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Deriving the Pontecorvo–Maki–Nakagawa–Sakata Matrix from Koide's Mass Formula and Brannen's Neutrino Mass Hypothesis: Resolving the Mystery of the θ_{13} Rotation

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Abstract: The Koide mass formula, proposed by Yoshio Koide, is known to describe the mass relationship of charged leptons. Carl A. Brannen hypothesized that this formula also applies to neutrinos. Assuming Brannen's hypothesis to be valid, I constructed two three-dimensional mass models based on his proposed neutrino masses. As a result, I discovered that the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix can be derived by introducing an intermediate set of hypothetical states, referred to as mass negative eigenstates $(\nu_{1-}, \nu_{2-}, \nu_{3-})$, which mediate the transformation between mass eigenstates and flavor eigenstates. The Tribimaximal mixing matrix represents the transformation between mass negative and flavor eigenstates.

$$\begin{bmatrix} \nu_{1-} \\ \nu_{2-} \\ i\nu_{3-} \end{bmatrix} = \begin{bmatrix} \cos\theta_{13} & 0 & i\sin\theta_{13} \\ 0 & 1 & 0 \\ i\sin\theta_{13} & 0 & \cos\theta_{13} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ i\nu_3 \end{bmatrix},$$

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \nu_{1-} \\ \nu_{2-} \\ i\nu_{3-} \end{bmatrix}.$$

Thus,

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \cos\theta_{13} & 0 & i\sin\theta_{13} \\ 0 & 1 & 0 \\ i\sin\theta_{13} & 0 & \cos\theta_{13} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ i\nu_3 \end{bmatrix},$$

where $\theta_{13} \approx -0.173478 \, [\text{rad}] \, (\approx -9.939583°)$.

Consequently, the PMNS matrix can be approximated as:

$$U_{\text{PMNS}} \approx \begin{bmatrix} 0.804241 & 0.577350 & -0.140935\,i \\ -0.402120 - 0.122054\,i & 0.577350 & 0.696493 + 0.070468\,i \\ -0.402120 + 0.122054\,i & 0.577350 & -0.696493 + 0.070468\,i \end{bmatrix}$$

Future research will determine whether this PMNS matrix and its predictions for neutrino oscillations are consistent with experimental results.

Keywords: The Koide formula; Carl A. Brannen; θ_{13}; CP violation; neutrino oscillation

1. Introduction

1.1. Koide's Mass Formula

In 1982, Yoshio Koide first proposed the Koide mass formula [1,2] based on the study by Harari, Haut, and Weyers [3]:

$$\frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3},$$

which elegantly describes the relationship among the masses of the three generations of charged leptons.

1.2. Interpretation by Carl A. Brannen

In 2006, Carl A. Brannen provided an interpretation of the Koide mass formula in his paper [4]. Let the masses of e^-, μ^-, and τ^- be denoted as m_1, m_2, and m_3, respectively. The masses are experimentally determined as follows [5]:

$$m_1 = 0.510999 \ [\text{MeV}],$$

$$m_2 = 105.658367 \ [\text{MeV}],$$

$$m_3 = 1776.84 \ [\text{MeV}].$$

According to Brannen, the square root of each mass is expressed as:

$$\sqrt{m_n} = 17.715992 \ (1 + \sqrt{2}\cos(0.222222 + \frac{2}{3}n\pi)) \ [\text{MeV}^{\frac{1}{2}}] \quad \text{for } n = 1, 2, 3.$$

Using this formula, the following relationships hold:

$$\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3} = 17.715992 \times 3 \ [\text{MeV}^{\frac{1}{2}}],$$

and $m_1 + m_2 + m_3 = 17.715992^2 \times (3 + 3) \ [\text{MeV}]$.

Thus,

$$\frac{m_1 + m_2 + m_3}{(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2} = \frac{17.715992^2 \times 3 \times 2}{17.715992^2 \times 3^2} = \frac{2}{3}.$$

This provides a deeper mathematical insight into the Koide mass formula.

1.3. Brannen's Neutrino Mass Hypothesis

Brannen hypothesized that a similar relationship holds for neutrinos:

$$\frac{m_1 + m_2 + m_3}{(-\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2} = \frac{2}{3}.$$

Let the masses of ν_1, ν_2, and ν_3, be denoted as m_1, m_2, and m_3, respectively. Brannen proposed the following expressions [4]:

$$-\sqrt{m_n} = \frac{17715.992}{3^{11}} \ (1 + \sqrt{2}\cos(0.222222 + \frac{1}{12}\pi + \frac{2}{3}n\pi)) \ [\text{eV}^{\frac{1}{2}}] \quad \text{for } n = 1,$$

the minus sign appears because the expression evaluates to a negative value when $n = 1$.

$$\sqrt{m_n} = \frac{17715.992}{3^{11}} \ (1 + \sqrt{2}\cos(0.222222 + \frac{1}{12}\pi + \frac{2}{3}n\pi)) \ [\text{eV}^{\frac{1}{2}}] \quad \text{for } n = 2, 3.$$

From these, the neutrino masses are calculated as:

$$m_1 = 0.383462 \ [\text{meV}],$$

$$m_2 = 8.913487 \ [\text{meV}],$$

$$m_3 = 50.711804 \ [\text{meV}].$$

1.4. Constructing Two Three-Dimensional Mass Models

Here, a question arises: while Brannen indicates that the square root of the mass of ν_1 is negative, what does it mean for the square root of a mass to be negative?

Could it imply that ν_1 is antimatter, or might it suggest that ν_1 travels faster than the speed of light, effectively moving backward in time? The observed ν_1 should correspond to the positive square root of the mass.

Assuming Brannen's hypothesis is valid, I propose that $-\sqrt{m_1}$ might be the origin of the θ_{13} rotation in the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [6]. Based on Brannen's hypothesis, I construct two three-dimensional mass models for neutrinos.

2. Method

2.1. Construction of the Neutrino Three-Dimensional Mass Models

Let the masses of ν_1, ν_2, and ν_3, be denoted as m_1, m_2, and m_3, respectively, satisfying:

$$m_1 + m_2 + m_3 = 60.008753 \ [\text{meV}] = 7.746532^2 \ [\text{meV}]$$

In three-dimensional space, let the origin be O.

Define the radius:

$$r = 7.746532 \ [\text{meV}^{\frac{1}{2}}],$$

where r represents the radius of the sphere described by $x^2 + y^2 + z^2 = r^2$.

Define the points:

$$P_-(-\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}),$$

$$P_+(\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}),$$

where both P_- and P_+ lie on the sphere.

Additionally, define three points on the sphere:

$$N_1(r, 0, 0),$$

$$N_2(0, r, 0),$$

$$N_3(0, 0, r).$$

The models are constructed in two patterns, based on the square roots of the neutrino masses:

1. The combination $-\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3},$

2. The combination $\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}.$

2.1.1. Case of the Combination $-\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}$

2.1.1.1. Vectors and Dot Products

Define the unit vector (I refer to this vector as the $\sqrt{m}$ unit vector):

$$\overrightarrow{OM_-} = \frac{\overrightarrow{OP_-}}{\|\overrightarrow{OP_-}\|}.$$

Define the following vectors originating from O:

$$\boldsymbol{M_-} = \overrightarrow{OM_-} = (-0.079938, 0.385404, 0.919279),$$

$$\boldsymbol{N_{1-}} = \overrightarrow{ON_1} = (r, 0, 0),$$

$$\boldsymbol{N_{2-}} = \overrightarrow{ON_2} = (0, r, 0),$$

$$\boldsymbol{N_{3-}} = \overrightarrow{ON_3} = (0, 0, r).$$

The dot products are calculated as follows:

$$\boldsymbol{N_{1-}} \cdot \boldsymbol{M_-} = -0.079938r = -0.61924 \ [\text{meV}^{\frac{1}{2}}] = -\sqrt{m_1},$$

$$\boldsymbol{N_{2-}} \cdot \boldsymbol{M_-} = 0.385404r = 2.985546 \ [\text{meV}^{\frac{1}{2}}] = \sqrt{m_2},$$

$$\boldsymbol{N_{3-}} \cdot \boldsymbol{M_-} = 0.919279r = 7.121222 \ [\text{meV}^{\frac{1}{2}}] = \sqrt{m_3}.$$

To align the direction of $\boldsymbol{M_-}$ with the x-axis, $\boldsymbol{M_-}$, $\boldsymbol{N_{1-}}$, $\boldsymbol{N_{2-}}$, and $\boldsymbol{N_{3-}}$ are rotated around the origin in three-dimensional space.

2.1.1.2. Initial Coordinates

The initial coordinates are expressed as:

$$\begin{bmatrix} \boldsymbol{M_-} \\ \boldsymbol{N_{1-}} \\ \boldsymbol{N_{2-}} \\ \boldsymbol{N_{3-}} \end{bmatrix} = \begin{bmatrix} -0.079938 & 0.385404 & 0.919279 \\ r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

2.1.1.3. Rotation in the xy-Plane

Using $\cos \alpha_- = -0.203091$ and $\sin \alpha_- = -0.979160$, corresponding to $\alpha_- \approx -1.775310$ [rad], a rotation in the xy-plane is applied:

$$\begin{bmatrix} \boldsymbol{M_-} \\ \boldsymbol{N_{1-}} \\ \boldsymbol{N_{2-}} \\ \boldsymbol{N_{3-}} \end{bmatrix} = \begin{bmatrix} 0.393607 & 0 & 0.919279 \\ -0.203091r & -0.979160r & 0 \\ 0.979160r & -0.203091r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

2.1.1.4. Rotation in the xz-Plane

Using $\cos \beta_- = 0.393607$ and $\sin \beta_- = -0.919279$, corresponding to $\beta_- \approx -1.166244$ [rad], a rotation in the xz-plane is applied:

$$
\begin{bmatrix} M_- \\ N_{1-} \\ N_{2-} \\ N_{3-} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.079938r & -0.979160r & 0.186697r \\ 0.385404r & -0.203091r & -0.900121r \\ 0.919279r & 0 & 0.393607r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
$$

2.1.1.5. Rotation in the yz-Plane

Using $\cos\gamma_- = 0.220094$ and $\sin\gamma_- = -0.975479$, corresponding to $\gamma_- \approx -1.348886$ [rad], a rotation in the yz-plane is applied (optional for visualization purposes to set the z-component of to N_{2-} to 0):

$$
\begin{bmatrix} M_- \\ N_{1-} \\ N_{2-} \\ N_{3-} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.079938r & -0.033388r & 0.996241r \\ 0.385404r & -0.922748r & 0 \\ 0.919279r & 0.383955r & 0.086631r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.
$$

The x-components of N_{1-}, N_{2-} and N_{3-} denote $-\sqrt{m_1}$, $\sqrt{m_2}$, and $\sqrt{m_3}$, respectively, thus associating each vector with the respective neutrino.

2.1.2. Case of the Combination $\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}$

2.1.2.1. Vectors and Dot Products

Define the unit vector:

$$
\overrightarrow{OM_+} = \frac{\overrightarrow{OP_+}}{\|\overrightarrow{OP_+}\|}.
$$

Define the following vectors originating from O:

$$
M_+ = \overrightarrow{OM_+} = (0.079938, 0.385404, 0.919279),
$$

$$
N_{1+} = \overrightarrow{ON_1} = (r, 0, 0),
$$

$$
N_{2+} = \overrightarrow{ON_2} = (0, r, 0),
$$

$$
N_{3+} = \overrightarrow{ON_3} = (0, 0, r).
$$

The dot products are calculated as follows:

$$
N_{1+} \cdot M_+ = 0.079938r = 0.61924 \; [\mathrm{meV}^{\frac{1}{2}}] = \sqrt{m_1},
$$

$$
N_{2+} \cdot M_+ = 0.385404r = 2.985546 \; [\mathrm{meV}^{\frac{1}{2}}] = \sqrt{m_2},
$$

$$
N_{3+} \cdot M_+ = 0.919279r = 7.121222 \; [\mathrm{meV}^{\frac{1}{2}}] = \sqrt{m_3}.
$$

To align the direction of M_+ with the x-axis, M_+, N_{1+}, N_{2+}, and N_{3+} are rotated around the origin in three-dimensional space.

2.1.2.2. Initial Coordinates

The initial coordinates are expressed as:

$$\begin{bmatrix} M_+ \\ N_{1+} \\ N_{2+} \\ N_{3+} \end{bmatrix} = \begin{bmatrix} 0.079938 & 0.385404 & 0.919279 \\ r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

2.1.2.3. Rotation in the xy-Plane

Using $\cos \alpha_+ = 0.203091$ and $\sin \alpha_+ = -0.979160$, corresponding to $\alpha_+ \approx -1.366283$ [rad], a rotation in the xy-plane is applied:

$$\begin{bmatrix} M_+ \\ N_{1+} \\ N_{2+} \\ N_{3+} \end{bmatrix} = \begin{bmatrix} 0.393607 & 0 & 0.919279 \\ 0.203091r & -0.979160r & 0 \\ 0.979160r & 0.203091r & 0 \\ 0 & 0 & r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

2.1.2.4. Rotation in the xz-Plane

Using $\cos \beta_+ = 0.393607$ and $\sin \beta_+ = -0.919279$, corresponding to $\beta_+ \approx -1.166244$ [rad], a rotation in the xz-plane is applied:

$$\begin{bmatrix} M_+ \\ N_{1+} \\ N_{2+} \\ N_{3+} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.079938r & -0.979160r & -0.186697r \\ 0.385404r & 0.203091r & -0.900121r \\ 0.919279r & 0 & 0.393607r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

2.1.2.5. Rotation in the yz-Plane

Using $\cos \gamma_+ = -0.220094$ and $\sin \gamma_+ = -0.975479$, corresponding to $\gamma_+ \approx -1.792707$ [rad], a rotation in the yz-plane is applied:

$$\begin{bmatrix} M_+ \\ N_{1+} \\ N_{2+} \\ N_{3+} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.079938r & 0.033388r & 0.996241r \\ 0.385404r & -0.922748r & 0 \\ 0.919279r & 0.383955r & -0.086631r \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The x-components of N_{1+}, N_{2+} and N_{3+} denote $\sqrt{m_1}$, $\sqrt{m_2}$, and $\sqrt{m_3}$, respectively, thus associating each vector with the respective neutrino.

3. Result

Figure 1 shows the results of the two neutrino three-dimensional mass models.
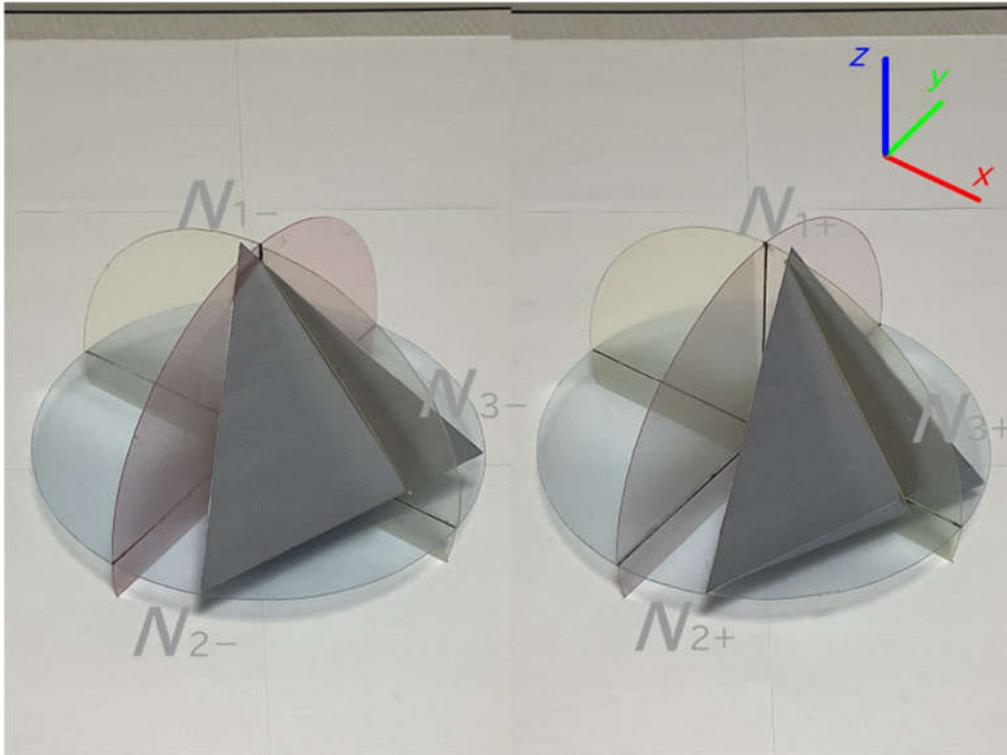
Figure 1. Two neutrino three-dimensional mass models.

The relationship between the two models can be expressed as:

$$\begin{bmatrix} \boldsymbol{N}_{1-} \\ \boldsymbol{N}_{2-} \\ \boldsymbol{N}_{3-} \end{bmatrix} = \begin{bmatrix} 0.984990 & 0.000000 & -0.172610 \\ 0.000000 & 1.000000 & 0.000000 \\ 0.172610 & 0.000000 & 0.984990 \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_{1+} \\ \boldsymbol{N}_{2+} \\ \boldsymbol{N}_{3+} \end{bmatrix}$$

This simplifies to:

$$\begin{bmatrix} \boldsymbol{N}_{1-} \\ \boldsymbol{N}_{2-} \\ \boldsymbol{N}_{3-} \end{bmatrix} = \begin{bmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} \\ 0 & 1 & 0 \\ -\sin\theta_{13} & 0 & \cos\theta_{13} \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_{1+} \\ \boldsymbol{N}_{2+} \\ \boldsymbol{N}_{3+} \end{bmatrix}$$

Here, $\theta_{13} \approx -0.173478$ [rad] $(\approx -9.939583°)$.

4. Discussion

4.1. Correspondence to CP Violation

Let each component be extended into a complex number to account for CP violation [7,8]. The extended vectors are given by:

$$\text{i}\boldsymbol{N}_{3-} = (0.919279r\,\text{i}, 0.38395r\,\text{i}, 0.086631r\,\text{i}),$$

$$\text{i}\boldsymbol{N}_{3+} = (0.919279r\,\text{i}, 0.383955r\,\text{i}, -0.086631r\,\text{i}).$$

Then, the relationship between the two matrices can be expressed as:

$$\begin{bmatrix} \boldsymbol{N}_{1-} \\ \boldsymbol{N}_{2-} \\ \text{i}\boldsymbol{N}_{3-} \end{bmatrix} = \begin{bmatrix} \cos\theta_{13} & 0 & \text{i}\sin\theta_{13} \\ 0 & 1 & 0 \\ \text{i}\sin\theta_{13} & 0 & \cos\theta_{13} \end{bmatrix} \begin{bmatrix} \boldsymbol{N}_{1+} \\ \boldsymbol{N}_{2+} \\ \text{i}\boldsymbol{N}_{3+} \end{bmatrix}$$

I distinguish between the two states: the state $(\nu_{1-}, \nu_{2-}, \nu_{3-})$ square root of the mass of ν_1 is negative, referred to as mass negative (ν_1, ν_2, ν_3) eigenstates , and the states where it is positive, referred to as mass eigenstates .

By associating each vector with the respective neutrino, the following relation can be written:

$$
\begin{bmatrix} \nu_{1-} \\ \nu_{2-} \\ i\nu_{3-} \end{bmatrix} = \begin{bmatrix} \cos\theta_{13} & 0 & i\sin\theta_{13} \\ 0 & 1 & 0 \\ i\sin\theta_{13} & 0 & \cos\theta_{13} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ i\nu_3 \end{bmatrix}
$$

4.2. Product with the Tribimaximal Mixing Matrix

I now consider the Tribimaximal mixing matrix [9].

The Tribimaximal mixing matrix U_{TBM} is defined by the product of two unitary matrices:

$$
U_{\mathrm{TBM}} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{\omega}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} \\ \frac{\omega^2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}i \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}i \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix},
$$

where $\omega = e^{\frac{2}{3}\pi i}$, which is the complex cube root of unity.

The Tribimaximal mixing matrix is regarded as a transformation matrix between the mass negative eigenstates and the flavor eigenstates of neutrinos:

$$
\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \nu_{1-} \\ \nu_{2-} \\ i\nu_{3-} \end{bmatrix}.
$$

Accordingly, the relationship between the mass negative eigenstates and the flavor eigenstates can be expressed as:

$$
\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \cos\theta_{13} & 0 & i\sin\theta_{13} \\ 0 & 1 & 0 \\ i\sin\theta_{13} & 0 & \cos\theta_{13} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ i\nu_3 \end{bmatrix}.
$$

The product of the θ_{13} rotation matrix and Tribimaximal mixing matrix can be approximated as follows:

$$
\begin{bmatrix} 0.804241 & 0.577350 & -0.140935\,i \\ -0.402120 - 0.122054\,i & 0.577350 & 0.696493 + 0.070468\,i \\ -0.402120 + 0.122054\,i & 0.577350 & -0.696493 + 0.070468\,i \end{bmatrix}
$$

Could this be interpreted as the PMNS matrix?

The absolute values of each component are:

$$
\begin{bmatrix} 0.804241 & 0.577350 & 0.140935 \\ 0.420235 & 0.577350 & 0.700049 \\ 0.420235 & 0.577350 & 0.700049 \end{bmatrix}
$$

The resulting values appear to closely match "Leptonic Mixing Matrix" provided by *NuFIT 5.3* [10].

4.3. Neutrino Oscillation

The validity of the PMNS matrix derived here depends on whether the neutrino oscillations [11] predictions calculated with this PMNS matrix agree with the experimental data.

4.3.1. Probability Calculation

The formula for the oscillation probabilities of each neutrino in neutrino oscillations can be derived as follows:

Let the flavor state before oscillation be ν_α and the flavor state after oscillation be ν_β. The calculation involves the following steps:

(1) Decomposing the flavor eigenstate ν_α into mass eigenstates (ν_1, ν_2, ν_3) using the PMNS matrix.

(2) Applying phase shifts due to the time evolution of each mass eigenstate.

(3) Reconstructing the flavor eigenstate ν_β from the mass eigenstates (ν_1, ν_2, ν_3) using the inverse PMNS matrix.

In step (2), the phase shift of each ν_n for $n = 1, 2, 3$ is shifted by θ_n, where θ_n depends on the neutrino mass, its propagation distance, and its energy. Since the estimated neutrino masses are known in this study, the calculation proceeds in a straightforward manner:

$$\theta_1 \approx \frac{-2.533866 \times 10^9 \,[/\mathrm{eV/km}] \cdot m_1{}^2 \cdot L}{E} \,[\mathrm{rad}] \approx \frac{-0.000373 \times 10^6 \,[\mathrm{eV/km}] \cdot L}{E} \,[\mathrm{rad}],$$

$$\theta_2 \approx \frac{-2.533866 \times 10^9 \,[/\mathrm{eV/km}] \cdot m_2{}^2 \cdot L}{E} \,[\mathrm{rad}] \approx \frac{-0.201316 \times 10^6 \,[\mathrm{eV/km}] \cdot L}{E} \,[\mathrm{rad}],$$

$$\theta_3 \approx \frac{-2.533866 \times 10^9 \,[/\mathrm{eV/km}] \cdot m_3{}^2 \cdot L}{E} \,[\mathrm{rad}] \approx \frac{-6.516309 \times 10^6 \,[\mathrm{eV/km}] \cdot L}{E} \,[\mathrm{rad}],$$

where L is the propagation distance, and E is the energy of the neutrino.

In step (3), the inverse of the PMNS matrix is required.

Representing the PMNS matrix as:

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ i\nu_3 \end{bmatrix},$$

and noting that the PMNS matrix is unitary, its inverse is simply its Hermitian adjoint matrix. Therefore:

$$\begin{bmatrix} \nu_1 \\ \nu_2 \\ i\nu_3 \end{bmatrix} = \begin{bmatrix} \overline{U_{e1}} & \overline{U_{\mu1}} & \overline{U_{\tau1}} \\ \overline{U_{e2}} & \overline{U_{\mu2}} & \overline{U_{\tau2}} \\ \overline{U_{e3}} & \overline{U_{\mu3}} & \overline{U_{\tau3}} \end{bmatrix} \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix}.$$

The probability of oscillation from ν_α to ν_β, denoted as $P(\nu_\alpha \to \nu_\beta)$, is given by:

$$P(\nu_\alpha \to \nu_\beta) = \left| \sum_{j=1}^{3} (U_{\alpha j} \, e^{i\theta_j} \, \overline{U_{\beta j}}) \right|^2 = \left| U_{\alpha 1} \, e^{i\theta_1} \, \overline{U_{\beta 1}} + U_{\alpha 2} \, e^{i\theta_2} \, \overline{U_{\beta 2}} + U_{\alpha 3} \, e^{i\theta_3} \, \overline{U_{\beta 3}} \right|^2.$$

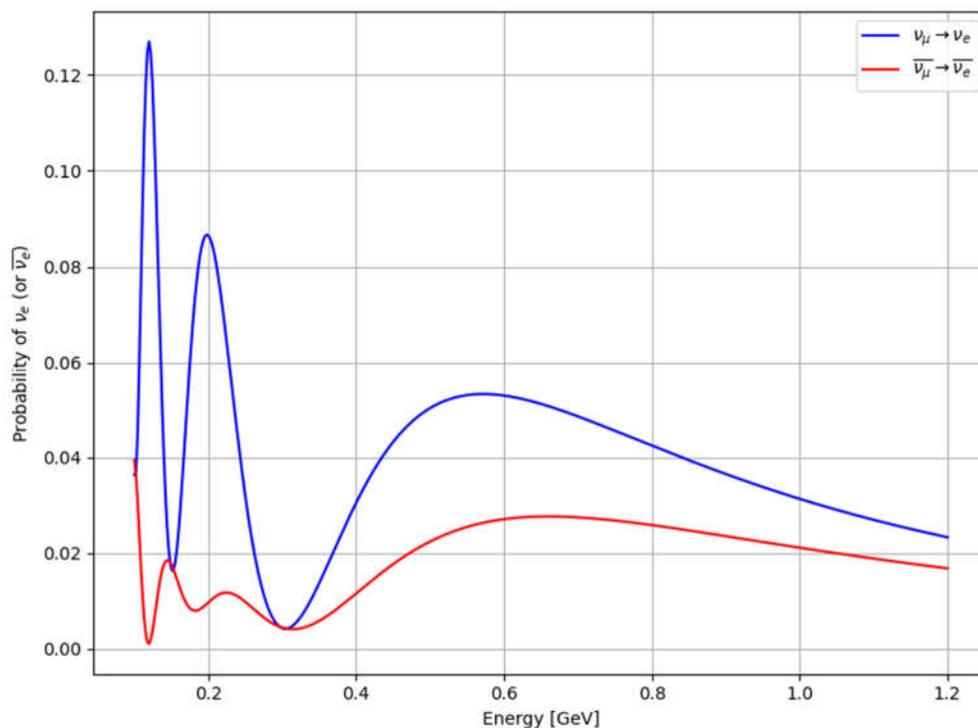For antineutrinos, the corresponding probability is:

$$P(\overline{\nu_\alpha} \to \overline{\nu_\beta}) = \left| \sum_{j=1}^{3} (\overline{U_{\alpha j}} \; e^{i\theta_j} \; U_{\beta j}) \right|^2,$$

or alternatively:

$$P(\overline{\nu_\alpha} \to \overline{\nu_\beta}) = \left| \sum_{j=1}^{3} (U_{\alpha j} \; e^{-i\theta_j} \; \overline{U_{\beta j}}) \right|^2.$$

Following the experimental setup of T2K (Tokai to Kamioka) [12,13], I calculate $P(\nu_\mu \to \nu_e)$ and $P(\overline{\nu_\mu} \to \overline{\nu_e})$.

When the propagation distance is fixed at $L = 295 \; [\mathrm{km}]$, the oscillation probabilities depend on the neutrino energy (see Figure 2).

Figure 2. Relation of the neutrino energy and the neutrino oscillation probability.

4.3.2. Energy Distribution of the Muon (Anti-)Neutrino Beam

Based on the experimental setup, the energy of the emitted ν_μ (or $\overline{\nu_\mu}$) beam is not precisely $0.6 \; [\mathrm{GeV}]$ but instead exhibits a spread in its distribution. Although I do not know the exact form of the beam energy distribution, I assume, for example, that it can be represented by a function such as:
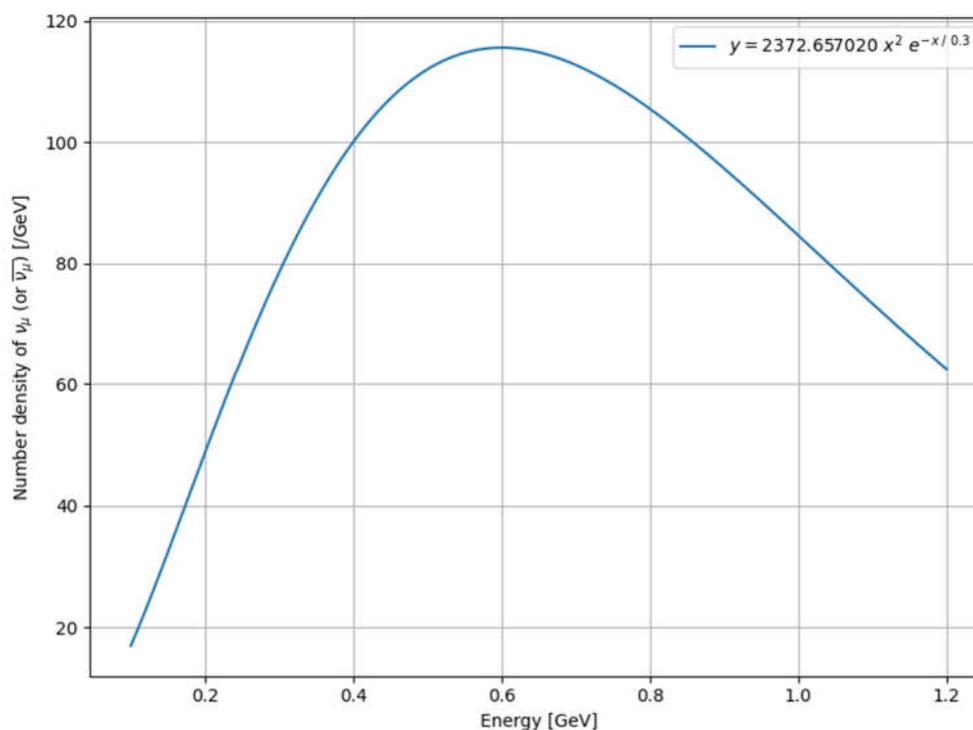
$$y = 2446.048140 \cdot x^2 \cdot e^{-\frac{x}{0.3}},$$

where $x \; [\mathrm{GeV}]$ represents the beam energy and $y \; [/\mathrm{GeV}]$ denotes the number density of the emitted ν_μ (or $\overline{\nu_\mu}$). This distribution is shown in Figure 3.

The function has its peak value at $x = 0.6$.

The expected number of emitted ν_μ (or $\overline{\nu_\mu}$) in the energy range $0.1 - 1.2 \; [\mathrm{GeV}]$ is given by:

$$\int_{0.1}^{1.2} y \ \mathrm{d}x \approx 100.000000$$

.

Figure 3. Relation of the neutrino beam energy and the number density of ν_e (or $\overline{\nu_e}$.)

4.3.3. Probability Density and Expected Number of (Anti-)Neutrinos

By combining the two functions, the probability density for ν_e (or $\overline{\nu_e}$) is obtained, as shown in Figure 4.

The shape of the graph in Figure 4 seems similar to the graph in the paper by the T2K Collaboration [13].

By integrating over the range $0.1 - 1.2 \ [\text{GeV}]$, the expected number of events can be estimated as follows:

Out of 100 observed neutrinos, the expected number of ν_e is:

approximately $3.921568 \ [\text{Events}]$.

Out of 100 observed antineutrinos, the expected number of $\overline{\nu_e}$ is:

approximately $2.023572 \ [\text{Events}]$.

However, various other conditions are involved in actual observations, making it difficult for an individual to verify whether the derived values of the PMNS matrix are correct.

Future research findings in the future are awaited.
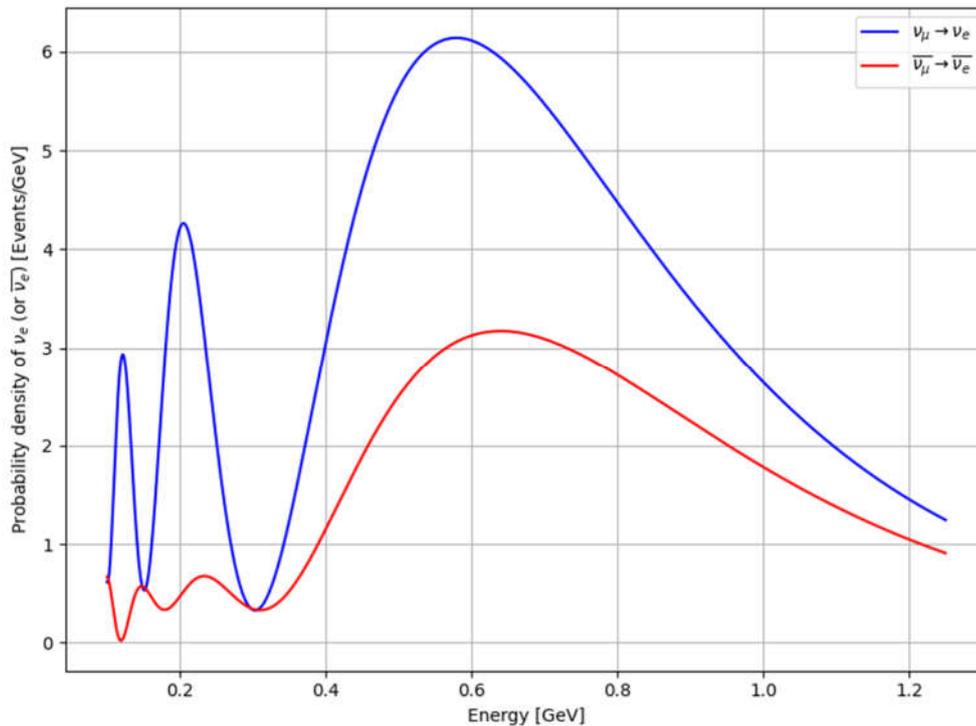
Figure 4. Relation of the neutrino energy and the probability density of ν_e (or $\overline{\nu_e}$.)

5. Conclusion

Assuming the correctness of Koide's mass formula and Carl A. Brannen's neutrino mass hypothesis, two three-dimensional mass models were constructed.

As a result, I discovered that the PMNS matrix can be derived by introducing an intermediate set of hypothetical states, referred to as mass negative eigenstates, which mediate the transformation between mass eigenstates and flavor eigenstates.

Based on this proposal, the PMNS matrix is derived as follows:

$$U_{\text{PMNS}} \approx \begin{bmatrix} 0.804241 & 0.577350 & -0.140935\,\mathrm{i} \\ -0.402120 - 0.122054\,\mathrm{i} & 0.577350 & 0.696493 + 0.070468\,\mathrm{i} \\ -0.402120 + 0.122054\,\mathrm{i} & 0.577350 & -0.696493 + 0.070468\,\mathrm{i} \end{bmatrix}$$

Whether this PMNS matrix and the neutrino oscillation expectations derived from it are correct remains to be verified by future research findings.

Revision Note: This is the fourth edition of the article, with minor textual errors corrected and additional references included. The original version was published on December 22, 2024.

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