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Elementary Proof of Beal's Conjecture

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Abstract: The 6 variable general equation of Beal's conjecture equation $x^a + y^b = z^c$, where x, y, z, a, b, and c are positive integers, and $a, b, c \geq 3$, is identified as an identity made by expansion of powers of binomials of integers x and y; where x, y and z have common prime factor. Here, a proof of the conjecture is presented in two folds: First, powers of binomials of integers x and y expand to all integer solutions of Beal's equation if they have common prime factor. Second, powers of binomials of coprime integers x, y expand in two terms such that if two of the three terms of the equation are perfect powers the third one is not a perfect power.

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Introduction

Beal's conjecture states that if $x^a + y^b = z^c$, where a, b, c, x, y and z are positive integers and a, b, c > 2, then x, y, and z have a common prime factor. The conjecture was made by math enthusiast Daniel Andrew Beal in 1997 [1]. It is a generalization of Fermat's Last Theorem (FLT) which states that no three positive integers a, b, c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2. FLT has been considered extensively in the literature [2–7] and was proved by Andrew Wiles [8]. Similar problems to Beal's conjecture have been suggested as early as the year 1914 [9] and the conjecture maybe referred to by different names in the literature [10,11]. So far a proof to the conjecture has been a challenge to the public as well as to mathematicians and no counterexample has been successfully presented to disprove it, i.e. Peter Norvig reported having conducted a series of numerical searches for counterexamples to Beal's conjecture. Among his results, he excluded all possible solutions having each of a, b, $c \leq 7$ and each of x, y, $z \leq 250{,}000$, as well as possible solutions having each of a, b, $c \leq 100$ and each of x, y, $z \leq 10{,}000$ [12]. In this paper, we prove Beal's conjecture by elementary approach.

Proof of the conjecture

The binomial identity describes the expansion of powers of a binomial as given in equation (1).

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \qquad (1)$$

$$(x+y)^n = x^n + \left(\sum t + y^n\right) \quad , n > 2 \qquad (1.1)$$

$$(x+y)^n = \left(\sum t + x^n\right) + y^n \quad , n > 2 \qquad (1.2)$$

Where $\sum t$ is the sum of the terms between x^n and y^n.

Lemma 1. For coprime positive integers x, y, the RHS of identity (1) produces a nonperfect power second term in Z+ if either x^n or y^n is held as perfect power of n.

Proof.

For the case of $(\sum t + y^n)$ and $(\sum t + x^n)$ to be expressed in the form of $\lambda^n y^n$ and $\lambda^n x^n$ respectively, the identities (1.1) and (1.2) ensure that the terms $(\sum t + y^n)$, $(\sum t + x^n)$ cannot be perfect power of n by FLT theorem, i.e. $(\sum t + y^n)$ cannot be reduced to $\lambda^n y^n$, neither $(\sum t + x^n)$

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can be reduced to $\lambda^n x^n$, where λ^n is perfect power of n positive integer. Therefore, such λ does not exist.
For the case of $(\sum t + y^n)$ and $(\sum t + x^n)$ to be expressed in the form of $\lambda^n y^n$ and $\lambda^n x^n$ respectively, the identities (1.1) and (1.2) ensures that the terms $(\sum t + y^n)$, $(\sum t + x^n)$ cannot be reduced to $y^\lambda y^n$, $x^\lambda x^n$ respectively to form a perfect power term because $\sum t$ always reduces to a composite number for $n \geq 3$ of coprime factors. To see this, let's expand the binomial $(x+y)^3$,

$$(x+y)^3 = x^3 + 3x^2y + 3y^2x + y^3 \qquad (2)$$

$$\sum t = 3x^2y + 3y^2x$$

$$\sum t = 3(x+y)xy \qquad (3)$$

The term $\sum t$ has coprime factors since the product of two coprime numbers is coprime with their sum therefore they cannot reduce to $y^\lambda y^n$ or $x^\lambda x^n$, where λ is a positive integer. This is simply because $\sum t$ of Equation (1) always gives a power of y or x that is less than n, as pertained by the expansion of binomials, and coefficients of composite numbers, i.e. $\sum t$ leaves the variable y with power 1 for the case of $n = 3$, which is less than the power 3 of the last term y^3, therefore, it cannot be combined to produce a perfect power term, i.e. the expression $3x^2y + 3y^2x + y^3$ on the RHS of equation (2) becomes $[3(x+y)x]y + y^3$, which cannot be combined to a perfect power term in y. This is because y and y^3 in the expression have different powers and the coefficient of y; $[3(x+y)x]$, has coprime factors that is different than y with the exception of $y = 3$, in which case the power of y becomes 2 and not 3 to be combined with y^3. This is always the case for higher n.
End of proof.

Example
Let $x = 2$ and $y = 3$. Equation (2) becomes,

$$5^3 = 2^3 + 3 * 2^2 * 3 + 3 * 3^2 * 2 + 3^3$$

Simplifying the term $\sum t$,

$$\sum t = 3 * 2^2 * 3 + 3 * 3^2 * 2 = 10 * 3^2$$

The expression $\sum t + y^n$ on the RHS of equation (2) becomes,

$$10 * 3^2 + 3^3$$

Which cannot be a perfect power of 3 because 10 is coprime with 3.

Lemma 2. For positive integers x, y, identity (1) produces all possible solutions of Beal's equation in three terms in Z+.

Proof.
On the RHS of identity (1), leaving x^n as perfect power term, $\sum t + y^n$ is a positive integer in Z+, and leaving y^n as perfect power term, $x^n + \sum t$ is a positive integer in Z+. Choosing all permutations of x, y over Z+ gives all possible solutions with the terms $x^n + \sum t$, $\sum t + y^n$ that include Beal's solutions with perfect power terms over Z+ with the proper choice of the common factor as pertained in Lemma1.
For powers different than n of x^non the RHS of equation (1.1), the identity fails to produce a second perfect power term on the RHS of the equation and describes a non-binomial identity as follows,

$$(x+y)^n = x^l + \sum t' \qquad (4)$$

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Where $l \neq n, l \geq 3$, $\sum t'$ is the sum of the rest of the terms on the RHS. $\sum t'$ is a composite number as pertained by the expansion of binomials of coprime variables x, y, therefore, $\sum t'$ cannot be perfect power integer by the methods of Lemma 1.
End of proof.

Proposition. Equation (1.1) can be simplified to,

$$(x+y)^n = x^n + ky^n \qquad (5)$$

Where k is composite number. We can introduce a common factor to simplify Equation (5) in two ways to obtain Beal's solutions.
First method:
For a common factor x, let $y = x$. Equation (5) becomes,

$$(2x)^n = x^n + kx^n \qquad (6)$$

To get Beal's solution, we set $x = k$,

$$(2k)^n = k^n + k^{n+1} \qquad (7)$$

Second method:
Set $ky^n = l$, l is a composite number, Equation (5) becomes

$$(x+y)^n = x^n + l \qquad (8)$$

Multiply Equation (8) by a common factor of l^n,

$$(x+y)^n l^n = x^n l^n + l^{n+1} \qquad (9)$$

Remark. The non-binomial identity, Equation (4) fails to form Beal's solution by multiplying the equation by a common factor because it does not comply with laws of exponents. Therefore, Lemma 2 holds over Z+.

Theorem. Expansion of powers of binomials produces an identity of three terms that requires a common factor for all three terms in Beal's equation to be perfect powers over Z+.

Proof.
From Lemmas 1, 2, the two terms on the RHS of equation (1) cannot be reduced to perfect power terms if x, y are coprime and we leave $\sum t$ in the RHS. If we move $\sum t$ to the LHS of the equation, LHS term cannot be reduced to perfect power term by same reasoning of Lemmas 1, 2.
End of proof.

Example. Let $x = 2$, $y = 3$ in Equation (2)

$$(2+3)^3 = 2^3 + 3*2^2*3 + 3*3^2*2 + 3^3$$

$(\sum t + y^n)$ produces the solution

$$5^3 = 2^3 + 117$$

We need to multiply the equation by the common factor $k^3$to produce all three perfect power terms.

$$(5k)^3 = (2k)^3 + 117k^3$$

Let $k = 117$, the solution with perfect power terms then is,

$$585^3 = 234^3 + 117^4$$

Let's set $y = x$ for a common factor x. Equation (2) becomes,

$$(2x)^3 = x^3 + 7x^3$$

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Taking the common factor $x = 7$, the equation becomes,

$$14^3 = 7^3 + 7^4$$

Remark. Generalization to Beal's equation where the bases share a common factor with infinitely many solutions are expressed in Equations (10)–(12),

$$3^{3n+2} = 3^{3n} + [2(3^n)]^3, \qquad n \geq 1 \qquad (10)$$

$$[a(a^n - b^n)^k]^n = [b(a^n - b^n)^k]^n + (a^n - b^n)^{kn+1}$$

$$a > b, \quad b \geq 1, \quad k \geq 1, \quad n \geq 3; \qquad (11)$$

$$(a^n + b^n)^{kn+1} = [a(a^n + b^n)^k]^n + [b(a^n + b^n)^k]^n$$

$$a \geq b, \quad b \geq 1, \quad k \geq 1, \quad n \geq 3; \qquad (12)$$

Equation (10) can be obtained from Equation (1) by setting $n = 2,\ x = 1,\ y = 2,$

$$(2 + 1)^2 = 2^2 + 2 * 2 * 1 + 1^2$$

to obtain the trivial equation,

$$3^2 = 1 + 2^3 \qquad (13)$$

Multiplying the equation by 3^{3n}, we get the generalized Equation (10)

$$3^{3n+2} = 3^{3n} + [2(3^n)]^3$$

Example. Multiply the trivial Equation (13) by 3^3,

$$3^5 = 3^3 + 6^3$$

Multiply Equation (13) by 3^9,

$$3^{11} = 54^3 + 3^9$$

Setting $x,\ y$, different than $1, 2;\quad n = 2$, gives different trivial equations,

Let $x = 2,\ x,\ y = 3,\ x,\ n = 2,$

$$(2 + 3)^2 = 2^2 + 2 * 2 * 3 + 3^2$$

$$5^2 = 2^2 + 21$$

Equation (11) can be derived from Equation (1) directly, while Equation (12) can be derived from Equation (1) by moving $\sum t$ to the LHS of Equation (1),

Example

$$(2 + 3)^2 = 2^2 + 2 * 2 * 3 + 3^2$$

moving $\sum t$ to the LHS,

$$5^2 - 12 = 2^2 + 3^2$$

Gives the trivial equation,

$$13 = 2^2 + 3^2$$

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Multiply by 13^2,

$$13^3 = 26^2 + 39^2$$

The same solution can be obtained by using the generalized Equation (12) by setting $a = 2$, $b = 3$, $n = 2$, $k = 1$.
For $n = 3$

$$(2+3)^3 = 2^3 + 3 * 2^2 * 3 + 3 * 3^2 * 2 + 3^3$$

moving $\sum t$ to the LHS,

$$5^3 - 90 = 2^3 + 3^3$$

Gives the trivial equation,

$$35 = 2^3 + 3^3$$

Multiply by 35^3,

$$35^4 = 70^3 + 105^3$$

The same solution can be obtained by using the generalized Equation (12) by setting $a = 2$, $b = 3$, $n = 3$, $k = 1$. If we let $k = 2$, we get the same equation from the trivial equation by multiplying by 35^6; multiplying by 1225^3. In other words, the generalized equations can be easily derived from Equation (1).

Conclusions

We have proved Beal's conjecture by identifying Beal's equation as an identity made by expansion of powers of binomials.

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