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Article

Canonical Number Systems in Multiple Dimensions and Their Representation Scope

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Abstract

We extend classical canonical number systems to an n-dimensional setting and investigate representation scope: when rings and modules admit digit expansions, with illustrative constructions and stability under products.

Keywords: canonical number systems; n-dimensional numeration; representation scope; finitely generated rings

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1. Introduction

Following Akiyama–Pethő [1, AkiyamaPetho], the pair $\{P(x), N\}$ is a canonical number system (CNS) if every $\alpha \in \mathbb{Z}[x]/(P(x))$, $\alpha \neq 0$, admits a unique finite expansion $\alpha = \sum_{j=0}^{\ell(\alpha)} a_j x^j$ with $a_j \in N$ and $a_{\ell(\alpha)} \neq 0$.

Polynomial/base specialization. Let $P(x) \in \mathbb{Z}[x]$ be monic with $|P(0)| \geq 2$, put $R = \mathbb{Z}[x]/(P)$, and let $\bar{x}$ be the class of x in R. With the digit set $\mathcal{N} = \{0, 1, \dots, |P(0)| - 1\}$ (a transversal of $R/\bar{x}R$), We say $(P, \mathcal{N})$ (or equivalently $(R, \bar{x}, \mathcal{N})$) is a CNS if every $\gamma \in R$ has a unique finite expansion

$$\gamma = \sum_{k=0}^{\ell} n_k \, \bar{x}^k, \qquad n_k \in \mathcal{N}.$$

If P is irreducible and α is a root, this is the same as unique finite α-expansions in $R \cong \mathbb{Z}[\alpha]$; in this case α is called a *CNS base*.

Example 1 (A concrete CNS expansion). *Take $P(x) = x + 2$ (here $d = 1$ and $p_0 = 2$). Then*

$$R = \mathbb{Z}[x]/(x+2)\mathbb{Z}[x], \qquad N = \{0, 1\}.$$

In R I have the relation $x \equiv -2$. Consider $\alpha = 23 \in R$. Define digits $a_0, \dots, a_6 \in N$ by

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6) = (1, 1, 0, 1, 0, 1, 1).$$

Then α has the (CNS-style) digit expansion

$$23 \; = \; 1 + x + x^3 + x^5 + x^6 \; = \; \sum_{j=0}^{6} a_j x^j, \qquad a_j \in N.$$

Indeed, using $x \equiv -2$ in R,

$$1 + x + x^3 + x^5 + x^6 = 1 + (-2) + (-2)^3 + (-2)^5 + (-2)^6 = 23.$$

Equivalently, if $\gamma = -2$ (a root of $x + 2$), this reads $23 = \sum_{j=0}^{6} a_j \gamma^j$ with $a_j \in \{0, 1\}$.

Canonical number systems (CNS) were developed in the polynomial setting by S. Akiyama and A. Pethő [2], and later studied over more general coefficient domains (for instance Euclidean domains) by A. Pethő and P. Varga [3]. In these works, a CNS is a positional representation in a quotient ring such as $R = \mathbb{Z}[x]/(P(x))$, where every element admits a finite expansion in powers of a single base (the class of x) using digits from a fixed finite set.

Related ideas also appear in the study of number systems arising from the Chinese Remainder Theorem, notably in work of van de Woestijne [4], where representations are constructed by decomposing rings into simpler components and recombining them through compatible digit systems. Moreover, earlier work of Kovács and Pethő [5,6] demonstrates that finite digit representations can exist in a variety of algebraic settings beyond the classical polynomial quotient framework, especially in integral domains and orders of algebraic number fields.

Motivated by these developments, in this paper we extend the classical one–base notion of CNS to an n–dimensional representation framework. Rather than fixing a single base, we allow several commuting shift maps to generate place values. Our primary interest is the *representation scope* of this extension: identifying broad classes of rings and modules for which such n–dimensional systems provide digit expansions for every element, and clarifying the conditions under which these expansions exist and are unique.

2. Preliminary

In the classical (one–base) setting, choosing an integer base $b \geq 2$ automatically provides the structural properties needed for a positional representation: the map $m \mapsto bm$ on $\mathbb{Z}$ is injective, powers b^k yield distinct place values, and there is only one shift so no ordering issues arise. In an n–dimensional (multi–base) setting, however, "place values" are generated by a family of $\mathbb{Z}$-endomorphisms $T_1, \dots, T_m$, and these properties are no longer automatic. Therefore we redefine what it means to be a *base* by imposing:

Definition 1 (Base family)**.** *Let L be a $\mathbb{Z}$-module and let $T_1, \dots, T_m \in \mathrm{End}_{\mathbb{Z}}(L)$. We call the ordered family $\mathbf{T} = (T_1, \dots, T_m)$ a* base *on L if it satisfies:*

1. ***Injective.*** *Each T_i is injective.*
2. ***Commuting.*** *The endomorphisms commute pairwise:*

$$T_i T_j = T_j T_i \qquad (1 \leq i, j \leq m).$$

3. ***Independent (no base is generated by the others).*** *For each $i \in \{1, \dots, m\}$ and each integer $a \geq 1$, the power T_i^a cannot be written as a composition of powers of the remaining maps. Equivalently, there do not exist exponents $\beta_j \in \mathbb{N}_0$ ($j \neq i$) such that*

$$T_i^a \;=\; \prod_{j \neq i} T_j^{\beta_j}.$$

(Under the commuting assumption, the product is well-defined independent of order.)

Example 2 (Classical base 10 on $\mathbb{Z}$)**.** *Let $L = \mathbb{Z}$ and define the (single) base map*

$$T : \mathbb{Z} \to \mathbb{Z}, \qquad T(m) = 10m.$$

Then T is injective. The associated place values are $T^k(1) = 10^k$. For instance,

$$372 = 2 + 10 \cdot 7 + 10^2 \cdot 3 = \sum_{k=0}^{2} T^k(d_k) \quad (d_k \in \{0, 1, \dots, 9\}).$$

Example 3. *(A multi-base family on $\mathbb{Z}[x_1, x_2]$: 10, x_1, x_2) Let $L = \mathbb{Z}[x_1, x_2]$ and define three $\mathbb{Z}$-endomorphisms by multiplication:*

$$T_1(f) = 10f, \qquad T_2(f) = x_1 f, \qquad T_3(f) = x_2 f.$$

Then each T_i is injective (since $\mathbb{Z}[x_1, x_2]$ is an integral domain), and they commute:

$$T_i T_j = T_j T_i \qquad (1 \le i, j \le 3),$$

because multiplication by $10, x_1, x_2$ commutes in the commutative ring $\mathbb{Z}[x_1, x_2]$.

Moreover, the family is independent in the sense that distinct exponent triples produce distinct place maps. Writing

$$T^{(a,b,c)} := T_1^a T_2^b T_3^c,$$

we have

$$T^{(a,b,c)}(1) = 10^a x_1^b x_2^c,$$

and if $T^{(a,b,c)} = T^{(a',b',c')}$ then $10^a x_1^b x_2^c = 10^{a'} x_1^{b'} x_2^{c'}$, hence $(a, b, c) = (a', b', c')$.

Finally, a polynomial can be expanded by writing each integer coefficient in base 10. For example,

$$f(x_1, x_2) = 123 + 45x_1 + 6x_2 + 78x_1^2 x_2$$

can be written as a finite sum of places $10^a x_1^b x_2^c$ with digits in $\{0, \dots, 9\}$:

$$\begin{aligned} f &= (3 + 10 \cdot 2 + 10^2 \cdot 1) + (5 + 10 \cdot 4)x_1 + (6)x_2 + (8 + 10 \cdot 7)x_1^2 x_2 \\ &= \sum_{a,b,c} d_{a,b,c}\, 10^a x_1^b x_2^c = \sum_{a,b,c} T_1^a T_2^b T_3^c(d_{a,b,c}), \qquad d_{a,b,c} \in \{0, \dots, 9\}, \end{aligned}$$

where only finitely many $d_{a,b,c}$ are nonzero.

Example 4 (A finite ring: $\mathbb{F}_p$ and the identity map)**.** *Let $L = \mathbb{F}_p$ (viewed as an abelian group). Take the base map*

$$T = \mathrm{id}_L.$$

Then T is injective and $T^k = \mathrm{id}_L$ for all k. Every element $a \in \mathbb{F}_p$ already has the trivial "expansion" $a = d_0$ (one digit). Remark: *$\mathbb{F}_p$ is not a free $\mathbb{Z}$-module, so this example is meant only to illustrate the idea of a base map (identity) in a finite setting.*

Example 5 (A non-commutative ring: $M_2(\mathbb{Z})$ with base -2)**.** *Let $L = M_2(\mathbb{Z})$, viewed as a free $\mathbb{Z}$-module of rank 4 with basis $E_{11}, E_{12}, E_{21}, E_{22}$. Define the base endomorphism*

$$T : L \to L, \qquad T(A) = (-2)A.$$

Then T is injective and $L/T(L) \cong M_2(\mathbb{Z})/2M_2(\mathbb{Z})$ is finite.

A natural digit set is

$$D = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \{0, 1\} \right\},$$

which is a complete set of representatives of $L/T(L)$ (entrywise reduction mod 2). Hence every $A \in M_2(\mathbb{Z})$ has a unique decomposition

$$A = d + T(W) = d - 2W, \qquad d \in D,\ W \in M_2(\mathbb{Z}),$$

and iterating yields a finite T-expansion

$$A = \sum_{k=0}^{\ell} T^k(d_k) = \sum_{k=0}^{\ell} (-2)^k d_k, \qquad d_k \in D.$$

For instance, for

$$A = \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix},$$

one possible expansion is

$$\begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (-2)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-2)^2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + (-2)^3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

with all digits in D.

Example 6. *(A one–base CNS on $L = \mathbb{Z} \oplus \mathbb{F}_2[x]$) Let $L := \mathbb{Z} \oplus \mathbb{F}_2[x]$ (as a $\mathbb{Z}$–module). Define the (single) base endomorphism*

$$T : L \to L, \qquad T(a, f) := (10a,\ xf).$$

Fix the base frame

$$\mathbf{b} = (b_1, b_2) := \big((1,0),\ (0,1)\big) \in L^2.$$

Then for each $k \geq 0$,

$$T^k(b_1) = (10^k, 0), \qquad T^k(b_2) = (0, x^k).$$

Let the digit alphabets be

$$\mathcal{D}_1 := \{0, 1, \ldots, 9\} \subset \mathbb{Z}, \qquad \mathcal{D}_2 := \{0, 1\} \subset \mathbb{Z},$$

and set the digit set

$$D := \mathcal{D}_1 \times \mathcal{D}_2 \subset \mathbb{Z}^2.$$

We redefine the digit set in the multi-base setting because, unlike the classical one–base case (where digits are representatives of $L/T(L)$), several base maps $T_1, \ldots, T_m$ contribute to the "higher-place" part, so digits must represent the quotient by the combined image $L/\sum_{i=1}^{m} T_i(L)$ in order to retain a unique remainder (coset) decomposition.

Definition 2 (Digit set for a base family). *Let L be a $\mathbb{Z}$-module and let $\mathbf{T} = (T_1, \ldots, T_m) \subset \mathrm{End}_{\mathbb{Z}}(L)$ be a base family. Set*

$$\mathbf{T}(L) \ := \sum_{i=1}^{m} T_i(L) \ \subseteq \ L.$$

A digit set *(for $\mathbf{T}$) is a* finite *subset $D \subset L$ such that $0 \in D$ and D is a complete set of representatives of the quotient $L/\mathbf{T}(L)$. Equivalently, every $v \in L$ admits a* unique *coset decomposition*

$$v \ = \ d + u, \qquad d \in D,\ \ u \in \mathbf{T}(L),$$

i.e. the natural projection $\pi : L \to L/\mathbf{T}(L)$ restricts to a bijection $D \xrightarrow{\sim} L/\mathbf{T}(L)$. In particular, for each $v \in L$ there exist $w_1, \ldots, w_m \in L$ such that

$$v \ = \ d + \sum_{i=1}^{m} T_i(w_i), \qquad d \in D.$$

Remark 1. *The remainder digit d is unique by construction. The tuple $(w_1, \dots, w_m)$ need not be unique unless additional hypotheses are imposed (e.g. $\sum_i T_i(L)$ is a direct sum). When $m = 1$, this reduces to the usual condition that D is a transversal of $L/T(L)$.*

Example 7 (Digit sets in common settings)**.**

1. ***Integers $\mathbb{Z}$ (one base).*** *Let $L = \mathbb{Z}$ and $T(m) = 10m$. Then $T(L) = 10\mathbb{Z}$ and*

$$L/T(L) \cong \mathbb{Z}/10\mathbb{Z}.$$

A standard digit set is

$$D = \{0, 1, \dots, 9\},$$

so every $z \in \mathbb{Z}$ has a unique decomposition $z = d + 10w$ with $d \in D$, $w \in \mathbb{Z}$.

2. ***Polynomials $\mathbb{Z}[x_1, x_2]$ (a base family).*** *Let $L = \mathbb{Z}[x_1, x_2]$ and define*

$$T_1(f) = 10f, \qquad T_2(f) = x_1 f, \qquad T_3(f) = x_2 f.$$

Then

$$\mathbf{T}(L) = T_1(L) + T_2(L) + T_3(L) = 10L + x_1 L + x_2 L.$$

Modulo $x_1 L + x_2 L$ only the constant term remains, and modulo $10L$ the constant term is taken modulo 10, hence

$$L/\mathbf{T}(L) \ \cong\ \mathbb{Z}/10\mathbb{Z}.$$

Thus one may again take the digit set

$$D = \{0, 1, \dots, 9\} \subset \mathbb{Z} \subset \mathbb{Z}[x_1, x_2],$$

and every $f \in \mathbb{Z}[x_1, x_2]$ admits a unique decomposition $f = d + u$ with $d \in D$ and $u \in 10L + x_1 L + x_2 L$.

3. ***Matrices $M_2(\mathbb{Z})$ (noncommutative ring, one base).*** *Let $L = M_2(\mathbb{Z})$ and $T(A) = (-2)A$. Then $T(L) = (-2)M_2(\mathbb{Z}) = 2M_2(\mathbb{Z})$ and*

$$L/T(L) \cong M_2(\mathbb{Z})/2M_2(\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^4$$

(as additive groups). A convenient digit set is

$$D = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \{0, 1\} \right\},$$

which is finite, contains 0, and represents each coset uniquely. Equivalently, every $A \in M_2(\mathbb{Z})$ has a unique decomposition

$$A = d + (-2)W, \qquad d \in D,\ \ W \in M_2(\mathbb{Z}).$$

4. *$\mathbb{Z} \oplus \mathbb{F}_2[x]$ **(two-component module, one base).** Let $L = \mathbb{Z} \oplus \mathbb{F}_2[x]$ and define the base endomorphism*

$$T : L \to L, \qquad T(a, f) = (10a,\ xf).$$

Then

$$T(L) = 10\mathbb{Z} \ \oplus\ x\mathbb{F}_2[x],$$

and hence

$$L/T(L) \ \cong\ (\mathbb{Z}/10\mathbb{Z}) \ \oplus\ \big(\mathbb{F}_2[x]/x\mathbb{F}_2[x]\big) \ \cong\ (\mathbb{Z}/10\mathbb{Z}) \ \oplus\ \mathbb{F}_2$$

(as additive groups). A convenient digit set is

$$D = \{0,1,\dots,9\} \times \{0,1\} \ \subset \ \mathbb{Z} \oplus \mathbb{F}_2[x],$$

which is finite, contains 0*, and represents each coset uniquely. Equivalently, every* $(a,f) \in \mathbb{Z} \oplus \mathbb{F}_2[x]$ *has a unique decomposition*

$$(a,f) = d + T(w), \qquad d \in D, \ \ w \in \mathbb{Z} \oplus \mathbb{F}_2[x].$$

Definition 3 (n-dimensional CNS (multi-base extension of [2]))**.** *Let* L *be a* $\mathbb{Z}$*-module and let* $\mathbf{T} = (T_1,\dots,T_n)$ *be a* base family *on* L*, i.e.* $T_i \in \mathrm{End}_{\mathbb{Z}}(L)$ *are* injective, pairwise commuting, *and* independent *in the sense that*

$$T^\alpha := T_1^{\alpha_1} \cdots T_n^{\alpha_n}, \qquad T^\alpha = T^\beta \Longrightarrow \alpha = \beta \quad (\alpha, \beta \in \mathbb{N}_0^n).$$

Set

$$I_{\mathbf{T}} := T_1(L) + \cdots + T_n(L) \ \subseteq \ L,$$

and assume the quotient $L/I_{\mathbf{T}}$ *is finite.*

A digit set *for* $\mathbf{T}$ *is a finite set* $D \subset L$ *with* $0 \in D$ *such that* D *is a complete set of representatives of* $L/I_{\mathbf{T}}$*; equivalently, for every* $v \in L$ *there exists a* unique $d \in D$ *with*

$$v \equiv d \pmod{I_{\mathbf{T}}}.$$

The pair $(\mathbf{T}, D)$ *is called an* n-dimensional number system *if every* $v \in L$ *admits a finite expansion of the form*

$$v \ = \ \sum_{\alpha \in F} T^\alpha(d_\alpha), \qquad d_\alpha \in D,$$

where $F \subset \mathbb{N}_0^n$ *is a finite set.*

Fixing once and for all a monomial order $\prec$ *on* $\mathbb{N}_0^n$*, we call* $(\mathbf{T}, D)$ *an* n-dimensional canonical number system (CNS) *if the above expansion is* unique *for every* $v \in L$ *when written in* normal form *(i.e.* F *consists exactly of those* α *with* $d_\alpha \neq 0$*, and* F *has a* $\prec$*-maximum).*

Example 8 (More detailed concrete expansions)**.**

1. $\mathbb{Z}$ ***(base*** 10***).***
   With $T(m) = 10m$ *and* $D = \{0,1,\dots,9\}$*,*

$$58042 \ = \ 2 + 10 \cdot 4 + 10^2 \cdot 0 + 10^3 \cdot 8 + 10^4 \cdot 5 \ = \ \sum_{k=0}^{4} T^k(d_k),$$

   so $(d_0,d_1,d_2,d_3,d_4) = (2,4,0,8,5)$*.*

2. $\mathbb{Z}[x_1,x_2]$ ***(bases*** $10, x_1, x_2$***).***
   Let $T_1(f) = 10f$*,* $T_2(f) = x_1 f$*,* $T_3(f) = x_2 f$*, and* $D = \{0,\dots,9\}$*. Consider*

$$f(x_1,x_2) = 123 + 45x_1 + 6x_2 + 7x_1x_2^3 + 98x_1^2x_2 + 305x_1^3x_2^2.$$

Writing each integer coefficient in base 10 *gives*

$$\begin{aligned} f(x_1,x_2) &= 123 + 45x_1 + 6x_2 + 7x_1x_2^3 + 98x_1^2x_2 + 305x_1^3x_2^2 \\ &= 3\cdot 10^0x_1^0x_2^0 + 2\cdot 10^1x_1^0x_2^0 + 1\cdot 10^2x_1^0x_2^0 \\ &\quad + 5\cdot 10^0x_1^1x_2^0 + 4\cdot 10^1x_1^1x_2^0 \\ &\quad + 6\cdot 10^0x_1^0x_2^1 \\ &\quad + 7\cdot 10^0x_1^1x_2^3 \\ &\quad + 8\cdot 10^0x_1^2x_2^1 + 9\cdot 10^1x_1^2x_2^1 \\ &\quad + 5\cdot 10^0x_1^3x_2^2 + 0\cdot 10^1x_1^3x_2^2 + 3\cdot 10^2x_1^3x_2^2. \end{aligned}$$

Equivalently,

$$f = \sum_{a,b,c} d_{a,b,c}\,10^a x_1^b x_2^c = \sum_{a,b,c} T_1^aT_2^bT_3^c(d_{a,b,c}), \qquad d_{a,b,c}\in D,$$

where the nonzero digits are precisely

$$\begin{aligned} &d_{0,0,0}=3,\ d_{1,0,0}=2,\ d_{2,0,0}=1; \\ &d_{0,1,0}=5,\ d_{1,1,0}=4; \\ &d_{0,0,1}=6; \\ &d_{0,1,3}=7; \\ &d_{0,2,1}=8,\ d_{1,2,1}=9; \\ &d_{0,3,2}=5,\ d_{2,3,2}=3 \quad (\textit{and } d_{1,3,2}=0). \end{aligned}$$

3. $M_2(\mathbb{Z})$ ***(base*** -2***).***
   With $T(A)=(-2)A$ *and* $D=\left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) : a,b,c,d\in\{0,1\}\right\}$, *the matrix*

$$A = \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}$$

admits the digit expansion

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (-2)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-2)^2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + (-2)^3\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

[A concrete element and its expansion] Consider the element

$$r = (23017,\ 1+x+x^4)\in\mathbb{Z}\oplus\mathbb{F}_2[x].$$

Write

$$23017 = 7 + 1\cdot 10 + 0\cdot 10^2 + 3\cdot 10^3 + 2\cdot 10^4, \qquad 1+x+x^4 = 1\cdot x^0 + 1\cdot x^1 + 1\cdot x^4.$$

Hence the digit sequence $\mathbf{d}_k = (d_k,\varepsilon_k)\in D=\{0,\dots,9\}\times\{0,1\}$ *is*

$$(\mathbf{d}_0,\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3,\mathbf{d}_4) = \big((7,1),\ (1,1),\ (0,0),\ (3,0),\ (2,1)\big).$$

With $T(a, f) = (10a, xf)$ *and* $\mathbf{b} = ((1, 0),(0, 1))$ *as in Example 6, we compute*

$$\begin{aligned} r &= \sum_{k=0}^{4} \langle \mathbf{d}_k,\ T^k \mathbf{b} \rangle \\ &= (7, 1) + T(1, 1) + T^2(0, 0) + T^3(3, 0) + T^4(2, 1) \\ &= (7, 1) + (10, x) + (0, 0) + (3000, 0) + (20000, x^4) \\ &= (23017,\ 1 + x + x^4). \end{aligned}$$

4. $\mathbb{Z} \oplus \mathbb{F}_2[x]$ ***(two-component module, one base).*** *Let* $L = \mathbb{Z} \oplus \mathbb{F}_2[x]$ *and define the base endomorphism*

$$T : L \to L, \qquad T(a, f) = (10a,\ xf).$$

Then

$$T(L) = 10\mathbb{Z} \ \oplus\ x\mathbb{F}_2[x], \qquad L/T(L) \ \cong\ (\mathbb{Z}/10\mathbb{Z}) \ \oplus\ \big(\mathbb{F}_2[x]/x\mathbb{F}_2[x]\big) \ \cong\ (\mathbb{Z}/10\mathbb{Z}) \ \oplus\ \mathbb{F}_2.$$

A convenient digit set is

$$D = \{0, 1, \dots, 9\} \times \{0, 1\} \ \subset\ \mathbb{Z} \oplus \mathbb{F}_2[x],$$

which is finite, contains 0*, and represents each coset uniquely. Equivalently, every* $(a, f) \in \mathbb{Z} \oplus \mathbb{F}_2[x]$ *has a unique decomposition*

$$(a, f) = d + T(w), \qquad d \in D,\ \ w \in \mathbb{Z} \oplus \mathbb{F}_2[x].$$

For example, for

$$r = (23017,\ 1 + x + x^4) \in \mathbb{Z} \oplus \mathbb{F}_2[x],$$

one digit expansion (in the sense of a finite T–expansion) is

$$r = (7, 1) + T(1, 1) + T^2(0, 0) + T^3(3, 0) + T^4(2, 1),$$

with all digits $(7, 1),(1, 1),(0, 0),(3, 0),(2, 1) \in D$*.*

Digit folding and a reduced-frame CNS. Lemma 1 shows that when a CNS is built from an expanded block frame $(b_i, Ub_i, \dots, U^{M-1}b_i)$, one can *fold* each block of M digits into a single polynomial digit in $E[u]/(u^M)$ (with u acting as U on L). Since $TU = UT$, this yields an equivalent CNS expansion using the smaller frame $\mathbf{b}^\circ$, reducing the coordinate dimension from mM to m at the cost of enlarging the digit alphabet. We refer to the resulting system as the *folded (dimension-reduced) CNS*.

Lemma 1 (Digit folding operation for an n–dimensional CNS). *Let E be a (not necessarily commutative) ring, L a left E–module, and let* $T, U \in \mathrm{End}_E(L)$ *satisfy* $TU = UT$*. Fix a finite base frame* $\mathbf{b}^\circ = (b_1, \dots, b_m) \in L^m$ *and an integer* $M \geq 1$*. Form the expanded frame*

$$\widetilde{\mathbf{b}} := \big(U^0 b_1, \dots, U^{M-1} b_1;\ \dots\ ;\ U^0 b_m, \dots, U^{M-1} b_m\big) \in L^{mM}.$$

For $\mathbf{c} = (c_1, \dots, c_s) \in E^s$ *and* $\mathbf{v} = (v_1, \dots, v_s) \in L^s$ *write*

$$\langle \mathbf{c}, \mathbf{v} \rangle \ :=\ \sum_{\ell=1}^{s} c_\ell\, v_\ell \ \in L \qquad \text{(left scalar action in } L\text{).}$$

Let $\mathcal{D}_1, \dots, \mathcal{D}_m \subset E$ *be finite digit alphabets and set*

$$\mathcal{D} := \mathcal{D}_1 \times \cdots \times \mathcal{D}_m, \qquad \widetilde{\mathcal{D}} := \mathcal{D}^M.$$

Assume $(L, T, \widetilde{\mathbf{b}}, \widetilde{\mathcal{D}})$ *is a CNS in the sense that every* $r \in L$ *has a unique finite expansion*

$$r = \sum_{k=0}^{\ell} \langle \widetilde{\mathbf{d}}_k,\ T^k \widetilde{\mathbf{b}} \rangle, \qquad \widetilde{\mathbf{d}}_k = (d_{k,i,j})_{1 \le i \le m,\ 0 \le j < M} \in \widetilde{\mathcal{D}}.$$

Define the folded digit alphabets as formal truncated polynomials

$$\mathcal{D}_i^\circ := \left\{ \sum_{j=0}^{M-1} d_j\, u^j \;\middle|\; d_j \in \mathcal{D}_i \right\} \subset E[u]_{<M} \quad (1 \le i \le m), \qquad \mathcal{D}^\circ := \mathcal{D}_1^\circ \times \cdots \times \mathcal{D}_m^\circ.$$

Let $E[u]_{<M}$ *act on* L *by evaluation at* U*:*

$$\left(\sum_{j=0}^{M-1} a_j u^j \right) \cdot v := \sum_{j=0}^{M-1} a_j\, U^j(v) \qquad (a_j \in E,\ v \in L).$$

(This is well-defined because $U \in \mathrm{End}_E(L)$ *is* E*–linear.)*

Then every $r \in L$ *admits a finite folded expansion with base frame* $\mathbf{b}^\circ$*:*

$$\boxed{r = \sum_{k=0}^{\ell} \langle \mathbf{e}_k,\ T^k \mathbf{b}^\circ \rangle, \qquad \mathbf{e}_k = (e_{k,1}, \ldots, e_{k,m}) \in \mathcal{D}^\circ, \quad e_{k,i} = \sum_{j=0}^{M-1} d_{k,i,j}\, u^j.}$$

If the original expansion with $(\widetilde{\mathbf{b}}, \widetilde{\mathcal{D}})$ *is unique and* T *is injective, then the folded expansion with* $(\mathbf{b}^\circ, \mathcal{D}^\circ)$ *is also unique.*

Example 9. *(*$\mathbb{Z} \oplus \mathbb{F}_2[x]$ *folded to one digit alphabet of size* 20*) Let*

$$L := \mathbb{Z} \oplus \mathbb{F}_2[x]$$

(viewed as a $\mathbb{Z}$*–module). Define the single base endomorphism*

$$T : L \to L, \qquad T(a, f) := (10a,\ xf).$$

Fix the base frame of length 2

$$\widetilde{\mathbf{b}} := (\widetilde{b}_1, \widetilde{b}_2) := \big((1,0),\ (0,1)\big) \in L^2.$$

Then for each $n \ge 0$*,*

$$T^n \widetilde{b}_1 = (10^n, 0), \qquad T^n \widetilde{b}_2 = (0, x^n).$$

(Expanded* 2*–coordinate digits). *Let the two coordinate digit alphabets be*

$$\mathcal{D}_1 := \{0, 1, \ldots, 9\} \subset \mathbb{Z}, \qquad \mathcal{D}_2 := \{0, 1\} \subset \mathbb{Z},$$

and set $\widetilde{\mathcal{D}} := \mathcal{D}_1 \times \mathcal{D}_2$*. Thus each expanded digit is a pair*

$$(d, \varepsilon) \in \{0, \ldots, 9\} \times \{0, 1\}.$$

Every element $(a, f) \in L$ *has a finite expansion*

$$(a, f) = \sum_{n=0}^{\ell} \left(d_n\, T^n \widetilde{b}_1 + \varepsilon_n\, T^n \widetilde{b}_2 \right) = \sum_{n=0}^{\ell} (d_n 10^n,\ \varepsilon_n x^n), \qquad (d_n, \varepsilon_n) \in \widetilde{\mathcal{D}},$$

obtained by the usual base–10 expansion of a *and the coefficient expansion of* f *in* $\mathbb{F}_2[x]$*.*

(Digit folding into one alphabet). *Define a* folding map

$$\phi : \{0,\dots,9\} \times \{0,1\} \longrightarrow \Delta, \qquad \phi(d,0) = d, \quad \phi(d,1) = id,$$

where

$$\Delta := \{0,1,\dots,9,\ i0,i1,\dots,i9\}.$$

Let $\psi : \Delta \to \{0,\dots,9\} \times \{0,1\}$ *be its inverse, i.e.*

$$\psi(d) = (d,0), \qquad \psi(id) = (d,1) \qquad (d \in \{0,\dots,9\}).$$

(So Δ *is just a relabeling of the product digit set* $\widetilde{\mathcal{D}}$*; this is exactly the "digit folding" idea: compress two coordinates into one larger alphabet.)*

Define the folded digit set

$$D^\circ := \Delta \subset (\text{symbols}).$$

Given a folded digit $\delta \in \Delta$*, write* $\psi(\delta) = (d(\delta), \varepsilon(\delta))$*.*

Then every $(a,f) \in L$ *admits a finite expansion using only* one digit alphabet Δ*:*

$$\boxed{(a,f) = \sum_{n=0}^{\ell} \Big(d(\delta_n)\, T^n \widetilde{b}_1 + \varepsilon(\delta_n)\, T^n \widetilde{b}_2 \Big), \qquad \delta_n \in \Delta.}$$

Equivalently, in coordinates,

$$(a,f) = \sum_{n=0}^{\ell} \big(d(\delta_n)10^n,\ \varepsilon(\delta_n)x^n\big).$$

Concrete computation. *Let*

$$(a,f) = (37245,\ 1 + x^2 + x^5) \in L.$$

Choose expanded digits (d_n, ε_n) *by*

$$(d_0,d_1,d_2,d_3,d_4,d_5) = (5,4,2,7,3,0), \qquad (\varepsilon_0,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5) = (1,0,1,0,0,1).$$

Fold them into $\delta_n := \phi(d_n, \varepsilon_n) \in \Delta$*:*

$$(\delta_0,\delta_1,\delta_2,\delta_3,\delta_4,\delta_5) = (i5,\ 4,\ i2,\ 7,\ 3,\ i0).$$

Then

$$\begin{aligned}(37245,\ 1 + x^2 + x^5) &= \sum_{n=0}^{5} \big(d(\delta_n)10^n,\ \varepsilon(\delta_n)x^n\big) \\ &= (5,1) + T(4,0) + T^2(2,1) + T^3(7,0) + T^4(3,0) + T^5(0,1),\end{aligned}$$

where the only *digits used are* $\delta_n \in \Delta$*.*

Uniqueness. *If*

$$\sum_{n=0}^{\ell} \big(d(\delta_n)10^n,\ \varepsilon(\delta_n)x^n\big) = (0,0),$$

then $\sum_n d(\delta_n)10^n = 0$ *forces all* $d(\delta_n) = 0$*, and* $\sum_n \varepsilon(\delta_n)x^n = 0$ *forces all* $\varepsilon(\delta_n) = 0$ *in* $\mathbb{F}_2[x]$*, hence all* δ_n *are the zero digit. Therefore the folded expansion is unique.*

Definition 4 (Digit–folding reduction; folding–irreducible CNS)**.** *Fix a class of digit–folding moves as in Lemma 1. A* digit–folding reduction *of an n–dimensional CNS is a finite sequence*

$$(L, T^{(0)}, \mathbf{b}^{(0)}, D^{(0)}) \longrightarrow (L, T^{(1)}, \mathbf{b}^{(1)}, D^{(1)}) \longrightarrow \cdots \longrightarrow (L, T^{(s)}, \mathbf{b}^{(s)}, D^{(s)}),$$

where each arrow is obtained by applying Lemma 1 with some commuting endomorphism U and some integer $M \geq 2$, so that the frame size strictly decreases at each step.

We call an n–dimensional CNS folding–irreducible *(or* fully folded*) if no nontrivial digit–folding move applies to it, i.e. there do not exist $U \in \mathrm{End}_E(L)$ commuting with the base shift(s) and an integer $M \geq 2$ for which Lemma 1 yields a further compression of the base frame.*

A fully folded form *of a given CNS is any folding–irreducible CNS obtained from it by a digit–folding reduction.*

Remark 2. *Every digit–folding reduction terminates: each application of Lemma 1 replaces an mM–frame by an m–frame with $M \geq 2$, hence strictly decreases the (positive integer) frame size. Therefore no infinite sequence of digit–folding reductions exists.*

Remark 3. *A fully folded form need not be unique in general: different choices of folding directions (choices of U and the associated finite tower structure) or different orders of folding may lead to non-isomorphic folding–irreducible presentations. If a canonical normal form is desired, one may fix a deterministic folding strategy (e.g. greedy in M, then lexicographic in the chosen towers).*

3. The Question and Some Clues

Remark 4 (Representation scope and guiding questions)**.** *A central motivation for introducing an n–dimensional number system (and its canonical variant) is to understand its* representation scope. *Informally, this asks:* for which modules (or rings) does the proposed positional mechanism actually represent every element by a finite digit expansion, and when is such an expansion unique? *To make this precise, we adopt the following terminology.*

Definition 5 (Representation scope)**.** *Fix $n \in \mathbb{N}$ and a class $\mathcal{C}$ of $\mathbb{Z}$–modules. For $L \in \mathcal{C}$, let $\mathbf{T} = (T_1, \ldots, T_n)$ be a base family on L and let $D \subset L$ be a finite digit set (a transversal of $L/I_{\mathbf{T}}$, where $I_{\mathbf{T}} = \sum_{i=1}^{n} T_i(L)$). Write $T^\alpha := T_1^{\alpha_1} \cdots T_n^{\alpha_n}$.*

We define the representation scope *of n–dimensional number systems on $\mathcal{C}$ to be*

$$\boxed{\begin{aligned}\mathrm{RS}_n(\mathcal{C}) := \Big\{ (L, \mathbf{T}, D) \;\Big|\; & L \in \mathcal{C},\ \forall v \in L,\ \exists F \subset \mathbb{N}_0^n \text{ finite}, \\ & \exists (d_\alpha)_{\alpha \in F} \subset D : \ v = \sum_{\alpha \in F} T^\alpha(d_\alpha) \Big\}.\end{aligned}}$$

In words, $(L, \mathbf{T}, D) \in \mathrm{RS}_n(\mathcal{C})$ iff every element of L admits a finite digit expansion in the places $\{T^\alpha\}$ with digits from D.

If a monomial order $\prec$ on $\mathbb{N}_0^n$ is fixed, we define the canonical representation scope *by*

$$\mathrm{RS}_n^{\mathrm{CNS}}(\mathcal{C}) := \Big\{ (L, \mathbf{T}, D) \in \mathrm{RS}_n(\mathcal{C}) \;\Big|\; \text{the } \prec\text{–normal form expansion is unique} \Big\}.$$

Remark. For notational convenience, we may adjoin $T_0 = \mathrm{Id}_L$ to a base family so the "0th place" is treated uniformly with shifted places. This does not change the digit theory: digits are still taken from a finite transversal of $L/\sum_{i=1}^{n} T_i(L)$ (we do not quotient by $T_0(L) = L$).

Definition 6. *(n–dimensional CNS$[\mathrm{Id}]$ (identity–adjoined form)) Let L be a $\mathbb{Z}$–module and let $(\mathbf{T}, D)$ be an n–dimensional CNS on L in the sense of Definition* [n–dimensional CNS]. *Define the* identity–adjoined *base family*

$$\mathbf{T}[\mathrm{Id}] \;:=\; (\mathrm{Id}_L,\ T_1, \ldots, T_n).$$

We call $(\mathbf{T}[\mathrm{Id}], D)$ *an* n–dimensional $\mathrm{CNS}[\mathrm{Id}]$ *if it satisfies the same expansion and uniqueness requirements as* $(\mathbf{T}, D)$*, with the understanding that the digit set is still chosen as a transversal of*

$$L \Big/ \sum_{i=1}^{n} T_i(L) \qquad (\text{so we do not quotient by } \mathrm{Id}_L(L) = L).$$

Remark 5. *The notation CNS*$[\mathrm{Id}]$ *is convenient when the digits are taken in a coefficient field, e.g.* $\mathbb{F}_p$*. In typical base families that include a prime* p *(so coefficients are reduced modulo* p*), one may view the digit alphabet as* $\mathbb{F}_p$ *itself (identified with* $\{0, 1, \ldots, p-1\}$*) and regard these coefficient digits as acting at the* 0*th place via* Id_L*.*

Definition 7. *(*$\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$ *(identity–closed representation scope)) Let* $\mathrm{RS}_n(\mathcal{C})$ *denote the representation scope of* n*–dimensional number systems on* $\mathcal{C}$ *(as defined previously). For each* $(L, \mathbf{T}, D) \in \mathrm{RS}_n(\mathcal{C})$ *with* $\mathbf{T} = (T_1, \ldots, T_n)$*, write*

$$\mathbf{T}[\mathrm{Id}] \;:=\; (\mathrm{Id}_L,\; T_1, \ldots, T_n).$$

We define the identity–closed representation scope *by*

$$\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}] \;:=\; \mathrm{RS}_n(\mathcal{C}) \,\cup\, \Big\{ (L, \mathbf{T}[\mathrm{Id}], D) \;\Big|\; (L, \mathbf{T}, D) \in \mathrm{RS}_n(\mathcal{C}) \Big\}.$$

Thus $\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$ *contains both the original* n*–dimensional number systems and their identity–adjoined presentations (CNS*$[\mathrm{Id}]$*).*

Proposition 1 (Direct products of n–dimensional number systems)**.** *Let* L_1, L_2 *be* $\mathbb{Z}$*–modules. Suppose*

$$(L_1, \mathbf{T}, D_1) \quad \textit{and} \quad (L_2, \mathbf{S}, D_2)$$

are n*–dimensional number systems, where*

$$\mathbf{T} = (T_1, \ldots, T_n) \subset \mathrm{End}_{\mathbb{Z}}(L_1), \qquad \mathbf{S} = (S_1, \ldots, S_n) \subset \mathrm{End}_{\mathbb{Z}}(L_2),$$

and $D_1 \subset L_1$*,* $D_2 \subset L_2$ *are digit sets (transversals of the corresponding quotients). Define*

$$L := L_1 \times L_2, \qquad \mathbf{U} = (U_1, \ldots, U_n) \subset \mathrm{End}_{\mathbb{Z}}(L) \quad \textit{by} \quad U_i := T_i \times S_i,$$

i.e. $U_i(x, y) = (T_i x,\; S_i y)$*, and set*

$$D := D_1 \times D_2 \subset L.$$

Then $(L, \mathbf{U}, D)$ *is an* n*–dimensional number system.*

Moreover, if $(L_1, \mathbf{T}, D_1)$ *and* $(L_2, \mathbf{S}, D_2)$ *are* n*–dimensional CNS (with respect to the same fixed monomial order on* $\mathbb{N}_0^n$*), then* $(L, \mathbf{U}, D)$ *is an* n*–dimensional CNS.*

Finally, if one (or both) of the factors is presented in the identity–adjoined form CNS$[\mathrm{Id}]$*, then the product can be presented as CNS*$[\mathrm{Id}]$ *on* L *by adjoining* Id_L*; forgetting* Id *recovers an ordinary CNS presentation.*

Proof. *Step 1:* $\mathbf{U}$ *is a base family.* Each U_i is injective because T_i and S_i are injective. For commutativity, for all i, j and $(x, y) \in L$,

$$U_i U_j(x, y) = (T_i T_j x,\; S_i S_j y) = (T_j T_i x,\; S_j S_i y) = U_j U_i(x, y),$$

since the T_i commute pairwise and the S_i commute pairwise.

For independence, suppose $U^\alpha = U^\beta$ where $U^\alpha := U_1^{\alpha_1} \cdots U_n^{\alpha_n}$. Applying to $(x, 0)$ gives

$$(T^\alpha x, 0) = U^\alpha(x, 0) = U^\beta(x, 0) = (T^\beta x, 0) \quad (\forall x \in L_1),$$

hence $T^\alpha = T^\beta$, so $\alpha = \beta$ by independence of $\mathbf{T}$. Thus $\mathbf{U}$ is independent.

Step 2: digit quotient and digit set. Let

$$I_{\mathbf{T}} := \sum_{i=1}^{n} T_i(L_1), \qquad I_{\mathbf{S}} := \sum_{i=1}^{n} S_i(L_2), \qquad I_{\mathbf{U}} := \sum_{i=1}^{n} U_i(L).$$

Then

$$I_{\mathbf{U}} = \sum_{i=1}^{n} \big(T_i(L_1) \times S_i(L_2)\big) = \Big(\sum_{i=1}^{n} T_i(L_1)\Big) \times \Big(\sum_{i=1}^{n} S_i(L_2)\Big) = I_{\mathbf{T}} \times I_{\mathbf{S}}.$$

Hence

$$L/I_{\mathbf{U}} \;\cong\; (L_1/I_{\mathbf{T}}) \times (L_2/I_{\mathbf{S}}),$$

so if D_1 and D_2 represent cosets uniquely in $L_1/I_{\mathbf{T}}$ and $L_2/I_{\mathbf{S}}$, then $D = D_1 \times D_2$ represents cosets uniquely in $L/I_{\mathbf{U}}$.

Step 3: existence of expansions. Take $(v_1, v_2) \in L$. Since $(L_1, \mathbf{T}, D_1)$ is an n–dimensional number system, there exist digits $(d_\alpha) \subset D_1$ such that

$$v_1 = \sum_{\alpha} T^{\alpha}(d_\alpha).$$

Similarly, there exist digits $(e_\alpha) \subset D_2$ such that

$$v_2 = \sum_{\alpha} S^{\alpha}(e_\alpha).$$

Therefore

$$(v_1, v_2) = \sum_{\alpha} \big(T^{\alpha}(d_\alpha),\; S^{\alpha}(e_\alpha)\big) = \sum_{\alpha} U^{\alpha}(d_\alpha, e_\alpha),$$

with $(d_\alpha, e_\alpha) \in D_1 \times D_2 = D$, proving that $(L, \mathbf{U}, D)$ is an n–dimensional number system.

Step 4: uniqueness (CNS case). Assume both factors are n–dimensional CNS (unique normal form with respect to the same monomial order). If

$$\sum_{\alpha} U^{\alpha}(\delta_\alpha) = \sum_{\alpha} U^{\alpha}(\delta'_\alpha) \qquad (\delta_\alpha, \delta'_\alpha \in D_1 \times D_2),$$

then comparing coordinates gives equality of the corresponding expansions in L_1 and in L_2. By uniqueness in each factor, $\delta_\alpha = \delta'_\alpha$ for all α, hence the product expansion is unique, so $(L, \mathbf{U}, D)$ is a CNS.

Step 5: the CNS[Id] *presentation.* If one wishes to work in the identity–adjoined notation, adjoin Id_L to $\mathbf{U}$ to obtain $\mathbf{U}[\mathrm{Id}] := (\mathrm{Id}_L, U_1, \ldots, U_n)$. This does not change the digit theory because digits are still taken modulo $\sum_{i=1}^{n} U_i(L) = I_{\mathbf{U}}$ (one does not quotient by $\mathrm{Id}_L(L) = L$). Thus the product admits a CNS[Id] presentation; forgetting Id_L recovers the ordinary CNS presentation. ☐

Remark 6. *(Examples in* $\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$ *and a guiding hypothesis) Let* $\mathcal{C}$ *be the class of* $\mathbb{Z}$*–modules (or rings viewed as* $\mathbb{Z}$*–modules). By the constructions above, the following standard objects admit* n*–dimensional number–system presentations (and hence lie in the identity–closed scope* $\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$*):*

$$\mathbb{Z}, \qquad \mathbb{Z}[x_1, \ldots, x_m], \qquad \mathbb{F}_p, \qquad \mathbb{F}_p[x],$$

together with finite direct sums of such systems (by the direct–product closure proposition). In particular, combinations such as

$$\mathbb{Z} \oplus \mathbb{F}_2[x]$$

can be represented by choosing compatible base maps on each component and taking the product digit set; adjoining Id *yields the corresponding CNS*[Id] *notation.*

Motivation of a new hypothesis. *Since many rings of interest are finitely generated* $\mathbb{Z}$*–algebras, there exists a surjective homomorphism*

$$\phi : \mathbb{Z}[x_1, \ldots, x_m] \to R,$$

and hence a standard presentation

$$R \;\cong\; \mathbb{Z}[x_1, \ldots, x_m] / \ker(\phi).$$

Moreover, the basic building blocks $\mathbb{Z}$*,* $\mathbb{Z}[x_1, \ldots, x_m]$*, and their typical quotients naturally support digit–expansion structures in our framework. It is therefore reasonable to expect that such a finite–generators presentation should place* R *inside the identity–closed representation scope* $\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$*. This leads to the following hypothesis.*

Assumption 1 (Guiding hypothesis). *If a ring* R *admits a presentation as a finitely generated* $\mathbb{Z}$*–algebra,*

$$R \;\cong\; \mathbb{Z}[x_1, \ldots, x_m] / \ker(\phi)$$

for some surjection $\phi : \mathbb{Z}[x_1, \ldots, x_m] \to R$*, then* R *admits an* n*–dimensional number–system presentation (possibly in the identity–adjoined form), i.e.*

$$R \;\in\; \mathrm{RS}_n(\mathcal{C})[\mathrm{Id}] \quad \text{for some } n.$$

Emphasize that $\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$ is not restricted to commutative settings: for instance, noncommutative rings such as $M_2(\mathbb{Z})$ (with base given by left multiplication by -2 and a finite digit set modulo $2M_2(\mathbb{Z})$) also admit CNS[Id] presentations, and hence lie in $\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$.
We also note that $\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$ does not automatically contain every ring one might consider: for example, noncommutative free algebras such as $\mathbb{Z}\langle x_1, x_2\rangle$ and fields like $\mathbb{Q}$ fall outside the present scope under our standing digit–set/quotient requirements.

Proposition 2. *(*$\mathbb{Q}$ *lies outside* $\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$*) For every* $n \geq 1$*, the field* $\mathbb{Q}$ *does not admit an* n*–dimensional number–system presentation of the type used in this paper (hence* $\mathbb{Q} \notin \mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$*).*

Proof. Let $f : \mathbb{Q} \to \mathbb{Q}$ be any $\mathbb{Z}$–endomorphism (additive group homomorphism). Set $q := f(1) \in \mathbb{Q}$. Then for every integer m we have $f(m) = mq$. Moreover, for $n \geq 1$,

$$q = f(1) = f\Big(n \cdot \frac{1}{n}\Big) = n\, f\Big(\frac{1}{n}\Big) \quad\Longrightarrow\quad f\Big(\frac{1}{n}\Big) = \frac{q}{n},$$

and hence for every $m/n \in \mathbb{Q}$,

$$f\Big(\frac{m}{n}\Big) = m\, f\Big(\frac{1}{n}\Big) = \frac{mq}{n} = q \cdot \frac{m}{n}.$$

Thus every $\mathbb{Z}$–endomorphism of $\mathbb{Q}$ is multiplication by a rational number.

In particular, if $T : \mathbb{Q} \to \mathbb{Q}$ is injective, then $T(x) = qx$ with $q \neq 0$, so T is surjective and $T(\mathbb{Q}) = \mathbb{Q}$. Therefore, for any base family $\mathbf{T} = (T_1, \ldots, T_n)$ of injective endomorphisms on $\mathbb{Q}$, we have

$$I_{\mathbf{T}} = \sum_{i=1}^{n} T_i(\mathbb{Q}) = \mathbb{Q}, \qquad \text{hence} \qquad \mathbb{Q}/I_{\mathbf{T}} = 0.$$

Consequently the only possible digit transversal is $D = \{0\}$, so any digit expansion evaluates to 0 and cannot represent $1 \in \mathbb{Q}$. Hence $\mathbb{Q}$ is not representable in the sense of $\mathrm{RS}_n(\mathcal{C})[\mathrm{Id}]$. □

Proposition 3 (The free algebra $\mathbb{Z}\langle x_1, x_2\rangle$ is not covered by the commuting multiplication–base model)**.** *Let $R = \mathbb{Z}\langle x_1, x_2\rangle$ be the (noncommutative) free $\mathbb{Z}$–algebra on two generators. Consider the natural "generator–multiplication" choice of base maps*

$$T_1(r) = x_1 r, \qquad T_2(r) = x_2 r \qquad (r \in R).$$

Then (T_1, T_2) cannot be a base family in the sense of Definition [Base family] *(hence R is not in the portion of* $\mathrm{RS}_n(\mathcal{C})$[Id] *arising from finitely many commuting multiplication–base maps).*

Proof. A base family is required to be finite and *pairwise commuting*. However, for every $r \in R$,

$$T_1T_2(r) = x_1(x_2 r) = (x_1x_2)r, \qquad T_2T_1(r) = x_2(x_1 r) = (x_2x_1)r.$$

Since R is the free algebra, the words x_1x_2 and x_2x_1 are distinct, hence $x_1x_2 \neq x_2x_1$ in R, and therefore $T_1T_2 \neq T_2T_1$. Thus the two independent "shift directions" coming from the two generators cannot simultaneously appear in a commuting base family. Consequently, the standard finite–base approach (using only finitely many independent generator multiplications) does not apply to $\mathbb{Z}\langle x_1, x_2\rangle$. ☐

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