Article

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Sixfold Discrete Symmetry Explains Dark Matter

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Abstract

Using the Pauli algebra version of the Dirac equation we show the existence of its six symmetric versions whose solutions are referred to as the six sectors of fermion fields. It is shown that the sectors are distinct, by showing that if a fermion field belongs to two sectors at once then its mass must be zero. Also shown is the lack electromagnetic interaction between different sectors, since each sector has its own unique matrix coupling its fermion field to its EM field. Altogether this predicts the ratio of dark matter to ordinary matter of 5 to 1, which is close to the observed ratio of 5.2 to 1 [1]. In order to fully justify this prediction, the sixfold discrete symmetry must first of all be extended to the electroweak sector, then to strong interactions. This article only deals with coupling of fermion fields to electromagnetic fields. This paper essentially depends on the full equivalence of the Pauli algebra version of the Dirac equation with its standard version, which is proven in (2.1). The sixfold symmetry can be seen in the Pauli algebra version of the Dirac equation, but is harder to see in its standard version.

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1. Introduction, Notation, and the Main Results

1.1. Introduction

The motivation for this article is the sixfold symmetry of fermion fields which is apparent in the Pauli algebra formulation [2] of the Dirac equation but is hidden in its standard four-component form. The solutions to the six symmetric versions of the Dirac equation form what is referred to as the six sectors of fermion fields, and the solutions to six versions of Maxwell's equations form the six sectors of the electromagnetic field.

The mathematical reason for the sixfold symmetry are the six representations of the Clifford algebra $\mathrm{Cl}_{1,2}$ into the algebra of two-by-two complex matrices $M_2(\mathbb{C})$. The key to this is the table in Section 6.

The group of discrete symmetries of fermion fields are the automorphisms of the first Pauli group G_1, with inner automorphisms containing the charge conjugation and the mass inversion [2]. The outer automorphisms are the subgroup of order two with the parity involution, taken direct product with the permutation group on three letters. (See 9.1).

We construct an operator that calculates the values of the electric and magnetic fields coupled to the fermion field out of the fermion field spinor, with all quantities taking values in the Pauli algebra of two by two complex matrices (4.1). The resulting second order wave equation has a coupling matrix which is unique for each of the six versions, obtained by applying the outer automorphisms of the Pauli algebra.

The six versions of the Dirac equation have the same scalar equations, which raises a question, will their solutions be distinct? This question is answered positively in (7) by showing that if a field belongs to two different sectors then its mass equals zero.

Another naturally arising question is, does there exist any electromagnetic transmission between different sectors? It is answered in the negative in (4.1). For each of the six sectors, there is a different coupling matrix connecting its electromagnetic field to its fermion field, thus precluding electromagnetic signals between different sectors.

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1.2. Notation

All quantities considered here are elements of the Pauli algebra of complex 2×2 matrices, written in the basis of Pauli matrices. Only lower indices are used and instead of upper and lower indices the change of sign of the vector component is indicated by the bar. σ_0 being identity is omitted.

$a = a_0 + \underline{a} = \sigma_0 a_0 + \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3$
$\bar{a} = a_0 - \underline{a} = \sigma_0 a_0 - \sigma_1 a_1 - \sigma_2 a_2 - \sigma_3 a_3$
$a\bar{a} = \bar{a}a = \det a$

Any element can be written as sum of its scalar component and its vector component. The "bar" operation switches the sign of the vector component leaving the scalar component unchanged. The "star" operation is taking the Hermitian conjugate. Their composition "bar-star", in either order, is an automorphism of the Pauli algebra, similar to complex conjugation. It is an outer automorphism since it replaces the determinant of an element with its complex conjugate. The subalgebra $\mathbb{H} \in M_2(\mathbb{C})$ fixed by the bar-star automorphism is called the subalgebra of real quaternions, and their products with i are called imaginary quaternions. Their sign is changed by the bar-star operation.

$$u, v \in \mathbb{H} \iff \overline{u}^* = u \qquad \overline{(iv)}^* = -iv \qquad \overline{u + iv}^* = u - iv$$

It is shown in [2] that the bar-star automorphism is the parity symmetry.

A real quaternion $u = u_0 + \underline{u}$ has a real scalar component u_0 and imaginary vector component $\underline{u}$, vice versa for imaginary quaternions.

The differential, where ∂_α stands for $\frac{\partial}{\partial x_\alpha}$:

$$\begin{aligned} \partial &= \partial_0 + \underline{\partial} = \sigma_0\partial_0 + \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3 \\ \bar{\partial} &= \partial_0 - \underline{\partial} = \sigma_0\partial_0 - \sigma_1\partial_1 - \sigma_2\partial_2 - \sigma_3\partial_3 \end{aligned} \tag{1.1}$$

Their composition in either order is the d'Alembertian, multiplied by the identity matrix:

$$\partial\bar{\partial} = \bar{\partial}\partial = \Box$$

The EM four-potential: $A = A_0 + \underline{A} = \sigma_0 A_0 + \sigma_1 A_1 + \sigma_2 A_2 + \sigma_3 A_3$

Inner and outer products of vector components:

$$\underline{a} \cdot \underline{b} := \frac{1}{2}(\underline{a}\,\underline{b} + \underline{b}\,\underline{a}) \qquad \underline{a} \wedge \underline{b} := \frac{1}{2}(\underline{a}\,\underline{b} - \underline{b}\,\underline{a}) \qquad \underline{a}\,\underline{b} = \underline{a} \cdot \underline{b} + \underline{a} \wedge \underline{b}$$

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad \underline{a} \wedge \underline{b} = \begin{vmatrix} i\sigma_1 & i\sigma_2 & i\sigma_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The electric and magnetic fields:

$$E = -\partial_0 \underline{A} - \underline{\partial} A_0 \qquad i\underline{B} = \underline{\partial} \wedge \underline{A} = \begin{vmatrix} i\sigma_1 & i\sigma_2 & i\sigma_3 \\ \partial_1 & \partial_2 & \partial_3 \\ A_1 & A_2 & A_3 \end{vmatrix}$$

The four-component spinor of the Dirac equation is written using letters rather than indices and is completed with the second column as follows:

$$\Psi = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b^* \\ b & -a^* \\ c & -d^* \\ d & c^* \end{bmatrix} = \begin{bmatrix} iv \\ \\ u \end{bmatrix}$$

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The lower square of the two-column matrix is an element u of the subalgebra of real quaternions $\mathbb{H}$ and is unchanged by the bar-star automorphism. The upper square iv is a product of a real quaternion with imaginary unit i and its sign is changed by the bar-star automorphism (this can be checked by direct calculation). Their sum is called the Pauli algebra spinor:

$$u = \begin{bmatrix} c & -d^* \\ d & c^* \end{bmatrix} \quad iv = \begin{bmatrix} a & b^* \\ b & -a^* \end{bmatrix} \quad \psi = u + iv \qquad \overline{\psi}^* = u - iv$$

1.3. The Main Results

a) Pauli algebra Dirac equation and the inhomogeneous wave equation for our sector, both with their bar-star conjugates, see Section 5.

b) The second order wave equation coupling the fermion field to EM field:

$$\left(\Box + \mathrm{m}^2 - \mathrm{q}^2 \det A\right) \psi \,(i\sigma_3) = \mathrm{q}\,(\bar{\partial}\bar{A} + A\partial)\,\psi$$

where $i\sigma_3$ is the coupling matrix, again specifically for our sector.

c) The six symmetric versions of all equations: they are obtained from the six different representations of the Clifford algebra $\mathrm{Cl}_{1,2}$ into the algebra of matrices $M_2(\mathbb{C})$. Each of these versions has its own coupling matrix that connects between the fermion field and the EM field.

d) The group of discrete symmetries of fermion fields: they correspond to the group of automorphisms of the first Pauli group G_1, with inner automorphisms being the charge conjugation, the mass inversion, and their composition, and the outer automorphisms being the group of order two containing the parity involution taken direct product with the group of permutations on three letters. (see 9.1)

2. Proof of Equivalence of the Pauli Algebra Version to the Standard Dirac equation

Proposition 2.1. *The standard Dirac equation taken together with its Hermitian conjugate is equivalent to the Pauli algebra Dirac equation taken together with its bar-star image.*

Proof. The original Dirac equation: $(i\gamma^\mu D_\mu - \mathrm{m})\Psi = 0$ is multiplied by $-i$ and is rewritten in two by two blocks with separate matrices for the differentials and for the EM interaction term. The left column of the equation below are the four scalar equations of the standard Dirac equation and the right column are the equations of the Hermitian conjugate Dirac equation, though they appear in different order. The matrices $i\sigma_3$ appear because without them the right column would have the opposite sign.

$$\begin{bmatrix} \partial_0 & \underline{\partial} \\ -\underline{\partial} & -\partial_0 \end{bmatrix} \begin{bmatrix} iv \\ u \end{bmatrix} + \mathrm{q} \begin{bmatrix} A_0 & -\underline{A} \\ \underline{A} & -A_0 \end{bmatrix} \begin{bmatrix} iv \\ u \end{bmatrix} \left[i\sigma_3 \right] + \mathrm{m} \begin{bmatrix} iv \\ u \end{bmatrix} \left[i\sigma_3 \right] = 0 \tag{2.1}$$

(It is useful at this point to rewrite the equation without blocks and check that the four equations in the left column are those of the Dirac equation and the four equations in the right column are of its Hermitian conjugate).

We now multiply the equation on the right by $i\sigma_3$ and rewrite it as two equations in Pauli algebra, and also switch the signs in the second equation:

$$\partial_0 iv(i\sigma_3) + \underline{\partial}\, u(i\sigma_3) - \mathrm{q}\, A_0 iv + \mathrm{q}\, \underline{A}u - \mathrm{m}\, iv = 0$$
$$\partial_0 u\,(i\sigma_3) + \underline{\partial}\, iv(i\sigma_3) - \mathrm{q}\, A_0 u + \mathrm{q}\, \underline{A}iv + \mathrm{m}\; u = 0$$

Next add and subtract the two equations recalling that $\psi = u + iv$, $\overline{\psi}^* = u - iv$:

$$\partial_0\psi\,(i\sigma_3) + \underline{\partial}\psi\,(i\sigma_3) - \mathrm{q}\,A_0\psi + \mathrm{q}\,\underline{A}\psi + \mathrm{m}\,\overline{\psi}^* = 0$$
$$\partial_0\overline{\psi}^*(i\sigma_3) - \underline{\partial}\overline{\psi}^*(i\sigma_3) - \mathrm{q}\,A_0\overline{\psi}^* - \mathrm{q}\,\underline{A}\overline{\psi}^* + \mathrm{m}\,\psi = 0$$

We obtain the Pauli algebra Dirac equation and its bar-star conjugate:

$$\partial\,\psi\,(i\sigma_3) - \mathrm{q}\,\bar{A}\psi + \mathrm{m}\,\overline{\psi}^* = 0 \qquad (2.2)$$
$$\bar{\partial}\,\overline{\psi}^*(i\sigma_3) - \mathrm{q}\,A\,\overline{\psi}^* + \mathrm{m}\,\psi = 0$$

These two equations are transformed one into the other by the bar-star automorphism, because ∂ and A are real. □

3. Obtaining the Maxwell's Equations From the Pauli algebra Dirac Equations

We apply the differential ∂ to the four-potential A:

$$\partial A = (\partial_0 + \underline{\partial})(A_0 + \underline{A}) = \partial_0 A_0 + \underline{\partial}\cdot\underline{A}\,| + \partial_0\underline{A} + \underline{\partial}A_0\,| + \underline{\partial}\wedge\underline{A} = L - \underline{E} + i\underline{B}$$

where the three summands are the scalar Lorentz sum, the electric field and the magnetic field.

Next apply the bar-star conjugate of the differential, whose composition with differential is the d'Alembertian, and that the current $J = \Box A$: (see Eq. 5.1)

$$J = \Box A = \bar{\partial}(\partial A) = (\partial_0 - \underline{\partial})(L - \underline{E} + i\underline{B}) =$$
$$= \partial_0 L - \partial_0\underline{E} + \partial_0 i\underline{B} - \underline{\partial}L + \underline{\partial}\cdot\underline{E} + \underline{\partial}\wedge\underline{E} - i\underline{\partial}\cdot\underline{B} - i\underline{\partial}\wedge\underline{B}$$

The summands in this equation are of four different types and we rewrite the four separate sub-equations for each type:

real scalar	$\partial_0 L + \underline{\partial}\cdot\underline{E} = 0$
imaginary scalar	$\underline{\partial}\cdot i\underline{B} = 0$
real vector	$\underline{J} + \partial_0\underline{E} + \underline{\partial}L + \underline{\partial}\wedge i\underline{B} = 0$
imaginary vector	$\partial_0\, i\underline{B} + \underline{\partial}\wedge\underline{E} = 0$

In order to see that these are the four Maxwell's equations we rewrite them in a more traditional order:

$$\begin{aligned}\underline{\partial}\cdot\underline{E} &= -\partial_0 L\\ \underline{\partial}\cdot i\underline{B} &= 0\\ \underline{\partial}\wedge\underline{E} &= -\partial_0\, i\underline{B}\\ \underline{\partial}\wedge i\underline{B} &= -\underline{J} - \partial_0\underline{E} - \underline{\partial}L\end{aligned}$$

The Lorentz sum L appears in the equations because there was no fixing of a gauge. Its time derivative is minus the charge density. The minus sign before the current in the last equation is because the wedge product introduces one more copy of i and $i^2 = -1$. See also [3].

4. Second Order Wave Equation Coupling the Fermion and the EM Fields

4.1. Reciprocal Expressions for the spinor and Its Bar-Star Image

The equations 2.2 allow to express $\overline{\psi}^*$ in terms of ψ and vice versa express ψ in terms of $\overline{\psi}^*$. Then combine to obtain second order equations for both:

$$\begin{aligned} \overline{\psi}^* &= -\frac{1}{\mathrm{m}}(\partial\,\psi\;\;(i\sigma_3) - \mathrm{q}\,\bar{A}\,\psi) \\ \psi &= -\frac{1}{\mathrm{m}}\left(\bar{\partial}\,\overline{\psi}^*(i\sigma_3) - \mathrm{q}\,A\,\overline{\psi}^*\right) \end{aligned}$$

Now plug in:

$$\begin{aligned} \psi &= \frac{1}{\mathrm{m}^2}\left(\bar{\partial}\,(\partial\,\psi\,(i\sigma_3) - \mathrm{q}\,\bar{A}\,\psi)\,(i\sigma_3) - \mathrm{q}\,A\,(\partial\,\psi\;\;(i\sigma_3) - \mathrm{q}\,\bar{A}\,\psi)\right) \\ \overline{\psi}^* &= \frac{1}{\mathrm{m}^2}\left(\partial\left(\bar{\partial}\,\overline{\psi}^*(i\sigma_3) - \mathrm{q}\,A\,\overline{\psi}^*\right)(i\sigma_3) - \mathrm{q}\,\bar{A}\left(\bar{\partial}\,\overline{\psi}^*(i\sigma_3) - \mathrm{q}\,A\,\overline{\psi}^*\right)\right) \end{aligned}$$

Simplify:

$$\begin{aligned} \psi &= \frac{1}{\mathrm{m}^2}\left(-\Box\psi\;-\;\mathrm{q}\,\bar{\partial}(\bar{A}\psi)(i\sigma_3)\;-\;\mathrm{q}\,A\partial\,\psi(i\sigma_3)\;+\;\mathrm{q}^2\det(A)\,\psi\right) \\ \overline{\psi}^* &= \frac{1}{\mathrm{m}^2}\left(-\Box\overline{\psi}^*-\mathrm{q}\,\partial(A\overline{\psi}^*)(i\sigma_3) - \mathrm{q}\,\bar{A}\bar{\partial}\,\overline{\psi}^*(i\sigma_3) + \mathrm{q}^2\det(A)\,\overline{\psi}^*\right) \end{aligned}$$

Rewriting we have:

$$\begin{aligned} \left(\Box + \mathrm{m}^2 - \mathrm{q}^2\det A\right)\psi &= -\mathrm{q}\,(\bar{\partial}\bar{A} + A\partial)\;\psi\;(i\sigma_3) \\ \left(\Box + \mathrm{m}^2 - \mathrm{q}^2\det A\right)\overline{\psi}^* &= -\mathrm{q}\,(\partial A + \bar{A}\bar{\partial})\;\overline{\psi}^*(i\sigma_3) \end{aligned}$$

Multiplying on the right by $i\sigma_3$ we have:

Proposition 4.1. *Coupling of Fermion and EM Fields*

$$\begin{aligned} \left(\Box + \mathrm{m}^2 - \mathrm{q}^2\det A\right)\psi\;(i\sigma_3) &= \mathrm{q}\,(\bar{\partial}\bar{A} + A\partial)\;\psi \\ \left(\Box + \mathrm{m}^2 - \mathrm{q}^2\det A\right)\overline{\psi}^*(i\sigma_3) &= \mathrm{q}\,(\partial A + \bar{A}\bar{\partial})\;\overline{\psi}^* \end{aligned}$$

Again, these two equations are transformed one into the other by the bar-star automorphism. Taking into account the placement of brackets in 4.1, the right sides should be understood as

$$\begin{aligned} \left(\Box + \mathrm{m}^2 - \mathrm{q}^2\det A\right)\psi\;(i\sigma_3) &= \mathrm{q}\,\bar{\partial}(\bar{A}\,\psi\,) + \mathrm{q}\,A(\partial\,\psi\,) \\ \left(\Box + \mathrm{m}^2 - \mathrm{q}^2\det A\right)\overline{\psi}^*(i\sigma_3) &= \mathrm{q}\,\partial(A\overline{\psi}^*) + \mathrm{q}\,\bar{A}(\bar{\partial}\,\overline{\psi}^*) \end{aligned} \tag{4.1}$$

5. The Pauli Algebra Lagrangian

The purpose of this section is to rewrite the Lagrangian of quantum electrodynamics in Pauli algebra form. Then the Lagrangian is rewritten in terms of Clifford algebra $\mathrm{Cl}_{1,2}$. Next we use the six representations of the Clifford algebra $\mathrm{Cl}_{1,2}$ into the algebra of two by two complex matrices $M_2(\mathbb{C})$ so as to construct the six symmetric forms of the Dirac Lagrangian and of the Dirac equation.

The main difference between the standard QED Lagrangian [4], [5], [6] :

$$\mathscr{L}_{QED} = \overline{\Psi}(i\gamma^\mu\partial^\mu - m)\Psi - qA_\mu\overline{\Psi}\gamma^\mu\Psi - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

and the Pauli algebra Lagrangian used here is use of the multiplicative structure of the algebra.

The other difference is that unlike the usual recovering the equations of motion from the Lagrangian using the Euler-Lagrange equations, another procedure is used, namely equating to zero the formal derivatives. This results in Pauli algebra Dirac equation [2], and the inhomogeneous wave equations.

Here is the Pauli algebra version of the QED Lagrangian:

$$\begin{aligned}\mathscr{L} &= \psi^* \partial\, \psi (i\sigma_3) - q\psi^* \bar{A}\, \psi + m\psi^* \overline{\psi}^* + \left(\overline{\partial A}\right) \partial A + \\ &+ \overline{\psi}\, \bar{\partial}\, \overline{\psi}^* (i\sigma_3) - q\overline{\psi} A\, \overline{\psi}^* + m\, \overline{\psi}\, \psi + \left(\overline{\bar{\partial}\bar{A}}\right) \bar{\partial}\bar{A}\end{aligned}$$

The Lagrangian is intentionally written in two lines because these lines are transformed one into the other by the bar-star automorphism of the Pauli algebra (see [2]).

The Lagrangian can also be written in terms of the conserved current J:

$$\begin{aligned}\mathscr{L} &= \psi^*\, \partial\, \psi (i\sigma_3) - \frac{1}{2}(\bar{A}\, J + \bar{J}\, A) + m\psi^* \overline{\psi}^* + \bar{A}\, \Box\, A \\ &+ \overline{\psi}\, \bar{\partial}\, \overline{\psi}^* (i\sigma_3) - \frac{1}{2}(A\, \bar{J} + J\, \bar{A}) + m\, \overline{\psi}\, \psi + A\, \Box\, \bar{A}\end{aligned}$$

The reason for this is that $\bar{\partial}\partial A = \partial\bar{\partial} A = \Box\, A$ and also (see 5.3)

$$q\psi^* \bar{A}\, \psi + q\overline{\psi} A\, \overline{\psi}^* = \frac{1}{2}(\bar{A}\, J + A\, \bar{J} + \bar{J}\, A + J\, \bar{A})$$

Recovering the equations of motion is done by equating to zero the formal derivatives of the Lagrangian with respect to ψ^*, $\overline{\psi}$, $\bar{A}$, A:

$$\frac{\partial \mathscr{L}}{\partial \psi^*} = 0 \qquad \frac{\partial \mathscr{L}}{\partial \overline{\psi}} = 0 \qquad \frac{\partial \mathscr{L}}{\partial \bar{A}} = 0 \qquad \frac{\partial \mathscr{L}}{\partial A} = 0$$

This results in four equations:

$$\begin{aligned}\partial\, \psi\, (i\sigma_3) - q\bar{A}\, \psi + m\overline{\psi}^* &= 0 \\ \bar{\partial}\, \overline{\psi}^* (i\sigma_3) - qA\, \overline{\psi}^* + m\, \psi &= 0 \\ -J + \Box\, A &= 0 \\ -\bar{J} + \Box\, \bar{A} &= 0\end{aligned} \tag{5.1}$$

The first two equations are the Pauli algebra Dirac equation [2], and the last two are the inhomogeneous wave equations [7].

The sign of the d'Alembertian used here is the opposite of the sign in [7].

The differential, the four-potential and the four-current are elements of the Pauli algebra $M_2(\mathbb{C})$ split into scalar and vector parts as follows:

$$\begin{aligned}\partial &= \partial_\mu \sigma_\mu = \partial_0 + \underline{\partial} & \bar{\partial} &= \partial_0 - \underline{\partial} \\ A &= A^\mu \sigma_\mu = A^0 + \underline{A} & \bar{A} &= A^0 - \underline{A} \\ J &= J^\mu \sigma_\mu = J^0 + \underline{J} & \bar{J} &= J^0 - \underline{J}\end{aligned}$$

If $u \in \mathbb{H}$ then $u^* = \bar{u}, \overline{u}^* = u$. For iv the opposite holds $(iv)^* = -i\bar{v}$.

For any $\psi \in M_2(\mathbb{C})$ $\bar{\psi}\psi = \det(\psi) I_2$

Both compositions of the differential and its bar are the d'Alembertian: $\partial\bar{\partial} = \bar{\partial}\partial = \Box$. Hence:

$$\begin{aligned}\overline{(\partial A)}\partial A &= \bar{A}\bar{\partial}\partial A = \bar{A}\, \Box\, A \\ \overline{(\bar{\partial}\bar{A})}\bar{\partial}\bar{A} &= A\partial\bar{\partial}\bar{A} = A\, \Box\, \bar{A}\end{aligned}$$

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The Lagrangian is rewritten separating the elements into their scalar and vector parts. This will be used when writing the representation independent form of the Lagrangian.

$$\begin{aligned}\mathscr{L} &= \psi^* (\partial_0 + \underline{\partial})\, \psi (i\sigma_3) + m\psi^* \overline{\psi}^* + \overline{\psi}\, (\partial_0 - \underline{\partial})\, \overline{\psi}^* (i\sigma_3) + m\, \overline{\psi}\, \psi \\ -\frac{1}{2}&[(A^0 - \underline{A})\, (J^0 + \underline{J}) + (J^0 - \underline{J})\, (A^0 + \underline{A})] + (A^0 - \underline{A}) \Box (A^0 + \underline{A}) \\ -\frac{1}{2}&[(A^0 + \underline{A})\, (J^0 - \underline{J}) + (J^0 + \underline{J})\, (A^0 - \underline{A})] + (A^0 + \underline{A}) \Box (A^0 - \underline{A})\end{aligned}$$

Taking into account that

$$\psi^* = \bar{u} - i\bar{v} \qquad \bar{\psi} = \bar{u} + i\bar{v} \qquad \overline{\psi}^* = u - iv \qquad \psi = u + iv$$

and the expressions for the current ([8], Proposition 8) which imply:

$$J^\alpha = \frac{1}{2}\,\mathrm{tr}(\psi^* \sigma^\alpha \psi) \quad J^0 = q(\bar{u}u + \bar{v}v)\sigma_0 \quad \underline{J} = J^k \sigma_k = iq(\bar{u}\sigma_k v - \bar{v}\sigma_k u)\sigma_k$$

the Lagrangian is rewritten in quaternionic functions u and v as follows:

$$\begin{aligned}\frac{1}{2}\mathscr{L} = (\bar{u}\partial_0 u + \bar{v}\partial_0 v)(i\sigma_3) - (\bar{u}\underline{\partial} v - \bar{v}\underline{\partial} u)\sigma_3 - \\ -qA_0(\bar{u}u + \bar{v}v) + iqA_k(\bar{u}\sigma_k v - \bar{v}\sigma_k u) + \\ +A_0 \Box A_0 - \underline{A} \Box \underline{A} + m(\bar{u}u - \bar{v}v)\end{aligned} \tag{5.2}$$

Equating to zero the derivatives

$$\frac{\partial \mathscr{L}}{\partial \bar{u}} = 0 \qquad \frac{\partial \mathscr{L}}{\partial \bar{v}} = 0$$

results in equations

$$\begin{aligned}\partial_0 u(i\sigma_3) - \underline{\partial} v \sigma_3 - qA^0 u + iq\underline{A} v + mu = 0 \\ \partial_0 v(i\sigma_3) + \underline{\partial} u \sigma_3 - qA^0 v - iq\underline{A} u - mv = 0\end{aligned}$$

which are equivalent to the first two equations of 5.1, which are the Pauli algebra Dirac equations. This can be checked by multiplying the second equation by i and then adding and subtracting with the first equation.

In order to have a representation-independent form of the Lagrangian, the quaternionic functions, the real u and the imaginary iv, need to be written first in terms of sigma matrices and then in terms of generators of the Clifford algebra:

$$u = \begin{bmatrix} c & -d^* \\ d & c^* \end{bmatrix} = \begin{bmatrix} c_0 + ic_1 & -d_0 + id_1 \\ d_0 + id_1 & c_0 - ic_1 \end{bmatrix} = c_0\sigma_0 + id_1\sigma_1 - id_0\sigma_2 + ic_1\sigma_3$$

$$iv = \begin{bmatrix} a & b^* \\ b & -a^* \end{bmatrix} = \begin{bmatrix} a_0 + ia_1 & b_0 - ib_1 \\ b_0 + ib_1 & -a_0 + ia_1 \end{bmatrix} = ia_1\sigma_0 + b_0\sigma_1 + b_1\sigma_2 + a_0\sigma_3$$

5.1. Proof that the Two Forms of Lagrangian Are the Same

Proposition 5.1. *The two sums forming the interaction terms in the Lagrangian are equal:*

$$q\psi^* \bar{A}\, \psi + q\overline{\psi} A\, \overline{\psi}^* = \frac{1}{2}(\bar{A}\, J + A\, \bar{J} + \bar{J}\, A + J\, \bar{A}) \tag{5.3}$$

Proof. The probability current J^μ is calculated out of the real quaternionic and imaginary quaternionic components of the Pauli algebra spinor ψ as follows:

$$J^\mu = u^*\sigma_\mu(iv) + (iv)^*\sigma_\mu u$$

(Here the components of the conserved current J are understood not as numbers, but as scalar two by two matrices).

This can be checked by using the two-column completion of Ψ as described in 1.2 and in [2], and then rewriting the usual formula for the probability current [6] with 2×2 blocks iv and u.

We need to calculate separately the scalar and the vector components of the current:

$$J_0 = q[u^*u + (iv)^*(iv)]\sigma_0 \qquad \underline{J} = J_k\sigma_k = q[u^*\sigma_k(iv) + (iv)^*\sigma_k u]\sigma_k$$

(Note that both $u^*u + (iv)^*(iv)$ and $u^*\sigma_k(iv) + (iv)^*\sigma_k u$ are real scalars)

To prove 5.3 we calculate separately the left side and the right side. The spinor $\psi = u + iv$ is decomposed as sum of a real quaternionic function u and an imaginary quaternionic function iv. We begin with the left side:

$$\begin{aligned}
&L.S. = q\psi^*\bar{A}\,\psi + q\overline{\psi}A\,\overline{\psi}^* = \\
&q\,[u^* + (iv)^*](A_0 - \underline{A})\,(u + iv) + q\,[u^* - (iv)^*](A_0 + \underline{A})\,(u - iv) = \\
&= q\,(\bar{u} - i\bar{v})(A_0 - \underline{A})\,(u + iv) + q\,(\bar{u} + i\bar{v})(A_0 + \underline{A})\,(u - iv) = \\
&= 2qA_0(\bar{u}u + \bar{v}v)\sigma_0 - 2iqA_k(\bar{u}\sigma_k v - \bar{v}\sigma_k u)\sigma_k = \\
&= 2qA_0[u^*u + (iv)^*(iv)]\sigma_0 - 2qA_k[u^*\sigma_k(iv) + (iv)^*\sigma_k u]\sigma_k
\end{aligned}$$

(Note that the expressions in square brackets are real scalars)

Now calculate the right side:

$$\begin{aligned}
&R.S. = \frac{1}{2}(\bar{A}\,J + A\,\bar{J} + \bar{J}\,A + J\,\bar{A}) = \\
&= \frac{1}{2}[(A_0 - \underline{A})(J_0 + \underline{J}) + (A_0 + \underline{A})(J_0 - \underline{J}) + \\
&+ (J_0 - \underline{J})(A_0 + \underline{A}) + (J_0 + \underline{J})(A_0 - \underline{A})] = \\
&= 2(A_0J_0 - 2\underline{A}\cdot\underline{J} - 2\underline{A}\wedge\underline{J} - 2\underline{J}\wedge\underline{A}) = \\
&= 2(A_0J_0 - 2\underline{A}\cdot\underline{J})
\end{aligned}$$

(the last equality because the outer product is anticommutative)

Now use the expressions for the current in (5.4):

$$R.S. = 2qA_0[u^*u + (iv)^*(iv)]\sigma_0 - 2qA_k[u^*\sigma_k(iv) + (iv)^*\sigma_k u]\sigma_k \tag{5.4}$$

$L.S. = R.S.$ □

6. The Clifford Algebra Lagrangian

The Pauli algebra $M_2(\mathbb{C})$ is isomorphic to the Clifford algebra $\mathrm{Cl}_{1,2}$ so that every element can be expressed in its three generators:

$$\mathrm{e}_0^2 = 1, \qquad \mathrm{e}_1^2 = -1, \qquad \mathrm{e}_2^2 = -1 \tag{6.1}$$

The table below defines six matrix representations of $\mathrm{Cl}_{1,2}$ into $M_2(\mathbb{C})$, of which the first is chosen to be the standard one.

The automorphisms of $M_2(\mathbb{C})$ are labelled by their action on the three cyclic subgroups of order 4 of the finite quaternion group Q_8. These automorphisms form a group isomorphic to the group of permutations of three letters, see [9], [10].

Here the finite quaternion group is (including their opposite signs):

$$Q_8 = \{\sigma_0,\ -i\sigma_1,\ -i\sigma_2,\ -i\sigma_3\} \leftrightarrow \{1,\ \mathbf{i},\ \mathbf{j},\ \mathbf{k}\}$$

This action can be seen in the columns corresponding to the generators

$$\mathrm{e}_1,\ \mathrm{e}_2,\ \mathrm{e}_1\mathrm{e}_2$$

The subscripts ijk in the following table show the isomorphism between the outer automorphisms of Q_8 and the group of permutations of three letters.

The morphisms between the representations are obtained by combining the inverse of one representation with another. The automorphism of $M_2(\mathbb{C})$ converting the DE_{123} into DE_{231} for example will be

$$\varphi_{231} \circ \varphi_{123}^{-1}$$

Here is the table for $\varphi_{ijk} : \mathrm{Cl}_{1,2} \to M_2(\mathbb{C})$:

φ_{ijk}	e_0	e_1	e_2	$\mathrm{e}_1\mathrm{e}_2$	$\mathrm{e}_0\mathrm{e}_1$	$\mathrm{e}_0\mathrm{e}_2$	$\mathrm{e}_0\mathrm{e}_1\mathrm{e}_2$
φ_{123}	σ_3	$-i\sigma_1$	$-i\sigma_2$	$-i\sigma_3$	σ_2	$-\sigma_1$	$-i\sigma_0$
φ_{231}	σ_1	$-i\sigma_2$	$-i\sigma_3$	$-i\sigma_1$	σ_3	$-\sigma_2$	$-i\sigma_0$
φ_{312}	σ_2	$-i\sigma_3$	$-i\sigma_1$	$-i\sigma_2$	σ_1	$-\sigma_3$	$-i\sigma_0$
φ_{213}	σ_3	$-i\sigma_2$	$-i\sigma_1$	$i\sigma_3$	$-\sigma_1$	σ_2	$i\sigma_0$
φ_{132}	σ_2	$-i\sigma_1$	$-i\sigma_3$	$i\sigma_2$	$-\sigma_3$	σ_1	$i\sigma_0$
φ_{321}	σ_1	$-i\sigma_3$	$-i\sigma_2$	$i\sigma_1$	$-\sigma_2$	σ_3	$i\sigma_0$

The action of the bar-star automorphism and of the bar and of "star" (Hermitian conjugate) anti-automorphisms on the generators of the Clifford algebra can be seen in the following table:

operation	e_0	e_1	e_2	$\mathrm{e}_1\mathrm{e}_2$	$\mathrm{e}_0\mathrm{e}_1$	$\mathrm{e}_0\mathrm{e}_2$	$\mathrm{e}_0\mathrm{e}_1\mathrm{e}_2$
bar-star	-1	1	1	1	-1	-1	-1
bar	-1	-1	-1	-1	-1	-1	1
star	1	-1	-1	-1	1	1	-1

Now use φ_{123}^{-1} to rewrite all the vector expressions in terms of generators of the Clifford algebra, according to the first line in the table. The σ_0 being identity is omitted.

$$\begin{aligned} \underline{\partial} &= -\mathrm{e}_0\mathrm{e}_2\,\partial_1 + \mathrm{e}_0\mathrm{e}_1\,\partial_2 + \mathrm{e}_0\,\partial_3 \\ \underline{A} &= -\mathrm{e}_0\mathrm{e}_2\,A^1 + \mathrm{e}_0\mathrm{e}_1\,A^2 + \mathrm{e}_0\,A^3 \\ \underline{J} &= -\mathrm{e}_0\mathrm{e}_2\,J^1 + \mathrm{e}_0\mathrm{e}_1\,J^2 + \mathrm{e}_0\,J^3 \end{aligned} \tag{6.2}$$

The other elements of the Lagrangian that depends on the representation are $i\sigma_3$ and σ_3, and according to the first line of the table, they should be replaced with $-\mathrm{e}_1\mathrm{e}_2$ and with e_0.

Now one can write the representation-independent Lagrangian, by substituting 6.2 into 5.2. In this article the substitution is done only for the case of the free Dirac equation .

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If the six representations are applied to all the expressions in the Lagrangian and in the resulting equations, then the morphisms between representations will transform solutions into solutions. Both the new Pauli algebra spinors and the corresponding new equations will be different but the underlying eight equations of the Dirac equation and its Hermitian conjugate will be exactly the same.

This raises the question, will the solutions to the transformed equations be the same as before, or will they be genuinely different? This question is answered in 7.1. Only the free Dirac equation is needed there, so here is its representation-independent form:

$$(\partial_0 - \mathrm{e}_0\mathrm{e}_2\,\partial_1 + \mathrm{e}_0\mathrm{e}_1\,\partial_2 + \mathrm{e}_0\,\partial_3)\psi + \mathrm{e}_0\mathrm{e}_1\mathrm{e}_2\,\mathrm{m}\,\overline{\psi}^*\mathrm{e}_0 = 0 \tag{6.3}$$

7. The Six Sectors of Fermion Fields Are Distinct

The method used to show this is the mass inversion symmetry, described in [2]. For convenience this short argument is reproduced here. Given a solution ψ of the free Dirac equation define $\psi' = \psi\sigma_3$. Then multiply this equation, which appears first, on the right by σ_3, and recall that $\overline{\sigma_3}^* = -\sigma_3$:

$$\partial\psi = i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 \quad \partial\psi\sigma_3 = i\,\mathrm{m}\,\overline{\psi}^*\sigma_3\sigma_3 \quad \partial\psi\sigma_3 = -i\,\mathrm{m}\,\overline{(\psi\sigma_3)}^*\sigma_3 \quad \partial\psi' = -i\,\mathrm{m}\,\overline{\psi'}^*\sigma_3$$

So ψ' is also a solution of the Dirac equation but with the mass of opposite sign.

Proposition 7.1. *Only massless spinors can be solutions to two versions of the free Dirac equation belonging to two different sectors.*

Proof. These are the six versions of the free Dirac equation obtained by applying the six representations to the representation-independent free Dirac equation :

$$\begin{array}{llcl}
123 & (\partial_0 + \sigma_1\,\partial_1 + \sigma_2\,\partial_2 + \sigma_3\,\partial_3)\psi - i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 & = & 0\\
231 & (\partial_0 + \sigma_2\,\partial_1 + \sigma_3\,\partial_2 + \sigma_1\,\partial_3)\psi - i\,\mathrm{m}\,\overline{\psi}^*\sigma_2 & = & 0\\
312 & (\partial_0 + \sigma_3\,\partial_1 + \sigma_1\,\partial_2 + \sigma_2\,\partial_3)\psi - i\,\mathrm{m}\,\overline{\psi}^*\sigma_1 & = & 0\\
213 & (\partial_0 - \sigma_2\,\partial_1 - \sigma_1\,\partial_2 + \sigma_3\,\partial_3)\psi + i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 & = & 0\\
132 & (\partial_0 - \sigma_1\,\partial_1 - \sigma_3\,\partial_2 + \sigma_2\,\partial_3)\psi + i\,\mathrm{m}\,\overline{\psi}^*\sigma_1 & = & 0\\
321 & (\partial_0 - \sigma_3\,\partial_1 - \sigma_2\,\partial_2 + \sigma_1\,\partial_3)\psi + i\,\mathrm{m}\,\overline{\psi}^*\sigma_2 & = & 0
\end{array} \tag{7.1}$$

Suppose that the spinor ψ is a solution of two different equations, with the same mass. We can always use the morphisms between representation to make the first of two equations the top line 123. Now let the second equation be any other except 213 (this case will be dealt with separately). For example suppose the second equation is 231. Multiply both equations on the right by σ_3:

$$\begin{array}{llcl}
123 & (\partial_0 + \sigma_1\,\partial_1 + \sigma_2\,\partial_2 + \sigma_3\,\partial_3)\psi' + i\,\mathrm{m}\,\overline{\psi'}^*\sigma_3 & = & 0\\
231 & (\partial_0 + \sigma_2\,\partial_1 + \sigma_3\,\partial_2 + \sigma_1\,\partial_3)\psi' - i\,\mathrm{m}\,\overline{\psi'}^*\sigma_2 & = & 0
\end{array} \tag{7.2}$$

So ψ' is a solution for the first equation with mass $-\mathrm{m}$ and for the second with mass m, which is only possible if the mass is zero. This argument applies to all lines except 213 where we do not have the anticommutation.

For the lines 123 and 213 suppose that $\mathrm{m} \neq 0$ and consider also their bar-star equations:

$$\begin{array}{lcl}
(\partial_0 + \sigma_1\,\partial_1 + \sigma_2\,\partial_2 + \sigma_3\,\partial_3)\psi - i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 & = & 0\\
(\partial_0 - \sigma_2\,\partial_1 - \sigma_1\,\partial_2 + \sigma_3\,\partial_3)\psi + i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 & = & 0\\
(\partial_0 - \sigma_1\,\partial_1 - \sigma_2\,\partial_2 - \sigma_3\,\partial_3)\overline{\psi}^* - i\,\mathrm{m}\,\psi\sigma_3 & = & 0\\
(\partial_0 + \sigma_2\,\partial_1 + \sigma_1\,\partial_2 - \sigma_3\,\partial_3)\overline{\psi}^* + i\,\mathrm{m}\,\psi\sigma_3 & = & 0
\end{array} \tag{7.3}$$

Subtracting the first two lines and the last two lines we get

$$\frac{1}{2}(\sigma_1+\sigma_2)(\partial_1+\partial_2)\,\psi = i\,\mathrm{m}\,\overline{\psi}^*\sigma_3 \quad \frac{1}{2}(\sigma_1+\sigma_2)(\partial_1+\partial_2)\,\overline{\psi}^* = -i\,\mathrm{m}\,\psi\sigma_3$$

Combining we get $\frac{1}{2\,\mathrm{m}^2}(\partial_1+\partial_2)^2\psi = \psi$ and with a change of coordinates we have an ODE

$$\frac{1}{2\,\mathrm{m}^2}\frac{\partial^2\psi}{\partial s^2} - \psi = 0$$

with real roots of its characteristic equation. So ψ is an exponential function and this is incompatible with it being normalizable. □

8. The Lack of Electromagnetic Interaction Between Different Sectors

We consider the six symmetric versions of the equation (4.1) that links the fermion field to the electromagnetic field. In the version labeled by the permutation 123 the linking matrix is $i\sigma_3$. A look at the fourth column of the table (6) shows that the representation independent expression is $-e_1e_2$. Its values under the six representations are all different, showing a specific coupling within each sector.

There are also the six symmetric versions of Maxwell's equations, see 3. The operator $\underline{\partial}$ which equals $\sigma_1\partial_1+\sigma_2\partial_2+\sigma_3\partial_3$ under the representation φ_{123} changes under the other representations according to the table 6. This produces six symmetric, but non-identical copies of Maxwell's equations.

9. The Group of Discrete Symmetries

Proposition 9.1. *The group of discrete symmetries of fermion fields is the group of automorphisms of the first Pauli group G_1 where the abelian normal subgroup of inner automorphisms consists of identity, the charge conjugation symmetry C, the mass inversion symmetry M and their composition. The group of outer automorphisms is a direct product of the group of order two generated by the parity symmetry, which is the bar-star automorphism, and the group of permutations on three letters, generated by permuting the three spatial derivates with the three sigma matrices.*

Proof. The proof consists of identifying the discrete symmetries with elements of the automorphism group $Aut(G_1)$.

The inner automorphisms of the first Pauli group G_1 are the conjugations by σ_1 which is the charge conjugation symmetry C, by σ_3 which is the mass inversion symmetry M, and by their product $i\sigma_2$, which is $CM = MC$.

(This appears in [2], except it appears as right multiplication by the sigma matrix rather than conjugation as here).

Rewriting the short exact sequence

$$1 \longrightarrow \mathrm{Inn}\,G_1 \longrightarrow \mathrm{Aut}\,G_1 \longrightarrow \mathrm{Out}\,G_1 \longrightarrow 1$$

we have

$$1 \longrightarrow C_2\times C_2 \longrightarrow \mathrm{Aut}G_1 \longrightarrow C_2\times S_3 \longrightarrow 1$$

where in the outer automorphisms the cyclic group of order two is generated by the parity transformation "bar-star" and the group of permutations permutes between spatial derivatives and the three sigma matrices.

The parity transformation inverts the signs of the three spatial derivatives and the three components of the vector potential. Recalling that there are two versions of the Pauli algebra Dirac equation related by the bar-star automorphism, this transformation essentially replaces the spinor ψ with its bar-star image $\overline{\psi}^*$ and vice versa. Since the choice between them is arbitrary, there are only six sectors of fermion fields and not twelve. The group of automorphisms of G_1 is described in [**?**]. □

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10. Conclusions and Open Questions

For the six-fold discrete symmetry to be a candidate to explain dark matter, two issues need to be addressed: first, show that the six sectors of free fermion fields are genuinely distinct from each other; and second, show the absence of electromagnetic interaction between different sectors. The first issue is addressed in (7.1). The second follows from the unique matrix coupling the fermion field to the EM field in (4.1).

What remains is the important caveat, that the six-fold symmetry must be extended to weak and strong interactions to be a reasonable explanation. This is not addressed in the present article.

According to [1] the universe consists of approximately 5% ordinary baryonic matter, $\sim 26\%$ dark matter, and $\sim 61\%$ dark energy. This gives the ratio of dark matter to ordinary matter of $5.2 : 1$ which is close to $5 : 1$, as predicted by the six-fold symmetry.

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