Hermite–Hadamard Framework for \((h,m)\)-Convexity

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Article

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Abstract

This work presents a generalization and extension of previous results by incorporating weighted integrals and a refined class of second-type (h, m)-convex functions. By utilizing classical inequalities, such as those of Hölder, Young and the Power mean, we establish new Hermite-Hadamard-type inequalities. The findings offer a broader and more flexible analytical framework, enhancing existing results in the literature. Potential applications of the developed inequalities are also explored.

Keywords: weighted integrals; integral inequalities; Hermite-Hadamard-type inequalities; fractional derivatives

1. Introduction

In [**?**], the following definitions were introduced.

Definition 1. *Let $h : [0,1] \to \mathbb{R}$ be a non-negative function, such that $h \neq 0$, and let $g : I = [0, +\infty) \to [0, +\infty)$. The function g is called (h, m)-convex modified of the first type on I if it satisfies*

$$g(\gamma\mu_1 + m(1-\gamma)\mu_2) \leq h^s(\gamma)g(\mu_1) + m(1 - h^s(\gamma))g(\mu_2), \tag{1}$$

for all $\mu_1, \mu_2 \in I$ and $\gamma \in [0,1]$, where $m \in [0,1]$ and $s \in [-1,1]$.

Definition 2. *Let $h : [0,1] \to \mathbb{R}$ be a non-negative function, such that $h \neq 0$, and let $g : I = [0, +\infty) \to [0, +\infty)$. The function g is called (h, m)-convex modified of the second type on I if it satisfies*

$$g(\gamma\mu_1 + m(1-\gamma)\mu_2) \leq h^s(\gamma)g(\mu_1) + m(1 - h(\gamma))^s g(\mu_2), \tag{2}$$

for all $\mu_1, \mu_2 \in I$ and $\gamma \in [0,1]$, where $m \in [0,1]$ and $s \in [-1,1]$.

Remark 1. *Definitions **??** and **??** enable us to define set $N^s_{h,m}[\mu_1, \mu_2]$, where $\mu_1, \mu_2 \in I$, as the set of modified (h, m)-convex functions, for which $\psi(\mu_1) \geq 0$. Here are some convexity classes-special cases described by the triple $(h(\gamma), m, s)$:*

1. *$(h(\gamma), 0, 0), (\gamma, 0, 1), (\gamma, 1, 1)$ and $(\gamma, 0, s)$ we have, respectively, the increasing starshaped classic convex on I and $s-$starshaped functions [**?**].*
2. *$(\gamma, 1, s)$ $s \in (0,1]$, then ψ is a $s-$convex* (see [**? ?**]) *and for $s \in [-1,1]$ extended $s-$convex on I (see [**?**]).*
3. *(γ^α, m, s) with $\alpha \in (0,1]$, then ψ is a $s-(\alpha, m)-$convex function on I [**?**]. If $\alpha = 1$, we will have (s, m)-convex function on I ([**?**]), but if $m = 1$, we get $(\alpha, s)-$convex function on I [**? ?**] and the last, if $s = 1$, we will have $(\alpha, m)-$convex function on I [**?**].*
4. *$(h(\gamma), m, 1)$, then ψ is a variant of an $(h, m)-$convex function on I citeOAS2016.*

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The weighted integral operators, which underpin our analysis are presented next ([? ?]).

Adding a particular weight function to the definition of an integral operator is a new and generic way to define an integral operator and start the process of generalizing a known result. This may be done as follows:

Definition 3. *(see [?]) Let $g \in L_1[\mu_1, \mu_2]$ and let $w : I \to \mathbb{R}$ be a continuous, positive function, whose first derivative is integrable in the interior of I. The weighted fractional integral operators are introduced as follows (right and left, respectively):*

$$\mathbf{J}^{w}_{\mu_1^+} g(\chi) = \int_{\mu_1}^{\chi} w'\left(\frac{\mu_2 - z}{\mu_2 - \mu_1}\right) g(z)\, dz, \quad \chi > \mu_1, \tag{3}$$

$$\mathbf{J}^{w}_{\mu_2^-} g(\chi) = \int_{\chi}^{\mu_2} w'\left(\frac{z - \mu_1}{\mu_2 - \mu_1}\right) g(z)\, dz, \quad \chi < \mu_2. \tag{4}$$

Remark 2. *The inclusion of the first derivative of the weight function w arises from the inherent nature of the problem. Alternatively, the second derivative, or a higher-order derivative, can also be considered.*

Remark 3. *Examine particular examples of the weight function w' to better demonstrate the reach of Definition **??**:*

(a) Setting $w'(z) \equiv 1$ recovers the classical Riemann integral.

(b) Choosing $w'(z) = \frac{z^{\alpha-1}}{\Gamma(\alpha)}$ leads to the Riemann-Liouville fractional integral.

(c) By selecting appropriate weight functions w', various fractional integral operators can be derived, such as the k-Riemann-Liouville integrals [?], right-sided fractional integrals of a function g relative to another function h on $[\mu_1, \mu_2]$ [?], and integral operators introduced in [? ? ? ?].

(d) Additional well-known integral operators, fractional or otherwise, can be retrieved as particular cases of the above formulation. Interested readers may consult [? ?].

The Caputo-Fabrizio definition's main basic feature can be explained (cf. [?]) with $0 < \alpha < 1$:

$$\left({}^{CF}_{\mu_1}\mathbf{I}^{\alpha} g\right)(\chi) = \frac{1-\alpha}{M(\alpha)} g(\chi) + \frac{\alpha}{M(\alpha)} \int_{\mu_1}^{\chi} g(z)\, dz, \tag{5}$$

$$\left({}^{CF}\mathbf{I}^{\alpha}_{\mu_2} g\right)(\chi) = \frac{1-\alpha}{M(\alpha)} g(\chi) + \frac{\alpha}{M(\alpha)} \int_{\chi}^{\mu_2} g(z)\, dz, \tag{6}$$

where $M(\alpha)$ is a normalization function, such that $M(0) = M(1) = 1$.

Caputo's Fractional Derivative is well known, given by the expression ([?]):

$$\left({}^{C}_{0}\mathbf{D}^{\alpha}_{\chi} g\right)(\chi) = \frac{1}{\Gamma(1-\alpha)} \int_0^{\chi} (\chi - z)^{-\alpha} g'(z)\, dz. \tag{7}$$

The idea comes from replacing the singular kernel $(\chi - z)^{-\alpha}$ in the Caputo fractional derivative, given in formula (**??**), with the kernel $\exp\left[-\frac{\alpha(\chi - z)}{1-\alpha}\right]$.

In the paper [?], the same authors proposed a more complete study of the operator (**??**) by presenting the definition of the adapted fractional integral operator ${}^{CF}_{0}\mathbf{I}^{\alpha}_{\chi}$, when $M(\alpha) = 1$.

$$\left({}^{CF}_{0}\mathbf{I}^{\alpha}_{\chi} g\right)(\chi) = \frac{1}{\alpha} \int_0^{\chi} \exp\left[-\frac{(1-\alpha)}{\alpha}(\chi - z)\right] g(z)\, dz. \tag{8}$$

As one can notice, this definition shows a big resemblance to the classical Riemann-Liouville fractional integral, as given by

$$\left({}^{RL}_{0}\mathbf{I}^{\alpha}_{\chi} g\right)(\chi) = \frac{1}{\Gamma(\alpha)} \int_0^{\chi} (\chi - z)^{\alpha-1} g(z)\, dz. \tag{9}$$

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In this work, we present some variants of the well-known Hermite-Hadamard inequality in the context of (h,m)-convex functions of the second kind using weighted integral operators. Our results include several well-known cases from the literature.

Definition 4. *Let $g \in L_1[\mu_1,\mu_2]$. The Riemann–Liouville integrals ${}^{RL}\mathbf{I}^{\alpha}_{\mu_1^+}g$ and ${}^{RL}\mathbf{I}^{\alpha}_{\mu_2^-}g$ of order $\alpha > 0$ are defined as:*

$$ {}^{RL}\mathbf{I}^{\alpha}_{\mu_1^+}g(\chi) = \frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\chi}(\chi - z)^{\alpha-1}g(z)\,dz,$$

$$ {}^{RL}\mathbf{I}^{\alpha}_{\mu_2^-}g(\chi) = \frac{1}{\Gamma(\alpha)}\int_{\chi}^{\mu_2}(z - \chi)^{\alpha-1}g(z)\,dz,$$

where $\Gamma(\alpha)$ is the Gamma function.

2. Generalizations

Theorem 1. *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $\mu_1,\mu_2 \in I^\circ$ with $\mu_1 < \mu_2$. Let $w : [0,1] \to \mathbb{R}$ be a continuous and positive function with first derivative integrable on $(0,1)$. Suppose that g is (h,m)-convex modified of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in \mathrm{Dom}(g)$, then it's true that*

$$\begin{aligned} g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) &\le h^s\left(\frac{1}{2}\right)\frac{r+1}{\mu_2-\mu_1}\mathbf{J}^{w}_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right) \\ &\quad + m\left(1-h\left(\frac{1}{2}\right)\right)^s\frac{m(r+1)}{\mu_2-\mu_1}\mathbf{J}^{w}_{\mu_2^-}g\left(\frac{r\mu_2+\mu_1}{m(r+1)}\right) \\ &\le h^s\left(\frac{1}{2}\right)\left[g(\mu_1)\mathbf{N}_1 + mg\left(\frac{\mu_2}{m}\right)\mathbf{N}_2\right] \\ &\quad + m\left(1-h\left(\frac{1}{2}\right)\right)^s\left[g(\mu_2)\mathbf{N}_3 + mg\left(\frac{\mu_1}{m^2}\right)\mathbf{N}_4\right], \end{aligned} \tag{10}$$

where $r \in \mathbb{N}\cup\{0\}$, $\mathbf{N}_1 = \int_0^1 w'(\gamma)\,h^s\left(\frac{r+\gamma}{r+1}\right)d\gamma$, $\mathbf{N}_2 = \int_0^1 w'(\gamma)\left(1-h\left(\frac{r+\gamma}{r+1}\right)\right)^s d\gamma$, $\mathbf{N}_3 = \int_0^1 w'(\gamma)\,h^s\left(\frac{r+\gamma}{m(r+1)}\right)d\gamma$ and $\mathbf{N}_4 = \int_0^1 w'(\gamma)\left(1-h\left(\frac{r+\gamma}{m(r+1)}\right)\right)^s d\gamma$.

Proof. By means of the (h,m)-convexity of g with $\gamma = \frac{1}{2}$, we have

$$g\left(\frac{x+y}{2}\right) \le h^s\left(\frac{1}{2}\right)g(x) + m\left(1-h\left(\frac{1}{2}\right)\right)^s g\left(\frac{y}{m}\right), \tag{11}$$

for $x,y \in I$.

Substituting $x = \frac{r+\gamma}{r+1}\mu_1 + \frac{1-\gamma}{r+1}\mu_2$ and $y = \frac{r+\gamma}{r+1}\mu_2 + \frac{1-\gamma}{r+1}\mu_1$ in (**??**), we get

$$\begin{aligned} g\left(\frac{\mu_1+\mu_2}{2}\right) &\le h^s\left(\frac{1}{2}\right)g\left(\frac{r+\gamma}{r+1}\mu_1 + \frac{1-\gamma}{r+1}\mu_2\right) \\ &\quad + m\left(1-h\left(\frac{1}{2}\right)\right)^s g\left(\frac{r+\gamma}{m(r+1)}\mu_2 + \frac{1-\gamma}{m(r+1)}\mu_1\right). \end{aligned} \tag{12}$$

Multiplying both sides of (**??**) by $w'(\gamma)$ and integrating over $[0,1]$, we obtain

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$$g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) \le h^s\left(\frac{1}{2}\right)\int_0^1 w'(\gamma)\, g\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma + m\left(1-h\left(\frac{1}{2}\right)\right)^s\int_0^1 w'(\gamma)\, g\left(\frac{r+\gamma}{m(r+1)}\mu_2+\frac{1-\gamma}{m(r+1)}\mu_1\right)d\gamma = h^s\left(\frac{1}{2}\right)\mathbf{L}_1 + m\left(1-h\left(\frac{1}{2}\right)\right)^s\mathbf{L}_2. \tag{13}$$

Rewriting the integrals, we find

$$\mathbf{L}_1 = -\left(\frac{r+1}{\mu_2-\mu_1}\right)\int_{\frac{r\mu_1+\mu_2}{r+1}}^{\mu_1} w'\left(\frac{\frac{r\mu_1+\mu_2}{r+1}-x}{\frac{\mu_2-\mu_1}{r+1}}\right)g(x)\,dx = \left(\frac{r+1}{\mu_2-\mu_1}\right)J^w_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right). \tag{14}$$

$$\mathbf{L}_2 = \left(\frac{m(r+1)}{\mu_2-\mu_1}\right)\int_{\frac{r\mu_2+\mu_1}{m(r+1)}}^{\frac{\mu_2}{m}} w'\left(\frac{y-\frac{r\mu_2+\mu_1}{m(r+1)}}{\frac{\mu_2-\mu_1}{m(r+1)}}\right)g(y)\,dy = \left(\frac{m(r+1)}{\mu_2-\mu_1}\right)J^w_{\frac{\mu_2}{m}^-}g\left(\frac{r\mu_2+\mu_1}{m(r+1)}\right). \tag{15}$$

From (**??**), (**??**) and (**??**), it follows that

$$g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) \le h^s\left(\frac{1}{2}\right)\frac{r+1}{\mu_2-\mu_1}\mathbf{J}^w_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right) + m\left(1-h\left(\frac{1}{2}\right)\right)^s\frac{m(r+1)}{\mu_2-\mu_1}\mathbf{J}^w_{\mu_2^-}g\left(\frac{r\mu_2+\mu_1}{m(r+1)}\right). \tag{16}$$

Employing again the (h,m)-convexity of g, we obtain

$$\int_0^1 w'(\gamma)\, g\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma \le g(\mu_1)\mathbf{N}_1 + mg\left(\frac{\mu_2}{m}\right)\mathbf{N}_2. \tag{17}$$

$$\int_0^1 w'(\gamma)\, g\left(\frac{r+\gamma}{m(r+1)}\mu_2+\frac{1-\gamma}{m(r+1)}\mu_1\right)d\gamma \le g(\mu_2)\mathbf{N}_3 + mg\left(\frac{\mu_1}{m^2}\right)\mathbf{N}_4. \tag{18}$$

By combining (**??**)–(**??**), we arrive at (**??**). □

Remark 4. *Setting $s=m=1$, $r=0$, $h(z)=z$ and $w'(z)=1$, we recover the classical Hermite-Hadamard inequality.*

Remark 5. *Considering s, m, r and $h(z)$ as in Remark* **??**, *but with $w'(z)=\frac{z^{\alpha-1}}{\Gamma(\alpha)}$, we obtain the Theorem 2 of [**?**].*

Remark 6. *Letting $m=1$, $n=0$, $h(z)=z$ and $w'(z)=\frac{z^{\alpha-1}}{\Gamma(\alpha)}$, we have*

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$$\begin{aligned}\frac{g\left(\frac{\mu_1+\mu_2}{2}\right)}{\Gamma(\alpha+1)} &\leq \frac{1}{2^s(\mu_2-\mu_1)\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\left(\frac{\mu_2-z}{\mu_2-\mu_1}\right)^{\alpha-1} g(z)\,dz \\ &\quad+\frac{1}{2^s(\mu_2-\mu_1)\Gamma(\alpha)}\int_{a}^{b}\left(\frac{z-\mu_1}{\mu_2-\mu_1}\right)^{\alpha-1} g(z)\,dz \\ &\leq \frac{1}{2^s}\left[\frac{g(\mu_1)}{(\alpha+s)\Gamma(\alpha)}+\frac{g(\mu_2)}{\Gamma(\alpha)}\int_0^1 z^{\alpha-1}(1-z)^s\,dz\right] \\ &\quad+\frac{1}{2^s}\left[\frac{g(\mu_2)}{(\alpha+s)\Gamma(\alpha)}+\frac{g(\mu_1)}{\Gamma(\alpha)}\int_0^1 z^{\alpha-1}(1-z)^s\,dz\right].\end{aligned} \tag{19}$$

Utilizing Definition **??** *in* (**??**), *we find*

$$\begin{aligned}\frac{g\left(\frac{\mu_1+\mu_2}{2}\right)}{\Gamma(\alpha+1)} &\leq \frac{1}{2^s(\mu_2-\mu_1)^\alpha}\left[{}^{RL}\mathbf{I}^{\alpha}_{\mu_1^+}g(\mu_2)+{}^{RL}\mathbf{I}^{\alpha}_{\mu_2^-}g(\mu_1)\right] \\ &\leq \frac{\alpha}{2^{s-1}\Gamma(\alpha+1)}\left(\frac{g(\mu_1)+g(\mu_2)}{2}\right)\left[\frac{1}{\alpha+s}+\frac{2}{\alpha+s}\left(1-\frac{1}{2^{\alpha+s}}\right)\right].\end{aligned} \tag{20}$$

Multiplying the three terms by $2^{s-1}\Gamma(\alpha+1)$ *in* (**??**), *we complete* (1) *of* [**?**].

Remark 7. *Under the same assumptions as before, but with* $w'(z)=1$, *we complete Theorem* 2.1 *of* [**?**].

Remark 8. *Maintaining the previous assumptions, but considering* $w'(z)=\frac{z^{\alpha-1}}{\Gamma(\alpha)}$, *we derive the Theorem* 3 *of* [**?**].

Remark 9. *Under the conditions of Remark* **??**, *but with* $w'(z)=\frac{\exp(-\varsigma z)}{\alpha}$, *where* $\varsigma=\frac{1-\alpha}{\alpha}$, *we retrieve the Theorem* 3.1 *of* [**?**].

Remark 10. *Substituting* $w(z)=\frac{z^\alpha}{\alpha}$, $r=0$, $m=s=1$ *and* $h(z)=z$ *in the previous result, it leads to the following inequality for the Riemann-Liouville fractional integral (it refers to Theorem* 2 *in* [**?**]*):*

$$g\left(\frac{\mu_1+\mu_2}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^\alpha}\left[{}^{RL}\mathbf{I}^{\alpha}_{\mu_2-}g(\mu_1)+{}^{RL}\mathbf{I}^{\alpha}_{\mu_1+}g(\mu_2)\right]\leq\frac{g(\mu_1)+g(\mu_2)}{2}.$$

Remark 11. *Theorem* 5 *in* [**?**] *(also see Theorem* 1 *in* [**?**]*), which is based on k-Riemann–Liouville fractional integrals, can be obtained from Theorem* **??** *by setting* $w(z)=z^{\frac{\alpha}{k}}$, $r=0$, $m=s=1$ *and* $h(z)=z$.

The above results form the foundation for deriving other inequalities by using different types of integral operators, as demonstrated in the following Remark.

Remark 12. *We consider s-convex functions,* $0<\alpha<1$, $m=1$, $h(z)=z$, *by putting* $r=0$ *in* (**??**), *and choosing* $w'(z)=1$, *we obtain:*

$$2^{s-1}g\left(\frac{\mu_1+\mu_2}{2}\right)\leq\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}g(z)\,dz\leq\frac{g(\mu_1)+g(\mu_2)}{s+1},$$

taking into account

 doi:10.20944/preprints202509.0395.v1

$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} g(z)\,dz = \frac{M(\alpha)}{\alpha(\mu_2-\mu_1)}\left[\frac{1-\alpha}{M(\alpha)}g(\chi)+\frac{\alpha}{M(\alpha)}\int_{\mu_1}^{\chi} g(z)\,dz \right.$$
$$\left. + \frac{1-\alpha}{M(\alpha)}g(\chi)+\frac{\alpha}{M(\alpha)}\int_{\chi}^{\mu_2} g(z)\,dz - \frac{2(1-\alpha)}{M(\alpha)}g(\chi)\right].$$

Using the last two results, we can derive easily the Theorem 2.1 *of [***?** *]. If additionally,* $s=1$*, from the above we can derive Theorem* 2 *of [***?** *].*

Theorem 2. *Let* g*,* w*,* r*,* μ_1 *and* μ_2 *as in the Theorem* **??**. *If* $g'\in L[\mu_1,\mu_2]$*, then*

$$\frac{\mu_2-\mu_1}{2}\int_0^1 [w(1-\gamma)-w(\gamma)]g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2\right)d\gamma$$
$$=(r+1)\left\{(w(0)-w(1))\left[g(\mu_1)+g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]-\left[\frac{\mathbf{J}^{w}_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right)+\mathbf{J}^{w}_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-}g(\mu_1)}{2}\right]\right\}. \tag{21}$$

Proof. Let us consider

$$\int_0^1 [w(1-\gamma)-w(\gamma)]g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2\right)d\gamma$$
$$=\int_0^1 w(1-\gamma)g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2\right)d\gamma-\int_0^1 w(\gamma)g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2\right)d\gamma$$
$$=\mathcal{I}_1-\mathcal{I}_2. \tag{22}$$

Integrating $\mathcal{I}_1$ by parts, we get

$$\mathcal{I}_1=\frac{r+1}{\mu_1-\mu_2}\left[w(0)g(\mu_1)-w(1)g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]$$
$$-\frac{r+1}{\mu_2-\mu_1}\int_0^1 w'(1-\gamma)\,g\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma. \tag{23}$$

Making change of variable $x=\frac{r+\gamma}{r+1}\mu_1+\frac{(1-\gamma)}{r+1}\mu_2$ in (**??**), we find

$$\mathcal{I}_1=\frac{r+1}{\mu_1-\mu_2}\left[w(0)g(\mu_1)-w(1)g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]$$
$$-\frac{r+1}{\mu_2-\mu_1}\int_{\mu_1}^{\frac{r\mu_1+\mu_2}{r+1}} w'\left(\frac{x-\mu_1}{\frac{\mu_2-\mu_1}{r+1}}\right)g\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma$$
$$=\frac{r+1}{\mu_1-\mu_2}\left[w(0)g(\mu_1)-w(1)g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]-\frac{r+1}{\mu_2-\mu_1}\mathbf{J}^{w}_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right). \tag{24}$$

Analogously for $\mathcal{I}_2$, we can prove

$$\mathcal{I}_2=\frac{r+1}{\mu_1-\mu_2}\left[w(1)g(\mu_1)-w(0)g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]-\frac{r+1}{\mu_2-\mu_1}\mathbf{J}^{w}_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-}g(\mu_1). \tag{25}$$

From (**??**), (**??**) and (**??**), we have

 doi:10.20944/preprints202509.0395.v1

$$\int_0^1 [w(1-\gamma)-w(\gamma)]g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma$$
$$=\frac{2(r+1)}{\mu_1-\mu_2}\left\{(w(0)-w(1))\left[g(\mu_1)+g\left(\frac{r\mu_1+\mu_2}{r+1}\right)\right]-\left[\frac{\mathbf{J}^w_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right)+\mathbf{J}^w_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-}g(\mu_1)}{2}\right]\right\}. \quad (26)$$

By multiplying both sides of (**??**) by $\frac{\mu_2-\mu_1}{2}$, we get the desired result. □

Remark 13. *Using convex functions, $r=0$ and $w(z)=z$, in this way the Theorem* **??** *becomes the following lemma:*

Lemma 1. *Let g be a real-valued function defined on a closed real interval $[\mu_1,\mu_2]$ and differentiable on (μ_1,μ_2). If $g'\in L_1[\mu_1,\mu_2]$, then the following equality holds*

$$\frac{g(\mu_1)+g(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_a^b g(u)\,du=\frac{\mu_2-\mu_1}{2}\int_0^1(1-2\gamma)g'(\gamma\mu_1+(1-\gamma)\mu_2)\,d\gamma,$$

which is Lemma 2.1 *of [***?** *], one of the most important results in the Theory of Integral Inequalities.*

Remark 14. *Establishing $r=0$ and $w(z)=z^{\frac{\lambda}{k}}$, Lemma* 2.1 *of [***?** *] is derived for $\lambda,k>0$.*

Theorem 3. *Let g, w, r, μ_1 and μ_2 be defined as before. Suppose that $|g'|$ is (h,m)-convex modified of the second type, the following inequality holds*

$$\left|(r+1)\left\{w(1)-w(0)-\left[\frac{\mathbf{J}^w_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right)+\mathbf{J}^w_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-}g(\mu_1)}{2}\right]\right\}\right|$$
$$\leq\frac{\mu_2-\mu_1}{2}\left[|g'(\mu_1)|\mathbf{W}_1+|g'(\mu_2)|\mathbf{W}_2\right], \quad (27)$$

where

$$\mathbf{W}_1=\int_0^1|w(1-\gamma)-w(\gamma)|\,h^s\left(\frac{r+\gamma}{r+1}\right)d\gamma,$$

$$\mathbf{W}_2=\int_0^1|w(1-\gamma)-w(\gamma)|\left(1-h\left(\frac{r+\gamma}{r+1}\right)\right)^s d\gamma.$$

Proof. By using Lemma (**??**) and the (h,m)-convexity of g, we have

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$$\begin{aligned}
&\left|(r+1)\left\{w(1)-w(0)-\left[\frac{\mathbf{J}^{w}_{\mu_1^+}g\left(\frac{r\mu_1+\mu_2}{r+1}\right)+\mathbf{J}^{w}_{\left(\frac{r\mu_1+\mu_2}{r+1}\right)^-}g(\mu_1)}{2}\right]\right\}\right| \\
&\leq \frac{\mu_2-\mu_1}{2}\int_0^1 |w(1-\gamma)-w(\gamma)|\left|g'\left(\frac{r+\gamma}{r+1}\mu_1+\frac{1-\gamma}{r+1}\mu_2\right)\right|d\gamma \\
&\leq \frac{\mu_2-\mu_1}{2}\left[|g'(\mu_1)|\int_0^1 |w(1-\gamma)-w(\gamma)|h^s\left(\frac{r+\gamma}{r+1}\right)d\gamma\right. \\
&\qquad \left.+|g'(\mu_2)|\int_0^1 |w(1-\gamma)-w(\gamma)|\left(1-h\left(\frac{r+\gamma}{r+1}\right)\right)^s d\gamma\right] \\
&= \frac{\mu_2-\mu_1}{2}\left[|g'(\mu_1)|\mathbf{W}_1+|g'(\mu_2)|\mathbf{W}_2\right].
\end{aligned}$$

The proof is finished. □

Remark 15. *Assuming the same conditions as in Remark* (**??**) *and invoking Lemma* (**??**)*, we recover Theorem* 2.2 *of* [**?**]*.*

Remark 16. *Under the same assumptions of the remark* (**??**)*, we retrieve Theorem* 6 *of* [**?**]*.*

Theorem 4. *Let g, w, n, μ_1 and μ_2 be defined as before. Suppose that g is (h,m)-convex modified of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in \mathrm{Dom}(g)$, then it is true that*

$$\begin{aligned}
g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) &\leq \frac{n}{\mu_2-\mu_1}\left[h^s\left(\frac{1}{2}\right)\mathbf{J}^{w}_{\left(\frac{n-1}{n}\mu_2+\frac{\mu_1}{n}\right)^+}g(\mu_2)\right. \\
&\qquad \left.+m^2\left(1-h\left(\frac{1}{2}\right)\right)^s\mathbf{J}^{w}_{\left(\frac{n-1}{n}\mu_1+\frac{\mu_2}{n}\right)^-}g\left(\frac{\mu_1}{m}\right)\right] \\
&\leq h^s\left(\frac{1}{2}\right)\left[g(\mu_1)\mathbf{N}_5+mg\left(\frac{\mu_2}{m}\right)\mathbf{N}_6\right] \\
&\qquad +m\left(1-h\left(\frac{1}{2}\right)\right)^s\left[g(\mu_2)\mathbf{N}_7+mg\left(\frac{\mu_1}{m^2}\right)\mathbf{N}_8\right], \qquad (28)
\end{aligned}$$

where

$$\mathbf{N}_1=\int_0^1 w'(\gamma)\,h^s\left(\frac{\gamma}{n}\right)d\gamma, \quad \mathbf{N}_2=\int_0^1 w'(\gamma)\left(1-h\left(\frac{\gamma}{n}\right)\right)^s d\gamma,$$

$$\mathbf{N}_3=\int_0^1 w'(\gamma)\,h^s\left(\frac{n-\gamma}{mn}\right)d\gamma, \quad \mathbf{N}_4=\int_0^1 w'(\gamma)\left(1-h\left(\frac{n-\gamma}{mn}\right)\right)^s d\gamma.$$

Proof. By means of the (h,m)-convexity of g with $\gamma=\frac{1}{2}$, we have

$$g\left(\frac{x+y}{2}\right)\leq h^s\left(\frac{1}{2}\right)g(x)+m\left(1-h\left(\frac{1}{2}\right)\right)^s g\left(\frac{y}{m}\right), \qquad (29)$$

for $x,y\in I$.
Substituting $x=\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2$ and $y=\frac{\gamma}{n}\mu_2+\frac{n-\gamma}{n}\mu_1$ in (**??**), we get

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$$g\left(\frac{\mu_1+\mu_2}{2}\right) \le h^s\left(\frac{1}{2}\right) g\left(\frac{\gamma}{n}\mu_1 + \frac{n-\gamma}{n}\mu_2\right) + m\left(1-h\left(\frac{1}{2}\right)\right)^s g\left(\frac{\gamma}{mn}\mu_2 + \frac{n-\gamma}{mn}\mu_1\right). \tag{30}$$

Multiplying both sides of (**??**) by $w'(\gamma)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned} g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) &\le h^s\left(\frac{1}{2}\right)\int_0^1 w'(\gamma)\, g\left(\frac{\gamma}{n}\mu_1 + \frac{n-\gamma}{n}\mu_2\right)d\gamma \\ &\quad + m\left(1-h\left(\frac{1}{2}\right)\right)^s \int_0^1 w'(\gamma)\, g\left(\frac{\gamma}{mn}\mu_2 + \frac{n-\gamma}{mn}\mu_1\right)d\gamma \\ &= h^s\left(\frac{1}{2}\right)\mathbf{L}_3 + m\left(1-h\left(\frac{1}{2}\right)\right)^s \mathbf{L}_4. \end{aligned} \tag{31}$$

Rewriting the integrals, we find

$$\begin{aligned} \mathbf{L}_3 &= -\left(\frac{n}{\mu_2-\mu_1}\right)\int_{\mu_2}^{\frac{(n-1)\mu_2+\mu_1}{n}} w'\left(\frac{x-\mu_2}{\frac{\mu_1-\mu_2}{n}}\right) g(x)\,dx \\ &= \left(\frac{n}{\mu_2-\mu_1}\right)\int_{\frac{(n-1)\mu_2+\mu_1}{n}}^{\mu_2} w'\left(\frac{\mu_2-x}{\frac{\mu_2-\mu_1}{n}}\right) g(x)\,dx \\ &= \left(\frac{n}{\mu_2-\mu_1}\right)\mathbf{J}^{w}_{\left(\frac{(n-1)\mu_2+\mu_1}{n}\right)^+} g(\mu_2), \end{aligned} \tag{32}$$

$$\begin{aligned} \mathbf{L}_4 &= \left(\frac{m\,n}{\mu_2-\mu_1}\right)\int_{\frac{\mu_1}{m}}^{\frac{\mu_2+(n-1)\mu_1}{m\,n}} w'\left(\frac{y-\frac{\mu_1}{m}}{\frac{\mu_2-\mu_1}{m\,n}}\right) g(y)\,dy \\ &= \left(\frac{m\,n}{\mu_2-\mu_1}\right)\mathbf{J}^{w}_{\left(\frac{\mu_2+(n-1)\mu_1}{m\,n}\right)^-} g\left(\frac{\mu_1}{m}\right). \end{aligned} \tag{33}$$

From (**??**)–(**??**), it follows that

$$\begin{aligned} g\left(\frac{\mu_1+\mu_2}{2}\right)(w(1)-w(0)) \le \frac{n}{\mu_2-\mu_1}\Bigg[& h^s\left(\frac{1}{2}\right)\mathbf{J}^{w}_{\left(\frac{(n-1)\mu_2+\mu_1}{n}\right)^+} g(\mu_2) \\ & + m\left(1-h\left(\frac{1}{2}\right)\right)^s \mathbf{J}^{w}_{\left(\frac{\mu_2+(n-1)\mu_1}{m\,n}\right)^-} g\left(\frac{\mu_1}{m}\right)\Bigg]. \end{aligned} \tag{34}$$

Employing again the (h,m)-convexity of g, we get

$$\int_0^1 w'(\gamma)\, g\left(\frac{\gamma}{n}\mu_1 + \frac{n-\gamma}{n}\mu_2\right)d\gamma \le g(\mu_1)\mathbf{N}_1 + m\, g\left(\frac{\mu_2}{m}\right)\mathbf{N}_2, \tag{35}$$

$$\int_0^1 w'(\gamma)\, g\left(\frac{\gamma}{m\,n}\mu_2 + \frac{n-\gamma}{m\,n}\mu_1\right)d\gamma \le g\left(\frac{\mu_2}{m}\right)\mathbf{N}_3 + m\, g\left(\frac{\mu_1}{m^2}\right)\mathbf{N}_4. \tag{36}$$

By combining (**??**)–(**??**), we arrive at (**??**). □

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Remark 17. *Specializing to the case g convex, $w(z) = z$ and $n = 1$, it yields the celebrated Hermite–Hadamard's inequality.*

Remark 18. *Considering $n = 1$, we get a new result for (h, m)-convex functions modified of the second type.*

Remark 19. *If g is a convex function and $n = 1$, by setting $w'(z) = z^{\alpha-1}$ with $\alpha > 0$, we derive expression (2.1) of Theorem 2 (see [?]).*

Indeed, applying Theorem ??, we obtain

$$\frac{1}{\alpha}g\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{1}{2(\mu_2-\mu_1)}\left[\mathbf{J}^{w}_{\mu_1^+}g(\mu_2) + \mathbf{J}^{w}_{\mu_2^-}g(\mu_1)\right] \leq \frac{g(\mu_1)+g(\mu_2)}{2\alpha}.$$

According to Definition ??, we have

$$\begin{aligned}\frac{1}{\alpha}g\left(\frac{\mu_1+\mu_2}{2}\right) &\leq \frac{1}{2(\mu_2-\mu_1)^{\alpha}}\left[\int_{\mu_1}^{\mu_2}(z-\mu_1)^{\alpha-1}g(z)\,dz + \int_{\mu_1}^{\mu_2}(\mu_2-z)^{\alpha-1}g(z)\,dz\right]\\ &\leq \frac{g(\mu_1)+g(\mu_2)}{2\alpha}.\end{aligned}$$

Given that $\Gamma(\alpha)$ is well-defined for $\alpha > 0$, it follows that

$$\begin{aligned}g\left(\frac{\mu_1+\mu_2}{2}\right) &\leq \frac{\alpha\Gamma(\alpha)}{2(\mu_2-\mu_1)^{\alpha}}\left[\frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}(z-\mu_1)^{\alpha-1}g(z)\,dz + \frac{1}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}(\mu_2-z)^{\alpha-1}g(z)\,dz\right]\\ &\leq \frac{g(\mu_1)+g(\mu_2)}{2}.\end{aligned}$$

From Definition ??, we conclude, that

$$g\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(\mu_2-\mu_1)^{\alpha}}\left[{}^{RL}\mathbf{I}^{\alpha}_{\mu_2^-}g(\mu_1) + {}^{RL}\mathbf{I}^{\alpha}_{\mu_1^+}g(\mu_2)\right] \leq \frac{g(\mu_1)+g(\mu_2)}{2}.$$

Remark 20. *With $w(z) = \frac{z^{\alpha}}{\alpha}$, $m = s = 1$, $r = 2$ and $h(z) = z$, the previous result simplifies to Theorem 4 in [?].*

For s-convex functions, using $w(z) = \frac{z^{\alpha}}{\alpha}$ and $n = 1$, we recover Theorem 2.1 from [?]. Additionally, Theorem 3 in [?], for $w(z) = z^{\alpha}$, provides further results. In this work, Theorem 5 for m-convex functions is also established under similar conditions and can be easly derived.

Remark 21. *By assigning $n = m = s = 1$ and $h(z) = z$ in (??), which corresponds to working with convex functions and choosing $w'(z) = \frac{\alpha z^{\frac{\alpha}{k}-1}}{kB(\alpha)\Gamma_k(\alpha)}$, the left-hand side yields*

$$\begin{aligned}g\left(\frac{\mu_1+\mu_2}{2}\right)\frac{1}{B(\alpha)\Gamma_k(\alpha)} \leq \frac{1}{2(\mu_2-\mu_1)^{\frac{\alpha}{k}}}\Bigg[&\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\mu_1}^{\mu_2}\frac{g(z)}{(z-\mu_1)^{1-\frac{\alpha}{k}}}\,dz\\ &+\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\mu_1}^{\mu_2}\frac{g(z)}{(\mu_2-z)^{1-\frac{\alpha}{k}}}\,dz\Bigg],\end{aligned}$$

$$\begin{aligned}g\left(\frac{\mu_1+\mu_2}{2}\right)\frac{2(\mu_2-\mu_1)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)} \leq &\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\mu_1}^{\mu_2}\frac{g(z)}{(z-\mu_1)^{1-\frac{\alpha}{k}}}\,dz\\ &+\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\mu_1}^{\mu_2}\frac{g(z)}{(\mu_2-z)^{1-\frac{\alpha}{k}}}\,dz. \qquad (37)\end{aligned}$$

 doi:10.20944/preprints202509.0395.v1

Adding the term $\frac{1-\alpha}{B(\alpha)}(g(\mu_1)+g(\mu_2))$ *on both sides of* (**??**) *and considering that* $g\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{g(\mu_1)+g(\mu_2)}{2}$, *we obtain*

$$g\left(\frac{\mu_1+\mu_2}{2}\right)\left[\frac{2(\mu_2-\mu_1)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)}+\frac{(1-\alpha)}{B(\alpha)}\right] \leq {}^{AB}I^{\alpha}_{\mu_1+}g(\mu_2)+{}^{AB}I^{\alpha}_{\mu_2-}g(\mu_1). \tag{38}$$

A similar approach applied to the right-hand side of (**??**), *gives*

$$\begin{aligned}&\frac{1}{2(\mu_2-\mu_1)^{\frac{\alpha}{k}}}\left[\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\mu_1}^{\mu_2}\frac{g(z)}{(z-\mu_1)^{1-\frac{\alpha}{k}}}dz+\frac{\alpha}{kB(\alpha)\Gamma_k(\alpha)}\int_{\mu_1}^{\mu_2}\frac{g(z)}{(\mu_2-z)^{1-\frac{\alpha}{k}}}dz\right]\\ &\leq \frac{1}{B(\alpha)\Gamma_k(\alpha)}\left(\frac{g(\mu_1)+g(\mu_2)}{2}\right).\end{aligned}$$

Multiplying both sides by $2(\mu_2-\mu_1)^{\frac{\alpha}{k}}$, *adding* $\frac{(1-\alpha)}{B(\alpha)}(g(\mu_1)+g(\mu_2))$ *and rearranging terms, we arrive at*

$${}^{AB}\mathbf{I}^{\alpha}_{\mu_1+}g(\mu_2)+{}^{AB}\mathbf{I}^{\alpha}_{\mu_2-}g(\mu_1) \leq \left[\frac{2(\mu_2-\mu_1)^{\frac{\alpha}{k}}}{B(\alpha)\Gamma_k(\alpha)}+\frac{(1-\alpha)}{B(\alpha)}\right]\left(\frac{g(\mu_1)+g(\mu_2)}{2}\right). \tag{39}$$

By combining (**??**) *and* (**??**), *we obtain a relation that closely resembles Theorem* 6 *in* [**?**]. *Moreover, setting* $k=1$ *in this expression yields a result comparable to Proposition* 2.1 *in* [**?**].

Remark 22. *Theorem* 7 *of* [**?**] *can be established by taking* $m=s=1$, $n=2$ *and* $w'(z)=z^{\frac{\lambda}{k}-1}$.

Lemma 2. *Let* g, w, n, μ_1 *and* μ_2 *be defined as before. If* $g' \in L[\mu_1,\mu_2]$, *then*

$$\begin{aligned}&\int_0^1 w(\gamma)\left[g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)-g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right]d\gamma\\ &=\frac{n}{\mu_2-\mu_1}\left[w(0)(g(\mu_1)+g(\mu_2))-w(1)\left(\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)+g\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right)\right]\\ &\quad+\left(\frac{n}{\mu_2-\mu_1}\right)^2\left[\mathbf{J}^{w}_{\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)^+}g(\mu_2)+\mathbf{J}^{w}_{\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)^-}g(\mu_1)\right].\end{aligned} \tag{40}$$

Proof. Let

$$\int_0^1 w(\gamma)g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)d\gamma-\int_0^1 w(\gamma)g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)d\gamma=\mathcal{I}_3-\mathcal{I}_4 \tag{41}$$

By integrating $\mathcal{I}_3$ by parts and making a change in the variables $x=\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2$, we have after some computations

$$\mathcal{I}_3=\frac{n}{\mu_2-\mu_1}\left[w(0)g(\mu_2)-w(1)g\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)\right]+\left(\frac{n}{\mu_2-\mu_1}\right)^2\mathbf{J}^{w}_{\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)^+}g(\mu_2). \tag{42}$$

Analogously for $\mathcal{I}_4$, we get

$$\mathcal{I}_4=\frac{n}{\mu_2-\mu_1}\left[w(1)g\left(\frac{\gamma}{n}\mu_2+\frac{n-\gamma}{n}\mu_1\right)-w(0)g(\mu_2)\right]+\left(\frac{n}{\mu_2-\mu_1}\right)^2\mathbf{J}^{w}_{\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)^+}g(\mu_1). \tag{43}$$

 doi:10.20944/preprints202509.0395.v1

From (??) and (??), it follows (??). □

Remark 23. *By setting $n = 2$ and $w(z) = z^{\alpha}$ with $\alpha > 0$, the Lemma 3 is derived from* [?].

Remark 24. *Lemma 3.1 in [?] may be derived by setting $m = s = 1$, $n = 2$ and $w'(z) = z^{\frac{\lambda}{k}}$.*

Remark 25. *By adopting a strategy similar to that utilized in Lemma (??), we establish a comparable result concerning the midpoint of the interval.*

Lemma 3. *Let g be a real-valued function defined on a closed real interval $[\mu_1, \mu_2]$, differentiable on (μ_1, μ_2) and w' is an integrable function on $[\mu_1, \mu_2]$. If $g' \in L_1[\mu_1, \mu_2]$, then the following equality holds*

$$\begin{aligned}
&\frac{1}{2(r+1)}\left[2w(1)g\left(\frac{\mu_1+\mu_2}{2}\right) - w(0)\left[g\left(\frac{(r+2)\mu_1+r\mu_2}{2(r+1)}\right)+g\left(\frac{r\mu_1+(r+2)\mu_2}{2(r+1)}\right)\right]\right]\\
&-\frac{1}{\mu_2-\mu_1}\left[\mathbf{J}^{w}_{\left(\frac{\mu_1+\mu_2}{2}\right)^{-}}g\left(\frac{(r+2)\mu_1+r\mu_2}{2(r+1)}\right)+\mathbf{J}^{w}_{\left(\frac{\mu_1+\mu_2}{2}\right)^{+}}g\left(\frac{r\mu_1+(r+2)\mu_2}{2(r+1)}\right)\right]\\
&=\frac{\mu_2-\mu_1}{4(r+1)^2}\int_0^1 w(t)\left[g'\left(\frac{r+\gamma}{r+1}\frac{\mu_1+\mu_2}{2}+\frac{1-\gamma}{r+1}\mu_1\right)-g'\left(\frac{r+\gamma}{r+1}\frac{\mu_1+\mu_2}{2}+\frac{1-\gamma}{r+1}\mu_2\right)\right]d\gamma, \qquad (44)
\end{aligned}$$

for $r \in \mathbb{N} \cup \{0\}$.

Below we present some remarks that show the breadth and generality of (??).

Remark 26. *By setting $w(z) = z^{\alpha}$ and $r = 0$, we recover the Lemma 2.1 of [?]. A similar result can be obtained very easily, for the k-Riemann-Liouville Integral of [?].*

Remark 27. *Letting $w(z) = z$ and $r = 0$, we find a result new in the framework of Riemann integral*

$$\begin{aligned}
&g\left(\frac{\mu_1+\mu_2}{2}\right)-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} g(z)\,dz\\
&=\frac{\mu_2-\mu_1}{4}\int_0^1 w(\gamma)\left[g'\left(\gamma\frac{\mu_1+\mu_2}{2}+(1-\gamma)\mu_1\right)-g'\left(\gamma\frac{\mu_1+\mu_2}{2}+(1-\gamma)\mu_2\right)\right]d\gamma.
\end{aligned}$$

Remark 28. *Considering $w(z)$ to be a linear function, but different for $\mathbb{I}_1$ and $\mathbb{I}_2$, and $r = 0$, we get*

$$\mathbb{I}=\int_0^1(\gamma-\lambda_1)g'\left(\gamma\frac{\mu_1+\mu_2}{2}+(1-\gamma)\mu_1\right)d\gamma-\int_0^1(\gamma-\lambda_2)g'\left(\gamma\frac{\mu_1+\mu_2}{2}+(1-\gamma)\mu_2\right)d\gamma,$$

where

$$\lambda_1,\lambda_2\in\mathbb{R},$$

$$\mathbb{I}_1=\int_0^1 w(\gamma)g'\left(\frac{r+\gamma}{r+1}\frac{\mu_1+\mu_2}{2}+\frac{1-\gamma}{r+1}\mu_1\right)d\gamma,$$

$$\mathbb{I}_2=\int_0^1 w(\gamma)g'\left(\frac{r+\gamma}{r+1}\frac{\mu_1+\mu_2}{2}+\frac{1-\gamma}{r+1}\mu_2\right)d\gamma.$$

From here we obtain

$$\frac{2-\lambda_1-\lambda_2}{2}g\left(\frac{\mu_1+\mu_2}{2}\right)+\frac{\lambda g(\mu_1)+\mu g(\mu_2)}{2}-\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} g(u)\,du=\frac{\mu_2-\mu_1}{4}\mathbb{I}.$$

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Given that

$$\int_0^1 (\gamma-\lambda_1)g'\left(\gamma\frac{\mu_1+\mu_2}{2}+(1-\gamma)\mu_1\right)d\gamma=\int_0^1(1-\gamma-\lambda_1)g'\left(\gamma\mu_1+(1-\gamma)\frac{\mu_1+\mu_2}{2}\right)d\gamma,$$

we retrieve Lemma 2.1 *of [***?** *].*

Remark 29. *Readers will have no difficulty in proving, in a similar manner, the following result:*

Lemma 4. *Let g be a real function defined on some closed real interval $[\mu_1,\mu_2]$, differentiable on (μ_1,μ_2) and w' is an integrable function on $[\mu_1,\mu_2]$. If $g'\in L_1[\mu_1,\mu_2]$, then we find the following equality*

$$\begin{aligned}&\frac{1}{n+1}\left[-w(1)\frac{g(\mu_1)+g(\mu_2)}{2}+w(0)\frac{g\left(\frac{n\mu_1+\frac{\mu_1+\mu_2}{2}}{n+1}\right)+g\left(\frac{n\mu_2+\frac{\mu_1+\mu_2}{2}}{n+1}\right)}{2}\right]\\&\quad+\frac{1}{\mu_2-\mu_1}\left[\mathbf{J}^w_{\mu_1+}g\left(\frac{n\mu_1+\frac{\mu_1+\mu_2}{2}}{n+1}\right)+\mathbf{J}^w_{\mu_2-}g\left(\frac{n\mu_2+\frac{\mu_1+\mu_2}{2}}{n+1}\right)\right]\\&=\frac{\mu_2-\mu_1}{4(n+1)^2}\int_0^1 w(\gamma)\left[g'\left(\frac{n+\gamma}{n+1}\frac{\mu_1+\mu_2}{2}+\frac{1-\gamma}{n+1}\mu_1\right)-g'\left(\frac{n+\gamma}{n+1}\frac{\mu_1+\mu_2}{2}+\frac{1-\gamma}{n+1}\mu_2\right)\right]d\gamma,\end{aligned}$$

for $n\in\mathbb{N}$.

Result that completes Lemma 2.1 *of [***?** *]. Of course, Remarks, similar to those presented above, can be derived.*

Theorem 5. *Let g, w, n, μ_1 and μ_2 be defined as before. If $|g'|$ is (h,m)-convex modified of the second type, then it is true that*

$$|\mathbf{L}|\le\left(|g'(\mu_1)|+|g'(\mu_2)|\right)\mathbf{W}_3+m\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|+\left|g'\left(\frac{\mu_2}{m}\right)\right|\right)\mathbf{W}_4, \tag{45}$$

where $\mathbf{L}$ *is the left-hand side of* (**??**), $\mathbf{W}_3=\int_0^1 w(\gamma)h^s\left(\frac{\gamma}{n}\right)d\gamma$ *and* $\mathbf{W}_4=\int_0^1 w(\gamma)\left(1-h\left(\frac{\gamma}{n}\right)\right)^s d\gamma$.

Proof. From Lemma **??**, by employing the properties of the modulus, we obtain

$$|\mathbf{L}|\le\int_0^1|w(\gamma)|\left[\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)\right|+\left|g'\left(\frac{n-\gamma}{n}\mu_1+\frac{\gamma}{n}\mu_2\right)\right|\right]d\gamma.$$

Utilizing the convexity property of $|g'|$, we get

$$\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)\right|\le h^s\left(\frac{\gamma}{n}\right)|g'(\mu_1)|+m\left(1-h\left(\frac{\gamma}{n}\right)\right)^s\left|g'\left(\frac{\mu_2}{m}\right)\right|,$$

and

$$\left|g'\left(\frac{n-\gamma}{n}\mu_1+\frac{\gamma}{n}\mu_2\right)\right|\le m\left(1-h\left(\frac{\gamma}{n}\right)\right)^s\left|g'\left(\frac{\mu_1}{m}\right)\right|+h^s\left(\frac{\gamma}{n}\right)|g'(\mu_2)|.$$

Summing the last two inequalities, we have

$$\begin{aligned}&\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{n-\gamma}{n}\mu_2\right)\right|+\left|g'\left(\frac{n-\gamma}{n}\mu_1+\frac{\gamma}{n}\mu_2\right)\right|\\&\le h^s\left(\frac{\gamma}{n}\right)\left(|g'(\mu_1)|+|g'(\mu_2)|\right)+m\left(1-h\left(\frac{\gamma}{n}\right)\right)^s\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|+\left|g'\left(\frac{\mu_2}{m}\right)\right|\right).\end{aligned}$$

Taking into account the accepted notations, we obtain (**??**). The proof is completed. □

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Remark 30. *If we consider the usual class of convex functions and $n = 2$, then from Theorem* **??**, *we obtain*

$$|\mathbf{L}| \leq \left(|g'(\mu_1)| + |g'(\mu_2)|\right) \int_0^1 w(\gamma)\, d\gamma.$$

*Here, if we take $w(z) = z$, then we get Theorem 2.2 from [**?**] and Theorem 5 from [**?**]. If we choose $w(z) = (1-z)$, then we have Theorem 2.2 in [**?**], and if $w(z) = z^{\alpha}$, then we get the inequality from [Remark of Theorem 1, for $w(z) = (1-z)^{\alpha}$] [**?**].*

Remark 31. *By adopting a strategy similar to that utilized in Theorem* (**??**) *and by employing Lemma* (**??**), *we establish a comparable result concerning the midpoint of the interval.*

Theorem 6. *Let $g : [\mu_1, \mu_2] \to \mathbb{R}$ be a differentiable function on (μ_1, μ_2), such that $g' \in L_1[\mu_1, \mu_2]$. If $|g'|$ is (h, m)-convex modified of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in \mathrm{Dom}(|g'|)$, then the following inequality holds:*

$$|\mathcal{L}(w, g, \mu_1, \mu_2, n)| \leq 2\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|\mathcal{H}_1 + m\left(\left|g'\left(\frac{\mu_1}{m}\right)\right| + \left|g'\left(\frac{\mu_2}{m}\right)\right|\right)\mathcal{H}_2,$$

where

$$\begin{aligned}
&\mathcal{L}(w, g, \mu_1, \mu_2, r) = \\
= \;& \frac{1}{2(r+1)}\left\{2w(1)g\left(\frac{\mu_1+\mu_2}{2}\right) - w(0)\left[g\left(\frac{(r+2)\mu_1 + r\mu_2}{2(r+1)}\right) + g\left(\frac{r\mu_1 + (r+2)\mu_2}{2(r+1)}\right)\right]\right\} \\
- \;& \frac{1}{\mu_2 - \mu_1}\left[J^{w}_{\left(\frac{\mu_1+\mu_2}{2}\right)^{-}} g\left(\frac{(r+2)\mu_1 + r\mu_2}{2(r+1)}\right) + J^{w}_{\left(\frac{\mu_1+\mu_2}{2}\right)^{+}} g\left(\frac{r\mu_1 + (r+2)\mu_2}{2(r+1)}\right)\right],
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_1 &= \int_0^1 w(\gamma) h^s\left(\frac{r+\gamma}{r+1}\right) d\gamma, \\
\mathcal{H}_2 &= \int_0^1 w(\gamma)\left[1 - h\left(\frac{r+\gamma}{r+1}\right)\right]^s d\gamma.
\end{aligned}$$

Corollary 1. *Under the assumptions of Theorem* **??**,

1. *If we choose $m = 1$, then we derive the following inequality*

$$|\mathcal{L}(w, g, \mu_1, \mu_2, r)| \leq 2\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|\mathcal{H}_1 + \left(|g'(\mu_1)| + |g'(\mu_2)|\right)\mathcal{H}_2,$$

   $\mathcal{H}_1$ and $\mathcal{H}_2$ are as before.
2. *If $s = m = 1$, then*

$$|\mathcal{L}(w, g, \mu_1, \mu_2, r)| \leq 2\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|\mathcal{H}_3(\gamma) + \left(|g'(\mu_1)| + |g'(\mu_2)|\right)\mathcal{H}_4(\gamma),$$

   where

$$\mathcal{H}_3 = \int_0^1 w(\gamma) h\left(\frac{r+\gamma}{r+1}\right) d\gamma, \quad \mathcal{H}_4 = \int_0^1 w(\gamma)\left[1 - h\left(\frac{r+\gamma}{r+1}\right)\right] d\gamma.$$

3. *If we take $w(z) = z$, $r = 0$, and $s = m = 1$, we get the following inequality, new for the Riemann Integral*

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$$\left|g\left(\frac{\mu_1+\mu_2}{2}\right)-\int_{\mu_1}^{\mu_2} g(z)\,dz\right| \le 2\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right| \int_0^1 \gamma\, h(\gamma)\,d\gamma + \left(\left|g'(\mu_1)\right|+\left|g'(\mu_2)\right|\right)\int_0^1 \gamma(1-h(\gamma))d\gamma.$$

4. *Putting* $w'(z)=\dfrac{z^{\alpha}}{\Gamma(\alpha+1)}$, $r=0$, *readers will have no difficulty in obtaining a new inequality for the Riemann–Liouville integral.*

Remark 32. *The generality of this result can be easily verified since, for different notions of convexity contained in our Definition* **??**, *with different values of r and for different kernels w′, new results can be derived under those conditions from Theorem* **??**.

Theorem 7. *Let* g, g', w, μ_1, μ_2 *and n as in the Theorem* (**??**). *Suppose that* $|g'|^q$ *is* (h,m)*-convex modified of the second type and* $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in Dom(|g'|^q)$, *then the inequality below is satisfied*

$$|\mathbf{U}| \le \frac{\mu_2-\mu_1}{n}\,\mathbf{W}_6\left[\left(|g'(\mu_1)|^q\,\mathbf{H}_1+m\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\mathbf{H}_2\right)^{\frac{1}{q}}+\left(|g'(\mu_2)|^q\,\mathbf{H}_1+m\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\mathbf{H}_2\right)^{\frac{1}{q}}\right],$$

where $p,q>1$, $\mathbf{U}$ *is the right-hand side of equation* (**??**), $\mathbf{W}_5=\left(\int_0^1 w^p(\gamma)\,d\gamma\right)^{\frac{1}{p}}$, $\mathbf{H}_1=\int_0^1 h^s\left(\frac{\gamma}{n}\right)d\gamma$ *and* $\mathbf{H}_2=\int_0^1\left(1-h\left(\frac{\gamma}{n}\right)\right)^s d\gamma$.

Proof. By adapting the approach used in the Theorem (**??**) but by employing Hölder's inequality instead, we arrive at

$$\begin{aligned}
|\mathbf{U}| &\le \frac{\mu_2-\mu_1}{n}\left[\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)\right|d\gamma+\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right|d\gamma\right]\\
&\le \frac{\mu_2-\mu_1}{n}\left(\int_0^1 w^p(\gamma)\,d\gamma\right)^{\frac{1}{p}}\left[\left(\int_0^1\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1\left|g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right]\\
&\le \frac{\mu_2-\mu_1}{n}\,\mathbf{W}_5\left[\left(|g'(\mu_1)|^q\,\mathbf{H}_1+m\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\mathbf{H}_2\right)^{\frac{1}{q}}+\left(|g'(\mu_2)|^q\,\mathbf{H}_1+m\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\mathbf{H}_2\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Therefore, the desired result has been established. □

Remark 33. *If* $w(z)=z^{\alpha}$ *and g is convex, we obtain the inequality to the Theorem 6 presented in [***?** *]*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\mu_2-\mu_1)^{\alpha}}\left[{}^{RL}\mathbf{I}^{\alpha}_{\left(\frac{\mu_1+\mu_2}{2}\right)^-}g(\mu_1)+{}^{RL}\mathbf{I}^{\alpha}_{\left(\frac{\mu_1+\mu_2}{2}\right)^+}g(\mu_2)\right]-g\left(\frac{\mu_1+\mu_2}{2}\right)\right|\\
&\le \frac{\mu_2-\mu_1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|g'(\mu_1)|^q}{4}+\frac{3|g'(\mu_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|g'(\mu_2)|^q}{4}+\frac{3|g'(\mu_1)|^q}{4}\right)^{\frac{1}{q}}\right]\\
&\le \frac{\mu_2-\mu_1}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left[|g'(\mu_1)|+|g'(\mu_2)|\right].
\end{aligned}$$

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Remark 34. *If $w(z) = 4z$, $q = \frac{p-1}{p}$ and g is convex, we get an inequality similar to the Theorem 2.3 presented in [?]*

$$\left|\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} g(z)\,dz - g\left(\frac{\mu_1+\mu_2}{2}\right)\right| \leq \frac{\mu_2-\mu_1}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|g'(\mu_1)|^{\frac{p-1}{p}}}{4}+\frac{3|g'(\mu_2)|^{\frac{p-1}{p}}}{4}\right)^{\frac{p}{p-1}} + \left(\frac{|g'(\mu_2)|^{\frac{p-1}{p}}}{4}+\frac{3|g'(\mu_1)|^{\frac{p-1}{p}}}{4}\right)^{\frac{p}{p-1}}\right].$$

Remark 35. *Utilizing a procedure parallel to that applied in Theorem (??) and invoking Lemma (??), we obtain an equivalent statement pertaining to the midpoint of the interval:*

Theorem 8. *Let $g : [\mu_1, \mu_2] \to \mathbb{R}$ be differentiable function on (μ_1, μ_2) such that $g' \in L_1[\mu_1, \mu_2]$. If $|g'|^q$ is (h, m)-convex modified of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in \mathrm{Dom}(|g'|^q)$ then it's true that:*

$$\begin{aligned}&|\mathcal{L}(w, g, \mu_1, \mu_2, r)| \\ &\leq \frac{\mu_2-\mu_1}{4(r+1)^2}\mathbf{W}_5\left\{\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_5 + m\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\right)\mathcal{H}_6\right]^{\frac{1}{q}}\right. \\ &\left.+\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_5 + m\left(\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\right)\mathcal{H}_6\right]^{\frac{1}{q}}\right\}.\end{aligned} \tag{46}$$

with $\frac{1}{p}+\frac{1}{q}=1$, $\mathcal{H}_5 = \int_0^1 h^s(\frac{r+t}{r+1})dt$, $\mathcal{H}_6 = \int_0^1 (1-h(\frac{r+t}{r+1}))^s dt$ and $\mathbf{W}_5$ defined as before.

Corollary 2. *Under the assumptions of Theorem **??**,*

1. *Choosing $m = 1$, then we obtain the following inequality*

$$\begin{aligned}&|\mathcal{L}(w, g, \mu_1, \mu_2, r)| \\ &\leq \frac{\mu_2-\mu_1}{4(r+1)^2}\left(\int_0^1 w^p(t)dt\right)^{\frac{1}{p}}\left\{\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_5 + \left(|g'(\mu_1)|^q\right)\mathcal{H}_6\right]^{\frac{1}{q}}\right. \\ &\left.+\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_5 + \left(|g'(\mu_2)|^q\right)\mathcal{H}_6\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

2. *If $s = m = 1$ then*

$$\begin{aligned}|\mathcal{L}(w, g, \mu_1, \mu_2, r)| &\leq \frac{\mu_2-\mu_1}{4(r+1)^2}\left(\int_0^1 w^p(t)dt\right)^{\frac{1}{p}} \\ &\times\left\{\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\int_0^1 h\left(\frac{r+t}{r+1}\right)dt + |g'(\mu_1)|^q\int_0^1\left(1-h\left(\frac{r+t}{r+1}\right)\right)dt\right]^{\frac{1}{q}}\right. \\ &\left.+\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\int_0^1 h\left(\frac{r+t}{r+1}\right)dt + |g'(\mu_2)|^q\int_0^1\left(1-h\left(\frac{r+t}{r+1}\right)\right)dt\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

3. *Bearing in mind Corollary **??**, items 3 and 4, we can derive new inequalities for Riemann and Riemann-Liouville integrals, respectively.*

Theorem 9. *Let $g, g', w, \mu_1, \mu_2, n, \mathbf{W}_3$ and $\mathbf{W}_4$ be as in the Lemma (??). Suppose that $|g'|^q$ is (h, m)-convex modified of the second type and $\frac{\mu_1}{m}, \frac{\mu_2}{m} \in Dom(|g'|^q)$ then the following result emerges*

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$$|\mathbf{U}| \leq \frac{\mu_2-\mu_1}{n}\mathbf{W}_6\left[\left(|g'(\mu_1)|^q\mathbf{W}_3+m\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\mathbf{W}_4\right)^{\frac{1}{q}}+\left(|g'(\mu_2)|^q\mathbf{W}_3+m\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\mathbf{W}_4\right)^{\frac{1}{q}}\right],$$

where $q \geq 1$, $\mathbf{W}_6 = \left(\int_0^1 w(\gamma)\,d\gamma\right)^{1-\frac{1}{q}}$ *and* $\mathbf{U}$ *is defined as before.*

Proof. Employing Lemma (**??**), triangle inequality, Power mean inequality and the definition (**??**) for $|g|^q$, we obtain

$$\begin{aligned}
|\mathbf{U}| &\leq \left|\frac{\mu_2-\mu_1}{n}\int_0^1 w(\gamma)\left[g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)-g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right]d\gamma\right| \\
&\leq \frac{\mu_2-\mu_1}{n}\left[\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)\right|d\gamma+\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right|d\gamma\right] \\
&\leq \frac{\mu_2-\mu_1}{n}\left(\int_0^1 w(\gamma)\,d\gamma\right)^{1-\frac{1}{q}}\left[\left(\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right. \\
&\quad \left.+\left(\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right] \\
&\leq \frac{\mu_2-\mu_1}{n}\mathbf{W}_6\left[\left(|g'(\mu_1)|^q\mathbf{W}_3+m\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\mathbf{W}_4\right)^{\frac{1}{q}}+\left(|g'(\mu_2)|^q\mathbf{W}_3+m\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\mathbf{W}_4\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Hence, the proof is finished. □

Remark 36. *Theorem* 8 *in [***?** *] follows as a consequence when the parameters* m, s, n *and the function* w' *are selected as in Remark* (**??**).

Remark 37. *In light of Theorem* (**??**) *and Lemma* (**??**), *we similarly obtain a result for the midpoint of the interval:*

Theorem 10. *Let* $g:[\mu_1,\mu_2]\to\mathbb{R}$ *be a differentiable function on* (μ_1,μ_2), *such that* $g'\in L_1[\mu_1,\mu_2]$. *If* $|g'|^q$ *is* (h,m)*-convex modified of the second type with* $q\geq 1$ *and* $\frac{\mu_1}{m},\frac{\mu_2}{m}\in \mathrm{Dom}(|g'|^q)$, *then it is true that*

$$\begin{aligned}
|\mathcal{L}(w,g,\mu_1,\mu_2,r)| &\leq \frac{\mu_2-\mu_1}{4(r+1)^2}\left(\int_0^1 w(t)\,dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left\{\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_1+m\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\right)\mathcal{H}_2\right]^{\frac{1}{q}}\right. \\
&\quad\left.+\left[\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_1+m\left(\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\right)\mathcal{H}_2\right]^{\frac{1}{q}}\right\},
\end{aligned} \tag{47}$$

where $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *are defined above in Theorem* (**??**).

Theorem 11. *Let* g, g', w, μ_1, μ_2, n, p, q, $\mathbf{U}$, $\mathbf{H}_1$ *and* $\mathbf{H}_2$ *be as defined in the preceding result. Suppose that* $|g'|^q$ *is* (h,m)*-convex modified of the second type and* $\frac{\mu_1}{m},\frac{\mu_2}{m}\in Dom(|g'|^q)$, *then it is true that*

 doi:10.20944/preprints202509.0395.v1

$$|\mathbf{U}| \leq \frac{\mu_2-\mu_1}{n}\left[\mathbf{W}_7+\left(|g'(\mu_1)|^q+|g'(\mu_2)|^q\right)\frac{\mathbf{H}_1}{q}+m\left(\left|g'\left(\frac{\mu_2}{m}\right)\right|^q+\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\right)\frac{\mathbf{H}_2}{q}\right], \tag{48}$$

where $\mathbf{W}_7 = 2\int_0^1 \frac{w(\gamma)}{p}\,d\gamma$.

Proof. Following a similar line of reasoning as in Theorem (**??**) but replacing the key inequality with that of Young, we get

$$\begin{aligned}
|\mathbf{U}| &\leq \frac{\mu_2-\mu_1}{n}\left[\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)\right|d\gamma+\int_0^1 w(\gamma)\left|g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right|d\gamma\right]\\
&\leq \frac{\mu_2-\mu_1}{n}\left[2\int_0^1 \frac{w^p(\gamma)}{p}\,d\gamma+\int_0^1 \frac{\left|g'\left(\frac{\gamma}{n}\mu_1+\frac{(n-\gamma)}{n}\mu_2\right)\right|^q}{q}\,d\gamma\right.\\
&\quad\left.+\int_0^1 \frac{\left|g'\left(\frac{\gamma}{n}\mu_2+\frac{(n-\gamma)}{n}\mu_1\right)\right|^q}{q}\,d\gamma\right]\\
&\leq \frac{\mu_2-\mu_1}{n}\left[\mathbf{W}_7+\left(|g'(\mu_1)|^q+|g'(\mu_2)|^q\right)\frac{\mathbf{H}_1}{q}+m\left(\left|g'\left(\frac{\mu_2}{m}\right)\right|^q+\left|g'\left(\frac{\mu_1}{m}\right)\right|^q\right)\frac{\mathbf{H}_2}{q}\right].
\end{aligned}$$

This concludes the proof. □

Remark 38. *If we consider the usual class of convex functions and* $n = 2$, *then from* (**??**) *we obtain*

$$|\mathbf{U}| \leq \frac{2(\mathbf{W}_5)^p}{p}+\frac{|g'(\mu_1)|^q+|g'(\mu_2)|^q}{q}.$$

Here, if we take $w(z) = z$, *then we get*

$$\left|\int_{\mu_1}^{\mu_2} g(z)\,dz - g\left(\frac{\mu_1+\mu_2}{2}\right)\right| \leq \frac{2}{p(p+1)}+\frac{|g'(\mu_1)|^q+|g'(\mu_2)|^q}{q}.$$

Remark 39. *By building upon the method employed in Theorem* (**??**) *and drawing on Lemma* (**??**), *we derive a parallel result concerning the midpoint of the interval.*

Theorem 12. *Let* $g:[\mu_1,\mu_2]\to\mathbb{R}$ *be a differentiable function on* (μ_1,μ_2) *such that* $g'\in L_1[\mu_1,\mu_2]$. *If* $|g'|^q$ *is* (h,m)*-convex modified of the second type with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $\frac{\mu_1}{m},\frac{\mu_2}{m}\in \mathrm{Dom}(|g'|^q)$, *then*

$$\begin{aligned}
|\mathcal{L}(w,g,\mu_1,\mu_2,r)| \leq \frac{\mu_2-\mu_1}{4(r+1)^2}\Bigg\{&\frac{2}{p}\int_0^1 w^p(t)\,dt+\frac{2}{q}\left|g'\left(\frac{\mu_1+\mu_2}{2}\right)\right|^q\mathcal{H}_5\\
&+\frac{m}{q}\left(\left|g'\left(\frac{\mu_1}{m}\right)\right|^q+\left|g'\left(\frac{\mu_2}{m}\right)\right|^q\right)\mathcal{H}_6\Bigg\},
\end{aligned} \tag{49}$$

holds, where $\mathcal{H}_5$ *and* $\mathcal{H}_6$ *are defined above in Theorem* **??**.

Remark 40. *The Remark* **??** *remains valid in these results.*

Remark 41. *Readers will have no difficulty in formulating the corresponding corollaries to Theorems* (**??**) *and* (**??**).

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3. Conclusions

This work focuses on the generalization and extension of existing results related to integral inequalities. The main results and contributions are Theorem 7, Theorem 17, Theorem 19, Theorem 21, and Theorem 25, which establish new inequalities for (h, m)-convex functions of second type using weighted integral operators. It also provides remarks showing how these new results generalize or connect with existing theorems in the literature by establishing specific parameters for s, m, h, and the weighting function w'.

In essence, we consider this work to contribute significantly to the theory of convex functions by providing a more generalized and flexible framework for Hermite-Hadamard-type inequalities through the introduction of weighted integrals and refined classes of (h, m)-convex functions.

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1. B. Ahmad, A. Alsaedi, M. Kirane, B. T. Torebek, Hermite-Hadamard, Hermite-Hadamard-Fejér, Dragomir-Agarwal and Pachpatte type inequalities for convex functions via new fractional integrals, Journal of Computational and Applied Mathematics 353 (2019), 120-129. https://doi.org/10.1016/j.cam.2018.12.015
2. A. Akkurt, M. E. Yildirim, H. Yildirim, On some integral inequalities for (k, h)-RiemannLiouville fractional integral, NTMSCI 4, No. 1, 138-146 (2016). http://dx.doi.org/10.20852/ntmsci.2016217824
3. M. A. Ali, J. E. Nápoles V., A. Kashuri, Z. Zhang, Fractional non conformable Hermite-Hadamard inequalities for generalized-convex functions, Fasciculi Mathematici, Nr 64 2020, 5-16. https://doi.org/10.21008/j.0044-4413.2020.0007
4. M. E. Amlashi, M. Hassani, More on the Hermite-Hadamard inequality, Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 2153-2159. http://dx.doi.org/10.22075/ijnaa.2021.21753.2293
5. M. A. Ardıc, A. O. Akdemir, H. K. Onalan, Integral inequalities for differentiable s-convex functions in the second sense via Atangana-Baleanu fractional integral operators, Filomat 37:18 (2023), 6229-6244. https://doi.org/10.2298/FIL2318229A
6. T. Abdeljawad, D. Baleanu, Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler non-singular kernel, J. Nonlinear Sci. Appl., 10 (2017), 1098-1107. http://dx.doi.org/10.22436/jnsa.010.03.20
7. A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: theory and application to Heat transfer model. Therm. Sci. 20, 763–769 (2016). https://doi.org/10.48550/arXiv.1602.03408
8. B. Bayraktar, J. E. Nápoles V., A note on Hermite-Hadamard integral inequality for (h, m)-convex modified functions in a generalized framework, submitted.
9. Bayraktar, B., and Nápoles Valdés, J. E., *New generalized integral inequalities via (h, m)-convex modified functions*, Izv. Inst. Mat. Inform., **60**(2022), no. 1, 3–15. https://doi.org/10.35634/2226-3594-2022-60-01
10. Bayraktar, B., and Nápoles Valdés, J. E., *Integral inequalities for mappings whose derivatives are (h, m, s)-convex modified of second type via Katugampola integrals*, Univ. Craiova Ser. Mat. Inform., **49**(2022), 371–383.
11. B. Bayraktar, Juan E. Nápoles, Rabossi Florencia (2022). *On Generalizations of Integral Inequalities.* Problemy Analiza - Issues of Analysis (2), 3-23. https://doi.org0.15393/j3.art.2022.11190
12. S. Bermudo, P. Kórus, J. E. Nápoles V., On q-Hermite-Hadamard inequalities for general convex functions, Acta Math. Hungar. 162, 364-374 (2020). https://doi.org/10.1007/s10474-020-01025-6
13. M. Bilal, M. Imtiaz, A. R. Khan, I. U. Khan, M. Zafran, Generalized Hermite-Hadamard inequalities for s-convex functions in the mixed kind, submitted.
14. W. W. Breckner, Stetigkeitsaussagen fur eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen, Publ. Inst. Math., 23 (1978), 13-20.
15. A. M. Bruckner, E. Ostrow, Some function classes related to the class of convex functions, Pacific J. Math. 12 (1962), 1203-1215.
16. M. Caputo, Linear Models of Dissipation whose Q is almost Frequency Independent-II, Geophys. J. R. Astr. Soc. 13 (1967), pp. 529-539.

17. M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progress in Fractional Differentiation & Applications, **1(2)**, 73-85, (2015).
18. M. Caputo and M. Fabrizio, *On the notion of fractional derivative and applications to the hysteresis phenomena*, Meccanica, **52**, 3043-3052, (2017).
19. S. S. Dragomir, R. P. Agarwal, Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula, Appl. Math. Lett. Vol. 11, No. 5, pp. 91-95, 1998.
20. S. S. Dragomir, S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense, Demonstratio Mathematica, Vol. 32, No. 4 (1999), 687-696.
21. S. S. Dragomir, J. Pecaric, L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math., 21 (1995), 335-241.
22. G. Farid, A. Javed, A. U. Rehman, M. I. Qureshi, On Hadamard-type inequalities for differentiable functions via Caputo k-fractional derivatives, Cogent Mathematics (2017), 4: 1355429. https://doi.org/10.1080/23311835.2017.1355429.
23. G. Farid, S. Naqvi, A. U. Rehman, A version of the Hadamard inequality for Caputo fractional derivatives and related results, RGMIA Research Report Collection, 2017, 11 pp, 20, Article 59.
24. A. E. Farissi, Simple proof and refinement of Hermite-Hadamard inequality, Journal of Mathematical Inequalities, Volume 4, Number 3 (2010), 365-369.
25. A. Fernandez, P. Mohammed, Hermite–Hadamard inequalities in fractional calculus defined using Mittag–Leffler kernels, Math. Meth. Appl. Sci. 2021, 44, 8414–8431. .
26. M. Gürbüz, A. O. Akdemir, S. Rashid, E. Set, Hermite-Hadamard inequality for fractional integrals of Caputo-Fabrizio type and related inequalities, Journal of Inequalities and Applications, 2020(1), 1-10, (2020).
27. P. M. Guzmán, J. E. Nápoles V., Y. Gasimov, Integral inequalities within the framework of generalized fractional integrals, Fractional Differential Calculus, Volume 11, Number 1 (2021), 69-84. https://doi.org/10.7153/fdc-2021-11-05
28. J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann, J. Math. Pures App. 9, 171-216 (1893).
29. C. Hermite, Sur deux limites d'une intégrale définie, Mathesis3, 82 (1883).
30. H. Hudzik, L. Maligranda, Some remarks on s-convex functions, Aequationes Math. 48 (1994), no. 1, 100-111.
31. F. Jarad, T. Abdeljawad, T. Shah, On the weighted fractional operators of a function with respect to another function, Fractals, Vol. 28, No. 8 (2020) 2040011 (12 pages). https://doi.org/10.1142/S0218348X20400113
32. F. Jarad, E. Ugurlu, T. Abdeljawad, D. Baleanu, On a new class of fractional operators, Adv. Differ. Equ. 2017, 2017, 247. https://doi.org/10.1186/s13662-017-1306-z
33. S. M. Kang, G. Farid, W. Nazeer, S. Naqvi, A version of the Hadamard inequality for Caputo fractional derivatives and related results. Journal of Computational Analysis & Applications, 27(6), 2019, 962-972. https://www.doi.org/10.30538/oms2021.0139
34. S. Kermausuor, E. R. Nwaeze, New Fractional Integral Inequalities via k-Atangana–Baleanu Fractional Integral Operators, Fractal and Fractional, 2023; 7(10):740. https://doi.org/10.3390/fractalfract7100740
35. T. U. Khan, M. A. Khan, Generalized conformable fractional integral operators, J. Comput. Appl. Math. 2019, 346, 378-389. http://doi.org/10.1016/j.cam.2018.07.018.
36. Khan, A. R., Khan, I. U., and Muhammad, S., *Hermite-Hadamard type fractional integral inequalities for s-convex functions of mixed kind*, Trans. Math. Comput. Sci., **1**(2021), no. 1, 25–37.
37. U. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comput., vol. 147, no. 5, pp. 137–146, 2004. http://doi.org/10.1016/S0096-3003(02)00657-4.
38. D. S. Marinescu, M. Monea, A Very Short Proof of the Hermite–Hadamard Inequalities, The American Mathematical Monthly, 127:9, 2020, 850-851. https://doi.org/10.1080/00029890.2020.1803648
39. M. Matloka, Hermite-Hadamard type inequalities for fractional integrals, RGMIA Res. Rep. Coll. 20 (2017), Art. 69. 11 pp. https://doi.org/10.1016/j.mcm.2011.12.048.
40. S. Mehmood, J. E. Nápoles Valdés, N. Fatima, B. Shahid, Some New Inequalities Using Conformable Fractional Integral of Order β, Journal of Mathematical Extension Vol. 15, SI-NTFCA, (2021) (33)1-22. https://doi.org/10.30495/JME.SI.2021.2184.
41. V. G. Mihesan, A generalization of the convexity, Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca (Romania) (1993).

42. L. N. Mishra, Q. U. Ain, G. Farid, A. U. Rehman, *k*-fractional integral inequalities for (h,m)-convex functions via Caputo k-fractional derivatives, Korean J. Math. 27 (2019), No. 2, pp. 357-374. https://doi.org/10.11568/kjm.2019.27.2.357.
43. S. Mubeen, G. M. Habibullah, *k*-fractional integrals and applications, Int. J. Contemp. Math. Sci. 7, 89-94 (2012).
44. J. E. Nápoles Valdés, On the Hermite-Hadamard type inequalities involving generalized integrals, Contrib. Math. 5 (2022) 45-51. https://doi.org/10.47443/cm.2022.020.
45. J. E. Nápoles Valdés, A Review of Hermite-Hadamard Inequality, Partners Universal International Research Journal (PUIRJ), Volume: 01 Issue: 04 | October-December 2022, 98-101. https://doi.org/10.5281/zenodo.7492608.
46. J. E. Nápoles Valdés, F. Rabossi, A. D. Samaniego, Convex functions: Ariadne's thread or Charlotte's spiderweb?, Advanced Mathematical Models & Applications Vol.5, No.2, 2020, pp.176-191.
47. J. E. Nápoles Valdés, J. M. Rodríguez, J. M. Sigarreta, On Hermite-Hadamard type inequalities for non-conformable integral operators, *Symmetry* **2019**, *11*, 1108.
48. J. Nasir, S. Qaisar, A. Qayyum, H. Budak, Newresults on Hermite–Hadamard type inequalities via Caputo-Fabrizio fractional integral for *s*-convex function, Filomat 37:15 (2023), 4943-4957 https://doi.org/10.2298/FIL2315943N.
49. M. A. Noor, K. I. Noor, M. U. Awan, GENERALIZED FRACTIONAL HERMITE-HADAMARD INEQUALITIES, Miskolc Mathematical Notes Vol. 21 (2020), No. 2, pp. 1001–1011. https://doi.org/10.18514/MMN.2020.1143.
50. M. E. Özdemit, A. O. Akdemri, E. Set, On (h,m)-convexity and Hadamard-type inequalities, Transylv. J. Math. Mech. 8(1), 51-58 (2016).
51. J. Park, Generalization of Ostrowski-type inequalities for differentiable real (s,m)-convex mappings, Far East J. Math. Sci., 49 (2011), 157-171.
52. M. Z. Sarikaya, F. Ertugral, On the generalized Hermite-Hadamard inequalities, Annals of the University of Craiova, Mathematics and Computer Science Series Volume 47(1), 2020, Pages 193-213.
53. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Basak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling 57 (2013) 2403–2407. http://doi.org/10.1016/j.mcm.2011.12.048.
54. M. Z. Sarikaya, H. Yildirim, ON HERMITE-HADAMARD TYPE INEQUALITIES FOR RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS, Miskolc Mathematical Notes Vol. 17 (2017), No. 2, pp. 1049–1059. https://doi.org/10.18514/MMN.2017.1197
55. E. Set, A. O. Akdemir, M. E. Özdemir, A. Karaoglan, M. A. Dokuyucu, New integral inequalities for Atangana-Baleanu fractional integral operators and various comparisons via simulations, Filomat 37:7 (2023), 2251–2267. https://doi.org/10.2298/FIL2307251S.
56. M. Z. Sarikaya, E. Set, H. Yaldiz, N. Başak, Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities, Mathematical and Computer Modelling 57 (2013) 2403–2407. https://doi.org/10.1016/j.mcm.2012.12.001.
57. E. Set, M. Z. Sarikaya, M. E. Özdemir, H. Yildirim, The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results, JAMSI, 10 (2014), No. 2, 69-83. https://doi.org/10.48550/arXiv.1112.6176.
58. S. Simic, Some refinements of Hermite-Hadamard inequality and an open problem, Kragujevac Journal of Mathematics Volume 42(3) (2018), Pages 349-356.
59. M. Tariq, H. Ahmad, A. G. Shaikh, S. K. Sahoo, K. M. Khedherand, T. N. Gia, New fractional integral inequalities for preinvex functions involving Caputo Fabrizio operator, AIMS Mathematics, 7(3): 3440-3455, (2021).
60. G. Toader, Some generalizations of the convexity, Proceedings of the Colloquium on Approximation and Optimization, University Cluj-Napoca, 1985, 329-338.
61. M. Tomar, E. Set, M. Z. Sarikaya, Hermite-Hadamard type Riemann-Liouville fractional integral inequalities for convex functions, AIP Conf. Proc. **1726** (2016), 020035. http://doi.org/10.1063/1.4945861.
62. M. Vivas-Cortez, M. U. Awan, M. Z. Javed, M. A. Noor, K. I. Noor, A Study of Uniform Harmonic-Convex Functions with respect to Hermite-Hadamard's Inequality and Its Caputo-Fabrizio Fractional Analogue and Applications, Journal of Function Spaces, 2021, (2021).

 doi:10.20944/preprints202509.0395.v1

63. J. R. Wang, X. Li, Y. Zhou, Hermite-Hadamard Inequalities Involving Riemann-Liouville Fractional Integrals via s-convex Functions and Applications to Special Means, Filomat 30:5 (2016), 1143–1150. https://doi.org/10.2298/FIL1605143W.
64. S. Wu, S. Iqbal, M. Aamir, M. Samraiz, A. Younus, On some Hermite–Hadamard inequalities involving k-fractional operators, Journal of Inequalities and Applications (2021) 2021:32. https://doi.org/10.1186/s13660-020-02527-1.
65. B. Y. Xi, D.D. Gao, F. Qi, Integral inequalities of Hermite-Hadamard type for (α, s)-convex and (α, s, m)-convex functions, Italian Journal of Pure and Applied Mathematics N.44-2020 (499-510).
66. B. Y. Xi, F. Qi, Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means, J. Nonlinear Convex. Anal. 16 (2015), no. 5, 873-890. https://doi.org/10.48550/arXiv.1406.5409.
67. B. Y. Xi, F. Qi, Some Integral Inequalities of Hermite-Hadamard Type for Convex Functions with Applications to Means, J. Funct. Spaces Appl. 2012 (2012), Article ID 980438, 14 pages. http://doi.org/10.1155/2012/980438.