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Article

Asymptotic Tail Moments of the Time Dependent Aggregate Risk Model

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Abstract: In this paper, we study an extension of the classical compound Poisson risk model with a dependence structure among the inter-claim time and the subsequent claim size. Under the underlying dependence structure proposed in [1], asymptotic tail moments for the aggregate claims are presented when the claim amounts are heavy tail distributed. Numerical examples are performed to validate the results we obtained.

Keywords: Compound Poisson risk model; Dependence; Sub-exponential distribution; Tail Conditional Expectation (TEC); Tail Variance (TV)

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1. Introduction

The classical compound Poisson risk model has been extensively analyzed in the actuarial literature. One of the key assumptions of this model is that the inter-claim times and the claim amounts are independent. This assumption can be rather restrictive in applications. For example, in the case of earthquake damages, it is usually believed that the longer the period between earthquakes, the greater the damages expected.

In this paper, we consider a compound Poisson risk model in which the inter-claim time and the subsequent claim size are statistically dependent. Specifically, we assume that the claim sizes X_i, $i = 1, 2, \ldots$ are non-negative independent and identically distributed (i.i.d.) random variables (rv's) with common distribution function (df) F_X. The claim arrival process $\{N(t), t \geq 0\}$ is modelled as a homogeneous Poisson process with intensity $\lambda > 0$. Let W_i, $i = 1, 2, \ldots$ denotes the ith inter-claim waiting time. Then they following i.i.d. exponential distribution with rate λ. Crucially, we assume that the bivariate random vectors (W_i, X_i) are mutually independent but that the r.v.'s W_i and X_i are no longer independent. As usual, the aggregate claim process $S(t)$ over a finite time horizon $(0, t]$ is defined as

$$S(t) = \sum_{i=1}^{N(t)} X_i \,. \tag{1}$$

Risk models that consider the dependence between the waiting time W_i and the claim size X_i have been studied extensively in the literature. For example, Boudreault et al. [2] introduced a dependence structure where the conditional density of $X_i|W_i$ is defined through a mixture of functions. They provided explicit expressions for quantities of interest, such as the ruin probability and the Gerber-Shiu function for a large class of claim size distributions. Asimit and Badescu [1] proposed a general dependence structure for (X_i, W_i) via the conditional tail probability of $X_i|W_i$. As stated in [3], this dependence structure is satisfied by several commonly used bivariate copulas and allows for both positive and negative dependencies. It is also very useful for analyzing the tail behavior of the sum or product of two dependent random variables. Under this dependence structure and assuming that the distribution of the claim amounts has a heavy tail, Asimit and Badescu [1] derived the asymptotic finite-time ruin probabilities and asymptotic results for Value at Risk (VaR) and Tail Conditional Expectation

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(TCE) of the aggregate losses. For other applications of this dependence structure in risk analysis and probability theory, one may refer to, for example, [3–5], among others. Bargès et al. [6] studied the moments of the compound Poisson sums when the dependence between the inter-claim time and the subsequent claim size is modelled by a Farlie-Gumbel-Morgenstern copula. Zhang and Chen [7] provided closed-form formulas for the densities of the discounted aggregate claims by assuming that the dependence is through mixing.

The moment (size-biased) transform of distributions, studied in [8], is a useful statistical tool, which has been exploited in many research areas. In risk management, for example, Furman and Landsman [9] applied moment transforms to compute the TCE. Further, Furman and Landsman [10] showed that the Tail Variance (TV) and other weighted risk measures can also be determined by moment transforms. More recently, Denuit [11] obtained the size-biased transform of compound sums and illustrated their applications in determining the TCE. Ren [12] studied the moment transform of both univariate and multivariate compound sums, and derived formulas to efficiently compute TCE, TV and higher tail moments.

In this paper, as detailed in Section 2, we assume that the dependence between the waiting time W_i and the claim size X_i is as proposed in [1]. We apply moment transforms to analyze TCE and TV of the risk process with dependence. Our approach generalizes that proposed in [1], which is based on extreme value theory. It allows us to derive the asymptotic results for the TCE, TV, and even higher tail moments. In addition, our numerical examples show that our asymptotic results provide more accurate values of TCE than those computed using the method in Asimit and Badescu [1].

The remainder of this paper is organized as follows. Section 2 provides some preliminary results and definitions needed. Section 3 presents asymptotic results for the first two tail moments of the aggregate claims with heavy-tailed claim amounts. Section 4 provides numerical examples with detailed computations to illustrate the results we obtained and compares with the existing results. Section 5 concludes.

2. Preliminaries

2.1. Model and Definitions

This section introduces the aggregate risk process, provides definitions, and reviews some well-known results that will be used to derive the main results.

2.1.1. The Dependent Compound Poisson Risk Process

Consider a Poisson process $N(t)$ with rate λ that represents the arrival of claims. Let W_i, $i = 1, \cdots, n$ be the waiting time for the ith claim. Then W_i's are i.i.d. and follow an exponential distribution with rate λ. For $i = 1, \cdots, n$, let $T_i = \sum_{k=1}^{i} W_k$ denote the arrival time of the ith claim. Then conditional on $N(t) = n$, the joint distribution of the claim arrival times random vector $\mathbf{T} = (T_1, \ldots, T_n)$ is identical to that of the order statistics of a sample of uniform random variables on $(0, t)$ of size n. That is,

$$f_{T_1,\ldots,T_n|N(t)=n}(t_1, \ldots, t_n) = \frac{n!}{t^n}, \quad 0 < t_1 < \ldots < t_n .$$

Equivalently, given $N(t) = n$, the joint p.d.f. of the waiting times random vector $\mathbf{W} = (W_1, \cdots, W_n)$ is given by

$$f_{W_1,\ldots,W_n|N(t)=n}(w_1, \ldots, w_n) = \frac{n!}{t^n} \tag{2}$$

defined on $D = \{(w_1, \ldots, w_n) : 0 \le w_1 \le t, \ldots, 0 \le w_n \le t - w_1 - \ldots - w_{n-1}\}$.

This implies that for any $i = 1, \ldots, n$, (see for example, Boudreault et al. [2])

$$f_{W_i|N(t)=n}(w_i) = \frac{n(t - w_i)^{n-1}}{t^n}, 0 < w_i < t.$$

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For $i, j, k = 1, \dots, n$, $i \neq j \neq k$, let $0 < w_i < t$, $0 < w_j \leq t - w_i$ and $0 < w_k \leq t - w_i - w_j$. Taking integrals of (2) leads to

$$f_{W_i, W_j | N(t)=n}(w_i, w_j) = \frac{n(n-1)(t - w_i - w_j)^{n-2}}{t^n}$$

and

$$f_{W_i, W_j, W_k | N(t)=n}(w_i, w_j, w_k) = \frac{n(n-1)(n-2)(t - w_i - w_j - w_k)^{n-3}}{t^n}.$$

Now, consider the risk model in which the size of the ith claim X_i depends on W_i and the pairs (W_i, X_i) are i.i.d.. Boudreault et al. [2] showed that

$$\mathbb{E}[S(t)] = \lambda \int_0^t \mathbb{E}[X_1 | W_1 = w] e^{-\lambda w} (1 + \lambda(t - w)) dw \,, \tag{3}$$

and

$$\begin{aligned} \mathbb{E}[S(t)^2] &= \lambda \int_0^t \mathbb{E}[X_1^2 | W_1 = w] e^{-\lambda w} (1 + \lambda(t - w)) dw \\ &+ \lambda^2 \int_0^t \int_0^{t-w} \mathbb{E}[X_1 | W_1 = w] \mathbb{E}[X_1 | W_1 = y] e^{-\lambda(w+y)} (2 + 4\lambda(t - w - y) + (\lambda(t - w - y))^2) dy dw \,. \end{aligned} \tag{4}$$

2.1.2. The Dependence Between Claim Waiting Time and Claim Size

In this paper, as in [1], we assume that the claim waiting time and claim severity satisfy the following assumption.

Assumption 1. *The bivariate random vectors (X_i, W_i), $i = 1, 2, \dots$ are mutually independent and have the same joint p.d.f. as a generic random vector (X, W). Moreover, there exists a positive and locally bounded function $g(\cdot)$ such that the relation*

$$\lim_{x \to \infty} \Pr(X > x | W = w) = \lim_{x \to \infty} \Pr(X > x) g(w)$$

holds uniformly for all $w \in (0, t]$. In other words,

$$\lim_{x \to \infty} \sup_{w \in (0, t]} \left| \frac{\Pr(X > x | W = w)}{\Pr(X > x) g(w)} - 1 \right| = 0 \,.$$

As mentioned in [1], a wide class joint distributions defined in terms of copulas satisfy Assumption 1.

Specifically, let $F_{X,W}$ be a joint distribution function of (X, W) with continuous margins F_X and F_W, by Sklar's Theorem (see [13]), there exists a unique copula $C(u, v)$ such that

$$F_{X,W}(x, w) = \Pr(X \leq x, W \leq w) = C(F_X(x), F_W(w)) \,.$$

Similarly,

$$\overline{F}_{X,W}(x, y) = \Pr(X > x, W > w) = \hat{C}(\overline{F}_X(x), \overline{F}_W(w)) \,,$$

where $\hat{C}$ is the survival copula satisfying

$$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v) \,, \quad (u, v) \in [0, 1]^2 \,,$$

For more details about copulas, one can refer to [14].

As mentioned in Asimit and Badescu [1], if $\hat{C}_2(u, v) = \frac{\partial \hat{C}(u,v)}{\partial v}$ exists, Assumption 1 can be rewritten as

$$\lim_{u \downarrow 0} \sup_{v \in [e^{-\lambda t}, 1)} \left| \frac{\hat{C}_2(u, v)}{u g(-\log v / \lambda)} - 1 \right| = 0 \,. \tag{5}$$

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Some examples of copulas that satisfy equation (5) are given below.

Example 1. *The Farlie-Gumbel-Morgenstern (FGM) copula*

$$C(u,v) = uv + \theta uv(1-u)(1-v), \quad \theta \in [-1,1],$$

with $g(w) = 1 + \theta(1 - 2e^{-\lambda w})$.

Example 2. *The Ali-Mikhail-Haq (AMH) copula*

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}, \quad \theta \in [-1,1],$$

with $g(w) = 1 + \theta(1 - 2e^{-\lambda w})$.

Example 3. *The Frank copula*

$$C(u,v) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right), \quad \theta \neq 0,$$

with $g(w) = \theta e^{\theta(1-e^{-\lambda w})}/(e^{\theta}-1)$.

For the detailed verification of Equation (5) as well as the examination of the three copula examples above, one can refer to Section 3 of [3].

Please note that parameter θ in Examples 1-3 controls the strength and direction of the dependence between variables. Specifically, $\theta < (>)0$ indicates negative (positive) dependence. The strength of the dependence increases when θ moves away from zero.

2.1.3. The assumptions for the distribution of claim sizes

In this paper, we assume that the distribution of the claim sizes belongs to the sub-exponential family, for which we give definitions and provide preliminary properties below. For details about the sub-exponential distributions, one is referred to [15].

Definition 1. *A random variable (r.v.) X with df F belongs to the sub-exponential family* $\mathcal{S}$ *(*$F \in \mathcal{S}$*) if*

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2,$$

where $\overline{F} = 1 - F$, F^{*n} *is the n-fold convolution of F.*

Examples of sub-exponential distributions include the Weibull, Pareto, and Lognormal distributions, among others.

According to Theorem 2.7 in [16] and Theorem 1 in [17], we have the following lemma.

Lemma 1. *Assume* $F \in \mathcal{S}$, *if the limit* $k_i = \lim_{x\to\infty} \frac{\overline{G_i}(x)}{\overline{F}(x)}$ *exists and is finite for* $i = 1,2$, *then*

$$\lim_{x\to\infty} \frac{\overline{G_1 * G_2}(x)}{\overline{F}(x)} = k_1 + k_2.$$

In addition, if $k_i > 0$, *then* $G_i \in \mathcal{S}$, $i = 1,2$.

Another useful property for sub-exponential distribution is given in Lemma 1.3.5 of [18]. It is stated below.

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Lemma 2. *If $F \in \mathcal{S}$, then given $\epsilon > 0$, there exists a finite constant K such that for all $n \geq 2$,*

$$\frac{\overline{F^{*n}}(x)}{\overline{F}(x)} \leq K(1+\epsilon)^n, \quad x > 0\,.$$

A well-known subclass of $\mathcal{S}$ is the set of regularly varying df's, $RV_{-\alpha}$, whose definition is provided below (Bingham et al. [19]).

Definition 2. *A r.v. X with df F belongs to the set of regularly varying df's $RV_{-\alpha}$ ($F \in RV_{-\alpha}$) if*

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad \alpha > 0\,. \tag{6}$$

Examples of regularly varying distribution include Pareto, Burr, and log-gamma, among others. One can refer to [19] for more details about heavy-tailed distributions.

2.1.4. Previous results on the risk measures of the dependent compound Poisson risk model.

We first introduce definitions of some important risk measures that would be used in this paper.

Definition 3. *The Value-at-Risk (VaR) of a random variable X at the $100q\%$ confidence level is defined as*

$$\mathrm{VaR}_q(X) = inf\{x_q : F(x_q) \geq q\}\,.$$

Definition 4. *The Tail Conditional Expectation (TCE) at level q of a continuous random variable X is given by*

$$\mathrm{TCE}_q(X) = \mathbb{E}[X|X > x_q]\,.$$

Definition 5. *The Tail Variance (TV) at level q of X is defined by [20]*

$$\mathrm{TV}_q(X) = \mathrm{Var}(X|X > x_q) = \mathbb{E}[(X - \mathrm{TCE}_q(X))^2|X > x_q]\,.$$

The following lemma, obtained in Theorem 3.1 of Asimit and Badescu [1], provides the asymptotic result of the tail probability of aggregate claims.

Lemma 3. *Consider the time dependent aggregate risk model with $F_X \in \mathcal{S}$. If Assumption 1 is satisfied, then*

$$\Pr(S(t) > x) \sim K_0 \Pr(X_1 > x), \quad x \to \infty\,,$$

where

$$K_0 = \lambda \int_0^t g(w) e^{-\lambda w}[\lambda(t-w)+1]dw\,.$$

The value-at-risk at confidence level $100q\%$ is then obtained as

$$\mathrm{VaR}_q(S(t)) = \mathrm{VaR}_{1-(1-q)/K_0}(X_1), \quad q \uparrow 1\,.$$

Asimit and Badescu [1] derived the tail conditional expectation through Extreme Value Theory, for which some background is now given. For more details, one may refer to [18] and [21].

Definition 6. *A r.v. X (the df F of X, or the distribution of X) belongs to the maximum domain of attraction (MDA) of the extreme value distribution H if there exist constants $c_n > 0$, $d_n \in \mathbb{R}$ such that*

$$\lim_{n\to\infty} F^n(c_n x + d_n) = H(x)$$

holds. We write $X \in MDA(H)$ ($F \in MDA(H)$).

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Given existence, H belongs to one of the following three df's:

$$\text{Gumble type: } \Lambda(x) = exp(-exp(-x)),\ x \in \mathbb{R}.$$
$$\text{Fréchet type: } \Phi_\alpha(x) = exp((-x)^{-\alpha}),\ x > 0,\ \alpha > 0,$$
$$\text{Weibull type: } \Psi_\alpha(x) = exp(-(-x)^{\alpha}),\ x < 0,\ \alpha > 0,$$

As mentioned in [1], sub-exponential df's maximal domain of attraction can be only the Gumble type or Fréchet type. For example, Weibull distribution belongs to Gumbel type, whereas Pareto distribution belongs to Fréchet type. The next lemma was obtained in [1]. It presents the asymptotic result of the tail conditional distribution of aggregate claims $S(t)$.

Lemma 4. *If $F_X \in MDA(\Phi_\alpha)$, then the* TCE *at level q is*

$$\mathrm{TCE}_q(S(t)) \sim \frac{\alpha}{\alpha - 1}\mathrm{VaR}_q(S(t)), \quad q \uparrow 1,\ \alpha > 1.$$

If $F_X \in MDA(\Lambda)$, then

$$\mathrm{TCE}_q(S(t)) \sim \mathrm{VaR}_q(S(t)), \quad q \uparrow 1.$$

2.2. Moment Transforms

In this subsection, we introduce some basic definitions and preliminary results related to moment transforms. More details can be found in Patil and Ord [8].

Definition 7. *Consider a non-negative r.v. X with distribution function F_X and moments $\mathbb{E}[X^\alpha] < \infty$ for some positive integer α. A random variable $\widetilde{X_\alpha}$ is said to be a copy of the αth moment transform of X if its cumulative distribution function is given by*

$$F_{\widetilde{X_\alpha}}(x) = \frac{\mathbb{E}[X^\alpha I(X \le x)]}{\mathbb{E}[X^\alpha]} = \frac{\int_0^x t^\alpha dF_X(t)}{\mathbb{E}[X^\alpha]}, \quad x > 0.$$

The first moment transform of X is commonly referred to as the size-biased transform. It is simply denoted by $\widetilde{X}$.

The relationship between risk measures and the moment transform of random variables has been studied extensively in the literature. See, for example, [9], [20], [11], and the references therein. Specifically, we have

$$\mathbb{E}[X^\alpha | X > x] = \mathbb{E}[X^\alpha]\frac{\Pr(\widetilde{X_\alpha} > x)}{\Pr(X > x)}.$$

Let

$$S_n = \sum_{i=1}^{n} X_i$$

be the summation of n i.i.d. random variables. Then as discussed in [11], one has

Lemma 5. *For $\alpha \ge 1$ and $i \in \{1, \dots, n\}$,*

$$\mathbb{E}[X_i^\alpha I(S_n > s)] = \mathbb{E}[X_i^\alpha]\Pr(S_n - X_i + \widetilde{X_{i,\alpha}} > s),$$

where $\widetilde{X_{i,\alpha}}$ is a copy of the αth moment transform of X_i. The random variables $\widetilde{X_{i,\alpha}}$ and X_i are mutually independent.

Lemma 6. *For $\alpha, \beta \ge 1$, $i, j \in \{1, \dots, n\}$ and $i \ne j$,*

$$\mathbb{E}[X_i^\alpha X_j^\beta I(S_n > s)] = \mathbb{E}[X_i^\alpha X_j^\beta]\Pr(S_n - X_i - X_j + \widetilde{X_{i,\alpha}} + \widetilde{X_{j,\beta}} > s),$$

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where $\widetilde{X_{i,\alpha}}$ is a copy of the αth moment transform of X_i, and $\widetilde{X_{j,\beta}}$ is that of the βth moment transform of X_j. The random variables $\widetilde{X_{i,\alpha}}$, $\widetilde{X_{j,\beta}}$ and X_i are mutually independent.

3. Main Results

In this section, we derive asymptotic results for the tail moments of the aggregate loss $S(t)$ defined in Equation (1).

3.1. The First Tail Moment

We start with the result for the first tail moment of $S(t)$.

Theorem 1. *Consider the compound Poisson model with $F_X \in \mathcal{S}$. If Assumption 1 is satisfied, then when $x \to \infty$, we have*

$$\mathbb{E}[S(t)I(S(t)>x)] \sim \Pr(X_1>x)\bigg\{\lambda\int_0^t \mathbb{E}[X_1|X_1>x,W_1=w]g(w)e^{-\lambda w}[\lambda(t-w)+1]dw$$
$$+\lambda^2\int_0^t\int_0^{t-w}\mathbb{E}[X_1|W_1=w]g(y)e^{-\lambda(w+y)}([\lambda(t-w-y)]^2+4\lambda(t-w-y)+2)dydw\bigg\}.$$

Proof. According to Lemma 5, given $\mathbf{W}=\mathbf{w}$ and $N(t)=n$, for $i=1,\ldots,n$, we have

$$\mathbb{E}[X_iI(S(t)>s)|\mathbf{W}=\mathbf{w},N(t)=n]=\mathbb{E}[X_i|W_i=w_i]\Pr(S(t)-X_i+\widetilde{X_i}>s|\mathbf{W}=\mathbf{w},N(t)=n)\,.$$

Therefore,

$$\begin{aligned}&\mathbb{E}[S(t)I(S(t)>x)|\mathbf{W}=\mathbf{w},N(t)=n]\\&=\sum_{i=1}^n\mathbb{E}[X_i|W_i=w_i]\Pr(S(t)-X_i+\widetilde{X_i}>x|\mathbf{W}=\mathbf{w},N(t)=n)\,,\end{aligned}$$

then

$$\begin{aligned}&\mathbb{E}[S(t)I(S(t)>x)|N(t)=n]\\&=\int_D\mathbb{E}[S(t)I(S(t)>x)|\mathbf{W}=\mathbf{w},N(t)=n]f_{\mathbf{W}|N(t)=n}(\mathbf{w})d\mathbf{w}\\&=\sum_{i=1}^n\int_D\mathbb{E}[X_i|W_i=w_i]\Pr(S(t)-X_i+\widetilde{X_i}>x|\mathbf{W}=\mathbf{w},N(t)=n)\frac{n!}{t^n}d\mathbf{w}\,,\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}[S(t)I(S(t)>x)]=\mathbb{E}[\mathbb{E}[S(t)I(S(t)>x)|N(t)]]\\&=\sum_{n=1}^\infty\frac{e^{-\lambda t}(\lambda t)^n}{n!}\sum_{i=1}^n\int_D\mathbb{E}[X_i|W_i=w_i]\Pr(S(t)-X_i+\widetilde{X_i}>x|\mathbf{W}=\mathbf{w},N(t)=n)\frac{n!}{t^n}d\mathbf{w}\,.\end{aligned}$$

According to Assumption 1, we have

$$\lim_{x\to\infty}\frac{\Pr(X_i>x|W_i=w_i)}{\Pr(X_i>x)}=g(w_i)\,,i=1,\ldots,n$$

which is finite.

Since

$$\Pr(\widetilde{X_i}>x|W_i=w_i)=\frac{\mathbb{E}[X_i|X_i>x,W_i=w_i]}{\mathbb{E}[X_i|W_i=w_i]}\Pr(X_i>x|W_i=w_i)\,,$$

we have

$$\lim_{x\to\infty}\frac{\Pr(\widetilde{X_i}>x|W_i=w_i)}{\Pr(X_i>x)}=\frac{\mathbb{E}[X_i|X_i>x,W_i=w_i]}{\mathbb{E}[X_i|W_i=w_i]}g(w_i)\,,\tag{7}$$

which is finite for $i=1,\ldots,n$. Thus, according to Lemma 1, $X_i|W_i=w_i\in\mathcal{S}$ and $\widetilde{X_i}|W_i=w_i\in\mathcal{S}$.

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Equation (7) indicates that $\Pr(\widetilde{X_i} > x|W_i = w_i)/Pr(X_1 > x)$ is finite for all $x > 0$. That is, there exists a constant M such that for all $x > 0$ and $w_i \in (0,t]$,

$$\Pr(\widetilde{X_i} > x|W_i = w_i) \leq M\Pr(X_1 > x)\,.$$

Let $\{Y_1, Y_2, \ldots\}$ be a sequence of non-negative i.i.d. rv's having the same p.d.f. as a generic random variable Y satisfying $\Pr(Y \leq x) = \max\{0, 1 - M\Pr(X_1 > x)\}$. Then

$$\Pr(Y > x) = \begin{cases} M\Pr(X_1 > x)\,, & M\Pr(X_1 > x) \leq 1 \\ 1\,, & M\Pr(X_1 > x) > 1 \end{cases}.$$

It is clear that for any $x > 0$,

$$\Pr(X_i > x|W_i = w_i) \leq \Pr(\widetilde{X_i} > x|W_i = w_i) \leq \Pr(Y > x) \leq M\Pr(X_1 > x)\,,$$

which indicates that

$$\Pr(S(t) - X_i + \widetilde{X_i} > x|\mathbf{W} = \mathbf{w}, N(t) = n) \leq \Pr(\sum_{i=1}^{n} Y_i > x).$$

Because X is sub-exponential and

$$\frac{\Pr(Y > x)}{\Pr(X_1 > x)} \leq M\,,$$

by Lemma 1, the df of Y is sub-exponential.

Applying Lemma 2, we obtain that given $\epsilon > 0$, there exists a finite constant K such that

$$\frac{\Pr(\sum_{i=1}^{n} Y_i > x)}{\Pr(Y > x)} \leq K(1+\epsilon)^n\,, \quad x > 0\,.$$

Then, there exists a finite constant A such that

$$\frac{\Pr(\sum_{i=1}^{n} Y_i > x)}{\Pr(X_1 > x)} \leq \frac{\Pr(\sum_{i=1}^{n} Y_i > x)}{\Pr(Y > x)/M} \leq A(1+\epsilon)^n\,, \quad x > 0\,.$$

Consequently,

$$\begin{aligned}
&\frac{\mathbb{E}[S(t)I(S(t) > x)]}{\Pr(X_1 > x)} \\
=& \sum_{n=1}^{\infty} e^{-\lambda t}\lambda^n \sum_{i=1}^{n} \int_D \mathbb{E}[X_i|W_i = w_i] \frac{\Pr(S(t) - X_i + \widetilde{X_i} > x|\mathbf{W} = \mathbf{w}, N(t) = n)}{\Pr(X_1 > x)} d\mathbf{w} \\
\leq& \sum_{n=1}^{\infty} e^{-\lambda t}\lambda^n \sum_{i=1}^{n} \int_D \mathbb{E}[X_i|W_i = w_i] \frac{\Pr(\sum_{i=1}^{n} Y_i > x)}{\Pr(X_1 > x)} d\mathbf{w} \\
\leq& \sum_{n=1}^{\infty} e^{-\lambda t}\lambda^n \sum_{i=1}^{n} \int_D \mathbb{E}[X_i|W_i = w_i] A(1+\epsilon)^n d\mathbf{w} \\
=& \sum_{n=1}^{\infty} e^{-\lambda t}\lambda^n n \int_0^t \mathbb{E}[X|W = w] A(1+\epsilon)^n \frac{(t-w)^{n-1}}{(n-1)!} dw \\
=& \int_0^t \mathbb{E}[X|W = w] \sum_{n=0}^{\infty} e^{-\lambda t}\lambda^{n+1}(n+1)A(1+\epsilon)^{n+1} \frac{(t-w)^n}{n!} dw \\
=& A\lambda(1+\epsilon)e^{\lambda(1+\epsilon)(t-w)-\lambda t} \int_0^t \mathbb{E}[X|W = w] \sum_{n=0}^{\infty} e^{-\lambda(1+\epsilon)(t-w)}(n+1)\frac{[\lambda(1+\epsilon)(t-w)]^n}{n!} dw \\
=& A\lambda(1+\epsilon)e^{\lambda(1+\epsilon)(t-w)-\lambda t} \int_0^t \mathbb{E}[X|W = w] dw < \infty\,.
\end{aligned}$$

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Therefore, $S(t)I(S(t) > x)/\Pr(X_1 > x)$ is integrable (thus bounded), which allows us to apply the Dominated Convergence Theorem. By applying Assumption 1, Lemma 1 and Fubini's theorem, we have

$$\begin{aligned}
&\lim_{x\to\infty}\frac{\mathbb{E}[S(t)I(S(t)>x)]}{\Pr(X_1>x)}\\
&=\lim_{x\to\infty}\sum_{n=1}^{\infty}e^{-\lambda t}\lambda^n\sum_{i=1}^{n}\int_D \mathbb{E}[X_i|W_i=w_i]\frac{\Pr(S(t)-X_i+\widetilde{X_i}>x|\mathbf{W}=\mathbf{w},N(t)=n)}{\Pr(X_1>x)}d\mathbf{w}\\
&=\sum_{n=1}^{\infty}e^{-\lambda t}\lambda^n\sum_{i=1}^{n}\int_D \mathbb{E}[X_i|W_i=w_i]\lim_{x\to\infty}\frac{\Pr(S(t)-X_i+\widetilde{X_i}>x|\mathbf{W}=\mathbf{w},N(t)=n)}{\Pr(X_1>x)}d\mathbf{w}\\
&=\sum_{n=1}^{\infty}e^{-\lambda t}\lambda^n\sum_{i=1}^{n}\int_D\left\{\mathbb{E}[X_i|W_i=w_i]\sum_{j=1,j\neq i}^{n}g(w_j)+\mathbb{E}[X_i|X_i>x,W_i=w_i]g(w_i)\right\}d\mathbf{w}\\
&=\int_D\sum_{n=1}^{\infty}e^{-\lambda t}\lambda^n\sum_{i=1}^{n}\left\{\mathbb{E}[X_i|W_i=w_i]\sum_{j=1,j\neq i}^{n}g(w_j)+\mathbb{E}[X_i|X_i>x,W_i=w_i]g(w_i)\right\}d\mathbf{w}\\
&=\int_D\sum_{n=1}^{\infty}e^{-\lambda t}\lambda^n\{n(n-1)\mathbb{E}[X_1|W_1=w_1]g(w_2)+n\mathbb{E}[X_1|X_1>x,W_1=w_1]g(w_1)\}dw_2dw_1\\
&=\int_0^t\int_0^{t-w}\mathbb{E}[X_1|W_1=w]g(y)\sum_{n=2}^{\infty}e^{-\lambda t}\lambda^n n(n-1)\frac{(t-w-y)^{n-2}}{(n-2)!}dydw\\
&\quad+\int_0^t\mathbb{E}[X_1|X_1>x,W_1=w]g(w)\sum_{n=1}^{\infty}e^{-\lambda t}\lambda^n n\frac{(t-w)^{n-1}}{(n-1)!}dw\\
&=\int_0^t\int_0^{t-w}\mathbb{E}[X_1|W_1=w]g(y)\lambda^2e^{-\lambda(w+y)}\sum_{n=0}^{\infty}e^{-\lambda(t-w-y)}(n^2+3n+2)\frac{[\lambda(t-w-y)]^n}{n!}dydw\\
&\quad+\int_0^t\mathbb{E}[X_1|X_1>x,W_1=w]g(w)\lambda e^{-\lambda w}\sum_{n=0}^{\infty}e^{-\lambda(t-w)}(n+1)\frac{[\lambda(t-w)]^n}{n!}dw\\
&=\lambda^2\int_0^t\int_0^{t-w}\mathbb{E}[X_1|W_1=w]g(y)e^{-\lambda(w+y)}([\lambda(t-w-y)]^2+4\lambda(t-w-y)+2)dydw\\
&\quad+\lambda\int_0^t\mathbb{E}[X_1|X_1>x,W_1=w]g(w)e^{-\lambda w}[\lambda(t-w)+1]dw\,.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}[S(t)I(S(t)>x)]\sim\Pr(X_1>x)\bigg\{&\lambda\int_0^t\mathbb{E}[X_1|X_1>x,W_1=w]g(w)e^{-\lambda w}[\lambda(t-w)+1]dw\\
&+\lambda^2\int_0^t\int_0^{t-w}\mathbb{E}[X_1|W_1=w]g(y)e^{-\lambda(w+y)}([\lambda(t-w-y)]^2+4\lambda(t-w-y)+2)dydw\bigg\}.
\end{aligned}$$

□

Remark 1. *Note that, when x is large enough, the term $\mathbb{E}[X_1|W_1=w]$ is negligible compared with $\mathbb{E}[X_1|X_1>x,W_1=w]$. Therefore, the result in Theorem 1 can be reduced to*

$$\lim_{x\to\infty}\frac{\mathbb{E}[S(t)I(S(t)>x)]}{\Pr(X_1>x)}\sim\lambda\int_0^t\mathbb{E}[X_1|X_1>x,W_1=w]g(w)e^{-\lambda w}[\lambda(t-w)+1]dw\,.$$

Considering Lemma 3, we then have

$$\lim_{x\to\infty}\frac{\mathbb{E}[S(t)I(S(t)>x)]}{\Pr(S(t)>x)}\sim\frac{\int_0^t\mathbb{E}[X_1|X_1>x,W_1=w]g(w)e^{-\lambda w}[\lambda(t-w)+1]dw}{\int_0^t g(w)e^{-\lambda w}[\lambda(t-w)+1]dw}\,.$$

Remark 2. *As a special case, if the claim amounts and the inter-claim times are independent, a direct extension to Proposition 1 in Section 3.3 of [11] can be applied, which results in*

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$$\mathbb{E}[S(t)I(S(t)>s)] = \mathbb{E}[N(t)]\mathbb{E}[X_1]\Pr\left(\sum_{i=1}^{\widetilde{N(t)}-1} X_i + \widetilde{X_1} > s\right),$$

where $\widetilde{N(t)} =_d N(t)+1$*. Similar results can also be found in Lemma 3 from [12]. Thus, we obtain*

$$\mathbb{E}[S(t)I(S(t)>s)] = \lambda t\mathbb{E}[X_1]\Pr(S(t)+\widetilde{X_1}>s)\,.$$

In other words,

$$\widetilde{S(t)} =_d S(t)+\widetilde{X_1}\,.$$

3.2. The Second Tail Moment

The second tail moment of $S(t)$ is given in the following theorem.

Theorem 2. *Consider the compound Poisson model with* $F_X \in \mathcal{S}$*. If Assumption 1 is satisfied, then when* $x \to \infty$*, we have the following asymptotic relationship*

$$\begin{aligned}
\mathbb{E}[S(t)^2 I(S(t)>x)] \sim \Pr(X_1>x)\Big\{&\lambda\int_0^t \mathbb{E}[X_1^2|X_1\ge x, W_1=w]g(w)e^{-\lambda w}[\lambda(t-w)+1]dw\\
&+\lambda^2\int_0^t\int_0^{t-w}(2\mathbb{E}[X_1|W_1=w]\mathbb{E}[X_1|X_1\ge x, W_1=y]+\mathbb{E}[X_1^2|W_1=w])g(y)e^{-\lambda(w+y)}\\
&\qquad\Big([\lambda(t-w-y)]^2+4\lambda(t-w-y)+2\Big)dydw\\
&+\lambda^3\int_0^t\int_0^{t-w}\int_0^{t-w-y}\mathbb{E}[X_1|W_1=w]\mathbb{E}[X_1|W_1=y]g(z)e^{-\lambda(w+y+z)}\\
&\qquad\Big([\lambda(t-w-y-z)]^3+9[\lambda(t-w-y-z)]^2+18\lambda(t-w-y-z)+6\Big)dzdydw\Big\}\,.
\end{aligned}$$

Proof. According to Lemmas 5 and 6, given $\mathbf{W}=\mathbf{w}$ and $N(t)=n$, for $i,j\in\{1,\dots,n\}$ and $i\neq j$, we have

$$\mathbb{E}[X_i^2 I(S(t)>s)|\mathbf{W}=\mathbf{w},N(t)=n] = \mathbb{E}[X_i^2|W_i=w_i]\Pr(S(t)-X_i+\widetilde{X_{i,2}}>s|\mathbf{W}=\mathbf{w},N(t)=n)\,,$$

and

$$\begin{aligned}
&\mathbb{E}[X_iX_jI(S(t)>s)|\mathbf{W}=\mathbf{w},N(t)=n]\\
&=\mathbb{E}[X_i|W_i=w_i]\mathbb{E}[X_j|W_j=w_j]\Pr(S(t)-X_i-X_j+\widetilde{X}_i+\widetilde{X}_j>s|\mathbf{W}=\mathbf{w},N(t)=n)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}[S(t)^2I(S(t)>x)|\mathbf{W}=\mathbf{w},N(t)=n]\\
&=\sum_{i=1}^n\sum_{j=1,j\neq i}^n \mathbb{E}[X_i|W_i=w_i]\mathbb{E}[X_j|W_j=w_j]\Pr(S(t)-X_i-X_j+\widetilde{X_i}+\widetilde{X_j}>x|\mathbf{W}=\mathbf{w},N(t)=n)\\
&\quad+\sum_{i=1}^n\mathbb{E}[X_i^2|W_i=w_i]\Pr(S(t)-X_i+\widetilde{X_{i,2}}>x|\mathbf{W}=\mathbf{w},N(t)=n)\,,
\end{aligned}$$

then

$$\mathbb{E}[S(t)^2I(S(t)>x)|N(t)=n] = \int_D \mathbb{E}[S(t)^2I(S(t)>x)|\mathbf{W}=\mathbf{w},N(t)=n]\frac{n!}{t^n}d\mathbf{w}\,,$$

and

$$\mathbb{E}[S(t)^2I(S(t)>x)] = \sum_{n=1}^\infty\int_D \mathbb{E}[S(t)^2I(S(t)>x)|\mathbf{W}=\mathbf{w},N(t)=n]e^{-\lambda t}\lambda^n d\mathbf{w}\,.$$

Since

$$\Pr(\widetilde{X_{i,2}}>x|W_i=w_i) = \frac{\mathbb{E}[X_i^2|X_i>x,W_i=w_i]}{\mathbb{E}[X_i|W_i=w_i]}\Pr(X_i>x|W_i=w_i)\,,$$

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together with Assumption 1, we have

$$\lim_{x\to\infty}\frac{\Pr(\widetilde{X_{i,2}}>x|W_i=w_i)}{\Pr(X_1>x)}=\frac{\mathbb{E}[X_i^2|X_i>x,W_i=w_i]}{\mathbb{E}[X_i|W_i=w_i]}g(w_i)\,,$$

exists finite for $i=1,\dots,n$.

For a Poisson distribution with p.d.f. $h(x)=e^{-\lambda}\lambda^k/k!$, it is easy to obtain its first three moments

$$\mathbb{E}[Z]=\lambda,\quad \mathbb{E}[Z^2]=\lambda+\lambda^2,\quad \mathbb{E}[Z^3]=\lambda+3\lambda^2+\lambda^3\,,$$

which will be used in the following steps.

The finiteness of $\mathbb{E}[S(t)^2 I(S(t)>x)]/\Pr(X_1>x)$ can be obtained by following the same procedure as in the proof of Theorem 1. We omit it here to avoid redundancy. Applying the Dominated Convergence Theorem, together with Assumption 1, Lemma 1 and Fubini's theorem yields

$$\begin{aligned}
&\lim_{x\to\infty}\frac{\mathbb{E}[S(t)^2 I(S(t)>x)]}{\Pr(X_1>x)}\\
&=\sum_{n=1}^{\infty}e^{-\lambda t}\lambda^n\int_D\Bigg\{\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\mathbb{E}[X_i|W_i=w_i]\mathbb{E}[X_j|W_j=w_j]\\
&\qquad\left(\sum_{k=1,k\neq i,k\neq j}^{\infty}g(w_k)+\frac{\mathbb{E}[X_i|X_i>x,W_i=w_i]}{\mathbb{E}[X_i|W_i=w_i]}g(w_i)+\frac{\mathbb{E}[X_j|X_j>x,W_j=w_j]}{\mathbb{E}[X_j|W_j=w_j]}g(w_j)\right)\\
&\quad+\sum_{i=1}^{n}\mathbb{E}[X_i^2|W_i=w_i]\left(\sum_{j=1,j\neq i}^{\infty}g(w_j)+\frac{\mathbb{E}[X_i^2|X_i>x,W_i=w_i]}{\mathbb{E}[X_i|W_i=w_i]}g(w_i)\right)\Bigg\}d\mathbf{w}\\
&=\int_0^t\int_0^{t-w}\int_0^{t-w-y}\mathbb{E}[X_1|W_1=w]\mathbb{E}[X_1|W_1=y]g(z)\sum_{n=3}^{\infty}e^{-\lambda t}\lambda^n n(n-1)(n-2)\frac{(t-w-y-z)^{n-3}}{(n-3)!}dzdydw\\
&\quad+\int_0^t\int_0^{t-w}2\mathbb{E}[X_1|W_1=w]\mathbb{E}[X_1|X_1>x,W_1=w]g(y)\sum_{n=2}^{\infty}e^{-\lambda t}\lambda^n n(n-1)\frac{(t-w-y)^{n-2}}{(n-2)!}dydw\\
&\quad+\int_0^t\int_0^{t-w}\mathbb{E}[X_1^2|W_1=w]g(y)\sum_{n=2}^{\infty}e^{-\lambda t}\lambda^n n(n-1)\frac{(t-w-y)^{n-2}}{(n-2)!}dydw\\
&\quad+\int_0^t\mathbb{E}[X_1^2|X_1>x,W_1=w]g(w)\sum_{n=1}^{\infty}e^{-\lambda t}\lambda^n n\frac{(t-w)^{n-1}}{(n-1)!}dw\\
&=\int_0^t\int_0^{t-w}\int_0^{t-w-y}\mathbb{E}[X_1|W_1=w]\mathbb{E}[X_1|W_1=y]g(z)\lambda^3 e^{-\lambda(w+y+z)}\sum_{n=0}^{\infty}(n^3+6n^2+11n+6)e^{-\lambda(t-w-y-z)}\\
&\quad\lambda^n\frac{(t-w-y-z)^n}{n!}dzdydw+\int_0^t\int_0^{t-w}(2\mathbb{E}[X_1|W_1=w]\mathbb{E}[X_1|X_1>x,W_1=w]+\mathbb{E}[X_1^2|W_1=w])\\
&\quad g(y)\lambda^2 e^{-\lambda(w+y)}\sum_{n=0}^{\infty}e^{-\lambda(t-w-y)}\lambda^n(n^2+3n+2)\frac{(t-w-y)^n}{n!}dydw\\
&\quad+\int_0^t\mathbb{E}[X_1^2|X_1>x,W_1=w]g(w)\lambda e^{-\lambda w}\sum_{n=0}^{\infty}e^{-\lambda(t-w)}\lambda^n(n+1)\frac{(t-w)^n}{n!}dw\\
&=\lambda^3\int_0^t\int_0^{t-w}\int_0^{t-w-y}\mathbb{E}[X_1|W_1=w]\mathbb{E}[X_1|W_1=y]g(z)e^{-\lambda(w+y+z)}\\
&\qquad\Big([\lambda(t-w-y-z)]^3+9[\lambda(t-w-y-z)]^2+18\lambda(t-w-y-z)+6\Big)dzdydw\\
&\quad+\lambda^2\int_0^t\int_0^{t-w}(2\mathbb{E}[X_1|W_1=w]\mathbb{E}[X_1|X_1\geq x,W_1=y]+\mathbb{E}[X_1^2|W_1=w])g(y)e^{-\lambda(w+y)}\\
&\qquad\Big([\lambda(t-w-y)]^2+4\lambda(t-w-y)+2\Big)dydw\\
&\quad+\lambda\int_0^t\mathbb{E}[X_1^2|X_1\geq x,W_1=w]g(w)e^{-\lambda w}[\lambda(t-w)+1]dw\,.
\end{aligned}$$

□

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Remark 3. *Again, when x is large enough, the terms $\mathbb{E}[X_1|W_1 = w]$ and $\mathbb{E}[X_1|X_1 \geq x, W_1 = y]$ are negligible compared with $\mathbb{E}[X_1^2|X_1 \geq x, W_1 = w]$. Therefore, the result in Theorem 2 can be reduced to*

$$\lim_{x\to\infty} \frac{\mathbb{E}[S(t)^2 I(S(t) > x)]}{\Pr(X_1 > x)} \sim \lambda \int_0^t \mathbb{E}[X_1^2|X_1 \geq x, W_1 = w]g(w)e^{-\lambda w}[\lambda(t-w)+1]dw\,.$$

Remark 4. *When the claim amounts and the inter-claim times are independent, we apply Theorem 2 in [12] and obtain*

$$\begin{aligned}\mathbb{E}[S(t)^2 I(S(t) > s)] = {}& \mathbb{E}[N(t)(N(t)-1)](\mathbb{E}[X_1])^2 \Pr\left(\sum_{i=1}^{\widetilde{N(t)}-2} X_i + \widetilde{X_1} + \widetilde{X_2} > s\right) \\ &+ \mathbb{E}[N(t)]\mathbb{E}[X_1^2]\Pr\left(\sum_{i=1}^{\widetilde{N(t)}-1} X_i + \widetilde{X_{1,2}} > s\right).\end{aligned}$$

Since $N(t)$ follows a Poisson distribution, we further have

$$\begin{aligned}\mathbb{E}[S(t)^2 I(S(t) > s)] = {}& \lambda^2 t^2 (\mathbb{E}[X_1])^2 \Pr\left(\sum_{i=1}^{N(t)-1} X_i + \widetilde{X_1} + \widetilde{X_2} > s\right) \\ &+ \lambda t \mathbb{E}[X_1^2]\Pr\Big(S(t) + \widetilde{X_{1,2}} > s\Big).\end{aligned}$$

Remark 5. *The methodology adopted in this paper is applicable to derive asymptotic results for the higher tail moments as well. They represent a generalization of [1], which obtained the first tail moment of the aggregate claims.*

4. Numerical Results

In this section, we present some numerical examples to examine the accuracy of the asymptotic results obtained in Section 3.

Asimit and Badescu [1] obtained results for $\mathrm{TCE}_q(S(t))$ when distribution of claim sizes belonging to the maximum domain of attraction of Gumbel and Fréchet types, respectively. To facilitate comparison with their results, we next present one example for each scenario.

4.1. Weibull Distributed Claim Size

Assume that X follows the Weibull distribution with df $F_X(x) = 1 - exp(-x^{1/\tau})$ for $x \geq 0$ and $\tau > 1$. As indicated in [18], this distribution is sub-exponential with a non-regularly varying tail, which is in the maximum domain of attraction of Gumbel type, i.e., $F_X \in MDA(\Lambda)$.

In this example, we select $\tau = 6$, the Poisson intensity $\lambda = 3$, time horizon $t = 100$. We assume that the inter-claim times and claim sizes are dependent through an FGM copula, with the parameter θ equal to $-0.5, 0, 0.5$ respectively, representing negative dependence, independence, and positive dependence between the claim waiting time and claim sizes.

For $\mathrm{VaR}_q(S(t))$, we apply Theorem 1 in [1]. For $\mathrm{TCE}_q(S(t))$ and $\mathrm{TV}_q(S(t))$, we utilize formulas derived in Theorem 1 and Theorem 2 in Section 3, respectively. Due to the complexity of obtaining exact expressions for these asymptotic results, we use R software to compute the numerical results.

Table 1 presents the asymptotic results of $\mathrm{VaR}_q(S(t))$, $\mathrm{TCE}_q(S(t))$ and $\mathrm{TV}_q(S(t))$ under different choices of q and θ.

We next present simulation studies to validate our asymptotic results. For each scenario, 10^7 rounds of simulations of the risk process are used.

To calculate the tail moment $\mathbb{E}[S(t)I(S(t) > x)]$, we simulated $\mathbb{E}[S(t)I(S(t) \leq x)]$ and use the relationship

$$\mathbb{E}[S(t)I(S(t) > x)] = \mathbb{E}[S(t)] - \mathbb{E}[S(t)I(S(t) \leq x)]\,,$$

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where the expression of $\mathbb{E}[S(t)]$ is provided in Equation (3) in Section 2.1. A similar approach was applied when simulating $\mathrm{TV}_q(S(t))$.

Table 1. Asymptotic results of $\mathrm{VaR}_q(S(t))$, $\mathrm{TCE}_q(S(t))$ and $\mathrm{TV}_q(S(t))$ (Weibull, $\tau = 6$).

	q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
$\mathrm{VaR}_q(S(t))$	99.00%	1.2009×10^6	1.2003×10^6	1.1997×10^6
	99.50%	1.7744×10^6	1.7736×10^6	1.7728×10^6
	99.90%	4.0251×10^6	4.0235×10^6	4.0219×10^6
$\mathrm{TCE}_q(S(t))$	99.00%	2.6331×10^6	2.6316×10^6	2.6300×10^6
	99.50%	3.6103×10^6	3.6084×10^6	3.6065×10^6
	99.90%	7.2235×10^6	7.2205×10^6	7.2174×10^6
$\mathrm{TV}_q(S(t))$	99.00%	5.3370×10^{12}	5.3145×10^{12}	5.3038×10^{12}
	99.50%	8.5383×10^{12}	8.5022×10^{12}	8.4665×10^{12}
	99.90%	2.2898×10^{13}	2.1999×10^{13}	2.1084×10^{13}

The simulated results for $\mathrm{VaR}_q(S(t))$, $\mathrm{TCE}_q(S(t))$ and $\mathrm{TV}_q(S(t))$ are shown in Table 2.

Table 2. Simulated results of $\mathrm{VaR}_q(S(t))$, $\mathrm{TCE}_q(S(t))$, $\mathrm{TV}_q(S(t))$.

	q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
$\mathrm{VaR}_q(S(t))$	99.00%	1.4172×10^6	1.4169×10^6	1.4153×10^6
	99.50%	1.9997×10^6	1.9983×10^6	1.9981×10^6
	99.90%	4.2536×10^6	4.2445×10^6	4.2338×10^6
$\mathrm{TCE}_q(S(t))$	99.00%	2.6317×10^6	2.5994×10^6	2.5868×10^6
	99.50%	3.5970×10^6	3.5900×10^6	3.5772×10^6
	99.90%	7.2199×10^6	7.2147×10^6	7.2073×10^6
$\mathrm{TV}_q(S(t))$	99.00%	5.0706×10^{12}	4.9663×10^{12}	4.8954×10^{12}
	99.50%	8.1521×10^{12}	8.0531×10^{12}	7.9265×10^{12}
	99.90%	2.2556×10^{13}	2.1545×10^{13}	2.0229×10^{13}

Remark 6. *The results in Tables 1 and 2 indicate that When the parameter θ in the FGM copula changes from negative to positive values (the correlation between inter-claim waiting time and subsequent claim size shifts from negative to positive), the tail risk measures $\mathrm{VaR}_q(S(t))$, $\mathrm{TCE}_q(S(t))$, and $\mathrm{TV}_q(S(t))$ slightly decrease. Intuitively, this may be because a negative dependence between claim waiting time and claim size leads to more chance of large claims occurring within a short period of time.*

To evaluate the accuracy of the asymptotic results, we report in Table 3 and 4 their relative errors with respect to the simulation results, which is defined as the absolute value of the ratio of asymptotic and simulated results minus 1.

Table 3. Relative errors of asymptotic and simulated results of $\mathrm{TCE}_q(S(t))$.

q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
99.00%	0.0108	0.0124	0.0167
99.50%	0.0037	0.0051	0.0082
99.90%	0.0005	0.0008	0.0014

As a comparison, we calculate the relative errors of the asymptotic results of $\mathrm{TCE}_q(S(t))$ in [1], which states that when $F_X \in MDA(\Lambda)$,

$$\mathrm{TCE}_q(S(t)) \sim \mathrm{VaR}_q(S(t)), \quad q \uparrow 1.$$

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I.e., asymptotically, the values of $\mathrm{TCE}_q(S(t))$ and $\mathrm{VaR}_q(S(t))$ are the same. The relative error values are reported in Table 5.

Table 4. Relative errors of asymptotic and simulated results of $\mathrm{TV}_q(S(t))$.

q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
99.00%	0.0525	0.0701	0.0834
99.50%	0.0474	0.0558	0.0681
99.90%	0.0152	0.0210	0.0423

Table 5. Relative errors of $\mathrm{TCE}_q(S(t))$ obtained using formulas in [1] and simulation.

q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
99.00%	0.5437	0.5382	0.5362
99.50%	0.5067	0.5060	0.5005
99.90%	0.4425	0.4423	0.4419

It is evident from Tables 3 and 4 that our asymptotic results for $\mathrm{TCE}_q(S(t))$ and $\mathrm{TV}_q(S(t))$ obtained using Theorems 1 and 2 are fairly accurate. Furthermore, the accuracy of the asymptotic results for $\mathrm{TCE}_q(S(t))$ improves gradually when the confidence level q is high. In addition, by comparing Tables 3 and 5, we see that the accuracy of our results is notably better than that obtained using the formulas in [1] for every level of q.

To facilitate a clearer comparison, we present the asymptotic results obtained in our study ("This paper"), the asymptotic results calculated through [1] ("Asimit and Badescu [1]"), and simulated results ("Simulation") of $\mathrm{VaR}_q(S(t))$, $\mathrm{TCE}_q(S(t))$ and $\mathrm{TV}_q(S(t))$ in Table 6. Specifically, we choose $q = 99.50\%$, $\theta = 0.5$, and consider $\tau = 3$, 6 and 9.

Table 6. Summary of asymptotic and simulated results (Weibull).

	Method	$\mathrm{VaR}_q(S(t))$	$\mathrm{TCE}_q(S(t))$	$\mathrm{TV}_q(S(t))$
$\tau = 3$	This Paper	1.0958×10^3	3.5714×10^3	3.9154×10^5
	Asimit and Badescu [1]	1.0958×10^3	1.0958×10^3	-
	Simulation	3.4871×10^3	3.8987×10^3	5.0900×10^5
$\tau = 6$	This Paper	1.7728×10^6	3.6065×10^6	8.4665×10^{12}
	Asimit and Badescu [1]	1.7728×10^6	1.7728×10^6	-
	Simulation	1.9981×10^6	3.5772×10^6	7.9265×10^{12}
$\tau = 9$	This Paper	2.3636×10^9	7.5122×10^9	2.7537×10^{20}
	Asimit and Badescu [1]	2.3636×10^9	2.3636×10^9	-
	Simulation	2.4283×10^9	7.4970×10^9	2.6981×10^{20}

From Table 6, it can be seen that accuracy of the asymptotic results for all quantities of interests $\mathrm{VaR}_q(S(t))$, $\mathrm{TCE}_q(S(t))$ and $\mathrm{TV}_q(S(t))$ improve as the parameter τ increases. This is because the tail of a Weibull distribution becomes heavier as the parameter τ increases and the asymptotic formulas work better for heavier-tailed distributions. For smaller values of τ, our results are more accurate than those obtained using [1].

4.2. Pareto Distributed Claim Size

Assume that X follows the Pareto (Type 1) distribution with df $F_X(x) = 1 - x^{-\alpha}$ for $x \geq 1$ and $\alpha > 0$. As mentioned in [18], this distribution is sub-exponential with a regularly varying tail, i.e., $\overline{F}_X \in RV_{-\alpha}$. It belongs to the maximum domain of attraction of Fréchet type, i.e., $F_X \in MDA(\Phi_\alpha)$.

In this example, we set $\alpha = 1.1$. The Poisson parameter and the dependence structure between the inter-claim times and claim sizes are set to the same values as in Section 4.1. Exact expressions and

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numerical evaluation of the asymptotic results of $\mathrm{TCE}_q(S(t))$ and $\mathrm{TV}_q(S(t))$ in Theorems 1 and 2 can be efficiently carried out using software Mathematica.

Table 7 presents the asymptotic results of $\mathrm{VaR}_q(S(t))$ and $\mathrm{TCE}_q(S(t))$ under different choices of q and θ. Please note that the variance and $\mathrm{TV}_q(S(t))$ are infinite for a Pareto distribution with parameter $\alpha = 1.1$. Thus, the values of $\mathrm{TV}_q(S(t))$ are not reported for this case.

Similar to Table 1, Table 7 shows that a transition from negative to positive dependence between inter-claim waiting time and subsequent claim size leads to slightly smaller values of risk measures.

Table 7. Asymptotic results of $\mathrm{VaR}_q(S(t))$ and $\mathrm{TCE}_q(S(t))$ (Pareto, $\alpha = 1.1$).

	q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
$\mathrm{VaR}_q(S(t))$	99.00%	1.1760×10^4	1.1751×10^4	1.1742×10^4
	99.50%	2.2085×10^4	2.2068×10^4	2.2051×10^4
	99.90%	9.5396×10^4	9.5323×10^4	9.5251×10^4
$\mathrm{TCE}_q(S(t))$	99.00%	1.3267×10^5	1.3256×10^5	1.3246×10^5
	99.50%	2.4624×10^5	2.4605×10^5	2.4585×10^5
	99.90%	1.0527×10^6	1.0519×10^6	1.0511×10^6

We simulated the values of $\mathrm{VaR}_q(S(t))$ and $\mathrm{TCE}_q(S(t))$ based on a sample size of 10^7. Their values under different choices of q and θ are shown in Table 8.

Table 8. Simulated results of $\mathrm{VaR}_q(S(t))$ and $\mathrm{TCE}_q(S(t))$ (Pareto, $\alpha = 1.1$).

	q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
$\mathrm{VaR}_q(S(t))$	99.00%	1.3800×10^4	1.3761×10^4	1.3737×10^4
	99.50%	2.4280×10^4	2.4177×10^4	2.3962×10^4
	99.90%	9.8901×10^4	9.7199×10^4	9.6345×10^4
$\mathrm{TCE}_q(S(t))$	99.00%	1.3158×10^5	1.3153×10^5	1.3143×10^5
	99.50%	2.4523×10^5	2.4521×10^5	2.4509×10^5
	99.90%	1.0525×10^6	1.0519×10^6	1.0512×10^6

In Table 9, we report the relative errors of the asymptotic results of $\mathrm{TCE}_q(S(t))$ presented in Table 7 with respect to the simulated results in Table 8.

Table 9. Relative errors of asymptotic and simulated results of $\mathrm{TCE}_q(S(t))$ (Pareto, $\alpha = 1.1$).

q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
99.00%	0.0083	0.0079	0.0078
99.50%	0.0041	0.0034	0.0031
99.90%	0.0002	0.0000	0.0001

Next, we compare our asymptotic results for $\mathrm{TCE}_q(S(t))$ with those in [1]. As mentioned in Lemma 4, when $F_X \in MDA(\Phi_\alpha)$,

$$\mathrm{TCE}_q(S(t)) \sim \frac{\alpha}{\alpha - 1}\mathrm{VaR}_p(S(t)), \quad p \uparrow 1. \tag{8}$$

The calculated values of $\mathrm{TCE}_q(S(t))$ using Equation (8) are presented in Table 10.

Table 10. Asymptotic results of $\mathrm{TCE}_q(S(t))$ calculated using [1] (Pareto, $\alpha = 1.1$).

q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
99.00%	1.2936×10^5	1.2926×10^5	1.2916×10^5
99.50%	2.4294×10^5	2.4275×10^5	2.4257×10^5
99.90%	1.0494×10^6	1.0486×10^6	1.0478×10^6

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Table 11 reports the relative errors of the asymptotic results of $\text{TCE}_q(S(t))$ in Table 10 with respect to the simulated results in Table 8.

Table 11. Relative errors of asymptotic and simulated results of $\text{TCE}_q(S(t))$ by [1] (Pareto, $\alpha = 1.1$).

q	$\theta = -0.5$	$\theta = 0$	$\theta = 0.5$
99.00%	0.0168	0.0172	0.0172
99.50%	0.0094	0.0100	0.0103
99.90%	0.0029	0.0031	0.0032

Tables 9 and 11 demonstrate that both our approach and the one presented in [1] yield rather accurate asymptotic results for $\text{TCE}_q(S(t))$. However, our approach results in smaller relative errors.

We next provide a summary of the asymptotic results obtained in our study, those calculated using [1], and simulated results of $\text{VaR}_q(S(t))$, $\text{TCE}_q(S(t))$ and $\text{TV}_q(S(t))$ (if applicable) in Table 12. Specifically, we select $q = 99.50\%$, $\theta = 0.5$, and consider $\alpha = 1.1\,, 1.6\,,$ and2.1.

Table 12. Summary of asymptotic and simulated results (Pareto).

	Method	$\text{VaR}_q(S(t))$	$\text{TCE}_q(S(t))$	$\text{TV}_q(S(t))$
	This Paper	2.2051×10^4	2.4585×10^5	-
α =1.1	Asimit and Badescu [1]	2.2051×10^4	2.4257×10^5	-
	Simulation	2.3962×10^5	2.4509×10^5	-
	This Paper	9.6754×10^2	3.3582×10^3	-
α =1.6	Asimit and Badescu [1]	9.6754×10^2	2.5801×10^3	-
	Simulation	1.7740×10^3	3.3983×10^3	-
	This Paper	1.8742×10^2	9.2976×10^2	3.8215×10^5
α =2.1	Asimit and Badescu [1]	1.8742×10^2	3.5780×10^2	-
	Simulation	4.7258×10^2	9.5955×10^2	4.8334×10^5

Table 12 shows that the asymptotic results in both this paper and those in [1] are more accurate when the tail of the claim size distribution becomes heavier. For larger values of α (lighter tail cases), the results in this paper provide more accurate values of $\text{TCE}_q(S(t))$.

Remark 7. *Combining the results from the two numerical examples we conducted, we claim that the moment transform technique serves as an efficient method for calculating the asymptotic tail moments for aggregate losses. Our approach extends the existing results presented in [1] in two key aspects. Firstly, it enables the derivation of asymptotic results for not only the first tail moment but also the second and higher tail moments of the aggregate claims. Secondly, our methodology enhances the accuracy of existing results.*

5. Conclusions

This paper studies the classical compound Poisson risk model in which the claim size distributions depend on the waiting time for the claims. We apply the concept of moment transform of distributions to derive asymptotic results for the TCE and TV of the aggregate claims. The accuracy of our results was verified through numerical examples.

For future research, we plan to develop mathematical formulas and computational methods for risk measures for compound risk models under more flexible dependence structures between claim frequencies and claim severities.

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