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Taylan Demir *

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Hartman and Lyapunov Inequalities

Taylan Demir

Department of mathematics, Atilim university, Turkey; demir.taylan96@gmail.com

Abstract

This undergraduate thesis addresses the Hartman and Lyapunov inequalities for second-order linear differential equations. First, the Green's function method for solving linear differential equations satisfying Dirichlet boundary conditions is comprehensively explained, and the integral forms of the solutions are obtained through this method. This approach facilitates both a clearer understanding of theoretical approaches and plays a significant role in the proof of inequalities. Hartman's inequality provides a lower bound on the definite integral of the potential function, depending on the behavior of the solution between zeros. Lyapunov's inequality, on the other hand, provides a tighter bound for the same problem, providing important information about the stability and behavior of the solution. The relationship between these two inequalities is analyzed in detail. A numerical example is also provided to demonstrate the validity of both Hartman and Lyapunov inequalities. Calculations based on this example demonstrate the accuracy of the theoretical results. The findings demonstrate how such inequalities can be used in the analysis of boundary value problems in the theory of differential equations.

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1. Introduction

Consider the second-order linear differential equation

$$x''(t) + q(t)x(t) = 0; \quad a \le t \le b, \tag{1}$$

where $q(t) \in L^1[a,b]$ is a real-valued function. If $x(t) \not\equiv 0$ is a solution of Eq. (1) having two consecutive zeros a and b, where $a, b \in \mathbb{R}$ with $a < b$ and $x(t) \neq 0$ for $t \in (a,b)$, then the inequality

$$\int_a^b |q(t)|dt > \frac{4}{b-a} \tag{2}$$

holds. This striking inequality was first proved by Lyapunov [1] and in the literature known as "Lyapunov inequality". Later Wintner [2] and thereafter some authors achieved to replace the function $|q(t)|$ in Ineq. (2) by the function $q^+(t)$ i.e., they obtained the following inequality:

$$\int_a^b q^+(t)dt > \frac{4}{b-a}, \tag{3}$$

where $q^+(t) = \max\{q(t), 0\}$, and the constant 4 in the right hand side of Ineq. (3) (and Ineq. (2)) is the best possible largest number (see [1] and [3, Thm. 5.1]). In [3], Hartman has obtained the more general inequality than both (2) and (3):

$$\int_a^b (b-t)(t-a)q^+(t)dt > b-a. \tag{4}$$

Since

$$(b-t)(t-a) \leq \frac{(b-a)^2}{4} \quad \text{for all } t \in [a,b], \tag{5}$$

In eq. (4) implies Ineq. (3), see Theorem 3.2.

2. Preliminaries

he Green's function method plays an important role for boundary value problems. Now, we observe that the solution of the second-order boundary value problem

$$x''(t) = -f(t); \qquad a < t < b \tag{6}$$

satisfying the Dirichlet boundary conditions

$$x(a) = x(b) = 0. \tag{7}$$

To express the solution of Prb. (6)–(7), we need the following lemma.

Lemma 2.1. *Solution of Eq. (6) satisfying the Dirichlet boundary conditions (7) can be expressed as the integral equation*

$$x(t) = \int_a^b G(t,s)f(s)ds, \tag{8}$$

where the function $G(t,s)$ is called the Green's function for the Prb. (6)-(7) and is defined as

$$G(t,s) = \frac{1}{b-a}\begin{cases}(b-t)(s-a) & if \;\; a \leq s \leq t; \\ (t-a)(b-s) & if \;\; t \leq s \leq b.\end{cases} \tag{9}$$

Proof. Integrating the both sides of Eq. (6), we get

$$x'(t) = x'(a) - \int_a^t f(s)ds; \qquad a \leq t \leq b. \tag{10}$$

Again integrating the both sides of Eq. (10), we get

$$\int_a^t x'(s)ds = (t-a)x'(a) - \int_a^t \int_a^\tau f(s)dsd\tau; \qquad a \leq t \leq b. \qquad (11) \tag{11}$$

Using Eq. (7) and Eq. (11), we obtain

$$x(t) = (t-a)x'(a) - \int_a^t \int_a^\tau f(s)dsd\tau; \qquad t \leq b. \tag{12}$$

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he double integral on the right hand side of Eq. (12) can be represented as

$$\int_a^t \int_a^\tau f(s)dsd\tau = \int\int_\Omega f(s)dsd\tau \qquad (13)$$ (13)

where Ω is the region defined as

$$\Omega := \{(t,s): a \le s \le t, a \le \tau \le t\}.$$

By changing the order of integration in the double integral (13), we get

$$\int\int_\Omega f(s)dsd\tau = \int_a^t f(s) \int_s^t d\tau ds = \int_a^t (t-s)f(s)ds. \qquad (14)$$

By using (14), Eq. (12) turns out to be

$$x(t) = (t-a)x'(a) - \int_a^t (t-s)f(s)ds. \qquad (15)$$

On the other hand, if we take $t = b$ in (15), then we obtain;

$$x(b) = (b-a)x'(a) - \int_a^b (b-s)f(s)ds. \qquad (16)$$ (16)

which implies that

$$x'(a) = \frac{1}{b-a}\int_a^b (b-s)f(s)ds. \qquad (17)$$

Using (15) and (17), we obtain

$$x(t) = \frac{t-a}{b-a}\int_a^b (b-s)f(s)ds - \int_a^t (t-s)f(s)ds$$

$$= \frac{t-a}{b-a}\int_a^t (b-s)f(s)ds + \frac{t-a}{b-a}\int_t^b (b-s)f(s)ds - \int_a^t (t-s)f(s)ds$$

$$= \int_a^t \frac{(t-a)(b-s)}{(b-a)} f(s)ds - \int_a^t \frac{(t-s)(b-a)}{b-a} f(s)ds + \int_t^b \frac{(t-a)(b-s)}{b-a} f(s)ds \qquad (18)$$

Since

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$$\frac{(t-a)(b-s)}{(b-a)} - \frac{(t-s)(b-a)}{b-a} = \frac{(b-t)(s-a)}{b-a},$$

(18) turns to

$$x(t) = \int_a^t \frac{(b-t)(s-a)}{b-a} f(s)ds + \int_t^b \frac{(t-a)(b-s)}{b-a} f(s)ds = \int_a^b G(t,s)f(s)ds, \tag{19}$$

where the function $G(t,s)$ is defined as in (9). ◎

3. Main Results

3.1. Hartman's Inequality

In what follows we shall assume that $p(t), q(t) \in L^1[a,b]$. To motivate the formulation of our main results, we state the Hartman and Lyapunov inequalities for the linear Eq. (1).

Theorem 3.1. (Hartman's Inequality): *If $x(t)$ is a non-trivial solution of Eq. (1) satisfying the Dirichlet boundary conditions (7), where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then Ineq. (4) holds.*

Proof. It is clear from (9) that

$$0 \le G(t,s) \le \frac{(b-s)(s-a)}{b-a}; \qquad s \in (a,b). \tag{20}$$

Now let $x(t)$ be a non-trivial solution of Eq. (1) satisfying Dirichlet boundary conditions (7), where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros. Without loss of generality, we may assume that $x(t) > 0$ for $t \in (a,b)$. In fact, if $x(t) < 0$ for $t \in (a,b)$, then we can consider $x(t)$, which is also a solution. Then by using (1) and (8), $x(t)$ can be expressed as,

$$x(t) = \int_a^b G(t,s)q(s)x(s)ds. \tag{21}$$

Let $x(c) = max_{t\in(a,b)} x(t)$. hen by (20) and (21),

$$x(c) = \int_a^b G(c,s)q(s)x(s)ds \le \frac{1}{b-a}\int_a^b (b-s)(s-a)q(s)x(s)ds. \tag{22}$$

Since $q^+(t) \ge q(t)$, (22) turns out to be

$$(b-a)x(c) \le x(c)\int_a^b (b-s)(s-a)q^+(s)ds.$$

Dividing both sides of the inequality above by $x(c) > 0$, we obtain the desired inequality (4). ◎

3.2. Lyapunov's Inequality

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Theorem 3.2. (Lyapunov's Inequality) *If $x(t)$ is a non-trivial solution of Eq. (1) satisfying the Dirichlet boundary conditions (7), where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then Ineq. (3) holds.*

Proof. Using In eq. (5), Ineq. (4) immediately turns out to be

$$b - a < \int_a^b (b-t)(t-a)q^+(t)dt < \frac{(b-a)^2}{4}\int_a^b q^+(t)dt. \tag{23}$$

Multiplying both sides of the inequality above by $4/(b-a)^2$, we obtain the desired inequality

$$\int_a^b q^+(t)dt > \frac{4}{b-a}.$$

□

3.3. An Example

Example 3.3.: *Consider the linear differential equation*

$$x''(t) + \lambda^2 x(t) = 0, \qquad \lambda \in \mathbb{R}^+. \tag{24}$$

Let $x(t)$ be a non-trivial solution of Eq. (24) satisfying the Dirichlet boundary conditions

$$x(0) = x(\pi/\lambda) = 0. \tag{25}$$

Then we have $a = 0, b = \pi/\lambda$ and $q(t) = \lambda^2$. Now we look for the integral,

$$\int_a^b (b-t)(t-a)q^+(t)dt = \int_0^{\pi/\lambda} \left(\frac{\pi}{\lambda} - t\right)(t-0)\lambda^2 dt$$
$$= \lambda^2 \int_0^{\pi/\lambda} \left(\frac{\pi t}{\lambda} - t^2\right) dt$$
$$= \lambda^2 \left(\frac{\pi t^2}{2\lambda} - \frac{t^3}{3}\right)\Big|_0^{\pi/\lambda}$$
$$= \lambda^2 \left(\frac{\pi^3}{2\lambda^3} - \frac{\pi^3}{3\lambda^3}\right)$$
$$= \frac{\pi^3}{6\lambda}$$

Since, $\frac{\pi^3}{(6\lambda)} > \frac{\pi}{\lambda} - 0 = b - a$, Hartman's inequality is verified for Eq. (24).
Now, if we evaluate the integral

$$\int_a^b q^+(t)dt = \int_0^{\pi/\lambda} \lambda^2 dt = \lambda\pi > \frac{4\lambda}{\pi} = \frac{4}{\frac{\pi}{\lambda} - 0} = \frac{4}{b-a},$$

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then we see that Lyapunov inequality is verified for Eq. (24). We note that $x(t) = sin(\lambda x)$ is the unique solution of Eq. (24).

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