form a new state. And for the outer ring, it expands by going up to a positive δr separation.

We have seen that this is not a whole dynamic analysis because that the pressure in the ring has also taken the effect of angular momentum conversion directly, but that does no harm for the conclusion that Keplerian shear, i.e., the viscous shear in the ring, leads to angular momentum conversion and rings split. One can make detailed study with Navier Stokes equations or the renovated ones.

A process of ring spreading around a source center is an irreversible process with entropy production.

10.1.3. Shearing Dissipation in Fluid Rings

This issue is carried out to estimate the shearing dissipation so that to discuss the evolution by Keplerian shear. Of course, the evolution would involve together with momentum conversion and ring split. But in the way of theoretical methodology, it is better to be investigated in two steps, the angular momentum conversion by Keplerian shear, and the energy momentum conservation during rings split.

Angular momentum conversion must be done across the interface between the two rings, so that the first step analysis is reasonable. I prefer to focus on the states before and after momentum conversion of two neighboring rings as has been shown in Figure 33. Before conversion, the neighboring rings have been separated by $\Delta r = r_{i+1} - r_i, (r_{i+1} > r_i)$, with velocities

$$v_i = \frac{(GM)^{0.5}}{r_i^{0.5}} \; and \; v_{i+1} = \frac{(GM)^{0.5}}{r_{i+1}^{0.5}} \quad (811)$$

The difference between v_i and v_{i+1} could be written after Eq. (791)

$$\Delta v = -\frac{(GM)^{0.5}}{2r^{1.5}} \Delta r \quad (812)$$

where, $r \approx 0.5(r_{i+1} + r_i)$, Δv is defined as minus quantity while Δr as plus. Namely,

$$v_{i+1} = v_i + \Delta v \quad (813)$$

Considering ring mass equals to each other, two conditions would be taken into study.

As angular momentum conversion happens, there are

$$v_{i,after} = v_i + \beta \Delta v \; and \; v_{i+1,after} = v_i + \Delta v - \beta \Delta v \quad (814)$$

As Δv has been defined minus, the inner ring is really slowed down and the outer ring is accelerated. Because of $0 \le \beta \le 1.0$, kinetic energy may not keep conservation. With original kinetic energy of

$$E_{before} = \frac{1}{2} m v_i^2 + \frac{1}{2} m (v_i + \Delta v)^2 \quad (815)$$

and the kinetic energy after conversion of

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$$E_{after} = \frac{1}{2}m(v_i + \beta\Delta v)^2 + \frac{1}{2}m(v_i + \Delta v - \beta\Delta v)^2 \quad (816)$$

Thus, there is the dissipation energy without considering potential variations

$$Dissipation_1 = E_{before} - E_{after} = \beta m\Delta v^2 - \beta^2 m\Delta v^2 \quad (817)$$

Considering the Eq. (812), it turns to be

$$Dissipation_1 = \beta(1-\beta)m\frac{GM}{4r^3}\Delta r^2 \quad (818)$$

Then go to the second step that after the first step of angular momentum conversion, the outer part of inner ring loses momentum so that provides driving force to drive the inner ring to inflow and the inner part of outer ring will be accelerated to higher momentum so that to squeeze the outer ring to outflow. This evolution will perform a little sophisticated in that the inflow ring should be accelerated by gravity and the outflow ring should be slow down by gravity. We know that the inflow of a ring will cause half potential release surplus with respect to the Keplerian energy requirement and the outflow will experience potential withdraw. And also, we know that the gravity is perpendicular to statistic velocity of the ring. These issues could be solved in fluid flow anyway, even though that must involve with the evolution of vortexes.

Now we are going to check the energy conservation of final state. After the inner ring inflows a separation δr and the outer ring outflows a separation δr, they come to a new state of Keplerian balance. Taking the inner ring for granted, before Keplerian shear, its total energy involves with Keplerian kinetic energy and potential

$$E_{i,before} = \frac{1}{2}m\frac{GM}{r_i} + 0 \quad (819)$$

After Keplerian shear and inflowing a separation δr, total energy becomes

$$E_{i,after} = \frac{1}{2}m\frac{GM}{r_i - \delta r} - m\frac{GM}{r_i^2}\delta r \quad (820)$$

and for outer ring, the energy before

$$E_{i+1,before} = \frac{1}{2}m\frac{GM}{r_{i+1}} + 0 \quad (821)$$

The energy of final state

$$E_{i+1,after} = \frac{1}{2}m\frac{GM}{r_{i+1} + \delta r} + m\frac{GM}{r_{i+1}^2}\delta r \quad (822)$$

Total energy difference before and after the two steps could be estimated by

$$E_{i,before} - E_{i,after} + E_{i+1,before} - E_{i+1,after}$$

$$= \left(\frac{1}{2}m\frac{GM}{r_i} - \frac{1}{2}m\frac{GM}{r_i - \delta r}\right) + \left(\frac{1}{2}m\frac{GM}{r_{i+1}} - \frac{1}{2}m\frac{GM}{r_{i+1} + \delta r}\right) + \left(\frac{GM}{r_i^2}\delta r - \frac{GM}{r_{i+1}^2}\delta r\right) \quad (823)$$

The first two items could be transformed as

$$\left(\frac{1}{2}m\frac{GM}{r_i} - \frac{1}{2}m\frac{GM}{r_i - \delta r}\right)$$

$$= \frac{1}{2}m\frac{GM}{r_i} - \frac{1}{2}mGM\left(\frac{1}{r_i} + \frac{1}{r_i^2}\delta r + \frac{1}{r_i^3}\delta r^2\right)$$

$$= GMm\left(-\frac{1}{2r_i^2}\delta r - \frac{1}{2r_i^3}\delta r^2\right) \quad (824)$$

The middle two items could be transformed as

$$\left(\frac{1}{2}m\frac{GM}{r_{i+1}} - \frac{1}{2}m\frac{GM}{r_{i+1} + \delta r}\right)$$

$$= \frac{1}{2}m\frac{GM}{r_{i+1}} - \frac{1}{2}mGM\left(\frac{1}{r_{i+1}} - \frac{1}{r_{i+1}^2}\delta r - \frac{1}{r_{i+1}^3}\delta r^2\right)$$

$$= GMm(\frac{1}{2r_{i+1}^2}\delta r + \frac{1}{2r_{i+1}^3}\delta r^2)$$

$$= GMm[\frac{1}{2(r_i+\Delta r)^2}\delta r + \frac{1}{2r_{i+1}^3}\delta r^2]$$

$$= GMm[\frac{1}{2r_i^2}(1-\frac{2\Delta r}{r_i})\delta r + \frac{1}{2r_{i+1}^3}\delta r^2] \quad (825)$$

so that the previous four items are

$$-GMm\frac{\Delta r\delta r}{r_i^3} = -GMm\beta\frac{\Delta r^2}{r_i^3} \quad (826)$$

The last two items are

$$(\frac{GMm}{r_i^2}\delta r - \frac{GMm}{r_{i+1}^2}\delta r)$$

$$= \frac{GMm}{r_i^2}\delta r - \frac{GMm}{(r_i+\Delta r)^2}\delta r$$

$$= \frac{GMm}{r_i^2}\delta r - \frac{GMm}{r_i^2}\delta r(1-\frac{2\Delta r}{r_i})$$

$$= 2\frac{GMm}{r_i^3}\Delta r\delta r$$

$$= 2GMm\beta\frac{\Delta r^2}{r_i^3} \quad (827)$$

The energy difference is total energy dissipation

$$Dissipation = E_{i,before} - E_{i,after} + E_{i+1,before} - E_{i+1,after}$$

$$= \beta GMm\frac{\Delta r^2}{r_i^3} \quad (828)$$

One can calculate the gravitational potential release of a single ring inflowing, although it does not sound reasonable for a comparison with dissipations because there is another ring outflowing. Anyway, that is

$$dW = m\frac{GM}{r^2}\delta r = \beta m\frac{GM}{r^2}\Delta r \quad (829)$$

Thus, we get the ratio of dissipation and potential release

$$\frac{Dissipation}{dW} = \frac{\Delta r}{r} \quad (830)$$

Or the ratio of dissipation-1 and potential release

$$\frac{Dissipation_1}{dW} = \frac{(1-\beta)\Delta r}{4r} \quad (831)$$

where $\beta \neq 0$.

Ring evolution dynamics indicates that the inner half of an accretion inclines to inflow and outer half inclines to outflow by and large. Additionally, the process of vortex evolution would dissipate, even though the two dissipations are not independent.

$$Dissipation_2 = Dissipation - Dissipation_1 = \frac{1}{4}\beta(3+\beta)m\frac{GM}{r^3}\Delta r^2 \quad (832)$$

It shows that the dissipation in vortex evolution might be greater than that in momentum exchange between rings.

We know that inter-ring separation Δr would be far less than the radius r of relative rings. It is said that in the condition of Keplerian shear, the dissipative energy is a higher rank infinitesimal quantity of that of potential release of inflow, no matter that inner ring potential release will support the outer ring potential withdrawing. Classical quasars could have luminosities of 10^{45} erg s^{-1} to 10^{46}

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erg s^{-1} [46]. It is unimaginable that those releases completely come from gravitational potential energy.

10.1.4. On Particle Rings

For solid particle rings, it is difficult to make a simple analysis on angular momentum conversions and on ring's evolutions. In fact, any head-to-head collisions, and partial collisions of edge-by-edge that might form additional rotations, between two particles, might cause great changes to their trajectories. It could be imagined that for numerous particles in neighboring rings, assuming very small velocity difference between any two Keplerian collision particles, we can predict quasi fluid behavior for Keplerian shear acting on neighboring rings, that in a whole, Keplerian shear help to make angular momentum conversion and ring split. One can also write down those equations of energy momentum just as that of fluid rings. But all in all, particle rings cannot form pressure so that they will only spread as a very thin disk with perfect and refined structures in the equator plane, unlike the fluid rings, which could form a bold 3-dimensional torus structures. I believe that numerical method could be expected for quasi fluid behavior simulation for Keplerian shear of particle rings, as the effects of the rings of Saturn have shown us.

10.2. Relativistic Release

Any efforts to conceiving the mechanism of relativistic release would encounter great difficulties, because of two reasons that relativistic release has not been well recognized and we still know less about the intrinsic structures of fundamental particles that might be revealed relativistic emissions. But on the other hand, tremendous of observations have shown incredible probabilities for intrinsic emissions and correlated pseudo redshifts, as well as intrinsic redshift of absorptions, which are so intensely corresponding to the inference of relativistic release that I cannot help to make a try.

A direct realization on this issue is that the conservation of electromagnetic momentum and invariance of electromagnetic wave speed will lead to the variations of efficiency of light moment transportation, then the interactions or so-called the forces between particles, especially that in intrinsic structures. The most complicated is that these variations would be anisotropic so that the dynamics must be even profound and comprehensive.

Despite the complexities of relativistic release for a massive matter inflow to the center source, I suggest a mathematic analysis based on an assumption of continuum release. It is still not a true expression of realities, but that may help for more understandings on that topic.

The apparent light speed c_0^1 could be employed to describe the concept of equivalent state in gravitational fields. It is the energy state corresponding to the total energy structure that forms the matter including the bound energy structures and sub-particles and the even sub-structures in them. Because the state of matters depends on the bounding capability and finally depends on the efficiency of momentum transportation so that they might vary case the matters inflow. Thus, the equivalent state is exactly equivalent to that of matters in very weak fields to keep congeneric structures. It may be a state after relativistic release, although this state should be not really stable because there should be energy deficiency in the other two dimensions.

One can discuss the momentum exchange efficiency under the conditions of different apparent light speeds so that to talk about the variation of equivalent state. Another important clue that indicates equivalent state is the fine structure constant. We know that fine structure constant determines the dimensions of condensed matters, of course that may determine the dimensions of intrinsic structures. We can imagine that the variation of apparent light speed could bring about the variation of fine structure constant so that to bring about the variation of the energy of equivalent state.

Maybe or not all of energy of a matter would take part in the relativistic release for the reasons of dynamics or intrinsic properties, for example, the most fundamental particles may be expected to be made up of quasi photons, in common recognitions, they maybe yes or not to be split anymore.

Thus, we would rather define the concept of releasable mass $\tilde{m}$. I cannot give an estimation whether the maximum releasable mass should be up to $0.9m_0$ or not, but it could be believed to be quite amount, where m_0 is total rest mass of original equivalent state, which corresponding to the state that the matters do not experience relativistic release.

Now for a center source field, the energy of equivalent state of matters is proposed to be

$$\tilde{E} = \tilde{m}(c_0^1)^2 \quad (833)$$

Matters located at different positions in the gravitational field perform different grades of equivalent states. As a matter goes a separation dr along the radial direction, the expression of exceeding energy is

$$d\tilde{E} = 2\tilde{m}c_0^1(c_0^1)'dr \quad (834)$$

As a matter goes from the farthest point to the position of radius r, the relativistic released energy could be integrated to be

$$\tilde{E}_{released} = \int_r^\infty d\tilde{E} = \tilde{m}c^2 - \tilde{m}(c_0^r)^2 = \tilde{m}c^2[1-(1-\frac{r^*}{r})^2] \quad (835)$$

It should be highlighted that the relativistic release is about the release of bound energy of bound matters rather than dynamic energy. Thus, the general mass equations may not be employed. Potential converted dynamic energy of matter is not internal energy except that would have been converted to be thermal energy. Thermal energy will not be considered release energy in our discussions.

Thereafter, we could also define a corresponding concept of the relativistic residual energy that, at a position, matters keep an amount of energy for subsequent releases,

$$\tilde{E}_{residual} = \tilde{m}c^2(1-\frac{r^*}{r})^2 \quad (836)$$

It seems that the residual energy may go zero as matters reaches r^*. It is just a solution of theoretical model. In practice, relativistic release will be intermittent, and matters perhaps to experience flattenizations as they close to r^* so that to bring about instabilities. As results, residual energy may not go zero at the end in practice. This effort is just to manage to perform kind of probabilities of relativistic release.

It can be calculated that most part of energy is kept before $10r^*$

$$\tilde{E}_{residual}(10r^*) = 0.81\tilde{m}c^2 \quad (837)$$

Now it is naturally to define the release rate by derivative of release energy along the radial direction to the center source. Released energy could be the integration of release distribution function from a position farthest to the position r or the minus inversely

$$\tilde{E}_{released} = \int_\infty^r edr = \int_r^\infty -edr \quad (838)$$

On the other hand, the residual energy could also be defined as an integration of release distribution from position r^* to position r or the minus form

$$\tilde{E}_{residual} = \int_{r^*}^r edr = -\int_r^{r^*} edr \quad (839)$$

Thus, the release distribution could be derived to be

$$e = 2\tilde{m}c^2(1-\frac{r^*}{r})\frac{r^*}{r^2} \quad (840)$$

It is easy to calculate that the peak value locates at the position

$$r_{emax} = \frac{3}{2}r^* \quad (841)$$

With the maximum release distribution

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$$e_{max} = \frac{8}{27}\frac{\tilde{m}c^2}{r^*} \quad (842)$$

One can calculate the release intensity of accretion inflow. Setting equivalent mass for every single ring in an accretion, the luminosity could be derived as

$$l = 2\tilde{\rho}^* v c^2 (1 - \frac{r^*}{r}) \frac{r^{*2}}{r^3} \quad (843)$$

where, $\tilde{\rho}^*$ is matter's assuming density at the position r^* and v is inflow velocity.

Suppose the accretion spreading sufficiently, it can be calculated that the most luminous area is at the position

$$r_{lmax} = \frac{4}{3} r^* \quad (844)$$

As well as maximum luminosity

$$l_{max} = \frac{27}{128}\frac{m\tilde{\rho}^* v c^2}{r^*} \quad (845)$$

Release distribution and accretion luminosity could be shown in Figure 34.

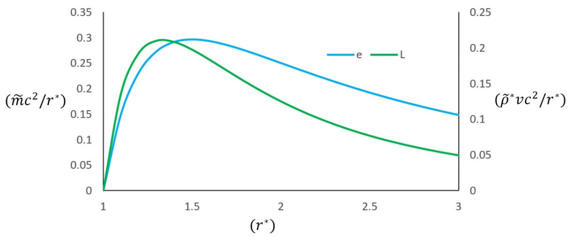

Figure 34. Release distribution and accretion luminosity.

The Eq. (840) and the Eq. (843) represents distribution and intensity of release at a specific position in an accretion disk. It is revealed that the peak luminosity should not happen at the edge of horizon event, but at a little outer position, just as that was shown in the event-horizon-scale images of M87 taken by the Event Horizon Telescope Collaboration with wavelength of 1.3mm [47].

10.3. Relativistic Emission Lines and Relativistic Redshift

Eyeing on the stimulated release of electrons in atoms, we could carry out a concept of exceeding ratio for relativistic release of an energy structure.

$$\sigma = \frac{\tilde{E}_{excited} - \tilde{E}_{released}}{\tilde{E}_{released}} \quad (846)$$

where $\tilde{E}_{excited}$ and $\tilde{E}_{released}$ are energy of excited state and released state of specific structure. Exceeding ratios for relativistic release may be more different from that of stimulated release of electrons, probably, they might be quite small or big. We know less about that.

In one source field, exceeding ratio would be variable with position as the so-called equivalent state has shown.

$$\sigma = \sigma_\infty (1 - \frac{r^*}{r})^2 \quad (847)$$

Thus, frequencies of specific emission rays could be written as

$$\nu_{rel} = \nu_\infty (1 - \frac{r^*}{r})^2 \quad (848)$$

Case $r \gg r^*$ it could be simplified to be

$$\nu_{rel} = \nu_\infty (1 - \frac{2r^*}{r}) \quad (849)$$

These emissions of course include but not limit to electron transition emissions that we are more familiar to that could be easily certificated comparing to stimulated release on the Earth. This kind of frequency will be quite different with which we have known well in weak field, in that it looks like redshifted after photons reach the Earth. But it is just a pseudo redshift because the emissions have not experienced real redshift at the emission time. The fact is that it is only verified to be redshifted by comparing to the spectrum characteristics of a matter in weak fields. In this case, it is just redshift seemingly. It could be called relativistic redshift.

In practice, emission lines will experience gravitational redshift to arrive a farthest position. The Eq. (571) could also be employed to interpret light mass energy variation in gravitational field, so that the detectable frequency could be expressed as

$$\nu_{det} = \nu_\infty(1-\frac{r^*}{r})^2 e^{-\frac{r^*}{2r}} \quad (850)$$

We have seen that the general mass equation employed for gravitational redshift is because that issue is of the issue of motions of light rays.

If a detector is at a position close to the Earth, one should furtherly calculate the gravitational frequency shift in gravitational field of the Earth. That will not be presented here.

Case $r \gg r^*$, the Eq. (850) could be simplified to be

$$\nu_{det} \approx \nu_\infty(1-\frac{r^*}{r})^2(1+\frac{r^*}{2r})^{-1}$$

$$= \nu_\infty(1-\frac{r^*}{r})^{2.5} \approx \nu_\infty(1-2.5\frac{r^*}{r}) \quad (851)$$

It is natural to propose a definition of relativistic frequency shift based on frequency

$$z_{rel} = \frac{\nu_\infty - \nu_{rel}}{\nu_{rel}} = (1-\frac{r^*}{r})^{-2} - 1 \quad (852)$$

Case $r \gg r^*$ it could be simplified to be

$$z_{rel} = \frac{\nu_\infty - \nu_{rel}}{\nu_{rel}} = \frac{2r^*}{r} \quad (853)$$

It is an intrinsic frequency shift rather than cosmological redshift. Note that ν_r keep invariant during propagation process if neglect the effect of gravitational redshift. ν_∞ is the comparative frequency of equivalent radiation in no gravity condition.

Detectable redshift should be calculated as

$$z_{det} = \frac{\nu_\infty - \nu_{det}}{\nu_{det}} = (1-\frac{r^*}{r})^{-2} e^{\frac{r^*}{2r}} - 1 \quad (854)$$

Case $r \gg r^*$ it could be simplified to be

$$z_{det} = (1-\frac{r^*}{r})^{-2.5} - 1 \approx \frac{2.5r^*}{r} \quad (855)$$

Relativistic frequency shift and detectable frequency shift could be calculated for comparison in Table 9.

Table 9. Relativistic and detectable frequency shifts at different positions.

Emission positions	Relativistic frequency shifts	Detectable frequency shifts
$1.3r^*$	17.778	26.585
$1.4r^*$	11.250	16.508
$1.5r^*$	8.000	11.561
$2.0r^*$	3.000	4.136
$3.0r^*$	1.250	1.658
$10.0r^*$	0.235	0.298
$20.0r^*$	0.108	0.136

The highest redshift we have observed on quasars is up to 16.4 [48,49]. It could be expected that higher redshift would be seen to be up to more than 20.0 in recent future. Perhaps higher redshifts have already been observed but not certificated.

Perhaps, up to now, we have found the probable way by which the accretions of quasars and active galactic nuclei release energy. It has been pointed out that the giant radiations of quasars and active galactic nuclei are not of thermal continuum. It has ever been believed that the radiations come from the energy of gravitational potential conversion [37], but we know that the energy of gravitational potential conversion must release radiation by the way of thermal black-body radiation. This will then lead to a contradiction. And in previous section, the dissipation of potential conversion of an accretion has been proved to be an infinitesimal of total conversion, that furtherly deduces the probabilities of potential origin of accretion radiation.

It is carried out that the detectable redshifts are just the property of relativistic emissions, which obviously rise against the theory of cosmological origin. The distribution statistics of observational high redshifts [50,51] might have indicated the redshifts are of property of the emissions rather than that of cosmic reasons.

Theoretically, it will be rarely probable that emission lines of multiple redshifts might be verified in some special observations in that the mechanism implies the probabilities although some of them may be ambiguous perhaps one of them dominates others as has been discussed. In fact, observations on frequency shift and line width and some other aspects of emissions might have present the performances of multiple redshifts. Celestial observations have always brought us surprises, multiple redshift emission might be the new one.

10.4. Broad Lines and Narrow Lines

In the emissions of quite amount of continuous inflow matters, the exciting width that the inflow involved sustains at radial direction may cause specific continuous emission distribution. As a result, that could be certificated to be a broadened line in spectrum diagram. For a massive accretion around a galactic nucleus, the Kepler shear may be employed to interpret the mechanism of inflow of a ring. If there is an inflow of a ring at position r with exciting width of Δr, the emission frequency might be distributed from ν_r to $\nu_{r+\Delta r}$. Considering the effect of gravitational redshift, one can get a line width expressed by frequency after Eq. (850) that

$$w_\nu = \nu_{det\,(r+\Delta r)} - \nu_{det\,(r)} = [(1-\frac{r^*}{r+\Delta r})^2 e^{-\frac{r^*}{2(r+\Delta r)}} - (1-\frac{r^*}{r})^2 e^{-\frac{r^*}{2r}}]\nu_\infty \quad (856)$$

Case $r \gg r^*$ and $r \gg \Delta r$ it could be simplified to be

$$w_\nu \approx (1-\frac{2.5r^*}{r+\Delta r} - 1 + \frac{2.5r^*}{r})\nu_\infty = (\frac{1}{r} - \frac{1}{r+\Delta r})2.5r^*\nu_\infty \approx \frac{2.5\Delta r r^*}{r^2}\nu_\infty \quad (857)$$

or by wave length

$$w_\lambda = \lambda_r - \lambda_{r+\Delta r} = c/\nu_{r\to d} - c/\nu_{r+\Delta r\to d}$$

$$= [(1-\frac{r^*}{r})^{-2} e^{\frac{r^*}{2r}} - (1-\frac{r^*}{r+\Delta r})^{-2} e^{\frac{r^*}{2(r+\Delta r)}}]\lambda_\infty \quad (858)$$

Case $r \gg r^*$ and $r \gg \Delta r$ it could be simplified to be

$$w_\lambda \approx (1+\frac{2.5r^*}{r} - 1 - \frac{2.5r^*}{r+\Delta r})\lambda_\infty \approx \frac{2.5\Delta r r^*}{r^2}\lambda_\infty \quad (859)$$

which could also be amount to a so-called Doppler velocity by frequency as

$$v_{D\nu} = \frac{w_\nu}{\nu_r}c \quad (860)$$

or Doppler velocity by wave length as

$$v_{D\lambda} = \frac{w_\lambda}{\lambda_r}c \quad (861)$$

where, c is light speed.

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Case $r \gg r^*$ it could be simplified to be

$$v_{D\nu} \approx v_{D\lambda} \quad (862)$$

These equations show that in the regions near to r^*, relativistic emissions should have broader line widths, and in the regions far off the r^*, relativistic emissions should have more narrow line widths. Calculations on line widths of various conditions will be presented in Table 10.

Table 10. Emission line widths at different positions with specific exciting widths.

Emission positions	Exciting widths	w_ν/ν_∞	$v_{D\nu}$ km/s	w_λ/λ_∞	$v_{D\lambda}$ km/s
$1.5r^*$	$0.1r^*$	0.02327	87681	2.8408	67850
$1.5r^*$	$0.05r^*$	0.01158	43634	1.5949	38093
$3.0r^*$	$0.1r^*$	0.01433	11425	0.09752	11006
$3.0r^*$	$0.05r^*$	0.007239	5772	0.05018	5663
$10.0r^*$	$0.1r^*$	0.002079	809	0.003492	807
$10.0r^*$	$0.05r^*$	0.001044	406	0.001756	406
$20.0r^*$	$0.1r^*$	0.0005706	194	0.0007360	194
$20.0r^*$	$0.05r^*$	0.0002860	97	0.0003690	97
$100.0r^*$	$0.1r^*$	0.00002455	7.6	0.00002582	7.6
$100.0r^*$	$0.05r^*$	0.00001228	3.8	0.00001291	3.8

This result forcefully matches the observational dimensions of broad line regions and narrow line regions. It is said that line widths depend on inflow exciting width, but in totally analysis, they highly relate to the inflow positions. The regions close to r^* may emit lines broader than that of farther regions. Observations show that the sizes of broad line regions are estimated within 0.01 or 0.1pc and line widths are about 1000 kms^{-1} to 10000 kms^{-1}, while the sizes of narrow line regions are about 0.1pc to 1kpc [50,52,53]. In fact, those certificated narrow lines incline to have been certificated as lower redshift lines. For radio emissions, some of them were observed to have compact cores and extended components [52]. The compact cores could be interpreted that equivalent state in inner areas of accretions would have lower equivalent energy to release lower frequency emissions, while some of the giant extended components of emission pictures might be the jet outflow who emit radio lines.

Relativistic release must perform in sophisticated conditions. The most probable condition is that the release in one of the rings could overwhelming all the others, especially that of the most inner ring. Thus, in most cases, releases of an active galactic nucleus might be certificated by the emissions of the sole ring. One can find that most of narrow lines have been certificated lower redshift, such as NGC 4151 with z=0.0033 [54] and MCG-5-23-16 with z=0.00849 [55] have narrow Fe Kα lines. But on the other hand, their spectra both involve with broad lines [56], that indicate controversies in emission line certifications.

The variability, asymmetry, wave length shift and so much as line broadening of emissions of quasars and active nuclei [51] indicate more sophisticated dynamic conditions than imagine.

10.5. Relativistic Absorption

Most of quasars and active galactic nuclei have giant accretion bodies spreading around their equatorial planes. Case a pulse of light ray goes through some parts of an accretion body, that light ray may experience different absorptions due to their passing positions. If a pulse of light crosses several separate rings or blocks of an accretion, it might experience multiple absorptions. Absorption frequencies depend on equivalent state of the matters that have experienced relativistic release, so that the absorption frequencies could be expressed as

$$\nu_{rel} = \nu_\infty(1 - \frac{r^*}{r})^2 \quad (863)$$

and the detectable frequency is

$$\nu_{det} = \nu_\infty(1 - \frac{r^*}{r})^2 e^{-\frac{r^*}{2r}} \quad (864)$$

where, ν_r is the frequency of absorption line in that the light rays cross throughout a ring located at a position of r, and ν_∞ is the absorption frequency case that structure would absorb in no-gravity fields. One can of course calculate the widths of absorption lines just like that of emission lines.

$$w_\nu = [(1-\frac{r^*}{r+\Delta r})^2 e^{-\frac{r^*}{2(r+\Delta r)}} - (1-\frac{r^*}{r})^2 e^{-\frac{r^*}{2r}}]\nu_\infty \quad (865)$$

or

$$w_\lambda = [(1-\frac{r^*}{r})^{-2} e^{\frac{r^*}{2r}} - (1-\frac{r^*}{r+\Delta r})^{-2} e^{\frac{r^*}{2(r+\Delta r)}}]\lambda_\infty \quad (866)$$

where, Δr is absorption width of a ring that the light ray passes across.
or expressed by Doppler velocity in frequency as

$$v_{D\nu} = \frac{w_\nu}{\nu_r} c \quad (867)$$

or Doppler velocity in wave length as

$$v_{D\lambda} = \frac{w_\lambda}{\lambda_r} c \quad (868)$$

But more different from emissions, absorption spectra must have more narrow lines and lower redshift than emissions, because the absorption rings are mostly at more outer regions with respect to the inner shining emission source, otherwise they would be difficult to be detected.

As has been discussed above, the inflow of inner rings of accretion may act as key role of entire emissions. So, we can image that brighter inner rings emit light rays, and on our view line, they go through some surrounding rings to be absorbed. It is said that, absorbing dusts might be the matters located at surrounding outer positions rather than those so-called insert bodies far away from the emission areas. This image may lead to the so-called relative blueshift, with respect to that of emission lines. Of course, there might be still seldom absorptions happening in inner areas so that we will also observe relative redshift absorption lines occasionally. Given a separation Δr between emission position and absorption position more exterior, the emission frequency could be calculated as

$$\nu_{det\ (em)} = \nu_\infty (1-\frac{r^*}{r})^2 e^{-\frac{r^*}{2r}} \quad (869)$$

with redshift of the line

$$z_{det\ (em)} = \frac{\nu_\infty - \nu_{det\ (em)}}{\nu_{det\ (em)}} \quad (870)$$

and absorption frequency varies because a separation Δr between absorption and emission positions

$$\nu_{det\ (ab)} = \nu_\infty (1-\frac{r^*}{r})^2 e^{-\frac{r^*}{2(r+\Delta r)}} \quad (871)$$

so that the redshift of absorption line is

$$z_{det\ (ab)} = \frac{\nu_\infty - \nu_{det\ (ab)}}{\nu_{det\ (ab)}} \quad (872)$$

And then, there is a blueshift of for an absorption line from the emission line

$$\Delta z = z_{det\ (ab)} - z_{det\ (em)} = (1-\frac{r^*}{r})^{-2} e^{\frac{r^*}{2(r+\Delta r)}} - (1-\frac{r^*}{r})^{-2} e^{\frac{r^*}{2r}} \quad (873)$$

Given $r = 2.5r^*$ and $\Delta r = 0.5r^*$, it can be calculated that $z_{det\ (em)} = 2.39$ and $z_{det\ (ab)}$ =1.66. Here, the absorption lines may show a blueshift of 0.73 with respect to the emission lines.

It was found that there are both absorption broad line region and absorption narrow line region [57] in single system in which those absorption lines are all corresponding to that of emission lines.

Case a continuum spectrum of light rays passes across multiple rings and arrive at an observatory on the Earth, one can then get a set of multiple frequency shift of absorptions. It is because absorptions only depend on the positions in accretions. This does interpret the multiple redshift absorption lines in tremendous of observations [50–52] in those quasars and active galactic nuclei. Especially for that of so-called Lyman-alpha forests [58], a series of regular Lyman-alpha absorption lines, with descending order of redshifts, queue up amazingly at the left side of the great main emission line. It could be predicted that, for Lyman-alpha absorption quasars, the angles between sight lines and the corresponding accretion planes of quasars might keep quite small degree in that emission light sources cannot pass through multiple surrounding rings to be absorbed in greater sight line angle accretions. It should be interesting to do correlation analysis upon the sight line angles of Lyman-alpha absorption quasars.

There is a special condition for a matter out flowing from center source, for example the center jet outflow. When a matter at ground state moves a separation toward outer direction, its structural energy should experience energy deficiency in that the energy of equivalent state gets increasing. We then see that the matter is in the state lack of energy, and generally, the state should be kept until surrounding environment happens to emit amount of specific light rays across, so as the result, the matter gets absorptions. It is said that, this kind of absorptions would be far different from normal relativistic absorptions as that have been discussed above, so that they could be called the super relativistic absorption. As for the mentioned absorption energy, it depends on the position it getting absorbing as well as the position it has ever been in ground state. Note that this kind of absorptions really distinguish from the general relativistic absorptions in that the super relativistic absorption energy depends on the difference between the final emission position and specific absorption position, while the general relativistic absorption is due to property of structure state. It could be calculated as

$$\Delta\nu = \nu_{ab} - \nu_{gr} \quad (874)$$

where, $\nu_{ab} = (1-\frac{r^*}{r_{ab}})^2 e^{\frac{-r^*}{2r_{ab}}}\nu_\infty$, $\nu_{gr} = (1-\frac{r^*}{r_{gr}})^2 e^{\frac{-r^*}{2r_{gr}}}\nu_\infty$, r_{ab} is the radius at absorption position, and r_{gr} is the radius of the position of equivalent ground state before outflow. This kind of absorption might need greater absorption energy than that of general relativistic absorption. Therefore, it should be called the super relativistic absorption.

To the observer at a farthest position, the absorption line will also perform special frequency shift that

$$z_{super} = \frac{\nu_\infty - \Delta\nu}{\Delta\nu} = \frac{1}{(1-\frac{r^*}{r_{ab}})^2 e^{\frac{-r^*}{2r_{ab}}} - (1-\frac{r^*}{r_{gr}})^2 e^{\frac{-r^*}{2r_{gr}}}} - 1 \quad (875)$$

Case the jet flow particles of an active galactic nucleus are assigned in a broad area, it could cause a broader absorption pit for a crossing light ray. One can calculate the average redshift and the width of absorption pit.

Additionally speaking on the topic of jet flow, massive particles may be easily accelerated up and keep approaching absolute light speed so as to escape from inner event horizon. On the other hand, apparent light speed may be at lower level in the nearby r^* area so that to form pseudo super luminal motions. A high pseudo super luminal motion may cause Cherenkov radiation, which could generally show polarized propagation. That might also be the reasons for some high energy light rays. One can imagine that the events of γ-ray burst detection delay in observations on SN1987A [59] and GW170817 [60] might be interpreted by pseudo super luminal propagations of neutrinos and gravitational waves.

Acknowledgments: This article has experience persistent revision that has greatly improved the readability whereas that was involved with tremendous calculations and deep going investigations. I appreciate anyone who would like to download and read through. As a while, I expect you to make more efforts to investigate part of or most part of the studies presented, that will indeed bring about new discoveries and comprehensive recognitions. I don't think those equations and solutions were too difficult to be checked and inspected, but some of them might really require reduplicative computations for finally confirmations because of complicated conditions involved. I don't think all of the equations have been quite closed to truth, just as well as you would find, but in fact I believe and expect some of them are. If then, they should be innegligible. Great many thanks!

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