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Posted Date: 26 January 2026

doi: 10.20944/preprints202601.1875.v1

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Article

Fractional Memory Meets Neural Oscillations: Analytical Foundations for the Delayed FitzHugh-Nagumo Model

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Abstract

This paper develops a rigorous and unified analytical framework for the study of implicit fractional-order dynamical systems incorporating distributed memory and discrete time delays. The core contribution is the derivation of novel *multivariate Grönwall–Wendroff-type integral inequalities* tailored to weakly singular fractional kernels and implicit nonlinear structures. These inequalities yield explicit *a priori* bounds for vector-valued solutions and constitute a fundamental analytical tool for controlling nonlocal dynamics induced by fractional memory and delayed feedback. Building on these estimates, we establish general existence, uniqueness, and continuous dependence results for a broad class of implicit fractional differential equations by means of fixed-point arguments in suitably weighted Banach spaces. Furthermore, we prove *Ulam–Hyers stability*, providing quantitative robustness estimates with respect to modeling uncertainties and perturbation errors, and extending classical stability notions to implicit systems with memory effects. The applicability of the proposed framework is demonstrated through a detailed analysis of the delayed fractional FitzHugh–Nagumo (FHN-α-τ) model, a canonical system in mathematical neuroscience. We derive explicit and verifiable conditions ensuring the existence of nontrivial, orbitally stable limit cycles corresponding to sustained neuronal spiking. In particular, we show that the interplay between subdiffusive fractional memory ($\alpha < 1$) and synaptic delay (τ) can induce and regulate self-sustained oscillations, yielding precise parameter thresholds for excitability and rhythmic activity. Overall, this work bridges a gap between abstract fractional calculus and applied dynamical systems theory, providing a versatile analytical toolkit for systems governed by long-range temporal correlations and delayed feedback, with potential applications in neurodynamics, complex biological networks, materials science, and control theory.

Keywords: fractional calculus; distributed memory systems; Grönwall–Wendroff inequalities; Ulam–Hyers stability; FitzHugh–Nagumo model

MSC: 34A08, 34K37, 45G10, 92C20

1. Introduction

Dynamical systems with long-range temporal dependencies, often referred to as systems with memory or nonlocal temporal behavior, are pervasive across scientific disciplines ranging from neurodynamics to material science. Unlike conventional ordinary differential equations, these systems exhibit evolution that depends intrinsically on their historical states, not merely on instantaneous conditions. Fractional calculus has emerged as a natural mathematical framework for such phenomena, replacing integer-order derivatives with integro-differential operators that weigh past states according to singular, power-law kernels [1,2].

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In this work, we consider a comprehensive class of implicit fractional differential systems with distributed memory, formally described by

$$ {}^{C}\mathcal{D}_0^{\alpha}\mathbf{x}(t) = \mathbf{F}\Big(t, \mathbf{x}(t), {}^{C}\mathcal{D}_0^{\alpha}\mathbf{x}(t), \mathcal{I}[\mathbf{x}](t)\Big), \quad t \in [0,T], \tag{1} $$

subject to the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Here, $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, ${}^{C}\mathcal{D}_0^{\alpha}$ denotes the vector-valued Caputo fractional derivative of order $\alpha \in (0,1]^n$ [3], and $\mathcal{I}[\mathbf{x}](t)$ represents a generic memory operator capable of incorporating fixed time delays, distributed integrals, or fractional integral terms. This implicit formulation, where the fractional derivative appears on both sides of the equation, captures self-regulatory mechanisms and intrinsic feedback often present in biological, physical, and engineering systems [4,5].

The analytical study of such systems presents significant challenges, particularly in establishing well-posedness, stability, and qualitative behavior under the combined effects of fractional memory, implicit structure, and distributed delays. While fractional versions of classical tools like the Grönwall inequality have been developed for simpler settings [6,7], a unified theory for implicit multivariate systems with compound memory remains largely unexplored. Recent interest in Ulam–Hyers stability for fractional equations [8,9] further motivates a rigorous investigation of robustness in this generalized context.

This work aims to bridge this gap by developing a coherent analytical framework based on novel fractional integral inequalities, fixed-point theory in weighted Banach spaces, and qualitative analysis techniques. We derive extended Grönwall–Wendroff-type inequalities tailored to implicit systems, prove general existence and uniqueness theorems, establish Ulam–Hyers stability criteria, and apply the theory to the fractional FitzHugh–Nagumo model with delay, a canonical neurodynamical system exhibiting both subdiffusive memory and discrete propagation delays [10]. Our results provide rigorous foundations for modeling and analyzing complex systems where history dependence and self-regulation play fundamental roles.

2. Fractional Wendroff-Type Integral Inequalities

In this section we establish a class of integral inequalities of Grönwall–Wendroff type adapted to fractional-order systems with memory and time delay. Such inequalities play a central role in the qualitative analysis of fractional functional differential equations, providing a priori bounds, stability estimates, and continuous dependence results. The presence of weakly singular kernels and delayed arguments requires a careful extension of classical techniques to the fractional setting.

Definition 1. *Let $k : [0,T] \times [0,T] \to \mathbb{R}_+$ be a weakly singular kernel of the form*

$$ k(t,s) = (t-s)^{\beta-1} g(t,s), \quad \beta > 0, \tag{2} $$

where g is continuous and bounded on $[0,T] \times [0,T]$. We define the associated fractional integral operator

$$ (\mathcal{K}u)(t) = \int_0^t k(t,s)\, \phi\big(s, u(s), u(s-\tau)\big)\, ds, \tag{3} $$

where ϕ is a continuous nonlinear function and $\tau > 0$ denotes a fixed time delay.

The kernel (2) naturally generalizes the Riemann–Liouville fractional integral kernel and allows the inclusion of variable coefficients, thereby covering a broad class of nonlocal memory effects.

Theorem 1 (Multivariate Fractional Grönwall–Wendroff Inequality). *Let $u, f \in C([0,T], \mathbb{R}^n)$ with $u(t) \geq 0$ and $f(t) \geq 0$ componentwise. Assume that*

$$ u(t) \leq f(t) + \int_0^t (t-s)^{\alpha-1} A(s) u(s)\, ds + \int_0^t (t-s)^{\beta-1} B(s) u(s-\tau)\, ds, \tag{4} $$

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for all $t \in [0,T]$, where $\alpha, \beta \in (0,1]$, $A, B \in L^\infty([0,T], \mathbb{R}^{n\times n})$, and $u(t) = \phi(t)$ for $t \in [-\tau, 0]$. Then there exists a constant $M > 0$, depending only on $\alpha, \beta, \|A\|_\infty, \|B\|_\infty$ and T, such that

$$u(t) \le M\Big(\|\phi\|_{[-\tau,0]} + \sup_{s\in[0,t]} f(s)\Big), \qquad \forall\, t \in [0,T]. \tag{5}$$

Proof. Define $v(t) = \sup_{s\in[0,t]} u(s)$, which is nonnegative and nondecreasing. From (4) and the boundedness of A and B, we obtain

$$u(t) \le f(t) + \|A\|_\infty \int_0^t (t-s)^{\alpha-1} v(s)\, ds + \|B\|_\infty \int_0^t (t-s)^{\beta-1} v(s)\, ds. \tag{6}$$

Since v is nondecreasing, $v(s) \le v(t)$ for all $s \le t$, yielding

$$v(t) \le \|f\|_\infty + v(t)\Big(\|A\|_\infty \int_0^t (t-s)^{\alpha-1} ds + \|B\|_\infty \int_0^t (t-s)^{\beta-1} ds\Big). \tag{7}$$

A direct computation gives

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha}, \tag{8}$$

$$\int_0^t (t-s)^{\beta-1} ds = \frac{t^\beta}{\beta}. \tag{9}$$

Therefore,

$$v(t) \le \|f\|_\infty + v(t)\left(\|A\|_\infty \frac{T^\alpha}{\alpha} + \|B\|_\infty \frac{T^\beta}{\beta}\right). \tag{10}$$

If T is sufficiently small such that

$$\|A\|_\infty \frac{T^\alpha}{\alpha} + \|B\|_\infty \frac{T^\beta}{\beta} < 1, \tag{11}$$

then $v(t)$ is uniformly bounded on $[0,T]$. For arbitrary T, the result follows by a standard partitioning of $[0,T]$ into finitely many subintervals and an iteration argument. □

3. Existence and Uniqueness for Implicit Fractional Systems

In this section we establish existence and uniqueness of solutions for a class of implicit fractional-order systems with memory effects. Unlike standard explicit fractional differential equations, the Caputo derivative appears both explicitly and implicitly through the nonlinear term, which prevents a direct application of classical fixed-point arguments. To overcome this difficulty, we recast the problem as a nonlinear Volterra-type integral equation and employ a contraction principle in a suitably weighted space.

We consider the general implicit system

$$^{C}\mathcal{D}_0^\alpha \mathbf{x}(t) = \mathbf{F}\Big(t, \mathbf{x}(t), {}^{C}\mathcal{D}_0^\alpha \mathbf{x}(t), \mathcal{I}[\mathbf{x}](t)\Big), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{12}$$

where $0 < \alpha \le 1$, $\mathbf{x} : [-\tau, T] \to \mathbb{R}^n$ and $\mathcal{I}$ denotes a linear memory operator.

We impose the following assumptions on the nonlinear term $\mathbf{F}$:

1. *Lipschitz continuity.* There exists $L > 0$ such that for all admissible arguments

$$\|\mathbf{F}(t, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1) - \mathbf{F}(t, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)\| \le L\big(\|\mathbf{x}_1 - \mathbf{x}_2\| + \|\mathbf{y}_1 - \mathbf{y}_2\| + \|\mathbf{z}_1 - \mathbf{z}_2\|\big). \tag{13}$$

2. *Continuity of the memory operator.* The operator $\mathcal{I}$ is linear and continuous on $C([-\tau, T], \mathbb{R}^n)$.

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Theorem 2 (Existence and Uniqueness). *Under the above assumptions, there exists a unique solution* $\mathbf{x} \in C([-\tau, T], \mathbb{R}^n)$ *of* (12) *for sufficiently small* $T > 0$.

Proof. Using the definition of the Caputo derivative, the implicit system (12) can be rewritten in the equivalent integral form

$$\mathbf{x}(t) = \mathbf{x}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbf{F}\Big(s, \mathbf{x}(s), {}^C\mathcal{D}_0^\alpha \mathbf{x}(s), \mathcal{I}[\mathbf{x}](s)\Big)\, ds. \tag{14}$$

Define the operator $\mathcal{T} : C([-\tau, T], \mathbb{R}^n) \to C([-\tau, T], \mathbb{R}^n)$ by

$$(\mathcal{T}\mathbf{x})(t) = \mathbf{x}_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbf{F}\Big(s, \mathbf{x}(s), {}^C\mathcal{D}_0^\alpha \mathbf{x}(s), \mathcal{I}[\mathbf{x}](s)\Big)\, ds. \tag{15}$$

We endow $C([-\tau, T], \mathbb{R}^n)$ with the exponentially weighted norm

$$\|\mathbf{x}\|_\rho = \sup_{t \in [0,T]} e^{-\rho t} \|\mathbf{x}(t)\|, \qquad \rho > 0, \tag{16}$$

which compensates for the growth induced by the weakly singular kernel.

Let $\mathbf{x}_1, \mathbf{x}_2 \in C([-\tau, T], \mathbb{R}^n)$. Using the Lipschitz property of $\mathbf{F}$ and linearity of $\mathcal{I}$, we obtain

$$\begin{aligned} \|(\mathcal{T}\mathbf{x}_1)(t) - (\mathcal{T}\mathbf{x}_2)(t)\| \le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ \times \Big(\|\mathbf{x}_1(s) - \mathbf{x}_2(s)\| + \|{}^C\mathcal{D}_0^\alpha(\mathbf{x}_1 - \mathbf{x}_2)(s)\| + \|\mathcal{I}[\mathbf{x}_1 - \mathbf{x}_2](s)\| \Big)\, ds. \end{aligned} \tag{17}$$

By continuity of $\mathcal{I}$ and standard estimates for the Caputo derivative, there exists $C > 0$ such that

$$\|(\mathcal{T}\mathbf{x}_1)(t) - (\mathcal{T}\mathbf{x}_2)(t)\| \le C \int_0^t (t-s)^{\alpha-1} e^{\rho s} \|\mathbf{x}_1 - \mathbf{x}_2\|_\rho\, ds. \tag{18}$$

Multiplying both sides by $e^{-\rho t}$ and choosing $\rho > 0$ sufficiently large, the operator $\mathcal{T}$ becomes a strict contraction on $C([-\tau, T], \mathbb{R}^n)$. The result follows from the Banach Fixed-Point Theorem. □

4. Ulam–Hyers Stability

In this section we investigate the stability of solutions to the implicit fractional system (12) in the sense of Ulam–Hyers. This notion of stability quantifies the robustness of exact solutions with respect to small perturbations in the governing equation, and is particularly relevant in fractional-order models, where memory effects and numerical approximations naturally introduce perturbations.

Definition 2. *The implicit fractional system* (12) *is said to be* Ulam–Hyers stable *if there exists a constant* $C > 0$ *such that for every function* $\mathbf{y} \in C([0, T], \mathbb{R}^n)$ *satisfying*

$$\Big\| {}^C\mathcal{D}_0^\alpha \mathbf{y}(t) - \mathbf{F}\Big(t, \mathbf{y}(t), {}^C\mathcal{D}_0^\alpha \mathbf{y}(t), \mathcal{I}[\mathbf{y}](t)\Big) \Big\| \le \varepsilon, \qquad \forall\, t \in [0, T], \tag{19}$$

there exists an exact solution $\mathbf{x}$ *of* (12) *such that*

$$\|\mathbf{y}(t) - \mathbf{x}(t)\| \le C\, \varepsilon, \qquad \forall\, t \in [0, T]. \tag{20}$$

This definition ensures that any ε-approximate solution remains uniformly close to a true solution, with an error proportional to the size of the perturbation.

Theorem 3 (Ulam–Hyers Stability). *Under the assumptions of Theorem 2, the implicit fractional system* (12) *is Ulam–Hyers stable on* $[0, T]$.

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Proof. Let $\mathbf{y}$ be an ε-approximate solution of (12) and define the perturbation function

$$\eta(t) = {}^{C}\mathcal{D}_0^{\alpha}\mathbf{y}(t) - \mathbf{F}\Big(t, \mathbf{y}(t), {}^{C}\mathcal{D}_0^{\alpha}\mathbf{y}(t), \mathcal{I}[\mathbf{y}](t)\Big),$$

which satisfies $\|\eta(t)\| \le \varepsilon$ for all $t \in [0, T]$. Using the integral representation of the Caputo derivative, we obtain

$$\mathbf{y}(t) = \mathbf{y}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[\mathbf{F}\Big(s, \mathbf{y}(s), {}^{C}\mathcal{D}_0^{\alpha}\mathbf{y}(s), \mathcal{I}[\mathbf{y}](s)\Big) + \eta(s)\Big] ds. \tag{21}$$

Let $\mathbf{x}$ denote the unique exact solution of (12) satisfying $\mathbf{x}(0) = \mathbf{y}(0)$. Subtracting the corresponding integral equation for $\mathbf{x}$ yields

$$\begin{aligned}\mathbf{y}(t) - \mathbf{x}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[&\mathbf{F}\Big(s, \mathbf{y}(s), {}^{C}\mathcal{D}_0^{\alpha}\mathbf{y}(s), \mathcal{I}[\mathbf{y}](s)\Big) \\ &- \mathbf{F}\Big(s, \mathbf{x}(s), {}^{C}\mathcal{D}_0^{\alpha}\mathbf{x}(s), \mathcal{I}[\mathbf{x}](s)\Big) + \eta(s)\Big] ds.\end{aligned} \tag{22}$$

Taking norms and applying the Lipschitz condition on $\mathbf{F}$, we obtain

$$\|\mathbf{y}(t) - \mathbf{x}(t)\| \le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\mathbf{y}(s) - \mathbf{x}(s)\|\, ds + \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds. \tag{23}$$

Since $\int_0^t (t-s)^{\alpha-1} ds = \frac{t^{\alpha}}{\alpha}$, the second term is uniformly bounded on $[0, T]$. An application of the fractional Grönwall–Wendroff inequality (Theorem 1) yields the estimate

$$\|\mathbf{y}(t) - \mathbf{x}(t)\| \le C\,\varepsilon, \qquad \forall\, t \in [0, T], \tag{24}$$

for some constant $C > 0$ depending only on the parameters of the system. □

5. Application to the Delayed Fractional FitzHugh–Nagumo Model

We now apply the abstract theory developed in Sections 2–4 to the delayed fractional-order FitzHugh–Nagumo (FHN-α-τ) model. This system serves as a canonical example where fractional memory effects and delayed feedback mechanisms interact to produce complex neurodynamical behavior.

5.1. Model Formulation and Well-Posedness

The FHN-α-τ system is given by:

$$\begin{cases} {}^{C}\mathcal{D}_0^{\alpha} v(t) = v(t) - \dfrac{v^3(t)}{3} - w(t) + I_{\text{ext}} + \lambda v(t-\tau), & t > 0, \\ {}^{C}\mathcal{D}_0^{\alpha} w(t) = \varepsilon\big(v(t) + a - b w(t)\big), & t > 0, \end{cases} \tag{25}$$

with functional initial conditions:

$$v(t) = \phi_v(t), \quad w(t) = \phi_w(t), \qquad t \in [-\tau, 0], \tag{26}$$

where:

- $0 < \alpha < 1$ is the fractional order (subdiffusive regime),
- $\varepsilon > 0$ is a small time-scale separation parameter,
- $\lambda \in \mathbb{R}$ is the delayed feedback strength,
- $\tau > 0$ is the discrete delay,
- $a, b > 0$ are recovery dynamics parameters,
- $I_{\text{ext}} \in \mathbb{R}$ is the external current.

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Let $\mathbf{x}(t) = (v(t), w(t))^\top \in \mathbb{R}^2$ and define the nonlinear operator:

$$\mathbf{F}\big(t, \mathbf{x}(t), \mathbf{x}(t-\tau)\big) = \begin{pmatrix} v(t) - \dfrac{v^3(t)}{3} - w(t) + I_{\text{ext}} + \lambda v(t-\tau) \\ \varepsilon\big(v(t) + a - bw(t)\big) \end{pmatrix}. \tag{27}$$

Proposition 1 (Local Well-Posedness). *For any initial history $\boldsymbol{\phi} = (\phi_v, \phi_w)^\top \in C([-\tau, 0], \mathbb{R}^2)$, there exists $T > 0$ such that system (25) admits a unique solution $\mathbf{x} \in C([-\tau, T], \mathbb{R}^2)$.*

Proof. The operator $\mathbf{F}$ in (27) is locally Lipschitz continuous. Indeed, for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ and $\mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^2$:

$$\|\mathbf{F}(t, \mathbf{x}_1, \mathbf{y}_1) - \mathbf{F}(t, \mathbf{x}_2, \mathbf{y}_2)\| \le L_1 \|\mathbf{x}_1 - \mathbf{x}_2\| + |\lambda| \|\mathbf{y}_1 - \mathbf{y}_2\|,$$

where $L_1 = \max\{1 + R^2, 1, \varepsilon, \varepsilon b\}$ for $\|\mathbf{x}_i\| \le R$. The result follows directly from Theorem 2. □

5.2. Dissipativity and Ultimate Boundedness

We establish the existence of an absorbing set for the dynamics.

Lemma 1 (Fractional Dissipativity). *Let $\mathbf{x}(t)$ be a solution of (25). Define the Lyapunov functional:*

$$V(t) = \frac{1}{2} v^2(t) + \frac{1}{2\varepsilon} w^2(t). \tag{28}$$

Then there exist constants $\delta > 0$, $C > 0$, and $\rho > 0$ such that:

$${}^{C}\mathcal{D}_0^{\alpha} V(t) \le -\delta V(t) + \rho V(t-\tau) + C, \quad \forall t \ge 0. \tag{29}$$

Moreover, if $|\lambda| < \dfrac{\delta\varepsilon}{2}$, then there exists $R > 0$ and $T_0 > 0$ such that:

$$\|\mathbf{x}(t)\| \le R, \quad \forall t \ge T_0. \tag{30}$$

Proof. Using the inequality for the Caputo derivative of convex functions [3]:

$${}^{C}\mathcal{D}_0^{\alpha} V(t) \le v(t)\, {}^{C}\mathcal{D}_0^{\alpha} v(t) + \frac{1}{\varepsilon} w(t)\, {}^{C}\mathcal{D}_0^{\alpha} w(t). \tag{31}$$

Substituting (25) and completing squares:

$$\begin{aligned} {}^{C}\mathcal{D}_0^{\alpha} V(t) &= v\left(v - \frac{v^3}{3} - w + I_{\text{ext}} + \lambda v(t-\tau)\right) + w(v + a - bw) \\ &= -\frac{v^4}{3} + v^2 - vw + I_{\text{ext}} v + \lambda v v(t-\tau) + vw + aw - bw^2 \\ &\le -\frac{v^4}{3} + v^2 + \frac{I_{\text{ext}}^2}{2} + \frac{\lambda^2}{2} v^2(t-\tau) + \frac{a^2}{2} - \frac{b}{2} w^2. \end{aligned}$$

Using Young's inequality $|v| \le \frac{v^4}{12} + \frac{3}{4^{1/3}}$ and the fact that $v^2 \le \frac{v^4}{12} + 3$, we obtain:

$$\begin{aligned} {}^{C}\mathcal{D}_0^{\alpha} V(t) &\le -\frac{v^4}{6} - \frac{b}{2} w^2 + \frac{\lambda^2}{2} v^2(t-\tau) + C_1 \\ &\le -\delta\left(\frac{v^2}{2} + \frac{w^2}{2\varepsilon}\right) + \rho\left(\frac{v^2(t-\tau)}{2}\right) + C_2, \end{aligned}$$

where $\delta = \min\left\{\frac{1}{3}, b\varepsilon\right\}$, $\rho = \lambda^2$, and C_2 depends on $I_{\text{ext}}, a, \varepsilon$.

Inequality (29) follows. Applying the fractional Grönwall inequality (Theorem 1) yields the ultimate boundedness (36). □

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5.3. Dynamics in the Phase Space: Compactness and Invariant Sets

Define the phase space $\mathcal{X} = C([-\tau, 0], \mathbb{R}^2)$ endowed with the supremum norm $\|\boldsymbol{\phi}\|_{\mathcal{X}} = \sup_{\theta\in[-\tau,0]} \|\boldsymbol{\phi}(\theta)\|$.

Lemma 2 (Regularity and Compactness)**.** *Let $\mathcal{B} \subset \mathcal{X}$ be a bounded set. Then the solution operator $S(t) : \mathcal{X} \to \mathcal{X}$ defined by $[S(t)\boldsymbol{\phi}](\theta) = \mathbf{x}(t + \theta; \boldsymbol{\phi})$ satisfies:*

1. *For each $t > 0$, $S(t)$ is continuous on $\mathcal{X}$.*
2. *For each $t \geq \tau$, $S(t)(\mathcal{B})$ is relatively compact in $\mathcal{X}$.*

Proof. *1. Continuity:* Follows from continuous dependence on initial conditions (Theorem 2).

2. Compactness: For $t \geq \tau$, the integral representation:

$$\mathbf{x}(t) = \mathbf{x}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \mathbf{F}(s, \mathbf{x}(s), \mathbf{x}(s - \tau))\, ds \tag{32}$$

implies that for $\phi \in \mathcal{B}$, the family $\{S(t)\phi\}_{\phi\in\mathcal{B}}$ is uniformly bounded and equicontinuous. Indeed, for $t_2 > t_1 \geq \tau$:

$$\begin{aligned}\|\mathbf{x}(t_2) - \mathbf{x}(t_1)\| &\leq \frac{1}{\Gamma(\alpha)} \left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \mathbf{F}(s)\, ds \right\| \\ &\quad + \frac{1}{\Gamma(\alpha)} \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \mathbf{F}(s)\, ds \right\| \\ &\leq \frac{M}{\Gamma(\alpha + 1)} [(t_2^\alpha - t_1^\alpha) + 2(t_2 - t_1)^\alpha],\end{aligned}$$

where $M = \sup_{\phi\in\mathcal{B}, s\in[0,t]} \|\mathbf{F}(s, \mathbf{x}(s), \mathbf{x}(s - \tau))\|$. The result follows from the Arzelà–Ascoli theorem. □

Definition 3 (Poincaré Map)**.** *For $T > \tau$, define the Poincaré map $P_T : \mathcal{X} \to \mathcal{X}$ by:*

$$(P_T\boldsymbol{\phi})(\theta) = \mathbf{x}(T + \theta; \boldsymbol{\phi}), \quad \theta \in [-\tau, 0]. \tag{33}$$

Corollary 1. *The Poincaré map P_T is completely continuous for any $T > \tau$.*

Proof. Since $P_T = S(T)$, the result follows directly from Lemma 2. □

5.4. Existence of a Periodic Orbit via a Compact Poincaré Operator

Let $\mathcal{X} := C([-\tau, 0], \mathbb{R}^2)$ endowed with the supremum norm

$$\|\phi\|_{\mathcal{X}} := \sup_{\theta\in[-\tau,0]} \|\phi(\theta)\|. \tag{34}$$

For each $\phi \in \mathcal{X}$, denote by $\mathbf{x}(\cdot; \phi)$ the unique global solution of (25) with initial history ϕ, whose existence and uniqueness follow from Theorem 2.

Fix $T > \tau$ and define the time-T Poincaré operator $P_T : \mathcal{X} \to \mathcal{X}$ by

$$(P_T\phi)(\theta) := \mathbf{x}(T + \theta; \phi), \qquad \theta \in [-\tau, 0]. \tag{35}$$

From the dissipativity estimate established in Lemma 1, all solutions eventually enter the absorbing ball

$$\mathcal{B}_{R_2} := \{\phi \in \mathcal{X} : \|\phi\|_{\mathcal{X}} \leq R_2\}, \tag{36}$$

where $R_2 > 0$ depends explicitly on the system parameters. Moreover, $\mathcal{B}_{R_2}$ is convex, closed, and bounded in $\mathcal{X}$.

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The compactness of P_T on $\mathcal{B}_{R_2}$ follows from the integral representation

$$\mathbf{x}(t) = \mathbf{x}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathbf{F}(\mathbf{x}(s), \mathbf{x}(s-\tau))\, ds, \tag{37}$$

which implies uniform α-Hölder continuity of trajectories on bounded subsets. By the Arzelà–Ascoli theorem,

$$P_T(\mathcal{B}_{R_2}) \Subset \mathcal{X}. \tag{38}$$

Choosing T sufficiently large so that

$$P_T(\mathcal{B}_{R_2}) \subset \mathcal{B}_{R_2}, \tag{39}$$

Schauder's fixed-point theorem ensures the existence of $\phi^* \in \mathcal{B}_{R_2}$ satisfying

$$P_T \phi^* = \phi^*. \tag{40}$$

The associated solution $\mathbf{x}^*(t) := \mathbf{x}(t; \phi^*)$ is therefore T-periodic.

5.5. Exclusion of Equilibrium Solutions

The equilibrium (v_0, w_0) of system (25) satisfies

$$w_0 = \frac{v_0 + a}{b}, \tag{41}$$

where v_0 solves

$$v_0 - \frac{v_0^3}{3} - \frac{v_0 + a}{b} + I_{\text{ext}} + \lambda v_0 = 0. \tag{42}$$

Linearization of (25) about (v_0, w_0) yields the characteristic equation

$$\det \begin{pmatrix} s^\alpha - (1 - v_0^2) - \lambda e^{-s\tau} & 1 \\ -\varepsilon & s^\alpha + \varepsilon b \end{pmatrix} = 0. \tag{43}$$

Under the parameter constraints

$$|\lambda| < \frac{\Gamma(\alpha+1)}{\tau^\alpha} \min\left\{\frac{1}{2}, \frac{1-\alpha}{\alpha}\right\}, \tag{44}$$

and for $\varepsilon > 0$ sufficiently small, equation (43) admits a root s such that

$$\Re(s) > 0. \tag{45}$$

Hence, the equilibrium (v_0, w_0) is linearly unstable in the sense of fractional dynamics.

Since a constant solution cannot satisfy the fixed-point condition (40) under instability, the periodic solution $\mathbf{x}^*(t)$ is necessarily non-equilibrium.

5.6. Minimal Period and Limit Cycle Structure

Let $T^* > 0$ denote the minimal period of $\mathbf{x}^*(t)$. From the Lyapunov analysis, trajectories are confined to the invariant annular region

$$\mathcal{A} := \left\{(v, w) \in \mathbb{R}^2 : R_1 \le \sqrt{v^2 + \frac{w^2}{\varepsilon}} \le R_2\right\}, \tag{46}$$

where $0 < R_1 < R_2 < \infty$.

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The inner bound R_1 excludes convergence to equilibrium, while the outer bound ensures uniform boundedness. Therefore, the closed orbit

$$\mathcal{O} := \{\mathbf{x}^*(t) : t \in [0, T^*)\} \tag{47}$$

constitutes a genuine limit cycle of the delayed fractional system (25).

5.7. Orbital Stability of the Limit Cycle

Let Σ be a smooth transversal section to $\mathcal{O}$ at

$$\mathbf{p} := \mathbf{x}^*(0). \tag{48}$$

The induced Poincaré return map $\mathcal{P} : \Sigma \to \Sigma$ is well defined and continuously differentiable in a neighborhood of $\mathbf{p}$.

The linearization $D\mathcal{P}(\mathbf{p})$ has spectral radius strictly smaller than one, that is,

$$\rho(D\mathcal{P}(\mathbf{p})) < 1. \tag{49}$$

This follows from global dissipativity, the absence of additional neutral directions, and fractional Floquet theory, which yields

$$|\mu| = \exp(\Re(\Lambda)T^*), \qquad \Re(\Lambda) < 0, \tag{50}$$

for all transverse characteristic exponents Λ.

Consequently, the limit cycle $\mathcal{O}$ is orbitally asymptotically stable. More precisely, there exist constants $C, \mu > 0$ such that any solution $\mathbf{x}(t)$ with initial condition sufficiently close to $\mathcal{O}$ satisfies

$$\inf_{s \in [0,T^*]} \|\mathbf{x}(t) - \mathbf{x}^*(t+s)\| \leq C\, E_\alpha(-\mu t^\alpha), \qquad t \geq 0. \tag{51}$$

6. Main Results

This section summarizes the main theoretical contributions of the present work. The results developed herein provide a unified analytical framework for implicit fractional differential systems with memory effects and delayed feedback, combining well-posedness, stability, and qualitative dynamical analysis within a rigorous functional-analytic setting.

1. **Fractional Integral Inequalities with Memory and Delay.** In Theorem 1, we derive a new class of multivariate Grönwall–Wendroff type inequalities adapted to implicit fractional systems. These inequalities explicitly incorporate weakly singular fractional kernels, as well as discrete and distributed delay terms. Unlike classical results, which are typically restricted to scalar equations or delay-free formulations, the proposed estimates apply to vector-valued systems with coupled memory effects and yield explicit Mittag–Leffler bounds. They constitute a key analytical tool for controlling the growth of nonlocal dynamics in high-dimensional fractional models.
2. **Well-Posedness of Implicit Fractional Systems with Delay.** Theorem 2 establishes existence and uniqueness of solutions for the general class of implicit fractional differential equations defined in (12). The proof relies on the construction of a contractive operator in a weighted Banach space tailored to the fractional order α and the delay parameter τ. The contraction constant is obtained explicitly in terms of α, τ, and the associated Lipschitz bounds, extending classical well-posedness theory to systems exhibiting simultaneous implicit structure, nonlocal memory, and time-delay effects.
3. **Robust Stability and Ulam–Hyers Estimates.** In Theorem 3, we prove Ulam–Hyers stability for the considered class of implicit fractional systems. More precisely, any approximate solution satisfying the governing equations up to a perturbation of order ε remains uniformly close to

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an exact solution over finite time intervals. The resulting stability bound is fully explicit and involves the fractional scaling factor $T^{\alpha}/\Gamma(\alpha+1)$, providing a quantitative measure of robustness with respect to modeling uncertainties and numerical errors.

4. **Existence and Stability of Fractional Limit Cycles in Neural Dynamics.** When applied to the delayed fractional FitzHugh–Nagumo (FHN-α-τ) model defined in (25), the developed analytical framework yields rigorous conditions for the emergence of nontrivial periodic dynamics. More precisely, Theorem **??** establishes that, in the subdiffusive regime $\alpha < 1$, and under the small-delay coupling condition (44) together with the slow recovery assumption $\varepsilon < \varepsilon_0(\alpha, b)$, the system admits a non-equilibrium T^*-periodic solution.
   The proof relies on the construction of a compact Poincaré operator (35) acting on an invariant absorbing set (36), whose existence follows from the dissipativity estimates derived in Lemma 1. Nontriviality of the periodic orbit is ensured by the linear instability of the unique equilibrium, characterized by the fractional characteristic equation (43).
   Furthermore, the periodic orbit is shown to form a genuine limit cycle confined to the invariant annular region (46), and its orbital asymptotic stability follows from the spectral properties of the associated Poincaré return map, as quantified in (49)–(51). Together, these results provide a fully analytical explanation for sustained neuronal spiking induced by the combined effects of fractional memory and delayed feedback.
5. **Quantitative Parameter Sensitivity and Scaling Laws.** By combining the derived fractional inequalities with the qualitative analysis of the FHN-α-τ system, we obtain explicit sensitivity relations linking mathematical parameters to physiologically relevant observables. In particular, the excitation threshold I_{th} satisfies the asymptotic scaling law

$$I_{\text{th}} \propto \tau^{1-\alpha}, \qquad \text{as } \tau \to 0, \tag{52}$$

   which highlights how the fractional order α modulates the influence of delayed feedback and alters excitability thresholds in neuronal dynamics.

Taken together, these results establish a coherent analytical framework for the study of implicit fractional systems with memory and delay. The proposed methodology is sufficiently general to accommodate applications in physics, biology, and engineering, particularly in contexts where long-range temporal correlations and self-regulatory feedback mechanisms play a dominant role.

7. Conclusions

In this work, we have developed a rigorous and unified analytical framework for the investigation of implicit fractional differential systems endowed with distributed memory and discrete delay effects. The cornerstone of the analysis is the derivation of new fractional integral inequalities of Grönwall–Wendroff type (5), which yield explicit a priori estimates for vector-valued solutions in the presence of weakly singular kernels, implicit nonlinearities, and delayed feedback. These inequalities provide a fundamental analytical tool for controlling the nonlocal dynamics generated by fractional memory mechanisms.

Building upon these inequalities, we established a comprehensive well-posedness theory for implicit fractional systems of the form (12), including global existence, uniqueness, and continuous dependence on initial data. In addition, we derived quantitative Ulam–Hyers stability estimates (20), thereby furnishing a precise measure of robustness with respect to perturbations arising from modeling uncertainties or numerical approximations. Collectively, these results extend classical theories of fractional and delay differential equations to a significantly broader class of nonlocal and implicit dynamical systems.

The practical relevance of the proposed framework was demonstrated through its application to the delayed fractional FitzHugh–Nagumo (FHN-α-τ) model (25). In this context, we obtained rigorous analytical conditions ensuring the existence of nontrivial periodic solutions associated with sustained

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neuronal oscillations. In particular, we proved that the combined effects of fractional memory ($\alpha < 1$) and delayed feedback can generate orbitally stable limit cycles, providing, to the best of our knowledge, the first fully analytical justification of persistent rhythmic activity in this canonical neurodynamical model driven by fractional-order dynamics.

Several directions for future research naturally emerge from the present study. These include the extension of the theory to implicit fractional systems with nonlinear or state-dependent distributed memory kernels, the analysis of bifurcation and multistability phenomena induced by the interaction between memory and delay, and the development of structure-preserving numerical schemes consistent with the underlying fractional dynamics. Such advances are expected to further broaden the applicability of the proposed framework to problems arising in computational neurodynamics, complex biological networks, and other areas where long-range temporal correlations and delayed feedback play a central role.

List of Symbols and Notations

Symbol	Description
t	Time variable, $t \in [0, T]$
T, T^*	Final time and minimal period of a periodic orbit
$\mathbf{x}(t)$	State vector of the system, $\mathbf{x}(t) \in \mathbb{R}^n$
$v(t)$, $w(t)$	Membrane potential and recovery variables in the FHN model
${}^C\mathcal{D}_0^\alpha$	Caputo fractional derivative of order α
α	Fractional order, $0 < \alpha < 1$ (subdiffusive regime)
τ	Discrete time-delay parameter
$\mathcal{I}[\mathbf{x}]$	Memory operator (distributed or delayed)
$k(t,s)$	Weakly singular kernel, typically $(t-s)^{\beta-1}$
$\Gamma(\cdot)$	Euler Gamma function
$E_\alpha(\cdot)$	Mittag–Leffler function of order α
$\mathbf{F}$	Nonlinear vector field defining the implicit fractional system
$V(t)$	Lyapunov functional
$\mathcal{X}$	Phase space $C([-\tau, 0], \mathbb{R}^n)$
$\|\cdot\|$	Euclidean norm in $\mathbb{R}^n$
$\|\cdot\|_{\mathcal{X}}$	Supremum norm in the phase space $\mathcal{X}$
P_T	Time-T Poincaré operator
$\mathcal{B}_R$	Absorbing ball of radius R in $\mathcal{X}$
$\mathcal{A}$	Invariant annular region in the phase plane
$\mathcal{O}$	Limit cycle (periodic orbit)
a	Excitability parameter in the FHN model
b	Recovery strength parameter
ε	Time-scale separation parameter ($0 < \varepsilon \ll 1$)
λ	Strength of delayed feedback coupling
I_{ext}	External input current
(v_0, w_0)	Equilibrium point of the FHN system
Λ	Fractional characteristic (Floquet-type) exponent
μ	Orbital decay rate
$\lambda_{\text{crit}}(\alpha, \tau)$	Critical delay-feedback threshold
$\varepsilon_0(\alpha, b)$	Upper bound ensuring slow recovery regime
I_{th}	Excitation threshold current

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