Article

Quantization & the Koopman-von Neumann Formulayion

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Abstract: In a previous work by the author it was shown that the Koopman-von Neumann formulation of classical mechanics (KvN) exhibits quantum interference. As such it was there claimed that KvN ought to be considered as a proper quantum mechanical theory in a similar sense as ordinary non-relativistic quantum mechanics (OQM) is. In this article this claim is made manifest by showing that KvN and OQM can be taken as merely inequivalent representations resulting from one and the same quantization scheme. Of course, as the typical notion of *quantization* necessitates the canonical commutation relations (CCR) to hold—which they do not in KvN —the concept needs to be properly revised. To justify such a revision the typical reasons for necessitation the CCR—reductionism and quantization as 'turning brackets to commutators'—are examined and found insufficient.

Keywords: Koopman-von Neumann formulation; quantization; reductionism; Groenewold-van Hove theorem

1. Introduction

KvN [1,2] is typically deemed 'classical' simply because in it the momentum $\widehat{P}$ and position operators $\widehat{Q}$ commute. This in contrast to how it is in OQM where they satisfy the CCR,

$$\left[\widehat{P}, \widehat{Q}\right] = i\hbar. \tag{1}$$

The claim is that KvN because of this cannot exhibit the hallmark quantum behaviour of quantum interference [3]. Nonetheless, in a previous article by the author [4] the actual validity of this claim was questioned and refuted. While it indeed is true that KvN cannot account for quantum interference in the actual double-slit experiment as done with electrons [3], it is not in this regard that quantum interference manifests in KvN. In [4] KvN was identified as classical statistical mechanics. In doing so statistical equilibria $|\lambda\rangle$, correspond to the eigenstates of the KvN generator of time evolution, i.e the Liouvillian $\mathcal{L}$. It follows that non-trivial superpositions of such,

$$\sum_{\lambda} c_{\lambda}|\lambda\rangle, \tag{2}$$

hence are non-equilibria, meaning that they are not time-invariant. In contrast, the corresponding mixed state,

$$\sum_{\lambda} |c_{\lambda}|^2 |\lambda\rangle\langle\lambda|, \tag{3}$$

to such a superposition (2) is time-invariant. This means that mixed states are distinguishable from pure ones—a characteristic quantum phenomenon—a manifestation of quantum interference. Hence, if this is the criterion of 'quantumness', then KvN is as 'quantum' as OQM. A point worth addressing here—which also may serve as a reason for wanting to fit KvN into a proper quantization scheme—is that in typical introductions to KvN [5] the formulation appears completely artificial, appearing more like a 'mathematical trick'. This might give the impression that the supposed quantum interference effect in KvN in reality nothing more than a mathematical defect because the Hilbert space formalism of quantum mechanics is not 'needed' for classical statistical mechanics. However, strictly speaking the Hilbert space formalism is not 'needed' in OQM neither in order to express quantum interference [6]. Hence this does not suffice as a dismissal of the in [4] suggested conclusion—that the foundational issues of quantum mechanics has to do with the foundations of probability in general—echoing the statements made by both Ballentine [7] and Koopman [8] that there is no such thing as 'classical' nor 'non-classical' probability, just probability. The purely probabilistic nature of quantum mechanics has been pushed before by the author in [9] as well as by others [10,11], to mention but a few. In contrast to this view, however, common 'folklore' dictates that there is a such difference in notions of *probability*. In classical statistical mechanics *probability* is taken as being 'classical' and non-problematic as Hamiltonian mechanics supposedly functions as a hidden-variable theory of classical statistical mechanics. *Probability* is taken as non-problematic in this case because it can then simply be interpreted as corresponding to a 'lack of knowledge' or fluctuations in these parameters constituting the phase space $\mathcal{P}$ of Hamiltonian mechanics. In contrast, no such hidden-variable theory is supposed to exists for quantum mechanics as it violates Bell-type inequalities, and hence *probability* supposedly cannot be interpreted in this way neihter. In these terms, what was essentially shown in [4] was that Hamiltonian mechanics does not function as a hidden-variable theory of classical statistical mechanics as the Liouvillian $\mathcal{L}$—taken as the observable of statistical equilibria—is not a function on the phase space. In light of this, if not as a hidden-variable theory, then in what sense does Hamiltonian mechanics relate to classical statistical mechanics?

In this article the idea put forth is that Hamiltonian mechanics relates to KvN the same way that it does to OQM, i.e via quantization. Hence, in conjunction with the previous article [4], KvN will have been put fourth as being equivalent to OQM both in terms of quantization and in terms of interpretations of *probability*. Of course, this means that the concept of *quantization* needs to be rethought. The typical view of quantization—in the sense of Dirac [12]—is as 'turning brackets to commutators',

$$\{\cdot,\cdot\} \mapsto -\frac{i}{\hbar}[\cdot,\cdot], \tag{4}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket. In this sense quantization prohibits KvN as resulting from it. The reason for asserting quantization as such is based on the analogous structure between the classical equations of motion

$$\frac{dF}{dt} = \{F, H\} \tag{5}$$

and their quantum version,

$$\frac{d\widehat{F}}{dt} = -\frac{i}{\hbar}\left[\widehat{F}, \widehat{H}\right], \tag{6}$$

in addition to the CCR, leading to one speculate whether quantization possibly more generally corresponds to a unitary representation of the Poisson structure of HM. The upshot of this is that would provide a natural—or 'canonical' —connection between classical and quantum physics, amongst other things, granting one the ability of interpreting the CCR as a kind of 'quantum version' of P being canonical conjugate tp Q in the classical Hamiltonian mechanical sense. However, the Groenewold-van Hove theorem [13–15] shows that no such unitary (irreducible) representation the Poisson algebra of the Hamiltonian mechanics exists. This results suffices to disprove the thesis of the CCR being the quantum version of canonical conjugancy, and hence the CCR cannot be taken as having to be enforced for that reason. This opens up the door for a notion of *quantization* that also encompasses KvN. In this article it will be suggested that quantization ought to be viewed in more generic representation theoretic manner, in terms of which KvN and OQM correspond to merely inequivalent representations of the same group theoretic structure.

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The structure of this article is as follows: In section 2 the overall structure of classical mechanics will be presented. Particular emphasis it put on how Lagrangian and Hamiltonian mechanics relate to one another. The reason for presenting this is that canonical quantization (CQ) in its formulation relies heavily on the 'going' from Lagrangian to the Hamiltonian mechanics while quantization often attempts at dealing with the generic formulation of Hamiltonian mechanics which is completely independent of the Lagrangian one. In that regard something like the Groenewold-van Hove theorem might not be that surprising. In section 3 CQ is defined and it is shown that it cannot be considered in the sense of Dirac as 'turning brackets to commutators'. This is done by proving an additional theorem of the Groenewold-van Hove-type showing that Dirac's view cannot capture the essence CQ even if weakening from requiring a Poisson algebra representation to a Lie algebra representation. Hence showing that even the most generous interpretation of CQ as 'brackets to commutators' does not work. In section 4 the generic representation theoretic view of quantization is presented and it will be shown that KvN and OQM merely correspond to inequivalent representations of one and the same Hamiltonian dynamics. In this section it will also be discussed how the conventional view of quantization—the supposed necessity of the CCR—is tied to a reductionist view of physics, why a non-reductionist hence should dismiss it and why the suggested revised concept of quantization ought to be taken in non-reductionist scheme of physics. Lastly, in section 5, the arguments and claims of the article are summarized.

2. The structure of modern classical mechanics

HM is in its formulation in terms of symplectic geometry a generalization of LM [16]. Nonetheless, it is in the typical textbook [17] introduced as something one 'gets' from Lagrangian mechanics via a Legendre transform. This is all standard stuff, but we will here recall how this story goes. It is instructive to do so because it is the author's impression that many seem to view CQ as something defined solely in terms of HM, forgetting that the explicit canonical coordinates in which this is done are defined in terms of a Lagrangian. This in contrast to Hamiltonian mechanics which has invariance under canonical transformations as a fundamental symmetry.

2.1. From Lagrangian to Hamiltonian mechanics

Lagrangian mechanics occurs on the tangent bundle $\mathcal{TM}$. For a point $(Q, \dot{Q}) \in \mathcal{TM}$, $\dot{Q}$ corresponds to spatial velocity and Q to spatial position. In Lagrangian mechanics the dynamics is described by the Lagrangian function

$$L : \mathcal{TM} \to \mathbb{R}, \tag{7}$$

from which—by means of the principle of least action—one obtains the corresponding equations of motion,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{Q}}\bigg|_{(\gamma(t),\dot{\gamma}(t))} = \frac{\partial L}{\partial Q}\bigg|_{(\gamma(t),\dot{\gamma}(t)}, \tag{8}$$

where

$$\gamma : \mathbb{R} \to \mathcal{M} \tag{9}$$

is the particle's spatial path and $\dot{\gamma}$ the derivative of that path.

One transitions to the Hamiltonian mechanics description by instead working on the cotangent bundle $\mathcal{T}^*\mathcal{M}$. For a point $(P, Q) \in \mathcal{T}^*\mathcal{M}$, P corresponds to conjugate momentum and Q to spatial position. *Conjugate momentum* is defined in terms of the specific dynamics [17]—i.e in terms of the particular Lagrangian—as

$$P := \frac{\partial L}{\partial \dot{Q}}. \tag{10}$$

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In HM the dynamics is determined by means of the Hamiltonian,

$$H : \mathcal{T}^*\mathcal{M} \to \mathbb{R}, \tag{11}$$

which one obtains by means of a Legendre transformation

$$H := P \cdot \dot{Q} - L. \tag{12}$$

That H, as defined (12), indeed satisfies (11) follows from (8), and in turn Hamilton's equations of motion follow

$$\begin{cases} \dot{P} = -\frac{\partial H}{\partial Q} \\ \dot{Q} = \frac{\partial H}{\partial P} \end{cases}. \tag{13}$$

Note that we hence have made the transition from Lagrangian to Hamiltonian mechanics in terms of a selection of particular coordinates (P, Q).

The generic formulation of Hamiltonian mechanics is more specifically a geometric formulation in terms of symplectic geometry [16,18]. That this is indeed so can be seen by constructing the Poisson bracket on $\mathcal{T}^*\mathcal{M}$ in terms of the explicit coordinates (P, Q). Let $C^\infty(\mathcal{T}^*\mathcal{M})$ be the set of smooth function on $\mathcal{T}^*\mathcal{M}$.

Definition 1. *The* **Poisson bracket**

$$\{\cdot, \cdot\} : (F, G) \in C^\infty(\mathcal{T}^*\mathcal{M}) \times C^\infty(\mathcal{T}^*\mathcal{M}) \mapsto F, G \in C^\infty(\mathcal{T}^*\mathcal{M}) \tag{14}$$

is defined as

$$\{F, G\} := \frac{\partial F}{\partial P}\frac{\partial G}{\partial Q} - \frac{\partial F}{\partial Q}\frac{\partial G}{\partial P}. \tag{15}$$

The utility of the Poisson bracket is that there associated to it is a 'canonical' symmetry, i.e certain coordinate transformations under which it is invariant. Let

$$(P, Q) \mapsto (P'(O, Q), Q'(P, Q)) \tag{16}$$

be an arbitrary coordinate transformation to from the coordinates (P, Q) to the coordinates (P', Q'). Let $\{\cdot, \cdot\}'$ denote an alternative Poisson bracket defined analogously to the original one in Definition 1 but instead with respect to the coordinates (P', Q').

Definition 2. *The coordinates* (P', Q') *are* **canonical coordinates** *and the coordinate transformation* (16) *is a* **canonical transformation** *if*

$$\{\cdot, \cdot\} = \{\cdot, \cdot\}'. \tag{17}$$

The following is a standard result [16]:

Theorem 1. (P', Q') *are canonical coordinates if and only if*

$$\{P', Q'\} = 1. \tag{18}$$

Now, Hamilton's equations of motion (13) may be written in terms of the Poisson bracket as

$$\begin{cases} \dot{P} = \{H, P\} \\ \dot{Q} = \{H, Q\} \end{cases}, \tag{19}$$

and for any canonical coordinates (P', Q') we have

$$\begin{cases} \dot{P}' = \{H, P'\} \\ \dot{Q}' = \{H, Q'\} \end{cases}, \tag{20}$$

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i.e they are of the same form. We have hence found a (certain) coordinate invariant way of expressing the equations of motion.

Remark 1. *Note that there is a certain subgroup of canonical coordinates—the* **point transformations** *[17]—which are defined as*

$$(P, Q) \mapsto (P'(P, Q), Q'(Q)), \tag{21}$$

where

$$P'_i := \frac{\partial L}{\partial \dot{Q}'_i}. \tag{22}$$

That these indeed are canonical coordinates follows by a direct calculation, having noted through the chain rule that

$$P'_i = \sum_j \left.\frac{\partial Q_j}{\partial Q'_i}\right|_{Q'(Q)} P_j. \tag{23}$$

Note furthermore that these form a particular subgroup of the group of canonical transformations, the **group of point transformations***.*

2.2. The Lagrangian mechanics-independent formulation of Hamiltonian mechanics

Note that the property of being canonical coordinates is still with respect to the specific Lagrangian from which we constructed the Hamiltonian mechanical system, as the construction of the Poisson bracket is still with respect to particular coordinates (P, Q), or any point transformation of these. Hamiltonian mechanics has hence at this point not yet been given a formulation that is completely indepedent of Lagrangian mechanics. The way we get such an indepedent formulation is by finding an inherently coordinate invariant way of defining the Poisson bracket. This is where Symplectic geometry [16] comes into play. We will not present this in its fullness—for that the reader is referred elsewhere [16,18]—here we will only reiterate how the general story goes, for that is all that is relevant for our purposes. For us the relevance are the following key points and standard results:

- A **Symplectic manifold** is a differentiable manifold $\mathcal{P}$ equipped with closed non-degenerate 2-form ω, called the **symplectic form** .
- Through the symplectic form one can associate to every function $F \in C^\infty(\mathcal{P})$, a vector field X_F called the **Hamiltonian vector field of** F.
- One can then replace Definition 1 as the definition of the *Poisson bracket* by the definition

$$\{F, G\} := \omega(X_F, X_G), \tag{24}$$

  for every $F, G \in C^\infty(\mathcal{P})$.
- It follows that symplectic manifolds always are even-dimensional.
- Darboux's theorem states that one can (locally) always find coordinates for which the Poisson bracket—as defined by (24)—takes the form as in Definition 1. These coordinates—called **Darboux coordinates** —are what in this generic formulation of Hamiltonian mechanics correspond to canonical coordinates.
- The cotangent bundle—the structure upon which we constructed the previous Lagrangian mechanics-dependent formulation of Hamiltonian mechanics—is a particular example of a symplectic manifold whose Poisson bracket as defined in terms of (24) is identical with the one in terms of Definition 1.
- The dynamics corresponds a Hamiltonian flow

$$U : t \in \mathbb{R} \mapsto U_t \in \mathrm{Sympl}(\mathcal{P}), \tag{25}$$

  which is a group homomorphism where $\mathrm{Sympl}(\mathcal{P})$ is the group of symplectomorphisms on $\mathcal{P}$—the diffeomorphisms on $\mathcal{P}$ that preserves its symplectic form.

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3. Canonical Quantization versus 'Turning brackets to commutators'

In this section we will prove that CQ cannot even in the weakest possible sense be characterized in the Diracian sense as 'turning brackets to commutators'.

3.1. Canonical Quantization

In the typical treatment of CQ it is implicit in its usage but often not emphasized enough that the prescription is really explicitly defined in terms of the particular canonical coordinates (P, Q). A notable exception to this is Gukow and Witten in [19], where it is also stated that CQ even is dependent on the particular choice of coordinates. Here CQ will be properly defined in this manner, making this potential coordinate dependence explicit. In addition, by performing a CQ-type quantization with respect to another choice of canonical coordinates it will be shown that that this yields an inequivalent quantum theory, and hence that CQ indeed is coordinate dependent.

Definition 3. *Let $(\mathcal{TM}, L)$ be a classical mechanical system described in terms of Lagrangian mechanics with associated Lagrangian L. We* **canonically quantize** *this system by representing the observables P and Q as quantum observables $\widehat{P}$ respectively $\widehat{Q}$ on some Hilbert space $\mathcal{H}$ such that their respective spectra concurs with the range of possible values of their classical counter parts—i.e $\mathbb{R}$—and such that the CCR hold,*

$$\left[\widehat{P}, \widehat{Q}\right] = i\hbar. \tag{26}$$

In addition, the generator of time evolution is set to be the operator

$$H\left(\widehat{P}, \widehat{Q}\right), \tag{27}$$

i.e the **quantum Hamiltonian***. The quantum Hamiltonian is furthermore interpreted as the quantum observable of energy of the system.*

Remark 2. *As is well-known, it follows from the Stone-von Neumann theorem [20] that the operators $\widehat{P}$ and $\widehat{Q}$ subjected to properties of CQ are uniquely defined up to a unitary equivalence.*

Furthermore, given a classical observable F on $\mathcal{T}^*\mathcal{M}$, its quantum counterpart is

$$F\left(\widehat{P}, \widehat{Q}\right) \tag{28}$$

inheriting its interpretation from F. This is not only the case for the Hamiltonian H—as explicitly stated in Definition (3)—but it is also the case for the quantum observable of angular momentum. Of course, there is the well-known issue of the inherent ordering ambiguities in the symbolic expressions (27) and more generically in (28). Based solely on this, it is clear that Definition 3 is not proper in the sense of being mathematically rigorous. However, it is 'proper enough' with regards to our purposes here. The point here is not to solve this issue of the ordering ambiguity but to point out that CQ occurs with respect to particular preferred set of canonical coordinates. As previously noted in [19], if one chooses to 'canonically quantize' with respect to a different set of canonical coordinates (P', Q'), say, then it is not generically also true that

$$\left[\widehat{P}, \widehat{Q}\right] = i\hbar. \tag{29}$$

Consider for instance the simple harmonic oscillator having the Hamiltonian

$$H_{\text{osc}}(P, Q) = \frac{1}{2}P^2 + \frac{1}{2}Q^2. \tag{30}$$

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We may perform a canonical transformation into action-angle coordinates (Θ, E), where

$$(P(\Theta, E), Q(\Theta, E)) = \left(\sqrt{2E}\cos\Theta, \sqrt{2E}\sin\Theta\right). \tag{31}$$

It is clear from this that $E = H_{\text{osc}}(P, Q)$. Assuming that this transformation makes sense also after CQ, we would then have

$$\langle E_m | \left[H_{\text{osc}}\left(\widehat{P}, \widehat{Q}\right), \Theta\left(\widehat{P}, \widehat{Q}\right)\right] | E_n \rangle = (E_m - E_n) \langle E_m | \Theta\left(\widehat{P}, \widehat{Q}\right) | E_n \rangle, \tag{32}$$

where $|E_n\rangle$ is a generic eigenstate of the quantum Hamiltonian. This means that the quantized action-angle variables cannot satisfy the CCR. Hence the CCR cannot be taken as 'quantum version' of *canonical coordinates* as it is not analogous to the version of it in Hamiltonian mechanics as given by Theorem 1. It also means that CQ does not possess invariance under canonical transformations.

3.2. *Quantization as 'turning brackets to commutators'*

In the textbook view *quantization* amounts to 'turning Poisson brackets into operator commutators'. In its most generous interpretation this formalizes to the following notion of *quantization*:

Definition 4. *Consider a symplectic manifold $\mathcal{P}$ with induced Poisson algebra $(C^\infty(\mathcal{P}), \{\cdot,\cdot\})$. A* **quantization** *$\mathcal{Q}$ of $\mathcal{P}$ is an irreducible unitary representation of a unital sub-Lie algebra $\mathfrak{g}$ of $C^\infty(\mathcal{P})$ on to a some Hilbert space $\mathcal{H}$, in the sense that $\mathcal{Q}(F)$ is self-adjoint for all $F \in \mathfrak{g}$ and that*

$$i\hbar\mathcal{Q}(\{F, G\}) = [\mathcal{Q}(F), \mathcal{Q}(G)]. \tag{33}$$

For convenience we will however set $\hbar = 1$ for the remainder of this section.

Indeed, the CQ of the simple harmonic oscillator corresponds to a quantization $\mathcal{Q}$ in the sense of Definition 4. In this case the sub-Lie algebra of the Poisson algebra is spanned by the observables P, Q, the unit function and H_{osc}—as defined by (30) —subject to the relations:

$$\begin{cases} \{P, Q\} = 1 \\ \{H_{\text{osc}}, P\} = -Q \\ \{H_{\text{osc}}, Q\} = P \end{cases}, \tag{34}$$

with the rest being zero. Then, because the operator

$$\mathcal{Q}(H_{\text{osc}}) - \frac{1}{2}\mathcal{Q}(P)^2 - \frac{1}{2}\mathcal{Q}(Q)^2 \tag{35}$$

commutes with $\mathcal{Q}(H_{\text{osc}})$, $\mathcal{Q}(P)$ and $\mathcal{Q}(Q)$, and because the representation is irreducible, we get from Schur's lemma [21] that

$$\mathcal{Q}(H_{\text{osc}}) = \frac{1}{2}\mathcal{Q}(P)^2 + \frac{1}{2}\mathcal{Q}(Q)^2 + c, \tag{36}$$

where $c \in \mathbb{R}$, as required for CQ. Of course, it works similarly for the free Hamiltonian or the trivial Hamiltonian.

However, as we will show next, CQ does not generically correspond to a quantization in the sense of Definition 4. We will show this by considering the particular Hamiltonian

$$H_c = \frac{1}{2}P^2 + \frac{Q^3}{3!}. \tag{37}$$

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In this case we have:

$$\begin{cases} \{P, Q\} = 1 \\ \{H_c, P\} = -\frac{Q^2}{2} \\ \{H_c, Q\} = P \end{cases} . \tag{38}$$

So in contrast to (30) these relations do not close the algebra $\mathfrak{g}$. Hence one must add Q^2 to $\mathfrak{g}$. But it does not stop here. Because we in turn get

$$\{H_c, Q^2\} = 2PQ, \tag{39}$$

meaning that we must enforce $PQ \in \mathfrak{g}$, and so on. It is this occurrence of more such elements that eventually leads to a contradiction, taken in conjuction with the requirement of

$$\mathcal{Q}(H_c) = \frac{1}{2}\mathcal{Q}(P)^2 + \frac{\mathcal{Q}(Q)^3}{3!} + C, \tag{40}$$

for some constant $C \in \mathbb{R}$, which is a necessity for $\mathcal{Q}$ to coincide with CQ. 155

Theorem 2. *There does not exist a quantization $\mathcal{Q}$ such that (40) holds.* 156

Proof. We prove this by showing that its existence would lead to a contradiction. Indeed, if (40) holds, then we compute that

$$\begin{aligned} \mathcal{Q}\left(Q^2\right) =& -2i[\mathcal{Q}(P), \mathcal{Q}(H_c)] \\ =& \mathcal{Q}(Q)^2 \end{aligned}, \tag{41}$$

and hence that

$$\begin{aligned} \mathcal{Q}(PQ) =& -i\left[\mathcal{Q}(H_c), \mathcal{Q}\left(\frac{Q^2}{2}\right)\right] \\ =& \mathcal{Q}(P)\mathcal{Q}(Q) + \frac{i}{2} \end{aligned}. \tag{42}$$

In turn this can be used to calculate

$$\begin{aligned} \mathcal{Q}\left(P^2 - \frac{1}{2}Q^3\right) =& -i[\mathcal{Q}(H_c), \mathcal{Q}(PQ)] \\ =& \mathcal{Q}(P)^2 - \frac{1}{2}\mathcal{Q}(Q)^3 \end{aligned} \tag{43}$$

and, in turn,

$$\begin{aligned} \mathcal{Q}\left(2P^2 + \frac{3}{2}Q^3\right) =& -i\left[\mathcal{Q}\left(P^2 - \frac{1}{2}Q^3\right), \mathcal{Q}(PQ)\right] \\ =& \mathcal{Q}(P)^2 + \frac{3}{2}\mathcal{Q}(Q)^3 \end{aligned}. \tag{44}$$

From (43) and (44) we obtain that

$$\begin{aligned} \mathcal{Q}\left(Q^3\right) =& \frac{2}{5}\mathcal{Q}\left(2P^2 + \frac{3}{2}Q^3\right) - \frac{4}{5}\mathcal{Q}\left(P^2 - \frac{1}{2}Q^3\right) \\ =& \mathcal{Q}(Q)^3 \end{aligned}. \tag{45}$$

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We can from (43) compute that

$$\begin{aligned} \mathcal{Q}\left(PQ^2\right) =& -i\frac{2}{5}\left[\mathcal{Q}(H_c), \mathcal{Q}\left(P^2 - \frac{1}{2}Q^3\right)\right] \\ =& \mathcal{Q}(P)\mathcal{Q}(Q)^2 - i\mathcal{Q}(Q) \end{aligned}. \tag{46}$$

From (46) we can compute that

$$\begin{aligned} \mathcal{Q}\left(2P^2Q - \frac{1}{2}Q^4\right) =& -i\left[\mathcal{Q}(H_c), \mathcal{Q}\left(PQ^2\right)\right] \\ =& 2\mathcal{Q}(P)^2\mathcal{Q}(Q) - \frac{1}{2}\mathcal{Q}(Q)^4 + 2i\mathcal{Q}(P). \end{aligned} \tag{47}$$

and from (43) and (46)

$$\begin{aligned} \mathcal{Q}\left(4P^2Q + \frac{3}{2}Q^4\right) =& -i\left[\mathcal{Q}\left(P^2 - \frac{1}{2}Q^3\right), \mathcal{Q}\left(PQ^2\right)\right] \\ =& 4\mathcal{Q}(P)^2\mathcal{Q}(Q) + \frac{3}{2}\mathcal{Q}(Q)^4 + 4i\mathcal{Q}(P) \end{aligned}. \tag{48}$$

From these two we then get that

$$\begin{aligned} \mathcal{Q}\left(P^2Q\right) =& \frac{3}{10}\mathcal{Q}\left(2P^2Q - \frac{1}{2}Q^4\right) + \frac{1}{10}\mathcal{Q}\left(4P^2Q + \frac{3}{2}Q^4\right) \\ =& \mathcal{Q}(P)^2\mathcal{Q}(Q) + i\mathcal{Q}(P) \end{aligned}. \tag{49}$$

From (47) we can compute that

$$\begin{aligned} \mathcal{Q}\left(2P^3 - 4PQ^3\right) =& -i\left[\mathcal{Q}(H_c), \mathcal{Q}\left(2P^2Q - \frac{1}{2}Q^4\right)\right] \\ =& 2\mathcal{Q}(P)^3 - 4\mathcal{Q}(P)\mathcal{Q}(Q)^3 + i6\mathcal{Q}(Q)^2. \end{aligned} \tag{50}$$

and from (43) and (47) that

$$\begin{aligned} \mathcal{Q}\left(4P^3 + 2PQ^3\right) =& -i\left[\mathcal{Q}\left(P^2 - \frac{1}{2}Q^3\right), \mathcal{Q}\left(2P^2Q - \frac{1}{2}Q^4\right)\right] \\ =& 4\mathcal{Q}(P)^3 + 2\mathcal{Q}(P)\mathcal{Q}(Q)^3 - i3\mathcal{Q}(Q)^2 \end{aligned}. \tag{51}$$

From these two we then get that

$$\begin{aligned} \mathcal{Q}\left(P^3\right) =& \frac{1}{10}\mathcal{Q}\left(2P^3 - 4PQ^3\right) + \frac{2}{10}\mathcal{Q}\left(4P^3 + 2PQ^3\right) \\ =& \mathcal{Q}(P)^3 \end{aligned}. \tag{52}$$

Now, we have

$$\begin{aligned} -\frac{i}{9}\left[\mathcal{Q}\left(P^3\right), \mathcal{Q}\left(Q^3\right)\right] =& \mathcal{Q}\left(P^2Q^2\right) \\ =& -\frac{i}{3}\left[\mathcal{Q}\left(P^2Q\right), \mathcal{Q}\left(PQ^2\right)\right]. \end{aligned} \tag{53}$$

However, we also have—because of (45) and (52)—that

$$-\frac{i}{9}\left[\mathcal{Q}\left(P^3\right), \mathcal{Q}\left(Q^3\right)\right] = \mathcal{Q}(P)^2\mathcal{Q}(Q)^2 - 2i\mathcal{Q}(P)\mathcal{Q}(Q) - \frac{2}{3}, \tag{54}$$

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and—because of (46) and (49)—that

$$-\frac{i}{3}\left[\mathcal{Q}\left(P^2Q\right),\mathcal{Q}\left(PQ^2\right)\right]=\mathcal{Q}(P)^2\mathcal{Q}(Q)^2-i\frac{2}{3}\mathcal{Q}(P)\mathcal{Q}(Q)+\frac{1}{3}. \tag{55}$$

(53) together with (54) and (55) imply a contradiction, and hence the proof is completed. □

From Theorem 2 it directly follows that:

Corollary 1. *CQ does not correspond to a quantization in the sense of Definition 4.*

Note that in Definition 4 $\mathfrak{g}$ is a sub-Lie algebra rather than a sub-Poisson algebra. The reason for excluding this case already form the start is that it has long been known that no quantization in that sense exists either. This is the statement of the Groenewold-van Hove theorem, whose proof also riles on showing that the same contradiction as in the proof of Theorem 2 unavoidably occurs. The reason for here not simply sticking to the the Groenewold-van Hove theorem is because we in subsection3 already had dismissed that the full Poisson structure supposedly played a crucial role also after CQ. This as CQ was shown to break the invariance under canonical transformations. This argument did however not dispute the claim that, given fixed coordinates, CQ still works by 'turning brackets to commutators'. In light of Corollary 1, however, we can safely say that 'turning brackets to commutators' is not how CQ works.

4. The need for a new perspective on quantization

In the context of classical statistical mechanics *probability* is typically seen as unproblematic. The reason for this is that allegedly Hamiltonian mechanics corresponds to a hidden-variable theory of it that tells us what is 'really going on'. As such it is then believed that probability safely can be interpreted as 'fluctuations' or 'lack of knowledge' of the parameters in phase space. Bell's theorem [22] addresses under what conditions a certain type of hidden-variable theories can be said to exist. As nature seems to violate these probabilistic conditions [23], *probability* does not generically seem to be so straight forwardly interpretable in the above alledged 'unproblematic' way. It is the claim of this article—based in previous work [4]—that classical statistical mechanics is a proper quantum theory as it exhibits the hallmark quantum phenomenon of quantum interference. Included in this claim is that *probability* in terms of classical statistical mechanics suffers the same interpretational issues as it does in OQM. This is still, however, in contrast to Einstein's view [24], where the existence of a hidden-variable theory of OQM and one for classical statistical mechanics is taken as a necessity. Here such a necessity does not exists. The claim is that the 'unproblematic' interpretation of probability does not work for classical statistical mechanics neither. Rather, the conceptual change in going from Hamiltonian mechanics to classical statistical mechanics is taken to be of similar kind as going from Hamiltonian mechanics to OQM. This conceptual change is the introduction of probability. This is what quantization is claimed to be about, translating deemed essential structures of Hamiltonian mechanics into the formalism of quantum mechanics. Indeed, as mentioned in the introduction, the observable $\mathcal{L}$ in KvN does not correspond to a random variable on the phase space of Hamiltonian mechanics. Hence Hamiltonian mechanics cannot be taken as a hidden-variable theory of classical statistical mechanics. Of course, this is not a proof of a no-go theorem preventing the existence of a hidden-variable theory of classical statistical mechanics but it is enough to show that Hamiltonian mechanics is not it. As Hamiltonian mechanics is typically taken as the archetypical hidden-variable theory in this regard, this not an inconsequential claim. Furthermore note that the occurrence of quantum interference itself is not a dismissal of realism [6], hence it not implied here that classical statistical mechanics is non-realist.

Note that *quantization* here hence has a different meaning than typical. Conventionally it refers more to a means by which one enforces the CCR, taking these to be what corresponds to 'the quantum condition' [12]. As such the notion of *quantization* considered here

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is of a different form then, say, deformation quantization [26], where the enforcement of the CCR is the whole point. There is however a disconnect between this 'quantum condition' and what constitutes 'the quantum' with regards to quantum foundations, where emphasis is put on Bell-typ theorems and quantum interference. In discussing these latter, one in quantum foundations permit the usage of finite dimensional Hilbert spaces. The CCR, however, necessitates infinite dimensional ones. Hence quantum foundations—at least as a practise—is independent of the CCR as 'the quantum condition'. That KvN exhibits quantum interference serves to stress this point of the non-centrality the CCR as 'the quantum condition' further, and thus also stressing the need for revising *quantization*. Closely tied to this 'the CCR as the quantum condition'-view is also the reductionistic view of classical physics as that which emerges at the $\hbar \to 0$-limit of OQM. In this view it is this limit that has actual physical contents, quantization merely being a formal process for ensuring that such a limit exists. Technically speaking however, no generic such limit exists, it only exists for certain states and Hamiltonians[27,28]. In contrast to this, *quantization* here is taken as something having its own intrinsic physical meaning regardless of the alleged 'classical limit'. Hence the notion of *quantization* argued for in this article should be seen in the light of non-reductionistic views of physics [11,29,30,32–34].

Remark 3. *Of course, ona can dissreagrd this convenional practice of quantum foundations by making this 'the CCR as the quantum condition'-view explicitly part of quantum mechanics [25], but at the cost of dismissing central parts of research in quantum foundation. One can even speculate whether, if really taking reductionism seriously, then quantum mechanics perhaps has to though of in such a way that makes the CCR manifest.*

More specifically, in this article it is suggested that *quantization* ought to be seen in the sense of representation theory, that KvN and OQM relate to Hamiltonian mechanics in a similar sense as elementary particles corresponds to irreducible representations of the Poincaré group in the Wigner classification [35]. In fact, though this will not be the way in which it is though of here, this analogy can be made more exact by noting that OQM corresponds to a projective representation of the Galilean group [36] while KvN (essentially) corresponding to an ordinary representation of the same group [37]. In fact Primas in [30] makes this analogy manifest by considering (projective) representations of the Galilean group as particles, which he refers to as 'Galileons'. Here, just as in the case of the Poincaré group, the mass of the particle can be identified as a label of the irreducible representation. The mass term corresponds to the central extension of the Galilean group, and hence it is by definition zero in the case of non-projective representations. For Primas, *mass* is identified as the the central extension, and hence he interprets the non-projective representation as a Galileon with zeor mass, which hence goes counter to the identification of the same representation as KvN made in [37]. There is however no a priori reason for *mass* to be ontically identified as such. Though this identification of the mass follows naturally in the case of projective representations, this is not so in the case of the non-projective one. Rather this identification must then be carried over to the non-projective one. So there is really no inconsistency in directly translating the Wigner classification to the Galilean group such that KvN corresponds to merely a different 'particle' than OQM.

Another sense in which KvN and OQM can be seen as merely inequivalent representations is as certain representations of the dynamics/Hamiltonian flow. Let us specify this further.

Definition 5. *Let U be a Hamiltonian flow associated with Hamiltonian function H on a phase space $\mathcal{P}$. Let (P, Q) be some choice of canonical coordinates and denote by $\mathtt{H}$ the function*

$$\mathtt{H} : (x, y) \in (P, Q)(\mathcal{P}) \mapsto \mathtt{H}(x, y) \tag{56}$$

such that

$$H = \mathtt{H} \circ (P, Q), \tag{57}$$

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i.e $\mathtt{H}$ *is the form* H *takes in terms of the coordinates* (P, Q). *A* **quantization** *of* U *with respect to* (P, Q) *is a unitary representation*

$$U_t \mapsto \widehat{U}_t \tag{58}$$

on a Hilbert space $\mathcal{H}$ *such that*

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{U}_t \widehat{P} \widehat{U}_{-t} = -\left.\frac{\partial \mathtt{H}}{\partial y}\right|_{(\widehat{P},\widehat{Q})} \tag{59}$$

and

$$\left.\frac{d}{dt}\right|_{t=0} \widehat{U}_t \widehat{Q} \widehat{U}_{-t} = \left.\frac{\partial \mathtt{H}}{\partial x}\right|_{(\widehat{P},\widehat{Q})}, \tag{60}$$

where $\widehat{P}$ *and* $\widehat{Q}$ *are a self-adjoint operators whose respective spectrum conforms with the range of* P *respectively* Q.

Because of (58) we may apply Stone's theorem [20] to conclude that there exists a self-adjoint operator $\widehat{T}$—the **generator of time-evolution**—such that

$$i\left[\widehat{T}, \widehat{P}\right] = -\left.\frac{\partial \mathtt{H}}{\partial y}\right|_{(\widehat{P},\widehat{Q})} \tag{61}$$

and

$$i\left[\widehat{T}, \widehat{Q}\right] = \left.\frac{\partial \mathtt{H}}{\partial x}\right|_{(\widehat{P},\widehat{Q})}, \tag{62}$$

where we leave ordering ambiguities in $\widehat{P}$ and $\widehat{Q}$ aside for the moment. In the case of OQM we have

$$\widehat{T} = \mathtt{H}\left(\widehat{P}, \widehat{Q}\right) \tag{63}$$

and for KvN

$$\widehat{T} = \mathcal{L}. \tag{64}$$

A sense in which KvN and OQM differ is that $\widehat{P}$ and $\widehat{Q}$ in OQM generate an irreducible algebra, so that we may apply Schur's lemma to conclude that $\mathtt{H}(\widehat{P}, \widehat{Q})$ must be the unique operator up to an additive constant satisfying (59) and (60). This because given any other operator satisfying these relations $\widehat{T}'$, then the operator

$$\widehat{T} - \widehat{T}' \tag{65}$$

must commute with both $\widehat{P}$ and $\widehat{Q}$, and hence

$$\widehat{T} - \widehat{T}' = c, \tag{66}$$

for some constant $c \in \mathbb{R}$. This is not the case for KvN where (59) and (60) only define $\widehat{T}$ up to a function $g(\widehat{P}, \widehat{Q})$. This does however not mean that $\widehat{T}$ is ambiguous in KvN, because by definition

$$\widehat{U}_t \psi := \psi \circ U_{-t}, \tag{67}$$

meaning that indeed (64) follows. Hence, in OQM (59) and (60) are necessary and sufficient while in KvN they are only necessary.

Addressing the issue of the potential ordering ambiguity in the expression $\mathtt{H}(\widehat{P}, \widehat{Q})$, this issue is not as pathological here as it for other notions of *quantization*. For one thing it is not here required that all such expressions—i.e classical observables—must be well-defined by one and the same ordering rule, nor that all such expressions even need to make sense quantum mechanically. Indeed, this is true conventionally as well, that not all self-adjoint operators correspond to physically meaningful quantum observables. This is indeed the purpose for introducing superselection rules [21,31] in the formalism of

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quantum mechanics. So even if it had been the case that every operator $F(\widehat{P}, \widehat{Q})$ made sense as a self-adjoint operator, it is still not certain that they would have any physical meaning because of that. On that point alone one could question whether the conventional view of quantization as 'turning brackets to commutators' (4) suitable from the start. In CQ a priori only $\mathtt{H}(\widehat{P}, \widehat{Q})$ really needs to be sensible. In the quantization sense of Definition (5) this point is however made even more explicit. There existence of the representation $\widehat{U}$ implies, in the case of OQM, that $\mathtt{H}(\widehat{P}, \widehat{Q})$ must be a well-defined self-adjoint operator under some ordering prescription. If $\mathtt{H}(\widehat{P}, \widehat{Q})$ cannot be defined as a self-adjoint operator, then the corresponding OQM representation does not exist. For the reductionist it is a potential problem if a particular classical Hamiltonian would not permit a quantization, because this would mean that it neither would be ultimately reducible to a quantum theory as some $\hbar \to 0$-limit, which is in conflict with the phrase 'quantum mechanics is more fundamental than classical mechanics'. For a non-reductionist, however, this is not an issue, at least not of the same kind.

The spacific type of reductionism considered here is really the so called 'physicit's reductionism' [32]. This is the sense in which it is a necessity that the more fundamental theory includes the less fundamental ona as a limiting case, e.g such as the non-relativistic $c \to \infty$-limit of special relativity and the (alleged) classical $\hbar \to 0$-limit of OQM. *Non-reductionism* here simply means the negation of this. Hence the issue of reductionism presented in the former paragraph is by definition not an issue for non-reductionism. This is however far from saying that non-reductionism is without its own issues. At least naively, the physicit's reductionism seems natural and intuitive and as such it is hard to think of an alternative to it. In spite of this seeming naturalness, however, it should not be taken as obvious that scientific theories can be put into a hierarchical structure of that kind [30,33,38]. Considering that the conventional view of quantization is closely linked to physicit's reductionism, dismissing this necessitates a reexamination of what *quantization* ought to correspond to. Here it has been suggested that quantization is a way of manifesting fundamental physical principles rather than that manifestation being what constitutes the 'fundamental'. More specifically, quantization has been attempted to be identified as the representation theory of groups, the level of these groups being where the fundamental physical principles lie. In this view OQM is no more fundamental than KvN. Instead they merely correspond to different manifestation of the same fundamental physical principles. In this case, the same dynamical law. One could speculate whether in this regard KvN and OQM can be considered as different 'superselection sectors' [31], with $\hbar$ corresponding to a 'superselectionobservable'. The particular values of $\hbar$ in this case interpreted in the similar sense as the 'classical limit' $\hbar \to 0$, i.e not as $\hbar$ actually taking on different values—as it is technically indeed a universal constant—but as it being 'comparatively small' with respect to other quantum numbers. More generally however, leaving this particular speculation a side, in this view, representation theory is promoted from merely something technical having great utility to something which is intrinsically related to physics, i.e how physical theories interrelate to one-another in a non-reductive manner. A proper meaning of *non-reductionism* has however not here been suggested. All that has been suggested is that representation theory—and in turn quantum mechanics—is a formalism of non-reductionism. In the author's mind this similar in kind to how the C^*-algebra relates to its representations in the framework of Relative onticity [34], though whether this this is truly so is still an open question. More generally whether this representation theoretic view can truly be but into proper ontology is also an open question and a topic of further research.

5. Summary

Based on a previous result [4], KvN exhibits quantum interference and hance ought to be considered as 'proper quantum mechanics' in the same sense as OQM is. This claim however goes counter to the common view that KvN is not proper quantum mechanics as the CCR are not satisfied by it. In this common view the CCR are seen as part of

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the 'quantum condition'. It is in light of this that quantization typically is seen—as a means of enforcing the CCR—and hence that quantization excludes KvN. This is the point that was addressed in this article, reexamining the notion of *quantization* in way such that KvN fits into it. The first part in doing so was showing that the relation between Hamiltonian mechanics and OQM is not as natural as a textbook take on CQ might have one believe. It was shown in section 3 that CQ cannot be seen in as 'turning brackets to commutators'. Indeed, it has long been known that the OQM cannot simply be seen as a unitary representation of the Poisson algebra of Hamiltonian mechanics, as implied by the Groenewold-van Hove theorem. Here, however, an even stronger result was shown in Theorem 2 —that CQ cannot even be identified as a certain type of Lie algebra representation—dismissing even the arguably most generous interpretation of quantization as 'turning brackets to commutators'. Given the central importance of the Poisson structure for Hamiltonian mechanics it is hence hard to argue that OQM is naturally connected to it in that regard. Another regard in which this alleged 'naturalness' can be claimed to manifest was addressed in section 4. This is the reductionist sense in which Hamiltonian mechanics is not only believed to correspond to a certain special limit case of OQM—the $\hbar \to 0$-limit—but the existence of a such is even a necessity. Neither in this sense is the relation between Hamiltonian mechanics and OQM as natural as is typically believed. For no generic such limit exists. it only exists for particular states and Hamiltonians. On top of this one can even question the validity of the general claim of the necessity of such a 'classical limit' of OQM in light of reductionism failing in general to characterize the hierarchy of physical and scientific theories [30,33,38]. Hence it is highly non-trivial to claim that OQM is 'natural' with respect to Hamiltonian mechanics even in this reductionistic regard. With this in mind it was in section 4 suggested that *quantization* ought to be viewed more generally in a representation theoretic sense. In particular OQM and KvN were both suggested to be seen as certain unitary representations of the Hamiltonian flow with respect to a fixed choice of canonical coordinates in terms of which the 'quantum version' of Hamilton's equations of motion still hold—see Definition 5—just unitarily inequivalent such. The general idea here is that the role and utility of representation theory in quantum mechanics is not only important for technical reasons but that it in addition—and perhaps this could turn out to be the reasons for it having such utility—has deeper meaning associated with it reflecting how theories of physics interrelate to one another in a non-reductionistic view of science.

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Abbreviations

The following abbreviations are used in this manuscript:

CCR	Canonical commutation relations
CQ	Canonical quantization
KvN	Koopman–von Neumann formalism of classical mechanics
OQM	Ordinary non-relativistic quantum mechanics

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