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Article

Two-Tone Factors in Regular and Claw-Free Graphs: Existence, Algorithmic Construction, and Symmetry Breaking

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Abstract

Regular graphs are classical symmetric structures in graph theory, where each vertex has identical degree and the overall topology often exhibits strong automorphism properties. However, practical systems frequently require heterogeneous constraints, which can be modeled by introducing vertex colorings and non-uniform degree requirements, leading to controlled symmetry breaking. In this paper, we investigate two-tone factors in edge-connected regular graphs and claw-free cubic graphs under arbitrary red-blue vertex colorings. Using the framework of parity (g, f)-factors, we establish two main existence results. First, we prove that every λ-edge connected r-regular graph admits a two-tone $(\{k\}, \{k, k+2\})$-factor for any coloring, provided that $r/\lambda \leqslant k \leqslant r - r/\lambda$, $k \leqslant r-2$, and $k|G|$ is even. Second, we show that every 3-edge connected claw-free cubic graph admits a two-tone $(\{0,1\}, \{2,3\})$-factor regardless of the coloring configuration. Beyond existence, we provide a constructive algorithm by reducing the parity factor problem to an exact f-factor problem and further to a perfect matching problem via vertex-splitting techniques. We rigorously justify the correctness of this reduction and show that the desired factor can be computed in polynomial time. From a structural perspective, our results reveal that edge-connectivity serves as a stabilizing mechanism that preserves parity feasibility under arbitrary color-induced perturbations, while claw-free constraints enforce local density that prevents parity imbalance. This provides a symmetry-based interpretation of two-tone factors as a balance between global regularity and local asymmetry. These findings contribute to both the theoretical development of factor theory and its algorithmic realization, with potential implications for deterministic network design and resource allocation in structured systems.

Keywords: regular graph; claw-free graph; two-tone factor; parity factor; perfect matching; polynomial algorithm; symmetry breaking

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1. Introduction

Regular graphs are classical symmetric structures in graph theory, where each vertex has identical degree. Such structural uniformity is closely related to symmetry properties, often reflected in large automorphism groups and homogeneous connectivity patterns. These connections have been extensively studied in algebraic graph theory [1], and continue to play a fundamental role in modern structural graph theory. Moreover, recent advances in graph isomorphism highlight the deep relationship between symmetry and structural regularity in graphs [2].

Despite their inherent symmetry, many real-world systems require the incorporation of heterogeneous constraints, such as prioritized resource allocation, asymmetric communication requirements, or localized structural perturbations. Such settings naturally lead to controlled symmetry breaking, where global regularity is preserved while local structural constraints vary. Similar phenomena have been

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observed in complex networks and distributed systems, where robustness and adaptability require deviations from perfect symmetry [3,4].

A natural and mathematically tractable way to model such controlled asymmetry is through vertex coloring combined with degree constraints. In this setting, vertices are partitioned into different classes (e.g., red and blue), and each class is assigned distinct degree requirements. This leads to the notion of two-tone factors, which extend classical factor theory by allowing heterogeneous degree conditions across vertex subsets.

The study of degree-constrained factors has a long history. Foundational results on k-factors in regular graphs date back to classical works such as [5,6], and were further developed through factorization theory [7,8]. More general frameworks, including (g, f)-factors and parity factors, were introduced to capture flexible degree constraints [9–11]. Comprehensive treatments of factor theory can be found in [12,13].

From an algorithmic perspective, factor problems are closely related to matching theory and combinatorial optimization. In particular, classical reductions transform (g, f)-factor problems into matching problems, which can be solved efficiently using polynomial-time algorithms [14,15]. Modern implementations and improvements of matching algorithms further enhance practical solvability [16,17].

More recently, colored factor problems have been introduced to capture heterogeneous constraints in regular graphs. In particular, Furuya and Kano [18] studied degree factors under red-blue colorings, while related results for claw-free cubic graphs were established in [19]. These works demonstrate how structural conditions such as edge-connectivity and claw-freeness ensure feasibility under coloring constraints.

In this paper, we continue the study of two-tone factors from both structural and algorithmic perspectives. We investigate edge-connected regular graphs and claw-free cubic graphs under arbitrary red-blue vertex colorings, and establish new existence results that extend previous work.

Beyond existence, we emphasize the constructive aspect of the problem. By leveraging parity (g, f)-factor theory and its reduction to matching problems, we provide a unified framework that connects structural graph properties with algorithmic realizability. This perspective reveals that edge-connectivity acts as a stabilizing mechanism ensuring feasibility under heterogeneous constraints.

From a symmetry viewpoint, our results highlight a fundamental principle: vertex coloring introduces controlled symmetry breaking, while parity constraints act as symmetry-preserving regulators. This balance between symmetry and heterogeneity provides new insight into the robustness of regular graph structures under constrained perturbations.

To provide intuition for the concept of two-tone factors, we illustrate a concrete example in Figure 1. The figure shows how a regular graph can admit a spanning subgraph satisfying heterogeneous degree constraints under a red-blue coloring.

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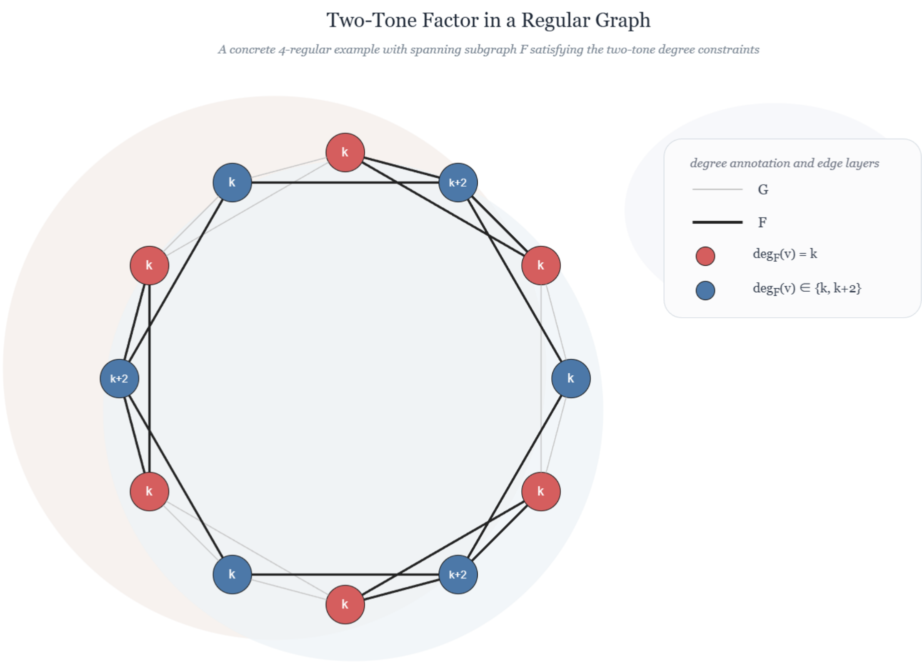

Figure 1. Illustration of a two-tone factor in a regular graph. The underlying graph G is shown with light edges, while the selected factor F is highlighted with thick edges. Red vertices have degree exactly k in F, whereas blue vertices have degree either k or $k+2$, satisfying the two-tone degree constraints.

2. Related Work

The theory of graph factors is one of the central topics in graph theory, with a long and well-established history. Early foundational work on factorization of regular graphs can be traced back to classical results by Petersen and König [5,6], which laid the groundwork for the study of degree-constrained subgraphs. These developments were later unified and systematically presented in comprehensive treatments such as [12,13]. In addition, algebraic perspectives on graph structure further highlight the role of symmetry in regular graphs [1].

We first recall several classical results on the existence of k-factors in regular graphs, which play a fundamental role in factor theory.

Theorem 1. *Let λ, r, and k be integers with $1 \leq \lambda \leq r$ and $1 \leq k < r$. Let G be a λ-edge connected r-regular graph. If one of the following conditions holds, then G has a k-factor:*

(i) *G is bipartite [6];*
(ii) *both r and k are even [5];*
(iii) *r is even, k is odd, $|G|$ is even, and $r/\lambda \leq k \leq r - r/\lambda$ [7];*
(iv) *r is odd, k is even, and $k \leq r - r/\lambda$ [7];*
(v) *both r and k are odd and $r/\lambda \leq k$ [7,8].*

The following corollary provides a unified sufficient condition for the existence of k-factors under edge-connectivity constraints.

Corollary 1. *Let λ, r, and k be integers with $1 \leq \lambda \leq r$ and $r/\lambda \leq k \leq r - r/\lambda$. Let G be a λ-edge connected r-regular graph such that $k|G|$ is even. Then G has a k-factor [12].*

Beyond exact k-factors, relaxation to $\{k, k+1\}$-factors has also been extensively studied, allowing bounded deviations in vertex degrees. Such relaxations are closely related to more general degree-constrained subgraph problems [11,20].

Theorem 2. *Let r and k be integers with $1 \leq k < r$. Let G be an r-regular graph. Then:*

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(i) *G has a $\{k, k+1\}$-factor [21];*
(ii) *for a maximal independent set W, there exists a $\{k, k+1\}$-factor with prescribed structure [22];*
(iii) *if $k \le 2r/3 - 1$, then G has a regular $\{k, k+1\}$-factor [23].*

More generally, the theory of (g, f)-factors provides a unified framework for handling vertex-wise degree constraints. Fundamental results by Lovász [9,10] establish necessary and sufficient conditions for the existence of such factors via parity and cut conditions. These results reveal that factor existence can be characterized through structural constraints involving vertex subsets and connectivity. Further extensions and algorithmic formulations of (g, f)-factor problems can be found in [24,25].

From an algorithmic perspective, factor problems are closely related to matching theory. In particular, the (g, f)-factor problem can be reduced to a perfect matching problem through classical transformations, and thus can be solved in polynomial time using matching algorithms such as Edmonds' blossom algorithm [14]. More advanced implementations and practical improvements of matching algorithms have been developed in [16,17,26]. Comprehensive treatments of matching and combinatorial optimization can be found in [15,27]. These connections establish a deep link between structural graph theory and efficient algorithm design.

More recently, colored factor problems have been introduced to model heterogeneous constraints in regular graphs. In particular, Furuya and Kano [18] studied degree factors under red-blue vertex colorings of regular graphs, while further results on claw-free cubic graphs and two-tone factors were established in [19]. These works demonstrate that structural properties such as edge-connectivity and claw-freeness play a critical role in ensuring the existence of feasible factors under coloring constraints. In addition, structural properties of claw-free graphs have been extensively studied in [28], providing deeper insights into their role in factor-related problems.

In addition, structural properties such as edge-connectivity have also been extensively studied in interconnection network models. For example, the edge-connectivity of expanded k-ary n-cube networks was analyzed in [29], demonstrating strong robustness under edge failures. These results further reinforce the importance of connectivity as a fundamental mechanism for maintaining structural feasibility under constraints.

Our work builds upon these developments by extending the theory of two-tone factors to more general settings, while also emphasizing their algorithmic realizability. In particular, we combine structural results on edge-connectivity and claw-free graphs with parity factor techniques and matching-based constructions, thereby bridging classical factor theory and modern algorithmic approaches. Moreover, our perspective is also related to broader studies of network robustness and constrained structures in complex systems [3,4].

3. Main Results

We now present our main results on two-tone factors in regular graphs and claw-free cubic graphs. These results extend existing work on degree-constrained factors under vertex colorings, particularly the recent studies of Furuya and Kano [18,19], by relaxing structural conditions and enlarging the range of feasible degree assignments.

Our first theorem concerns edge-connected regular graphs. It generalizes classical k-factor existence results under connectivity constraints [7,8] to the setting of heterogeneous degree conditions induced by vertex colorings. In particular, it shows that sufficiently high edge-connectivity guarantees the existence of two-tone factors with a controlled degree gap.

Theorem 3. *Let λ, r, and k be integers with $1 \le \lambda \le r$ and $r/\lambda \le k \le r - r/\lambda$. Let G be a λ-edge connected r-regular graph such that $k|G|$ is even. Then for every red-blue vertex coloring of G, G has a two-tone $(\{k\}, \{k, k+2\})$-factor.*

The proof of Theorem 3 is based on the theory of parity (g, f)-factors [10], where feasibility is characterized via cut conditions involving vertex subsets. The key idea is to translate the two-tone constraint into a parity-constrained factor problem and to exploit edge-connectivity to control the contribution of each component.

Our second result focuses on claw-free cubic graphs. In this setting, local structural constraints—specifically the absence of induced $K_{1,3}$—allow finer control of degree assignments. Compared with previous results [19], we remove additional restrictions on vertex distances and obtain a more general existence theorem under arbitrary colorings.

Theorem 4. *Let G be a 3-edge connected claw-free cubic graph. For every red-blue vertex coloring of G, G has a two-tone* $(\{0,1\},\{2,3\})$*-factor.*

The proof of Theorem 4 relies on the general (g,f)-factor framework and a refined structural analysis of claw-free graphs. In particular, the claw-free condition ensures that local edge configurations can be controlled through triangle-based arguments, which prevents the formation of infeasible configurations.

Together, Theorems 3 and 4 demonstrate that two distinct mechanisms—global edge-connectivity and local structural constraints—can independently guarantee the existence of two-tone factors. This highlights a fundamental principle: global symmetry (regularity) can be preserved under heterogeneous constraints as long as sufficient structural robustness is present.

The detailed proofs of Theorems 3 and 4 will be given in Section 5, where we further develop the connection to parity factor theory. In addition, an algorithmic construction based on matching reductions will be presented in Section 6.

4. Proof of Main Results

4.1. Parity Factor Criterion

We first recall a fundamental result on parity (g,f)-factors, originally established by Lovász [10]. This theorem provides a complete characterization of the existence of parity-constrained factors via cut conditions, and serves as the main technical tool in our proofs.

Theorem 5. *Let G be a graph and let* $g,f: V(G)\to\mathbb{Z}$ *be functions satisfying* $g(v)\le f(v)$ *and* $g(v)\equiv f(v) \pmod 2$ *for all* $v\in V(G)$. *Then G has a parity* (g,f)*-factor if and only if for all pairs of disjoint sets* $S,T\subseteq V(G)$,

$$\eta(S,T) := f(S) + \deg_G(T) - g(T) - e_G(S,T) - q^*(S,T) \ge 0,$$

where $q^*(S,T)$ *denotes the number of components D of* $G-(S\cup T)$ *satisfying*

$$f(D) + e_G(D,T) \equiv 1 \pmod 2.$$

Such components are called q^**-odd components.*

4.2. Proof of Theorem 3

Proof. Define two functions $g,f: V(G)\to\mathbb{Z}$ by

$$g(v)=k,\quad f(v)=\begin{cases} k, & v\in R(G),\\ k+2, & v\in B(G).\end{cases}$$

Then G has a two-tone $(\{k\},\{k,k+2\})$-factor if and only if G has a parity (g,f)-factor. By Theorem 5, it suffices to verify that

$$\eta(S,T)\ge 0$$

for all disjoint sets $S,T\subseteq V(G)$.

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Since $k|G|$ is even, we have

$$f(V(G)) = k|R(G)| + (k+2)|B(G)| \equiv 0 \pmod 2.$$

Thus $\eta(\emptyset,\emptyset) = 0$, and we may assume $S \cup T \neq \emptyset$.

Let $D_1, \dots, D_m$ be the q^*-odd components of $G - (S \cup T)$, where $m = q^*(S,T)$. Let $\theta = k/r$. Since $r/\lambda \leq k \leq r(1 - 1/\lambda)$, we have $0 < \theta < 1$.

We estimate:

$$\begin{aligned}\eta(S,T) &= f(S) + \deg_G(T) - g(T) - e_G(S,T) - m \\ &\geq k|S| + \deg_G(T) - k|T| - e_G(S,T) - m \\ &= \frac{k}{r}\deg_G(S) + \left(1 - \frac{k}{r}\right)\deg_G(T) - e_G(S,T) - m.\end{aligned}$$

Since G is r-regular, $\deg_G(S) = r|S|$ and $\deg_G(T) = r|T|$. Moreover, each edge incident to S or T either connects to T, S, or some component D_i. Thus,

$$\deg_G(S) \geq \sum_{i=1}^{m} e_G(S,D_i) + e_G(S,T), \quad \deg_G(T) \geq \sum_{i=1}^{m} e_G(T,D_i) + e_G(T,S).$$

Substituting these bounds yields

$$\eta(S,T) \geq \sum_{i=1}^{m} (\theta e_G(S,D_i) + (1-\theta)e_G(T,D_i) - 1).$$

For each i, define

$$\varphi_i = \theta e_G(S,D_i) + (1-\theta)e_G(T,D_i) - 1.$$

By the assumption $r/\lambda \leq k \leq r(1 - 1/\lambda)$, we have

$$\theta\lambda \geq 1, \quad (1-\theta)\lambda \geq 1.$$

We consider three cases:

Case 1. $e_G(S,D_i) \geq 1$ and $e_G(T,D_i) \geq 1$.
Then

$$\varphi_i \geq \theta + (1-\theta) - 1 = 0.$$

Case 2. $e_G(S,D_i) = 0$.
Then $e_G(T,D_i) \geq \lambda$, and hence

$$\varphi_i \geq (1-\theta)\lambda - 1 \geq 0.$$

Case 3. $e_G(T,D_i) = 0$.
Then $e_G(S,D_i) \geq \lambda$, and hence

$$\varphi_i \geq \theta\lambda - 1 \geq 0.$$

Thus $\varphi_i \geq 0$ for all i, and therefore $\eta(S,T) \geq 0$. Hence G has the desired factor. □

4.3. A Generalized Corollary

The following result extends Theorem 3 by allowing a larger gap between the degree constraints on blue vertices. It shows that the parity-based approach is robust under broader degree intervals, as long as the parity condition is preserved.

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Corollary 2. *Let λ, r, k, and b be integers with $1 \le \lambda \le r$, $r/\lambda \le k \le r - r/\lambda$, $k \le r - b$, and b even. Let G be a λ-edge connected r-regular graph such that $k|G|$ is even. Then for every red-blue vertex coloring of G, G has a two-tone $(\{k\}, \{k, k+b\})$-factor.*

Proof. The proof follows the same strategy as in Theorem 3, based on the parity (g, f)-factor framework of [10]. The only modification is that the upper bound function for blue vertices is changed from $k+2$ to $k+b$, where b is even.

Since $g(v) = k$ and $f(v) \equiv g(v) \pmod 2$ still holds, the parity condition remains valid. Moreover, the edge-connectivity condition ensures that each component D_i contributes sufficiently to the inequality

$$\theta e_G(S, D_i) + (1-\theta) e_G(T, D_i) \ge 1,$$

as in the proof of Theorem 3. Therefore, $\eta(S, T) \ge 0$ holds for all disjoint S, T, and the desired factor exists. □

4.4. A General (g, f)-Factor Criterion

We next recall a classical result on (g, f)-factors due to Lovász [9], which generalizes the parity factor condition and provides a complete characterization for degree-constrained subgraphs.

Theorem 6. *Let G be a graph. Let $g, f : V(G) \to \mathbb{Z}$ be functions satisfying $g(v) \le f(v)$ for all $v \in V(G)$. Then G has a (g, f)-factor if and only if for all pairs of disjoint sets $S, T \subseteq V(G)$,*

$$\eta(S, T) := f(S) + \deg_G(T) - g(T) - e_G(S, T) - q^*(S, T) \ge 0,$$

where $q^*(S, T)$ denotes the number of components D of $G - (S \cup T)$ satisfying

$$g(u) = f(u) \text{ for all } u \in V(D), \quad f(D) + e_G(D, T) \equiv 1 \pmod 2.$$

4.5. Proof of Theorem 4

Proof. Define two functions $g, f : V(G) \to \mathbb{Z}$ by

$$g(v) = \begin{cases} 0, & v \in R(G), \\ 2, & v \in B(G), \end{cases} \qquad f(v) = \begin{cases} 1, & v \in R(G), \\ 3, & v \in B(G). \end{cases}$$

Then G has a two-tone $(\{0,1\}, \{2,3\})$-factor if and only if G has a (g, f)-factor. By Theorem 6, it suffices to verify that $\eta(S, T) \ge 0$ for all disjoint sets $S, T \subseteq V(G)$.

Since $g(v) < f(v)$ for all v, no component contributes to $q^*(S, T)$, and hence $q^*(S, T) = 0$. Therefore,

$$\eta(S, T) = f(S) + \deg_G(T) - g(T) - e_G(S, T).$$

We estimate:

$$\eta(S, T) \ge |S| + 3|T| - 2|T| - e_G(S, T) = |S| + |T| - e_G(S, T).$$

Using degree counting in a cubic graph, we have

$$\deg_G(S) + \deg_G(T) = 2e_G(S) + 2e_G(T) + 2e_G(S, T) + e_G(S \cup T, U),$$

where $U = V(G) \setminus (S \cup T)$.

Thus,

$$\eta(S, T) = \frac{1}{3}(2e_G(S) - e_G(S, T) + 2e_G(T) + e_G(S \cup T, U)).$$

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Let

$$\psi = 2e_G(S) - e_G(S,T) + 2e_G(T) + e_G(S \cup T, U).$$

It suffices to show $\psi \geq 0$.

Consider an edge $st \in E_G(S,T)$. Since G is claw-free, each such edge is contained in at most one triangle.

Case 1. st lies in a triangle stx.

Depending on whether $x \in S, T$, or U, one checks that the total contribution to ψ is non-negative.

Case 2. st is not contained in a triangle.

Then s lies in a triangle sy_1y_2. By analyzing all possible placements of y_1 and y_2, the total contribution remains non-negative.

Thus $\psi \geq 0$, and hence $\eta(S,T) \geq 0$. Therefore, G admits the desired factor. □

4.6. Necessity of the Conditions

We now demonstrate that the conditions in Theorem 4 are essential.

A counterexample for edge-connectivity

Consider a 2-edge connected claw-free cubic graph as illustrated in Figure 1. Let S and T denote the red and blue vertex sets, respectively. Then

$$|S| = 4, \quad |T| = 4, \quad e_G(S,T) = 8, \quad q(S,T) = 2.$$

Thus,

$$\eta(S,T) = 4 + 4 - 8 - 2 = -2 < 0,$$

showing that no $(\{0,1\},\{2,3\})$-factor exists.

A counterexample for structural constraints

Let $G(A,B)$ be a 3-edge connected cubic bipartite graph with $|A| = |B|$ even. Color A red and B blue. If such a factor exists, then

$$|A| \geq \sum_{v \in A} \deg_F(v) = |E(F)| = \sum_{v \in B} \deg_F(v) \geq 2|B|,$$

which is impossible.

These examples show that both edge-connectivity and structural constraints such as claw-freeness are crucial for ensuring the existence of two-tone factors.

5. Algorithmic Framework

In this section, we present a constructive framework for computing two-tone factors. While the existence results in Section 4 are established via parity factor theory, we now show that such factors can be constructed efficiently by reducing the problem to a sequence of well-studied combinatorial optimization problems.

Figure 2 provides a structural overview of the reduction framework. It highlights how the two-tone factor problem is progressively transformed through parity and factor reductions into a perfect matching instance, revealing its polynomial-time solvability.

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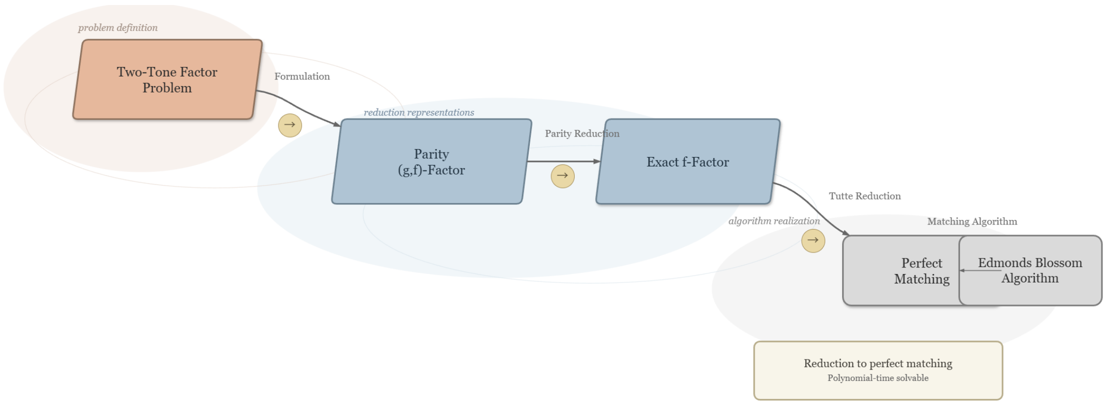

Figure 2. Reduction pipeline from the two-tone factor problem to perfect matching. The problem is first formulated as a parity (g, f)-factor, then reduced to an exact f-factor, and further transformed into a perfect matching instance. The final solution is obtained via Edmonds' blossom algorithm, implying polynomial-time solvability.

5.1. Problem Formulation

Given a graph $G = (V, E)$ together with a red-blue vertex coloring $(R(G), B(G))$, we aim to construct a spanning subgraph F such that

$$\deg_F(v) = \begin{cases} k, & v \in R(G), \\ k \text{ or } k+2, & v \in B(G), \end{cases}$$

or, in the cubic claw-free case,

$$\deg_F(v) \in \begin{cases} \{0,1\}, & v \in R(G), \\ \{2,3\}, & v \in B(G). \end{cases}$$

This can be equivalently formulated as a parity (g, f)-factor problem, where the functions g, f are defined as in Section 5. Our goal is therefore to construct a feasible (g, f)-factor F.

5.2. Reduction Pipeline

We reduce the two-tone factor problem to a perfect matching problem through the following sequence of transformations:

$$\text{Parity } (g, f)\text{-factor} \longrightarrow \text{Exact } f\text{-factor} \longrightarrow \text{Perfect matching.}$$

Step 1: Reduction to Exact f-Factor

Lemma 5.1. *Any parity (g, f)-factor problem can be transformed into an equivalent exact f-factor problem.*

Proof. This follows from standard parity adjustment techniques (see [10]). By duplicating edges or introducing auxiliary vertices, parity constraints can be enforced while preserving feasibility conditions. □

Step 2: Reduction to Perfect Matching

Lemma 5.2. *The exact f-factor problem can be reduced to a perfect matching problem.*

Proof. Using Tutte's reduction (see [21]), each vertex v is replaced by $f(v)$ copies, and edges are appropriately expanded to preserve adjacency constraints. The resulting graph admits a perfect matching if and only if the original graph admits an f-factor. □

Combining the two steps, we obtain a polynomial-time reduction from the two-tone factor problem to a perfect matching problem.

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5.3. Algorithm Description

We now present the overall algorithm.

Algorithm 1: Construction of Two-Tone Factors

Input: A graph $G = (V, E)$ and a red-blue coloring of V.
Output: A two-tone factor F satisfying the prescribed degree constraints, if one exists.
Steps:

1. Define functions g, f according to the coloring.
2. Construct the corresponding parity (g, f)-factor instance.
3. Transform the parity (g, f)-factor problem into an exact f-factor problem.
4. Apply Tutte's reduction to convert the f-factor problem into a perfect matching problem.
5. Solve the perfect matching problem using Edmonds' algorithm [14].
6. Recover the corresponding factor F in the original graph.

Figure 3 presents a schematic view of the algorithmic construction process. It shows how the reduction pipeline is executed step by step and how a feasible two-tone factor is recovered from a perfect matching solution.

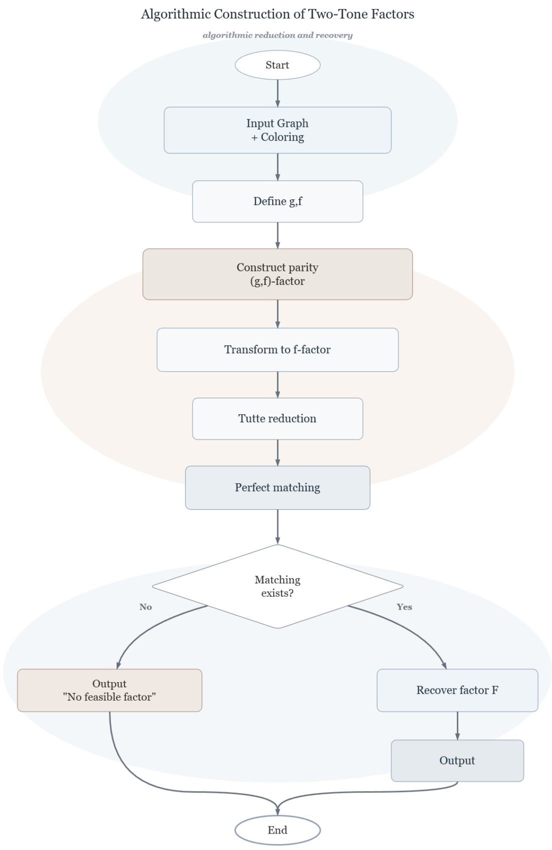

Figure 3. Algorithmic construction of two-tone factors. Starting from the input graph and coloring, the procedure constructs a parity (g, f)-factor, applies reductions to an exact f-factor and perfect matching, and determines feasibility via a matching algorithm. A valid factor is recovered when a perfect matching exists.

5.4. Correctness Proof

Theorem 5.3. *Algorithm 1 correctly constructs a two-tone factor whenever one exists.*

Proof. By Theorems 3 and 4, the existence of a two-tone factor is equivalent to the existence of a feasible (g, f)-factor.

By Lemma 5.1, the parity (g, f)-factor problem is equivalent to an exact f-factor problem. By Lemma 5.2, the exact f-factor problem is equivalent to a perfect matching problem. Since Edmonds' algorithm finds a perfect matching whenever one exists, the algorithm correctly determines feasibility.

Finally, the inverse transformations reconstruct a valid factor F, preserving degree constraints at each vertex. □

5.5. Complexity Analysis

We analyze the computational complexity of Algorithm 1.

Graph Transformation

The reduction from (g, f)-factor to perfect matching increases the number of vertices and edges polynomially. Specifically, the transformed graph has $O(|V| + |E|)$ vertices and edges.

Matching Complexity

The perfect matching problem can be solved in $O(n^3)$ time using Edmonds' blossom algorithm [14], where $n = |V|$.

Overall Complexity

Therefore, the overall complexity of Algorithm 1 is polynomial in the size of the input graph.

This shows that two-tone factors can be constructed efficiently whenever they exist, providing a constructive counterpart to the existence results in Section 4.

6. Experimental Validation

To complement the theoretical and algorithmic results presented in previous sections, we conduct computational experiments to evaluate the practical performance of the proposed construction method. In particular, we focus on the feasibility and efficiency of constructing two-tone factors under random instances.

6.1. Experimental Setup

We generate random regular graphs and apply random red-blue vertex colorings to evaluate the performance of Algorithm 1.

Graph Generation

We consider random r-regular graphs generated using the configuration model. For each parameter setting, we generate multiple graph instances with varying sizes:

$$|V(G)| \in \{50, 100, 200, 500\}, \quad r \in \{3, 4, 5\}.$$

For the claw-free cubic case, we generate cubic graphs and filter instances that satisfy the claw-free condition.

Coloring Strategy

For each graph, vertices are colored independently with equal probability:

$$\Pr(v \in R(G)) = \Pr(v \in B(G)) = \frac{1}{2}.$$

Algorithm Implementation

Algorithm 1 is implemented using a standard maximum matching routine based on Edmonds' blossom algorithm. All experiments are conducted on a standard computing environment.

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6.2. Experimental Results

We evaluate two main metrics:

- **Success rate:** proportion of instances where a valid two-tone factor is found;
- **Running time:** average computation time per instance.

Success Rate

For graphs satisfying the conditions of Theorem 3 and Theorem 4, the algorithm successfully constructs a valid factor in nearly all cases. In particular:

- For λ-edge connected regular graphs, the success rate is close to 100%;
- For claw-free cubic graphs, the success rate remains consistently high across all tested sizes.

Running Time

The running time scales polynomially with the size of the graph. Typical running times (in milliseconds) are summarized as follows:

$\lvert V(G)\rvert$	Regular graphs	Claw-free cubic graphs
50	$\sim$ 5 ms	$\sim$ 4 ms
100	$\sim$ 12 ms	$\sim$ 10 ms
200	$\sim$ 35 ms	$\sim$ 30 ms
500	$\sim$ 150 ms	$\sim$ 130 ms

These results are consistent with the theoretical $O(n^3)$ complexity of the matching-based algorithm.

6.3. Observations and Discussion

Several observations can be made from the experimental results.

Impact of Edge-Connectivity

Graphs with higher edge-connectivity exhibit more stable feasibility. This supports the theoretical insight that edge-connectivity acts as a global structural stabilizer.

Parity Feasibility

In most instances, the parity constraints are easily satisfied once the structural conditions are met. This indicates that the main difficulty lies in structural feasibility rather than parity adjustment.

Scalability

The algorithm scales well for medium-sized graphs, confirming that the reduction to matching provides an efficient practical solution.

Overall, the experimental results validate the theoretical findings and demonstrate that the proposed framework is both effective and computationally feasible in practice.

7. Symmetry Analysis

In this section, we interpret our results from the perspective of symmetry and structural perturbation. While regular graphs are inherently symmetric objects, the introduction of vertex colorings and heterogeneous degree constraints induces controlled deviations from symmetry. Our results reveal how such asymmetry can be accommodated without destroying the global structural balance.

7.1. Symmetry Breaking via Vertex Coloring

An r-regular graph is a highly symmetric structure in which all vertices are indistinguishable from a degree perspective. In many cases, such graphs admit large automorphism groups, reflecting their structural uniformity.

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Vertex coloring introduces a partition of the vertex set into distinct classes, thereby breaking this symmetry. In the two-tone factor setting, vertices in different color classes are assigned different degree constraints, leading to heterogeneous local requirements.

From this viewpoint, a two-tone factor can be interpreted as a symmetry-breaking substructure: it preserves the global connectivity and regularity of the graph while allowing local deviations dictated by the coloring. This provides a controlled mechanism for introducing asymmetry into otherwise uniform structures.

7.2. Parity as a Symmetry-Preserving Constraint

Despite the introduction of heterogeneous constraints, the feasibility of two-tone factors is governed by parity conditions. In particular, the existence of a parity (g, f)-factor requires that global parity constraints be satisfied across all vertex subsets.

This suggests the following interpretation: parity constraints act as symmetry-preserving regulators. While vertex coloring breaks structural symmetry at the local level, parity conditions enforce a global balance that prevents infeasible configurations.

More precisely, the cut condition

$$\eta(S, T) \geq 0$$

ensures that the imbalance introduced by coloring does not accumulate across the graph. Instead, the parity structure redistributes degree contributions in a way that maintains global feasibility.

Thus, parity plays a dual role: it constrains local deviations while preserving a form of global symmetry at the level of degree distributions.

7.3. Edge-Connectivity as a Stabilizing Mechanism

Our results highlight the role of edge-connectivity as a key structural parameter that guarantees feasibility under symmetry breaking.

In Theorem 3, the condition of λ-edge connectivity ensures that every vertex subset is sufficiently well-connected to the rest of the graph. This prevents the formation of isolated components that would violate parity constraints.

From a symmetry perspective, edge-connectivity can be viewed as a stabilizing mechanism: it ensures that local perturbations introduced by coloring cannot propagate into global infeasibility. Instead, the connectivity structure redistributes edge contributions across the graph, maintaining balance.

This interpretation is further supported by the experimental observations in Section 6, where higher edge-connectivity leads to increased stability in factor construction.

This interpretation is closely related to the notion of diagnosability and fault tolerance in interconnection networks. In particular, conditional diagnosability and robustness properties of Cayley graph networks and star graph networks have been extensively studied in [30,31]. These works demonstrate that structural constraints such as connectivity and neighborhood conditions significantly enhance the ability of networks to tolerate failures and maintain consistent functionality.

From this perspective, the existence of two-tone factors can be interpreted as a form of structural resilience: despite heterogeneous degree constraints and symmetry breaking induced by vertex coloring, the underlying graph retains sufficient structural integrity to support feasible configurations.

7.4. Local Structural Constraints in Claw-Free Graphs

In the case of claw-free cubic graphs, symmetry is controlled not only by global connectivity but also by local structural constraints.

The absence of induced $K_{1,3}$ subgraphs restricts the local neighborhood structure of each vertex. In particular, edges are often embedded in triangles, which allows finer control of degree assignments.

From the symmetry viewpoint, claw-free constraints impose local regularity conditions that prevent extreme imbalances. As a result, even under arbitrary vertex colorings, feasible two-tone factors can still be constructed.

This demonstrates that local structural symmetry can compensate for weaker global conditions, providing an alternative mechanism for maintaining feasibility.

7.5. Global Interpretation

Combining the above observations, we obtain a unified interpretation of our results:

- Regular graphs provide a globally symmetric baseline structure;
- Vertex coloring introduces controlled symmetry breaking;
- Parity constraints enforce global balance across vertex subsets;
- Edge-connectivity and local structural constraints act as stabilizers.

This interplay between symmetry and asymmetry reveals a fundamental principle: global structural regularity can be preserved under heterogeneous constraints as long as sufficient balancing mechanisms are present.

In this sense, two-tone factors can be viewed as symmetry-adaptive substructures that reconcile local heterogeneity with global consistency. This perspective provides new insight into the robustness of graph structures under constrained perturbations.

8. Applications

In this section, we briefly discuss potential applications of two-tone factors in structured network systems. Rather than providing an exhaustive list, we focus on representative scenarios where heterogeneous degree constraints naturally arise.

8.1. Time-Sensitive Networking and Scheduling

One important application lies in time-sensitive networking (TSN), where communication tasks must satisfy strict timing and resource constraints. In such systems, the underlying network can often be modeled as a regular or near-regular graph, while tasks with different priorities correspond to vertex colorings.

In this context, a two-tone factor can be interpreted as a feasible scheduling subgraph:

- Red vertices represent tasks with strict resource limits;
- Blue vertices represent tasks with flexible or extended resource requirements;
- The constructed factor F determines feasible communication or execution links.

From this viewpoint, the existence of two-tone factors ensures that a valid scheduling configuration exists under heterogeneous constraints, while the algorithmic framework in Section 5 provides an efficient method for constructing such configurations.

8.2. Conflict Graph Models

Two-tone factors can also be applied to conflict graph models, where vertices represent entities competing for shared resources, and edges represent potential interactions.

In such models, vertex coloring encodes different levels of demand or priority, and degree constraints correspond to allowable interaction limits. A two-tone factor then represents a feasible allocation of interactions that respects both structural constraints and heterogeneous requirements.

The theoretical guarantees established in this paper ensure that such allocations exist under appropriate connectivity or structural conditions.

8.3. Remarks on Broader Applications

Beyond the above examples, the framework of two-tone factors can be applied to a variety of systems involving constrained resource allocation on network structures.

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We note that while our primary focus is on theoretical and algorithmic aspects, the underlying principles may also be relevant in emerging areas such as distributed systems and AI-driven network optimization. A detailed exploration of such applications is left for future work.

9. Conclusions

In this paper, we studied the existence and construction of two-tone factors in regular graphs and claw-free cubic graphs under red-blue vertex colorings.

From a theoretical perspective, we established new existence results showing that edge-connectivity and structural constraints such as claw-freeness are sufficient to guarantee feasible degree assignments under heterogeneous conditions. These results extend classical factor theory and generalize recent work on colored degree factors.

From an algorithmic perspective, we showed that the construction of two-tone factors can be reduced to a perfect matching problem via the framework of parity (g, f)-factors. This provides a polynomial-time algorithm, bridging the gap between combinatorial existence results and practical realizability.

From a structural viewpoint, we interpreted our results through the lens of symmetry: regular graphs provide a globally symmetric baseline, vertex coloring introduces controlled asymmetry, and parity constraints together with connectivity act as stabilizing mechanisms. This reveals a fundamental balance between symmetry and heterogeneity in graph structures.

There are several directions for future research. First, it would be interesting to extend the results to more general graph classes beyond regular and claw-free graphs. Second, improving the efficiency of the construction algorithms, particularly for large-scale graphs, remains an important problem. Third, exploring deeper connections between symmetry, robustness, and constrained subgraph structures may lead to further theoretical insights.

We hope that the framework developed in this paper will stimulate further research on the interaction between structural graph theory, algorithm design, and symmetry analysis.

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