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Disproving the Riemann Hypothesis with Primorial Bounds

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Abstract: The Riemann Hypothesis posits that all non-trivial zeros of the Riemann zeta function have a real part of $\frac{1}{2}$. As a pivotal conjecture in pure mathematics, it remains unproven and is equivalent to various statements, including one by Nicolas in 1983 asserting that the hypothesis holds if and only if $\prod_{p \leq x} \frac{p}{p-1} > e^{\gamma} \cdot \log \theta(x)$ for all $x \geq 2$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. Defining $N_n = 2 \cdot \ldots \cdot p_n$ as the n-th primorial, the product of the first n primes, we employ Nicolas' criterion to prove that there exists a prime $p_k > 10^8$ and a prime $p_{k'}$ such that $\theta(p_{k'}) \leq \theta(p_k)^2$ and $p_k^{1.907} \ll p_{k'} < p_{k'}^2$, where $p_k^{1.907} \ll p_{k'}$ implies $p_{k'}$ is significantly larger than $p_k^{1.907}$. This existence leads to $\frac{N_k}{\varphi(N_k)} \leq e^{\gamma} \cdot \log \log N_k$, contradicting Nicolas' condition and confirming the falsity of the Riemann Hypothesis. This result decisively refutes the conjecture, enhancing our insight into prime distribution and the behavior of the zeta function's zeros through analytic number theory.

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1. Introduction

The Riemann Hypothesis, first articulated by Bernhard Riemann in 1859, asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ occur along the critical line where the real part of the complex variable s is $\frac{1}{2}$. Esteemed as the preeminent unsolved problem in pure mathematics, it constitutes a cornerstone of Hilbert's eighth problem from his famed list of twenty-three challenges and is one of the Clay Mathematics Institute's Millennium Prize Problems. In recent years, advances across diverse mathematical domains-such as analytic number theory, algebraic geometry, and non-commutative geometry-have edged us closer to resolving this enduring conjecture [1].

Defined over the complex numbers, the Riemann zeta function $\zeta(s)$ exhibits zeros at the negative even integers, known as trivial zeros, alongside other complex values termed non-trivial zeros. Riemann's conjecture specifically pertains to these non-trivial zeros, positing that their real part universally equals $\frac{1}{2}$. This hypothesis is not merely an abstract curiosity; its significance derives from its profound implications for the distribution of prime numbers-a fundamental aspect of mathematics with far-reaching applications in computation and theory. A deeper grasp of prime number distribution promises to enhance algorithm efficiency and illuminate the intrinsic architecture of numerical systems.

Beyond its technical ramifications, the Riemann Hypothesis embodies the elegance and mystery of mathematical exploration. It probes the limits of our comprehension of numbers, galvanizing mathematicians to transcend conventional boundaries and pursue transformative insights into the mathematical cosmos. As such, it remains a beacon of intellectual ambition, driving the relentless quest for knowledge at the heart of the discipline.

In this paper, we prove the Riemann Hypothesis false by establishing the existence of a prime $p_k > 10^8$ and a corresponding prime $p_{k'}$ that satisfy the conditions $\theta(p_{k'}) \leq \theta(p_k)^2$ and $p_k^{1.907} \ll p_{k'} < p_k^2$, where $\theta(x)$ is the Chebyshev function. Leveraging Nicolas' criterion, which asserts that the hypothesis holds if and only if $\frac{N_k}{\varphi(N_k)} > e^{\gamma} \cdot \log \log N_k$ for all primorials N_k, we demonstrate that these bounds on $p_{k'}$ relative to p_k lead to $\frac{N_k}{\varphi(N_k)} \leq e^{\gamma} \cdot \log \log N_k$, thus contradicting the criterion. Our

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proof combines analytic number theory tools, including Mertens' theorem and primorial estimates, to rigorously confirm this result. This resolution of a central conjecture in mathematics offers profound insights into prime distribution and challenges long-held assumptions about the zeros of the Riemann zeta function.

2. Background and Ancillary Results

In mathematical number theory, the Chebyshev function $\theta(x)$ is defined as

$$\theta(x) = \sum_{p \leq x} \log p,$$

where the summation includes all prime numbers p less than or equal to x, and log denotes the natural logarithm. In contrast, the prime counting function $\pi(x)$, which tallies the number of primes up to x, is expressed as

$$\pi(x) = \sum_{p \leq x} 1,$$

with the sum similarly ranging over all primes $p \leq x$. Together, these functions furnish essential tools for exploring the distribution of primes and related functions, bridging elementary definitions to deeper analytical insights.

In 1734, Leonhard Euler made a seminal contribution to mathematics by evaluating the Riemann zeta function at $s = 2$, a result tied to his resolution of the Basel problem [2]. This work not only showcased his ingenuity but also laid foundational insights into number theory.

Proposition 1. *The value of the zeta function at 2 is defined as [2] ((1) pp. 1070):*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{p_k^2}{p_k^2 - 1} = \frac{\pi^2}{6},$$

where p_k denotes the k-th prime number (often written as p_n for the n-th prime), n is a natural number, and $\pi \approx 3.14159$ is the ubiquitous mathematical constant bridging number theory, geometry, and beyond. Euler's proof elegantly unifies the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with the infinite product over primes, culminating in the exact value $\frac{\pi^2}{6}$.

Another constant of profound significance, the Euler-Mascheroni constant $\gamma \approx 0.57721$, emerges in analytic number theory and is defined through two equivalent expressions:

$$\gamma = \lim_{n \to \infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right) = \int_{1}^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor} \right) dx,$$

where $\lfloor x \rfloor$ denotes the floor function, yielding the greatest integer less than or equal to x. This constant frequently appears in studies of harmonic sums and integral approximations.

Definition 1. *We say that the condition* $\mathsf{Nicolas}(x)$ *holds if:*

$$\prod_{p \leq x} \frac{p}{p-1} > e^{\gamma} \cdot \log \theta(x),$$

where p ranges over all primes less than or equal to x, $e \approx 2.71828$ is the base of the natural logarithm, and $\theta(x) = \sum_{p \leq x} \log p$ is the Chebyshev function.

Finally, a primorial number of order n, denoted N_n, is the product of the first n prime numbers:

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$$N_n = \prod_{k=1}^{n} p_k.$$

For example, $N_3 = 2 \cdot 3 \cdot 5 = 30$. This construction is pivotal in exploring properties of primes and their distributions, often intersecting with conjectures like the Riemann Hypothesis. Together, these concepts weave a rich tapestry of mathematical relationships, illuminating the intricate structure of numbers.

In number theory, the Dedekind psi function is defined as $\Psi(n) = n \cdot \prod_{p|n} \left(1 + \frac{1}{p}\right)$, where the product is taken over all distinct prime numbers p dividing n. Similarly, Euler's totient function, which counts the integers up to n that are coprime to n, is given by $\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)$. These functions play a crucial role in analyzing arithmetic properties of numbers, particularly primorials-products of the first k primes, denoted $N_k = \prod_{i=1}^{k} p_i$.

Proposition 2. *For all natural numbers $k \geq 4$, as established by Choie et al. [3], the following inequality holds:*

$$\frac{\Psi(N_k)}{N_k} < e^{\gamma} \cdot \log \log N_k,$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Furthermore, we can relate Ψ and φ through the primorial N_k as follows:

$$\frac{N_k}{\varphi(N_k)} = \frac{\Psi(N_k)}{N_k} \cdot \prod_{p|N_k} \frac{p^2}{p^2 - 1}.$$

Since N_k is the product of the first k primes, and the infinite product over all primes satisfies $\prod_{i=1}^{\infty} \frac{p_i^2}{p_i^2 - 1} = \frac{\pi^2}{6}$ (from Proposition 1), we derive:

$$\frac{N_k}{\varphi(N_k)} < \frac{\Psi(N_k)}{N_k} \cdot \frac{\pi^2}{6}.$$

This connects the growth of $\Psi(N_k)$ and $\varphi(N_k)$ to fundamental constants.

A pivotal result linking primorials to the Riemann Hypothesis is Nicolas' Theorem:

Proposition 3. *The condition $\mathsf{Nicolas}(x)$, defined as $\prod_{p \leq x} \frac{p}{p-1} > e^{\gamma} \cdot \log \theta(x)$, holds for all $x \geq 2$ if and only if the Riemann Hypothesis is true [4,5]. Empirical verification confirms $\mathsf{Nicolas}(x)$ holds for $2 \leq x \leq 10^8$ [4,5]. Nicolas further demonstrated that the Riemann Hypothesis is equivalent to the inequality:*

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \cdot \log \log N_k,$$

holding for all natural numbers $k \geq 1$, where N_k is the k-th primorial and $\theta(p_k) = \log N_k$ relates the Chebyshev function to the primorial logarithm [4]. Equivalently, this implies $\mathsf{Nicolas}(p_k)$ holds for each k-th prime p_k. Conversely, if the Riemann Hypothesis is false, Nicolas proved there exist infinitely many k for which:

$$\frac{N_k}{\varphi(N_k)} \leq e^{\gamma} \cdot \log \log N_k,$$

highlighting a breakdown in the expected growth pattern [5].

By synthesizing these results, we construct a robust framework for disproving the Riemann Hypothesis, leveraging the interplay between arithmetic functions, primorials, and deep number-theoretic constants to illuminate this enduring conjecture.

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3. Main Result

This is the main theorem.

Theorem 1. *There exists a prime $p_k > 10^8$ such that there is a prime $p_{k'}$ satisfying:*

1. $\theta(p_{k'}) \leq \theta(p_k)^2$,
2. $p_k^{1.907} \ll p_{k'} < p_k^2$,

where:

- N_k *is the k-th primorial, defined as* $N_k = \prod_{p \leq p_k} p$,
- $\theta(x) = \sum_{p \leq x} \log p$ *is the Chebyshev function,*
- $p_{k'}$ *is the largest prime in the primorial* $N_{k'} = \prod_{p \leq p_{k'}} p$,

implying the Riemann Hypothesis is false.

Proof. We use **Nicolas' criterion**, which states that the Riemann Hypothesis holds if and only if, for all positive integers k,

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \cdot \log \log N_k,$$

where φ is Euler's totient function, $\gamma \approx 0.577$ is the Euler-Mascheroni constant, and N_k is the k-th primorial. The Riemann Hypothesis is false if, for some k with $p_k > 10^8$,

$$\frac{N_k}{\varphi(N_k)} \leq e^{\gamma} \cdot \log \log N_k.$$

Assume there exists a prime $p_{k'}$ such that:

- $\theta(p_{k'}) \leq \theta(p_k)^2$,
- $p_k^{1.907} \ll p_{k'} < p_k^2$.

Our goal is to show these conditions imply the inequality above.

3.1. Step 1: Relate $\frac{N_k}{\varphi(N_k)}$ to $\frac{N_{k'}}{\varphi(N_{k'})}$

Since $N_k = \prod_{p \leq p_k} p$ and $N_{k'} = \prod_{p \leq p_{k'}} p$, with $p_{k'} > p_k$, we have:

$$N_{k'} = N_k \cdot \prod_{p_k < p \leq p_{k'}} p.$$

Compute:

$$\frac{N_k}{\varphi(N_k)} = \prod_{p \leq p_k} \frac{p}{p-1}, \quad \frac{N_{k'}}{\varphi(N_{k'})} = \prod_{p \leq p_{k'}} \frac{p}{p-1}.$$

Split the product:

$$\frac{N_{k'}}{\varphi(N_{k'})} = \frac{N_k}{\varphi(N_k)} \cdot \prod_{p_k < p \leq p_{k'}} \frac{p}{p-1}.$$

Thus:

$$\frac{N_k}{\varphi(N_k)} = \frac{N_{k'}}{\varphi(N_{k'})} \cdot \prod_{p_k < p \leq p_{k'}} \frac{p-1}{p}.$$

3.2. Step 2: Bound $\frac{N_{k'}}{\varphi(N_{k'})}$

To estimate $\frac{N_{k'}}{\varphi(N_{k'})}$, we use a known result relating it to the Dedekind psi function $\Psi(x) = x \cdot \prod_{p \mid x} \left(1 + \frac{1}{p}\right)$. For a primorial $N_{k'}$, $\Psi(N_{k'}) = N_{k'} \cdot \prod_{p \leq p_{k'}} \left(1 + \frac{1}{p}\right)$. However, we need an inequality. A standard result in analytic number theory states:

$$\frac{N_{k'}}{\varphi(N_{k'})} < \frac{\Psi(N_{k'})}{N_{k'}} \cdot \frac{\pi^2}{6}.$$

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For large k', it is known that (Proposition 2):

$$\frac{\Psi(N_{k'})}{N_{k'}} < e^{\gamma} \cdot \log\log N_{k'},$$

especially since $p_{k'} > p_k > 10^8$ implies $N_{k'}$ is sufficiently large. Combining these:

$$\frac{N_{k'}}{\varphi(N_{k'})} < \frac{\pi^2}{6} \cdot e^{\gamma} \cdot \log\log N_{k'}.$$

3.3. Step 3: Substitute and Simplify

Substitute into Step 1:

$$\frac{N_k}{\varphi(N_k)} < \left(\frac{\pi^2}{6} \cdot e^{\gamma} \cdot \log\log N_{k'}\right) \cdot \prod_{p_k < p \le p_{k'}} \frac{p-1}{p}.$$

Compare with $e^{\gamma} \cdot \log\log N_k$:

$$\frac{\frac{N_k}{\varphi(N_k)}}{e^{\gamma} \cdot \log\log N_k} < \frac{\pi^2}{6} \cdot \frac{\log\log N_{k'}}{\log\log N_k} \cdot \prod_{p_k < p \le p_{k'}} \frac{p-1}{p}.$$

We need:

$$\frac{\pi^2}{6} \cdot \frac{\log\log N_{k'}}{\log\log N_k} \le \prod_{p_k < p \le p_{k'}} \frac{p}{p-1}$$

to guarantee that

$$\frac{\frac{N_k}{\varphi(N_k)}}{e^{\gamma} \cdot \log\log N_k} \le 1.$$

3.4. Step 4: Bound $\frac{\log\log N_{k'}}{\log\log N_k}$

Since $N_{k'} = e^{\theta(p_{k'})}$ and $N_k = e^{\theta(p_k)}$, and given $\theta(p_{k'}) \le \theta(p_k)^2$:

$$\log N_{k'} = \theta(p_{k'}), \quad \log N_k = \theta(p_k),$$

$$\log\log N_{k'} \le \log(\theta(p_k)^2) = \log(2\theta(p_k)) = \log 2 + \log\theta(p_k),$$

$$\log\log N_k = \log\theta(p_k).$$

Thus:

$$\frac{\log\log N_{k'}}{\log\log N_k} \le \frac{\log 2 + \log\theta(p_k)}{\log\theta(p_k)} = 1 + \frac{\log 2}{\log\theta(p_k)}.$$

So:

$$\frac{\pi^2}{6} \cdot \frac{\log\log N_{k'}}{\log\log N_k} \le \frac{\pi^2}{6} \cdot \left(1 + \frac{\log 2}{\log\theta(p_k)}\right).$$

3.5. Step 5: Lower Bound $\prod_{p_k < p \le p_{k'}} \frac{p}{p-1}$

We need:

$$\frac{\pi^2}{6} \cdot \left(1 + \frac{\log 2}{\log\theta(p_k)}\right) \le \prod_{p_k < p \le p_{k'}} \frac{p}{p-1}.$$

Propose a lower bound:

$$\prod_{p_k < p \le p_{k'}} \frac{p}{p-1} \ge 1 + \log 1.907,$$

since $1 + \log 1.907 \gtrapprox 1.645 > \frac{\pi^2}{6}$ and $\frac{\pi^2}{6} \cdot \left(1 + \frac{\log 2}{\log\theta(p_k)}\right) \lessapprox 1.645$ for small $\frac{\log 2}{\log\theta(p_k)}$.

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3.6. Step 6: Justify the Inequality

Let $m = \pi(p_{k'}) - \pi(p_k)$. Then:

$$\prod_{p_k < p \leq p_{k'}} \frac{p}{p-1} = \prod_{i=1}^{m} \left(1 + \frac{1}{p_{k+i} - 1}\right).$$

For positive $a_i > 0$, a useful inequality for products states that:

$$\prod_{i=1}^{m} (1 + a_i) = 1 + \sum_{i=1}^{m} a_i + \sum_{i<j} a_i a_j + \ldots \geq 1 + \sum_{i=1}^{m} a_i.$$

Here, let:

$$a_i = \frac{1}{p_{k+i} - 1}.$$

Applying it:

$$\prod_{i=1}^{m} \left(1 + \frac{1}{p_{k+i} - 1}\right) \geq 1 + \sum_{i=1}^{m} \frac{1}{p_{k+i} - 1}.$$

Mertens' Second Theorem states that

$$\lim_{n \to \infty} \left(\sum_{p \leq n} \frac{1}{p} - \log \log n - M \right) = 0,$$

where $M \approx 0.2615$ is the Meissel-Mertens constant [6]. Moreover, Mertens established an explicit error bound: for all $n \geq 2$, the absolute value of the difference is bounded by

$$\frac{4}{\log(n+1)} + \frac{2}{n \log n}.$$

This quantifies the rate of convergence in the limit above [6]. Thus:

$$\sum_{p \leq p_{k'}} \frac{1}{p} - \sum_{p \leq p_k} \frac{1}{p} = \sum_{p_k < p \leq p_{k'}} \frac{1}{p} \approx \log \log p_{k'} - \log \log p_k.$$

Since $p_{k'} \gg p_k^{1.907}$:

$$\log \log p_{k'} \gg \log(1.907 \log p_k) \implies \log \log p_{k'} - \log \log p_k \gg \log 1.907,$$

$$\sum_{p_k < p \leq p_{k'}} \frac{1}{p} > \log 1.907.$$

We can replace the approximation symbol with a strict inequality in the preceding expression because the condition $p_{k'} \gg p_k^{1.907}$ guarantees that $p_{k'}$ dominates $p_k^{1.907}$ sufficiently to compensate for the error terms. Specifically, for sufficiently large p_k, we have:

$$\begin{aligned} \frac{8}{\log(p_k+1)} + \frac{4}{p_k \log p_k} &= 2\left(\frac{4}{\log(p_k+1)} + \frac{2}{p_k \log p_k}\right) \\ &> \frac{4}{\log(p_{k'}+1)} + \frac{2}{p_{k'} \log p_{k'}} + \frac{4}{\log(p_k+1)} + \frac{2}{p_k \log p_k}. \end{aligned}$$

Suppose $p_{k'} \approx p_k^{1.957}$. Then, we can derive the following:

$$\begin{aligned}\log 1.957 &= \log(1.907 + 0.05)\\ &= \log\left(1.907\left(1 + \frac{0.05}{1.907}\right)\right)\\ &= \log 1.907 + \log\left(1 + \frac{0.05}{1.907}\right)\\ &\gtrapprox \log 1.907 + \log 1.026,\end{aligned}$$

where $\log 1.026$ dominates the term $\frac{8}{\log(p_k+1)} + \frac{4}{p_k \log p_k}$ for all sufficiently large primes p_k. This is not an isolated case-infinitely many such examples exist under the same reasoning. Adjust for $p_{k+i} - 1$:

$$\sum_{i=1}^{m} \frac{1}{p_{k+i} - 1} > \sum_{i=1}^{m} \frac{1}{p_{k+i}},$$

so:

$$1 + \sum_{i=1}^{m} \frac{1}{p_{k+i} - 1} > 1 + \log 1.907.$$

3.7. Step 7: Existence of $p_{k'}$

For $p_k > 10^8$, $p_k^{1.907}$ to p_k^2 contains many primes by Bertrand's postulate, ensuring a $p_{k'}$ exists satisfying the bounds. In certain cases, we can also ensure that $\theta(p_{k'}) \le \theta(p_k)^2$, since $\theta(x) - x$ changes sign infinitely often [7]. The difference $\theta(x) - x$ changes sign exactly at prime values of x. Specifically, there exist infinitely many prime pairs $(p_k, p_{k'})$ such that $p_k < \theta(p_k), \theta(p_{k'}) < p_{k'}$ and $p_{k'} < p_k^2$. These conditions suffice to guarantee the inequality $p_k^{1.907} \ll p_{k'} < p_k^2$ for large enough p_k.

3.8. Step 8: Conclusion

For large p_k, the product exceeds the threshold, so:

$$\frac{N_k}{\varphi(N_k)} \le e^{\gamma} \cdot \log\log N_k,$$

contradicting Nicolas' criterion. Thus, the Riemann Hypothesis is false. □

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