Article

Not peer-reviewed version

Thermodynamic Compactness and InformationGeometric Bounds in Excluded-Volume Systems

Angelo Plastino *

Posted Date: 26 February 2026

doi: 10.20944/preprints202602.1729.v1

Keywords: thermodynamics; information theory; partition functions; compact support; admissible macrostates

Article

Thermodynamic Compactness and Information-Geometric Bounds in Excluded-Volume Systems

Angelo Plastino

Instituto de Física, Universidad Nacional de La Plata, Argentina; angeloplastino@fisica.unlp.edu.ar-mail@e-mail.com

Abstract

We show that thermodynamic consistency in systems with finite excluded volume implies compact support of the grand canonical particle-number distribution. Understanding whether fundamental bounds on information and matter content can arise purely from statistical-mechanical principles — independent of gravitational dynamics — is of central interest in thermodynamics, information theory, and cosmology. For any nonzero excluded volume parameter b, the partition function vanishes identically beyond $N_{\max} = V/b$, enforcing a strict upper bound on admissible macrostates. We demonstrate that this compactness induces bounded particle-number fluctuations and finite Fisher information with respect to the chemical potential, thereby rendering the associated statistical manifold effectively finite-dimensional. This informational compactness provides a structural mechanism limiting distinguishability of macrostates independently of gravitational considerations. We argue that such thermodynamically enforced bounds are compatible with entropy bounds and holographic scaling principles, suggesting that informational finiteness may arise from statistical-mechanical consistency alone. Cosmological implications are discussed cautiously: infinite matter content at fixed volume is incompatible with compact support induced by finite excluded volume.

Keywords: thermodynamics; information theory; partition functions; compact support; admissible macrostates

---

1. Introduction

The relationship between entropy, information, and the number of physically admissible degrees of freedom has become a central theme in modern theoretical physics. In gravitational settings, the *Bekenstein bound* establishes that the entropy contained within a bounded region is finite and proportional to the area of its boundary rather than its volume [2]. This insight culminated in the *Holographic Principle*, according to which the fundamental degrees of freedom of a spatial region may be encoded on its boundary [3,4]. Such results suggest that physical systems possess an intrinsic informational compactness: the space of distinguishable states is restricted more strongly than naive volume counting would indicate.

Area scaling is not exclusive to gravitational physics. In quantum many-body systems, ground states of local Hamiltonians satisfy entanglement entropy area laws, whereby the entanglement entropy of a spatial region scales with its boundary [5]. This behavior reflects the effective redundancy of bulk degrees of freedom under locality constraints. Both gravitational entropy bounds and many-body area laws thus indicate that microscopic consistency conditions can reduce the effective dimensionality of state space.

These observations motivate a broader structural question:

Can informational compactness arise purely from thermodynamic consistency, independently of gravitational dynamics?

In this work we show that it can. We demonstrate that the presence of a strictly positive excluded volume parameter $b > 0$ in a thermodynamic system enforces compact support of the grand canonical particle-number distribution. Specifically, thermodynamic consistency requires that $N \leq V/b$,

 doi:10.20944/preprints202602.1729.v1

implying that the partition function vanishes identically beyond a finite $N_{\max}$. The statistical manifold therefore contains only a finite number of admissible macrostates.

This compactness has direct information-geometric consequences. The Fisher information with respect to the chemical potential is proportional to the particle-number variance, $F_{\mu\mu} = \beta^2 \mathrm{Var}(N)$ [6]. Since the support of $P(N)$ is finite, fluctuations are bounded and Fisher information remains finite. Distinguishability along the chemical-potential direction is therefore intrinsically limited. The resulting statistical manifold is bounded and effectively finite-dimensional.

We argue that this mechanism provides a structural realization of informational finiteness that is independent of gravitational considerations yet compatible with entropy bounds and holographic scaling principles. While we do not derive gravitational entropy bounds, we show that thermodynamic compactness enforces a restriction of admissible states analogous in spirit to boundary-controlled degree-of-freedom reduction.

Cosmological implications are discussed cautiously. If matter obeys finite excluded volume constraints, infinite matter content at fixed volume becomes thermodynamically inconsistent. Our analysis therefore suggests that informational finiteness may emerge from statistical-mechanical consistency alone, providing a complementary perspective on holographic behavior.

Novelty and scope.

The central novelty of this work is the identification of a model-independent geometric consequence of excluded-volume constraints: any grand-canonical system with a finite particle-number cutoff induces a Fisher–Rao statistical manifold of finite diameter in the chemical-potential direction. While bounds on fluctuations in finite systems are elementary, we show that compact support in particle number implies (i) uniform boundedness of the Fisher metric, (ii) finiteness of total thermodynamic length, and (iii) bounded scalar curvature. These results establish a previously unnoticed link between microscopic occupancy constraints and global information-geometric structure. The argument is independent of specific interaction details, equations of state, or approximations, and therefore reveals a structural property of thermodynamic ensembles with excluded volume rather than a feature of any particular model. To our knowledge, the finite-diameter and curvature-boundedness implications of particle-number compactness have not been formulated in the information-geometric literature.

2. Preliminaries

In equilibrium statistical mechanics, particle-number fluctuations are described by the grand canonical ensemble, where the number of particles N is controlled by the chemical potential μ. For an ideal gas, the probability distribution $P(N)$ has infinite support, since no restriction prevents additional particles from being added to the system.

When short-range repulsive interactions are introduced, as in the van der Waals model, the configurational space is no longer unbounded. Each particle excludes a finite volume b from the total available space, so that the effective volume accessible to the system becomes [6-8]

$$V_{\mathrm{eff}} = V - Nb. \tag{1}$$

The condition $V_{\mathrm{eff}} > 0$ then implies a strict upper bound

$$N \le N_{\max} = \frac{V}{b}. \tag{2}$$

This limitation is not a mathematical artifact but a reflection of the model's internal consistency: once all available volume is filled by hard cores, no further microstates exist.

One might perhaps speculate that such configurational bounds could bear on the finiteness of the Universe, but that conclusion would concern only the internal logic of equilibrium statistical mechanics.

 doi:10.20944/preprints202602.1729.v1

The existence of a cutoff in N is a statement about the *finite-domain consistency* of excluded-volume models, not about cosmological global properties.

An important conceptual precedent for our present analysis is provided by Bekenstein's celebrated entropy bound [2,10], which establishes a universal constraint on the entropy that can be stored within any finite physical region. In its simplest form, the bound reads

$$S \leq \frac{2\pi k_B R E}{\hbar c}, \tag{3}$$

where S denotes the entropy of the system, R its characteristic linear size, and E its total energy. This inequality asserts that any bounded system—that is, one occupying a finite spatial domain and possessing finite energy—admits only a limited information capacity. Physically, the saturation of this bound corresponds to the formation of a black hole, whose Bekenstein–Hawking entropy $S_{\mathrm{BH}} = k_B c^3 A/(4G\hbar)$ provides the maximal entropy consistent with general relativity. From the standpoint of statistical mechanics, this implies that entropy is not merely a measure of disorder but also a geometric quantity intrinsically linked to spacetime curvature and boundary constraints [6-8]. In the context of the present work, such finite-domain limitations resonate with the appearance of particle-number cutoffs in excluded-volume fluids, suggesting a deep connection between thermodynamic consistency, spatial confinement, and informational bounds in statistical physics. In an analogous manner, the van der Waals equation [1-3] embodies a finite-domain restriction at the microscopic level. When formulated within the grand canonical ensemble, the excluded volume parameter b effectively limits the admissible number of particles through the condition $V_{\mathrm{eff}} = V - Nb > 0$, leading to the intrinsic cutoff $N_{\mathrm{Max}} = V/b$. This constraint mirrors the spirit of the Bekenstein bound: both express the impossibility of unlimited configurational occupation within a finite region of phase space. In the van der Waals case, the cutoff originates from short-range repulsive correlations that prevent particles from overlapping, enforcing a geometric restriction in configuration space [9]; in the Bekenstein scenario, the bound arises from spacetime curvature and gravitational self-energy. The parallel suggests that the emergence of bounded support in the probability distribution $P(N)$ is not an artifact of approximation, but a manifestation of a more general thermodynamic principle linking finite volume, finite energy, and finite informational capacity. Consequently, the finite-domain cutoff N_{Max} may be viewed as a classical statistical analogue of Bekenstein's entropy limit, reinforcing the consistency between excluded-volume thermodynamics and the broader framework of physical information bounds.

3. Derivation of the Particle-Number Distribution [6-8]

For a system of non-ideal particles with short-range repulsion and mean-field attraction, the canonical partition function may be written to first order in the Mayer cluster expansion as

$$Z_N(T,V) = \frac{1}{N!}\left(\frac{V-Nb}{\lambda^3}\right)^N \exp\left(\beta a \frac{N^2}{V}\right), \tag{4}$$

where a and b are the van der Waals parameters, $\lambda = h/\sqrt{2\pi m k_B T}$ is the thermal wavelength, and $\beta = 1/(k_B T)$.

The grand partition function then reads

$$\Xi(T,V,\mu) = \sum_{N=0}^{V/b} \frac{z^N}{N!}\left(\frac{V-Nb}{\lambda^3}\right)^N \exp\left(\beta a \frac{N^2}{V}\right), \tag{5}$$

with fugacity $z = e^{\beta\mu}$. The summation terminates naturally at $N_{\mathrm{max}} = V/b$, ensuring that configurations with overlapping excluded volumes are automatically excluded.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 February 2026 doi:10.20944/preprints202602.1729.v1

The normalized particle-number probability distribution is then

$$P(N) = \frac{1}{\Xi(T,V,\mu)} \frac{z^N}{N!} \left(\frac{V - Nb}{\lambda^3} \right)^N \exp\left(\beta a \frac{N^2}{V} \right), \qquad 0 \le N \le \frac{V}{b}. \tag{6}$$

This expression follows directly from the grand-canonical construction applied to the van der Waals equation of state, without heuristic assumptions. Its compact support embodies the configurational saturation imposed by the excluded-volume correction.

4. Thermodynamic Compactness and Informational Finiteness

We now formalize the structural content of the argument.

4.1. Compact Support Induced by Excluded Volume

Consider a classical system of volume V in the grand canonical ensemble with inverse temperature β and chemical potential μ. Let each particle occupy a finite excluded volume $b > 0$. Thermodynamic consistency requires that the total occupied volume does not exceed the system volume:

$$Nb \le V. \tag{7}$$

It follows that admissible particle numbers satisfy

$$N \le N_{\max} := \left\lfloor \frac{V}{b} \right\rfloor. \tag{8}$$

Proposition 1 (Compact Support)**.** *If $b > 0$, then the grand canonical probability distribution $P(N)$ has compact support:*

$$P(N) = 0 \quad \textit{for} \quad N > N_{\max}. \tag{9}$$

Proof. The canonical partition function Z_N necessarily vanishes for $N > N_{\max}$, since configurations violating $Nb \le V$ are physically inadmissible. Therefore the grand partition function

$$\Xi(\mu, \beta) = \sum_{N=0}^{\infty} e^{\beta \mu N} Z_N \tag{10}$$

truncates at $N_{\max}$, establishing compact support. □

4.2. Bounded Fluctuations and Fisher Rank

The Fisher information with respect to the chemical potential is

$$F_{\mu\mu} = \left\langle \left(\partial_\mu \log P(N) \right)^2 \right\rangle = \beta^2 \mathrm{Var}(N). \tag{11}$$

Since the support of $P(N)$ is finite,

$$0 \le \mathrm{Var}(N) \le N_{\max}^2, \tag{12}$$

and therefore

$$F_{\mu\mu} \le \beta^2 N_{\max}^2. \tag{13}$$

Proposition 2 (Informational Finiteness)**.** *Compact support of $P(N)$ implies bounded Fisher information and finite distinguishability of macrostates along the μ-direction of the statistical manifold.*

Thus, the statistical manifold endowed with the Fisher metric is effectively finite-dimensional and bounded in extent.

4.3. Structural Interpretation

Compact support restricts the number of physically distinguishable macrostates. Infinite matter content at fixed volume would require unbounded support in N, which is incompatible with finite excluded volume. Informationally, unbounded distinguishability would correspond to divergent Fisher information. Thermodynamic compactness therefore enforces informational finiteness independently of gravitational considerations.

The argument does not rely on the detailed form of the interaction, but only on the existence of a strictly positive excluded volume parameter $b > 0$.

5. Compatibility with Entropy Bounds and Holographic Scaling

The compactness derived above has a natural interpretation in the context of entropy bounds. In systems with finite excluded volume $b > 0$, the admissible particle-number sector is strictly bounded,

$$N \leq N_{\text{max}} = \left\lfloor \frac{V}{b} \right\rfloor. \tag{14}$$

Consequently, the number of distinguishable macrostates is finite, and the associated entropy is necessarily bounded.

Entropy bounds are central to gravitational physics, most notably in the *Bekenstein bound* and the *Holographic Principle*, which suggest that the maximal entropy contained in a spatial region scales not with its volume but with its boundary area. While the present argument does not rely on gravitational dynamics, it identifies an independent statistical-mechanical mechanism that enforces informational compactness.

The essential structural element is not gravity but the existence of a microscopic constraint preventing arbitrarily large occupation numbers. Once such a constraint is present, the statistical manifold acquires compact support, Fisher information becomes bounded, and distinguishability of macrostates is intrinsically finite.

In holographic scenarios, area scaling emerges because bulk degrees of freedom become effectively redundant beyond a certain threshold. In the present framework, redundancy arises because particle configurations exceeding V/b are physically inadmissible. Both mechanisms reduce the effective dimensionality of the space of admissible states.

We therefore interpret thermodynamic compactness as structurally compatible with holographic scaling: in both cases, the space of physically distinguishable configurations is restricted by a boundary-like constraint. The analysis presented here does not derive gravitational entropy bounds, but shows that informational finiteness can arise from thermodynamic consistency alone.

This suggests that holographic behavior may be understood, at least partially, as a manifestation of constrained statistical support rather than exclusively as a dynamical property of spacetime geometry.

Connection to Entanglement area Laws

Area scaling also appears in non-gravitational quantum many-body systems through entanglement entropy. Ground states of local Hamiltonians are known to satisfy entanglement entropy area laws, whereby the entropy of a spatial region grows proportionally to the size of its boundary rather than its volume. This behavior reflects the fact that correlations are predominantly short-ranged, rendering bulk degrees of freedom effectively redundant beyond a boundary layer.

The thermodynamic compactness discussed here exhibits an analogous structural feature. The excluded-volume constraint eliminates bulk overcounting by restricting the admissible particle-number sector, thereby limiting the number of distinguishable configurations that can occupy a region of fixed volume. Although the mechanism is classical and does not rely on quantum entanglement, both settings share a common informational characteristic: effective reduction of accessible state space due to local constraints.

 doi:10.20944/preprints202602.1729.v1

In this sense, area-law behavior and thermodynamic compactness may be viewed as distinct realizations of a broader principle in which microscopic consistency conditions restrict the dimensionality of physically meaningful degrees of freedom.

6. Finite-Domain Consistency and Extensivity

The compact-support condition $N \leq V/b$ expresses the internal consistency of excluded-volume thermodynamics. It ensures that equilibrium statistical mechanics does not assign nonzero weight to configurationally inadmissible states. This bound is a property of finite systems with short-range repulsive interactions and should be interpreted strictly within that domain.

The existence of $N_{\max}$ does not contradict the standard thermodynamic limit. When $V \to \infty$ and $N \to \infty$ with fixed density N/V, the ratio $b/V \to 0$ and the truncation becomes irrelevant at fixed density. Extensivity is therefore preserved. The cutoff operates only as a finite-domain consistency condition and does not alter the conventional large-system behavior of the model.

Any extrapolation to gravitating systems or cosmology would require a fundamentally different statistical framework, since gravity is long-ranged and non-extensive [12]. The present analysis does not address such systems. Its purpose is solely to clarify how excluded-volume thermodynamics enforces intrinsic occupancy limits within ordinary equilibrium statistical mechanics.

Thus, the bound $N \leq V/b$ should be understood as a local consistency constraint rather than a statement about the global structure of the Universe. “‘

Theorem 1 (Thermodynamic Compactness Implies Fisher-Manifold Compactness)**.** *Let a grand canonical ensemble be defined on particle numbers $N \in \{0, 1, \ldots, N_{\max}\}$ with $N_{\max} < \infty$, and probability distribution*

$$P_\mu(N) = \frac{e^{\beta\mu N} Z_N}{\Xi(\mu)}, \qquad \Xi(\mu) = \sum_{N=0}^{N_{\max}} e^{\beta\mu N} Z_N,$$

where $Z_N \geq 0$ and at least one $Z_N > 0$ for $0 \leq N \leq N_{\max}$.

Then the statistical manifold $\mathcal{M} = \{P_\mu\}_{\mu \in \mathbb{R}}$ endowed with the Fisher metric

$$g_{\mu\mu}(\mu) = F_{\mu\mu} = \beta^2 \mathrm{Var}_\mu(N)$$

is geodesically complete and has finite total Fisher length. Specifically,

$$0 \leq g_{\mu\mu}(\mu) \leq \frac{\beta^2}{4} N_{\max}^2 \quad \forall \mu,$$

and the total Fisher length along the entire chemical-potential axis satisfies

$$\mathcal{L} = \int_{-\infty}^{+\infty} \sqrt{g_{\mu\mu}(\mu)}\, d\mu < \infty.$$

Hence the Fisher–Rao manifold associated with the chemical potential direction is metrically compact.

Proof. Since $0 \leq N \leq N_{\max}$, the variance of any distribution supported on this finite interval satisfies the sharp bound

$$\mathrm{Var}(N) \leq \frac{1}{4} N_{\max}^2,$$

with equality only for a two-point distribution at the endpoints. Therefore

$$g_{\mu\mu}(\mu) = \beta^2 \mathrm{Var}(N) \leq \frac{\beta^2}{4} N_{\max}^2.$$

To establish finiteness of total Fisher length, note that

$$\frac{d}{d\mu}\langle N\rangle = \beta\,\mathrm{Var}(N).$$

Hence

$$\sqrt{g_{\mu\mu}}\,d\mu = \beta\sqrt{\mathrm{Var}(N)}\,d\mu \leq d\langle N\rangle.$$

Integrating from $\mu = -\infty$ to $+\infty$ yields

$$\mathcal{L} \leq \int_{\langle N\rangle(-\infty)}^{\langle N\rangle(+\infty)} d\langle N\rangle = N_{\max},$$

since $\langle N\rangle$ interpolates monotonically between 0 and $N_{\max}$ as μ ranges over $\mathbb{R}$.

Thus the total Fisher distance between the empty and saturated configurations is finite and bounded by $N_{\max}$. □

Corollary 1. *The Fisher–Rao diameter of the manifold satisfies*

$$\mathrm{diam}(\mathcal{M}) \leq N_{\max}.$$

Therefore no infinite statistical distinguishability can arise in any finite excluded-volume system.

7. Applicability and Limitations

It is important to distinguish between short-range fluids, where the van der Waals description is valid, and systems governed by long-range forces such as gravity. Gravitational interactions are non-extensive, and their equilibrium statistical mechanics must be formulated differently—typically through mean-field or microcanonical treatments [12]. Therefore, the present results should not be viewed as applying to gravitating matter or cosmology. Our goal is to clarify how equilibrium consistency alone imposes internal occupancy limits within the ordinary thermodynamic domain.

8. Information-Theoretic Characterization

The particle-number distribution $P(N)$ carries structural information about the configurational state of the system. Two complementary measures are useful:

$$S[P] = -\sum_N P(N)\ln P(N), \tag{15}$$

$$I[P] = \sum_N P(N)\left[\frac{\partial \ln P(N)}{\partial N}\right]^2. \tag{16}$$

The ratio

$$\mathcal{R}[P] = \frac{I[P]}{S[P]} \tag{17}$$

quantifies the rigidity of the probability structure. As the occupancy approaches the configurational limit $N \to V/b$, the distribution narrows, $I[P]$ increases, and $S[P]$ decreases, leading to larger $\mathcal{R}$. This monotonic rise provides a scalar diagnostic of the transition from a flexible to a rigid configurational regime, complementing traditional thermodynamic indicators such as compressibility.

9. Analytic Example: Discrete Gaussian (Quadratic Effective Action)

Assume that, near its dominant region, the canonical weight Z_N can be approximated by a quadratic expansion in N,

$$\log Z_N \approx A - \frac{(N-N_0)^2}{2\sigma_0^2},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 February 2026 doi:10.20944/preprints202602.1729.v1

with constants A, N_0 and width $\sigma_0 > 0$ determined by microscopic parameters. In the grand canonical ensemble with fugacity $z = e^{\beta\mu}$,

$$P(N) \propto z^N Z_N \quad \Rightarrow \quad \log P(N) = \beta\mu N + \log Z_N + \text{const} \approx \beta\mu N - \frac{(N-N_0)^2}{2\sigma_0^2} + \text{const}.$$

Treating N as continuous when the distribution is well localized, $P(N)$ is approximately Gaussian,

$$P(N) \approx C \exp\left[-\frac{(N-N^*)^2}{2\sigma_0^2}\right],$$

where the peak N^* is found by extremizing the exponent:

$$0 = \frac{\partial}{\partial N}\left(-\frac{(N-N_0)^2}{2\sigma_0^2} + \beta\mu N\right) \quad \Rightarrow \quad N^* = N_0 + \beta\mu\,\sigma_0^2.$$

The normalization constant (ignoring boundary truncation) is $C \approx (2\pi\sigma_0^2)^{-1/2}$.

Mean and variance:

$$\langle N\rangle \approx N^* = N_0 + \beta\mu\,\sigma_0^2, \qquad \mathrm{Var}(N) \approx \sigma_0^2.$$

Fisher information with respect to the chemical potential:

$$\partial_\mu \ln P = \beta\,(N - \langle N\rangle) \quad \Rightarrow \quad F_{\mu\mu} = \langle(\partial_\mu \ln P)^2\rangle = \beta^2\,\mathrm{Var}(N) \approx \beta^2\sigma_0^2.$$

Validity and remarks:

- This approximation requires the Gaussian peak to be well separated from the hard cutoff $N_{\max} = V/b$ and from $N = 0$: $\sigma_0 \ll \min(N^*, N_{\max} - N^*)$.
- The parameters N_0 and σ_0 are obtained by expanding $\log Z_N$ to second order around its maximum (from the microscopic partition function or cluster expansion).
- The result illustrates boundedness: for fixed σ_0 and β, $\langle N\rangle$, $\mathrm{Var}(N)$ and $F_{\mu\mu}$ are finite.

10. Explicit Curvature in the Discrete Gaussian Approximation

To obtain an explicit closed-form curvature expression, we consider the quadratic (discrete Gaussian) approximation to the canonical weights. Assume that near its dominant region the canonical partition function satisfies

$$\log Z_N(\beta) \approx A(\beta) - \frac{(N-N_0(\beta))^2}{2\sigma_0(\beta)^2}, \tag{18}$$

where $N_0(\beta)$ and $\sigma_0(\beta) > 0$ are smooth functions of the inverse temperature.

In the grand canonical ensemble with fugacity $z = e^{\beta\mu}$,

$$P(N) \propto \exp\left[\beta\mu N - \frac{(N-N_0)^2}{2\sigma_0^2}\right]. \tag{19}$$

Completing the square gives an approximate Gaussian form,

$$P(N) \approx \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left[-\frac{(N-N^*)^2}{2\sigma_0^2}\right], \tag{20}$$

with shifted mean

$$N^* = N_0 + \beta\mu\sigma_0^2. \tag{21}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 February 2026 doi:10.20944/preprints202602.1729.v1

Thus

$$\langle N \rangle = N^*, \qquad \mathrm{Var}(N) = \sigma_0^2. \tag{22}$$

10.1. Natural-Parameter Representation

Introduce natural parameters

$$\theta_1 = \beta\mu, \qquad \theta_2 = -\frac{1}{2\sigma_0^2}. \tag{23}$$

Then the distribution can be written as a two-parameter quadratic exponential family,

$$P(N) \propto \exp\Big[\theta_1 N + \theta_2 (N - N_0)^2\Big]. \tag{24}$$

The Massieu potential (log-partition function) is

$$\psi(\theta_1, \theta_2) = \log \Xi = A + \frac{\theta_1^2}{4|\theta_2|} - \frac{1}{2}\log|\theta_2|. \tag{25}$$

Using $|\theta_2| = 1/(2\sigma_0^2)$, this becomes

$$\psi = A + \frac{1}{2}\sigma_0^2\theta_1^2 + \frac{1}{2}\log(2\sigma_0^2). \tag{26}$$

10.2. Fisher Metric

The Fisher–Rao metric is the Hessian of ψ:

$$g_{ij} = \partial_i \partial_j \psi. \tag{27}$$

Direct differentiation yields

$$g_{\theta_1\theta_1} = \sigma_0^2, \tag{28}$$

$$g_{\theta_1\theta_2} = -\frac{\theta_1}{2\theta_2^2}, \tag{29}$$

$$g_{\theta_2\theta_2} = \frac{1}{2\theta_2^2}. \tag{30}$$

The determinant is

$$\det g = \frac{\sigma_0^2}{2\theta_2^2} - \frac{\theta_1^2}{4\theta_2^4}. \tag{31}$$

Substituting $\theta_2 = -1/(2\sigma_0^2)$ gives

$$\det g = \sigma_0^6 \Big(1 - \sigma_0^2\theta_1^2\Big). \tag{32}$$

10.3. Scalar Curvature

For two-dimensional Hessian metrics generated by quadratic exponential families, the scalar curvature is constant. A direct computation yields

$$\boxed{R = -\frac{1}{\sigma_0^2}.} \tag{33}$$

10.4. Interpretation

The discrete Gaussian approximation therefore induces a constant negative curvature manifold. The curvature magnitude is controlled entirely by the variance parameter σ_0^2 and is independent of μ.

In particular:

doi:10.20944/preprints202602.1729.v1

- The manifold is hyperbolic ($R < 0$).
- Curvature increases in magnitude as fluctuations shrink.
- In the rigidity limit $\sigma_0 \to 0$,
$$R \to -\infty.$$

Thus configurational rigidity corresponds geometrically to hyperbolic curvature blow-up. The model provides an explicit realization of how suppressed fluctuations translate into large statistical curvature, while remaining consistent with the global compactness results established above.

11. Geodesics of the Discrete Gaussian Fisher–Rao Manifold

We now compute explicitly the geodesics of the Fisher–Rao metric associated with the discrete Gaussian approximation introduced above. Since the scalar curvature is constant and negative, the manifold is locally isometric to a two–dimensional hyperbolic space. Nevertheless, it is instructive to obtain the geodesics directly.

11.1. Metric and Coordinates

In the natural parameters (θ_1, θ_2), the Fisher metric reads

$$g_{ij} = \begin{pmatrix} \sigma_0^2 & -\dfrac{\theta_1}{2\theta_2^2} \\ -\dfrac{\theta_1}{2\theta_2^2} & \dfrac{1}{2\theta_2^2} \end{pmatrix}, \tag{34}$$

with $\theta_2 = -1/(2\sigma_0^2) < 0$. For clarity, we introduce the positive coordinate

$$y = \sigma_0 > 0, \qquad x = \theta_1 = \beta\mu. \tag{35}$$

In these variables the metric simplifies to

$$ds^2 = y^2\, dx^2 + \frac{1}{y^2}\, dy^2. \tag{36}$$

This form makes the hyperbolic structure manifest.

11.2. Christoffel Symbols

From Eq. (36), the nonvanishing Christoffel symbols are

$$\Gamma^x_{xy} = \Gamma^x_{yx} = \frac{1}{y}, \tag{37}$$

$$\Gamma^y_{xx} = -y^3, \qquad \Gamma^y_{yy} = -\frac{1}{y}. \tag{38}$$

11.3. Geodesic Equations

Let s denote an affine parameter. The geodesic equations are

$$\frac{d^2x}{ds^2} + \frac{2}{y}\frac{dx}{ds}\frac{dy}{ds} = 0, \tag{39}$$

$$\frac{d^2y}{ds^2} - y^3\left(\frac{dx}{ds}\right)^2 - \frac{1}{y}\left(\frac{dy}{ds}\right)^2 = 0. \tag{40}$$

doi:10.20944/preprints202602.1729.v1

11.4. First Integrals

Because the metric is independent of x, the conjugate momentum

$$p_x = g_{xx}\frac{dx}{ds} = y^2\frac{dx}{ds} \tag{41}$$

is conserved:

$$y^2\frac{dx}{ds} = C, \tag{42}$$

with constant C.

The normalization of the tangent vector,

$$g_{ij}\frac{d\theta^i}{ds}\frac{d\theta^j}{ds} = 1, \tag{43}$$

gives

$$y^2\left(\frac{dx}{ds}\right)^2 + \frac{1}{y^2}\left(\frac{dy}{ds}\right)^2 = 1. \tag{44}$$

Using Eq. (42), we obtain

$$\left(\frac{dy}{ds}\right)^2 = y^2 - C^2. \tag{45}$$

11.5. Explicit Solution

Solving for $y(s)$ yields

$$y(s) = \sqrt{C^2 + (s - s_0)^2}, \tag{46}$$

where s_0 is an integration constant.

Integrating Eq. (42) then gives

$$x(s) = x_0 + \frac{C}{|C|}\ln\!\left(\frac{s - s_0 + \sqrt{C^2 + (s - s_0)^2}}{|C|}\right), \tag{47}$$

with constant x_0.

Eliminating the affine parameter s, the geodesics satisfy

$$(x - x_0)^2 + y^2 = \text{const}. \tag{48}$$

11.6. Geometric Interpretation

Equation (48) shows that the geodesics are semicircles orthogonal to the boundary $y = 0$, together with vertical straight lines ($C = 0$). Thus the Fisher–Rao manifold of the discrete Gaussian model is isometric to the Poincaré upper half-plane.

In physical terms:

- $y = \sigma_0$ measures the strength of fluctuations,
- $x = \beta\mu$ parametrizes occupancy bias,
- geodesics represent optimal statistical interpolation between macrostates.

Approaching the rigidity limit $\sigma_0 \to 0$ corresponds to reaching the hyperbolic boundary at infinite statistical distance, consistent with the divergence of curvature derived above.

12. Discussion and Conceptual Parallels

The boundedness of $P(N)$ parallels, in spirit, other consistency relations in physics that connect physical size, energy, and information content. For example, Bekenstein's bound constrains the entropy of bounded systems but does not imply global cosmological finiteness. Similarly, the excluded-volume condition constrains admissible configurations within a finite system without extrapolating to the

Universe as a whole. Both express the principle that interaction and confinement jointly delimit the range of physically meaningful states.

From an information-geometric standpoint [6], both the Bekenstein bound and the excluded-volume cutoff $N_{\text{Max}} = V/b$ can be interpreted as manifestations of *thermodynamic rigidity*. In each case, a finite-domain constraint suppresses the proliferation of accessible microstates, thereby enhancing the sensitivity of the system to parametric variations. Within our framework, this effect is quantitatively captured by the information-theoretic ratio $\mathcal{R}[P] = I[P]/S[P]$, where $I[P]$ measures the curvature—or Fisher sensitivity—of the probability distribution $P(N)$, and $S[P]$ quantifies its configurational entropy. As N approaches N_{Max}, $S[P]$ decreases while $I[P]$ grows, driving $\mathcal{R}[P]$ to large values that signal the onset of microscopic rigidity. This behavior parallels the tightening of informational bounds in Bekenstein's inequality: both phenomena reflect a geometric restriction in the underlying state space [6], where the system's capacity for fluctuation is progressively replaced by structural order. Hence, $\mathcal{R}[P]$ serves as a unifying scalar diagnostic that bridges information geometry, excluded-volume thermodynamics, and fundamental entropy bounds [6-8].

13. Conclusions

We have shown that a finite particle-number cutoff in the grand canonical ensemble implies a global geometric restriction on the associated Fisher–Rao manifold: the chemical-potential direction possesses finite statistical diameter and bounded curvature. These results follow directly from compact support in particle number and are independent of interaction details or specific equations of state.

In this sense, excluded-volume constraints do not merely limit fluctuations locally, but determine the global information-geometric structure of thermodynamic parameter space. Finite-domain consistency—expressed through the condition $N \le V/b$—translates into bounded statistical distinguishability along occupancy directions.

The analysis identifies a structural link between microscopic configurational constraints and macroscopic informational limits. Such informational compactness arises here within ordinary equilibrium statistical mechanics, without appeal to gravitational or holographic arguments.

More generally, the results illustrate how geometric restrictions in configuration space induce global bounds in statistical manifolds, providing a concrete connection between excluded-volume physics, fluctuation theory, and information

Funding: not applicable.

Data Availability Statement: there is no data to share

Conflicts of Interest: There are no conflicts.

References

1. M. B. Soley and E. J. Heller, Chem. Rev. 124, 12631 (2024).
2. J. D. Bekenstein, "Universal upper bound on the entropy-to-energy ratio for bounded systems," Phys. Rev. D **23**, 287 (1981).
3. G. 't Hooft, "Dimensional reduction in quantum gravity," arXiv:gr-qc/9310026.
4. L. Susskind, "The world as a hologram," J. Math. Phys. **36**, 6377 (1995).
5. J. Eisert, M. Cramer, and M. B. Plenio, "Colloquium: Area laws for the entanglement entropy," Rev. Mod. Phys. **82**, 277 (2010).
6. R. A. Fisher, "Theory of statistical estimation," Proc. Camb. Phil. Soc. **22**, 700 (1925).
7. H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics*, 2nd ed. (Wiley, 1985).
8. R. K. Pathria and P. D. Beale, *Statistical Mechanics*, 3rd ed. (Elsevier, 2011).
9. K. Huang, *Statistical Mechanics*, 2nd ed. (Wiley, 1987).
10. J. D. Bekenstein, "Black holes and entropy," *Phys. Rev. D* **7**, 2333–2346 (1973).
11. G. Ruppeiner, "Riemannian geometry in thermodynamic fluctuation theory," *Rev. Mod. Phys.* **67**, 605 (1995).

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 26 February 2026 doi:10.20944/preprints202602.1729.v1

12. T. Padmanabhan, "Statistical mechanics of gravitating systems," *Phys. Rep.* **188**, 285–362 (1990).
13. A. Plastino, F. Pennini, and A. R. Plastino, "Fisher information and thermodynamics," *Physica A* **258**, 446–457 (1998).