

Article

Not peer-reviewed version

Deriving Fundamental Physics via Entropy Maximization Under Linear Constraints

[Alexandre Harvey-Tremblay](#) *

Posted Date: 1 October 2024

doi: 10.20944/preprints202404.1009.v6

Keywords: foundations of quantum physics



Preprints.org is a free multidiscipline platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

Deriving Fundamental Physics via Entropy Maximization Under Linear Constraints

Alexandre Harvey-Tremblay

Independent Scientist; aht@protonmail.ch

Abstract: We present a novel derivation of fundamental physical laws by solving a maximization problem on the Shannon entropy of all possible measurements relative to a system's initial state, subject to specific linear constraints. By introducing appropriate linear constraints, we create probability measures that adhere to particular underlying mathematical structures, enabling us to recover various physical theories within a unified framework. Specifically, imposing a $U(1)$ group constraint leads to the emergence of quantum mechanics by incorporating complex probability amplitudes and interference effects. Extending this approach, we apply a $\text{Spin}^c(3,1)$ group constraint to derive a relativistic quantum theory that naturally includes Lorentz symmetry. Remarkably, in $3 + 1$ dimensions, this method uniquely results in the metric tensor of general relativity through a double-copy mechanism applied to the Dirac current. Furthermore, it inherently incorporates the $SU(3) \times SU(2) \times U(1)$ gauge symmetries of the Standard Model, providing a unified description of fundamental interactions. These findings highlight the power of entropy maximization under linear constraints to reveal the deep connections between probability theory and the mathematical structures underlying fundamental physics, offering new insights into the emergence of spacetime dimensions and symmetry structures in our universe.

1. Introduction

Quantum mechanics (QM) is traditionally founded on five principal axioms [1,2]:

QM Axiom 1 of 5 **State Space:** Every physical system is associated with a complex Hilbert space, and its state is represented by a ray (an equivalence class of vectors differing by a non-zero scalar multiple) in this space.

QM Axiom 2 of 5 **Observables:** Physical observables correspond to Hermitian (self-adjoint) operators acting on the Hilbert space.

QM Axiom 3 of 5 **Dynamics:** The time evolution of a quantum system is governed by the Schrödinger equation, where the Hamiltonian operator represents the system's total energy.

QM Axiom 4 of 5 **Measurement:** Measuring an observable projects the system into an eigenstate of the corresponding operator, yielding one of its eigenvalues as the measurement result.

QM Axiom 5 of 5 **Probability Interpretation:** The probability of obtaining a specific measurement outcome is given by the squared magnitude of the projection of the state vector onto the relevant eigenstate (Born rule).

While these axioms provide a robust framework for QM, they raise foundational questions: Can quantum mechanics be derived from deeper principles, similar to how statistical mechanics emerges from entropy maximization? Is there a more parsimonious approach that naturally leads to these axioms and potentially extends to encompass other fundamental theories?

Statistical mechanics (SM) derives its probability distribution by maximizing entropy subject to a constraint, establishing a profound connection between information theory and physics. Specifically, SM maximizes the Boltzmann entropy under the constraint of a fixed average energy $\bar{E}$:

$$\bar{E} = \sum_i \rho_i E_i. \tag{1}$$

By applying the method of Lagrange multipliers, an optimization problem is formulated to maximize the Boltzmann entropy while satisfying both the normalization and average energy constraints [3,4]. The Lagrangian is:

$$\mathcal{L} = -k_B \sum_i \rho_i \ln \rho_i + \lambda \left(1 - \sum_i \rho_i \right) + \beta \left(\bar{E} - \sum_i \rho_i E_i \right), \quad (2)$$

where λ and β are Lagrange multipliers enforcing the normalization and average energy constraints. Solving this optimization problem yields the Gibbs distribution:

$$\rho_i = \frac{1}{Z} \exp(-\beta E_i), \quad (3)$$

where $Z = \sum_i \exp(-\beta E_i)$ is the partition function.

Inspired by this parsimonious and foundational approach, we propose an extension to derive quantum mechanics, relativity, and other elements of fundamental physics from an entropy maximization principle. Instead of constraining the average energy, we impose a linear constraint involving $n \times n$ matrices $\mathbf{M}_i$:

Definition 1 (Linear Constraint).

$$\bar{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i. \quad (4)$$

We then construct the following optimization problem, aiming to maximize the relative Shannon entropy [5,6] between the initial preparation probabilities p_i and the final measurement probabilities ρ_i , subject to the normalization and linear constraints:

Definition 2 (Lagrangian for Fundamental Physics).

$$\mathcal{L} = -\sum_i \rho_i \ln \frac{\rho_i}{p_i} + \lambda \left(1 - \sum_i \rho_i \right) + \tau \operatorname{tr} \left(\bar{\mathbf{M}} - \sum_i \rho_i \mathbf{M}_i \right), \quad (5)$$

where λ and τ are Lagrange multipliers enforcing the normalization and linear constraints, respectively. In the resulting probability measure, the matrices $\mathbf{M}_i$ introduce phase factors, and the trace ensures that these phases vanish upon measurement.

Theorem 1. The solution to the entropy maximization problem is:

$$\rho_i = \frac{p_i \det \exp(-\tau \mathbf{M}_i)}{\sum_j p_j \det \exp(-\tau \mathbf{M}_j)}. \quad (6)$$

Proof. We solve the maximization problem by setting the derivative of the Lagrangian with respect to ρ_i to zero:

$$\frac{\partial \mathcal{L}}{\partial \rho_i} = -\ln \frac{\rho_i}{p_i} - 1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i = 0. \quad (7)$$

$$\implies \ln \frac{\rho_i}{p_i} = -1 - \lambda - \tau \operatorname{tr} \mathbf{M}_i. \quad (8)$$

$$\implies \rho_i = p_i \exp(-1 - \lambda) \exp(-\tau \operatorname{tr} \mathbf{M}_i). \quad (9)$$

Normalizing the probabilities using $\sum_i \rho_i = 1$, we find:

$$1 = \sum_i \rho_i = \exp(-1 - \lambda) \sum_i p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i), \quad (10)$$

$$\Rightarrow \exp(1 + \lambda) = \sum_j p_j \exp(-\tau \operatorname{tr} \mathbf{M}_j). \quad (11)$$

Substituting back, we obtain:

$$\rho_i = \frac{p_i \exp(-\tau \operatorname{tr} \mathbf{M}_i)}{\sum_j p_j \exp(-\tau \operatorname{tr} \mathbf{M}_j)}. \quad (12)$$

Finally, using the identity $\det \exp(\mathbf{M}) = \exp \operatorname{tr} \mathbf{M}$ for square matrices $\mathbf{M}$, we get:

$$\rho_i = \frac{p_i \det \exp(-\tau \mathbf{M}_i)}{\sum_j p_j \det \exp(-\tau \mathbf{M}_j)}. \quad (13)$$

□

1.1. Statistical Mechanics

To recover statistical mechanics, we consider the case where the matrices $\mathbf{M}_i$ are 1×1 , i.e., scalars. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = [E_i], \quad (14)$$

and take $p_i \rightarrow 1$. Multiplying by the Boltzmann constant k_B , our solution reduces to the Gibbs distribution:

$$\rho_i = \frac{1}{Z} \exp(-\tau E_i), \quad (15)$$

where τ corresponds to β in traditional statistical mechanics. This demonstrates that our solution generalizes SM—the scalar case.

1.2. Quantum Mechanics

By choosing $\mathbf{M}_i$ to generate the U(1) group, we derive the fundamental equations of quantum mechanics from our entropy maximization principle. Specifically, we set:

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i, \quad \text{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}, \quad (16)$$

where E_i are energy levels. This choice yields a quantum probability measure that includes a unitarily invariant ensemble and the Born rule, satisfying all five axioms of QM. The matrices $\mathbf{M}_i$ introduce complex phases corresponding to the U(1) group. In the results section, we will detail how this choice leads to the recovery of quantum mechanics as a special case of our optimization problem.

1.3. Fundamental Physics

Extending our approach, we choose $\mathbf{M}_i$ to be 4×4 matrices representing the generators of the $\text{Spin}^c(3,1)$ group. Specifically, we consider multivectors of the form $\mathbf{u} = \mathbf{f} + \mathbf{b}$, where $\mathbf{f}$ is a bivector and $\mathbf{b}$ is a pseudoscalar of the geometric algebra $\text{GA}(3,1)$. The matrix representation of $\mathbf{M}_i$ is:

$$\mathbf{M}_i = \begin{bmatrix} f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & -f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02} \end{bmatrix}, \quad (17)$$

where $f_{01}, f_{02}, f_{03}, f_{12}, f_{13}, f_{23}$, and b correspond to the generators of the $\text{Spin}^c(3,1)$ group, which includes both Lorentz transformations and $U(1)$ phase rotations. This choice leads to a relativistic quantum probability measure:

$$\rho_i = \frac{p_i \det \exp\left(-\frac{1}{2}\zeta \mathbf{M}_i\right)}{\sum_j p_j \det \exp\left(-\frac{1}{2}\zeta \mathbf{M}_j\right)}, \quad (18)$$

where ζ emerges as a parameter generating boosts, rotations, and phase transformations.

Within this framework, the Dirac current arises naturally and is automatically invariant under the gauge symmetries of the Standard Model, specifically $SU(3) \times SU(2) \times U(1)$. Furthermore, we will show that the metric tensor of general relativity emerges via a double-copy mechanism applied to the Dirac current. This suggests a deep connection between quantum mechanics, relativity, and the structure of spacetime within our entropy maximization approach.

1.4. Dimensional Obstructions

Our general solution yields valid probability measures only in specific cases. Beyond the instances of statistical mechanics (with a scalar constraint) and quantum mechanics (with a $U(1)$ group constraint), the entropy maximization technique yields a consistent solution only in $3 + 1$ dimensions under a $\text{Spin}^c(3,1)$ group constraint.

In other configurations, various obstructions arise—such as the absence of a real matrix algebra isomorphism or the emergence of negative probabilities—thereby violating the axioms of probability theory. The following table summarizes the cases and their obstructions:

<i>Dimensions</i>	<i>Obstruction</i>	
GA(0)	Unobstructed $\implies$ statistical mechanics	(19)
GA(0,1)	Unobstructed $\implies$ quantum mechanics	(20)
GA(1,0)	Negative probabilities	(21)
GA(2,0)	Unobstructed $\implies$ toy model	(22)
GA(1,1)	Negative probabilities	(23)
GA(0,2)	Not isomorphic to a real matrix algebra	(24)
GA(3,0)	Not isomorphic to a real matrix algebra	(25)
GA(2,1)	Not isomorphic to a real matrix algebra	(26)
GA(1,2)	Not isomorphic to a real matrix algebra	(27)
GA(0,3)	Not isomorphic to a real matrix algebra	(28)
GA(4,0)	Not isomorphic to a real matrix algebra	(29)
GA(3,1)	Unobstructed $\implies$ quantum gravity $\wedge SU(3) \times SU(2) \times U(1)$	(30)

$$\text{GA}(2,2) \quad \text{Negative probabilities} \quad (31)$$

$$\text{GA}(1,3) \quad \text{Not isomorphic to a real matrix algebra} \quad (32)$$

$$\text{GA}(0,4) \quad \text{Not isomorphic to a real matrix algebra} \quad (33)$$

$$\text{GA}(5,0) \quad \text{Not isomorphic to a real matrix algebra} \quad (34)$$

$$\vdots \quad \vdots$$

$$\text{GA}(6,0) \quad \text{No probability measure as a self-product} \quad (35)$$

$$\vdots \quad \vdots$$

$$\infty \quad (36)$$

We will first investigate the unobstructed cases and then demonstrate these obstructions in Section 2.4.

2. Results

2.1. Quantum Mechanics

In statistical mechanics (SM), the central observation is that energy measurements of a thermally equilibrated system tend to cluster around an average value. In contrast, quantum mechanics (QM) is characterized by the presence of interference effects in measurement outcomes. To capture these features within an entropy maximization framework, we introduce the following special case of the linear constraint:

Definition 3 (U(1) Generating Constraint). *We reduce the linear constrain to the generator of the U(1) group. Specifically, we replace*

$$\overline{\mathbf{M}} = \sum_i \rho_i \mathbf{M}_i \quad \text{with} \quad \mathbf{M}_i = \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} \quad (37)$$

Here, E_i are scalar values (e.g., energy levels), ρ_i are the probabilities of outcomes, and the matrix generates the U(1) group.

The general solution of the maximization problem

$$\rho_i = \frac{1}{\sum_i p_i \det \exp(-\tau \mathbf{M}_i)} \det \exp(-\tau \mathbf{M}_i) p_i \quad (38)$$

likewise reduces as follows

$$\rho_i = \frac{1}{\sum_i p_i \det \exp\left(-\tau \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right)} \det \exp\left(-\tau \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix}\right) p_i \quad (39)$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of quantum mechanics, as we will demonstrate.

To align our results with conventional quantum mechanical notation, we translate the matrices to complex numbers. Specifically, we consider that:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \leftrightarrow a + ib. \quad (40)$$

Then, we note the following equivalence with the complex norm:

$$\det \exp \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r^2 \det \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \text{ where } r = \exp a \quad (41)$$

$$= r^2 (\cos^2(b) + \sin^2(b)) \quad (42)$$

$$= \|r(\cos(b) + i \sin(b))\| \quad (43)$$

$$= \|r \exp(ib)\| \quad (44)$$

Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_B T)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial preparation into :

$$\rho_i = \frac{1}{\underbrace{\sum_i p_i \|\exp(-itE_i/\hbar)\|}_{\text{Unitarily Invariant Partition Function}}} \underbrace{\|\exp(-itE_i/\hbar)\|}_{\text{Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \quad (45)$$

We are now in a position to explore the solution space.

The wavefunction emerges by decomposing the complex norm into a complex number and its conjugate. It is then visualized as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_i p_i \|\exp(-itE_i/\hbar)\| = Z = \langle \psi | \psi \rangle \quad (46)$$

where

$$\begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} = \begin{bmatrix} \exp(-itE_1/\hbar) & & \\ & \ddots & \\ & & \exp(-itE_n/\hbar) \end{bmatrix} \begin{bmatrix} \psi_1(0) \\ \vdots \\ \psi_n(0) \end{bmatrix} \quad (47)$$

We clarify that p_i represents the probability associated with the initial preparation of the wavefunction, where $p_i = \langle \psi_i(0) | \psi_i(0) \rangle$.

We also note that Z is invariant under unitary transformations.

Let us now investigate how the axioms of quantum mechanics are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle \psi | \psi \rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as physical states associate to the probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates 1.
- In Z , an observable must satisfy:

$$\bar{O} = \sum_i p_i O_i \|\exp(-itE_i/\hbar)\| \quad (48)$$

Since $Z = \langle \psi | \psi \rangle$, then any self-adjoint operator satisfying the condition $\langle \mathbf{O} \psi | \phi \rangle = \langle \psi | \mathbf{O} \phi \rangle$ will equate the above equation, simply because $\langle \mathbf{O} \rangle = \langle \psi | \mathbf{O} | \psi \rangle$. This demonstrates 1.

- Upon transforming Equation (47) out of its eigenbasis through unitary operations, we find that the energy, E_i , typically transforms in the manner of a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar) |\psi(0)\rangle \quad (49)$$

The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle) \quad (50)$$

$$= -i\mathbf{H}/\hbar \exp(-it\mathbf{H}/\hbar)|\psi(0)\rangle \quad (51)$$

$$= -i\mathbf{H}/\hbar|\psi(t)\rangle \quad (52)$$

$$\implies \mathbf{H}|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle \quad (53)$$

which is the Schrödinger equation. This demonstrates 1.

- From Equation (47) it follows that the possible microstates E_i of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from ρ , with the measured state being the occupied microstate i . Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability measure is:

$$\rho_i(t) = \frac{1}{\langle\psi|\psi\rangle}(\psi_i(t))^\dagger \psi_i(t). \quad (54)$$

In scenarios where the probability measure $\rho_i(\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 \quad (55)$$

Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observables, this demonstrates 1.

- Finally, since the probability measure (Equation (45)) replicates the Born rule, 1 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, the $U(1)$ group constraint is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. The following Lagrange multiplier equation

$$\mathcal{L} = -\sum_i \rho_i \ln \frac{\rho_i}{p_i} + \lambda \left(1 - \sum_i \rho_i\right) + \tau \text{tr} \left(\begin{bmatrix} 0 & \bar{E} \\ \bar{E} & 0 \end{bmatrix} - \frac{1}{2} \sum_i \rho_i \begin{bmatrix} 0 & -E_i \\ E_i & 0 \end{bmatrix} \right) \quad (56)$$

becomes the formulation's new singular foundation, and QM Axioms 1, 2, 3, 4, and 5 are now promoted to theorems.

2.2. RQM in 2D

In this section, we investigate a toy model that lives in 2D which provides a valuable starting point before addressing the more complex 3+1D case. In RQM 2D, the fundamental Lagrange Multiplier Equation is:

$$\mathcal{L} = -\sum_i \rho_i \ln \frac{\rho_i}{p_i} + \lambda \left(1 - \sum_i \rho_i\right) + \theta \text{tr} \left(\bar{\mathbf{M}} - \frac{1}{2} \sum_i \rho_i \mathbf{M}_i \right) \quad (57)$$

where λ and θ are the Lagrange multipliers, and where $\mathbf{M}_i$ is the 2×2 matrix representation of the multivectors of $\text{GA}(2)$.

In general a multivector $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ of $\text{GA}(2)$, where a is a scalar, $\mathbf{x}$ is a vector and $\mathbf{b}$ a pseudo-scalar, is represented as follows:

$$\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \cong a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y \quad (58)$$

This holds for any 2×2 matrix and any multivectors of $\text{GA}(2)$.

The basis elements are defined as:

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_x \wedge \sigma_y = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (59)$$

To investigate this case in more detail, we introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 4 (Multivector conjugate). *Let $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ be a multi-vector of the geometric algebra over the reals in two dimensions $\text{GA}(2)$. The multivector conjugate is defined as:*

$$\mathbf{u}^\dagger = a - \mathbf{x} - \mathbf{b} \quad (60)$$

The determinant of the matrix representation of a multivector can be expressed as a self-product:

Theorem 2 (Determinant as a Multivector Self-Product).

$$\mathbf{u}^\dagger \mathbf{u} = \det \mathbf{M} \quad (61)$$

Proof. Let $\mathbf{u} = a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y$, and let $\mathbf{M}$ be its matrix representation $\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix}$. Then:

$$1 : \mathbf{u}^\dagger \mathbf{u} \quad (62)$$

$$= (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y)^\dagger (a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) \quad (63)$$

$$= (a - x\sigma_x - y\sigma_y - b\sigma_x \wedge \sigma_y)(a + x\sigma_x + y\sigma_y + b\sigma_x \wedge \sigma_y) \quad (64)$$

$$= a^2 - x^2 - y^2 + b^2 \quad (65)$$

$$2 : \det \mathbf{M} \quad (66)$$

$$= \det \begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \quad (67)$$

$$= (a+x)(a-x) - (y-b)(y+b) \quad (68)$$

$$= a^2 - x^2 - y^2 + b^2 \quad (69)$$

□

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 5 (Multivector Conjugate Transpose). *Let $|V\rangle \in (\text{GA}(2))^n$:*

$$|V\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \quad (70)$$

The multivector conjugate transpose of $|V\rangle$ is defined as first taking the transpose and then the element-wise multivector conjugate:

$$\langle\langle V| = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \quad (71)$$

Definition 6 (Bilinear Form). Let $|V\rangle$ and $|W\rangle$ be two vectors valued in $\text{GA}(2)$. We introduce the following bilinear form:

$$\langle\langle V|W\rangle\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a_1 + \mathbf{x}_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a_n + \mathbf{x}_n + \mathbf{b}_n) \quad (72)$$

Theorem 3 (Inner Product). Restricted to the even sub-algebra of $\text{GA}(2)$, the bilinear form is an inner product.

Proof.

$$\langle\langle V|W\rangle\rangle_{\mathbf{x} \rightarrow 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \quad (73)$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \sigma_x \wedge \sigma_y$. $\square$

Let us now solve the optimization problem for the even multivectors of $\text{GA}(2,0)$, whose inner product is positive-definite.

We take $a \rightarrow 0, \mathbf{x} \rightarrow 0$ then $\mathbf{M}$ reduces as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}|_{a \rightarrow 0, \mathbf{x} \rightarrow 0} = \mathbf{b} \implies \mathbf{M} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \quad (74)$$

The Lagrange multiplier equation can be solved as follows:

$$0 = \frac{\partial \mathcal{L}[\rho_1, \dots, \rho_n]}{\partial \rho_i} \quad (75)$$

$$= -\ln \frac{\rho_i}{p_i} - p_i - \lambda - \theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \quad (76)$$

$$= \ln \frac{\rho_i}{p_i} + p_i + \lambda + \theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \quad (77)$$

$$\implies \ln \frac{\rho_i}{p_i} = -p_i - \lambda - \theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \quad (78)$$

$$\implies \rho_i = p_i \exp(-p_i - \lambda) \exp\left(-\theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (79)$$

$$= \frac{1}{Z(\theta)} p_i \exp\left(-\theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (80)$$

The partition function $Z(\theta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_i p_i \exp(-p_i - \lambda) \exp\left(-\theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (81)$$

$$\implies (\exp(-p_i - \lambda))^{-1} = \sum_i p_i \exp\left(-\theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (82)$$

$$Z(\theta) := \sum_i p_i \exp\left(-\theta \text{tr} \frac{1}{2} \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix}\right) \quad (83)$$

Consequently, the least biased probability measure that connects an initial preparation p_i to a final measurement ρ_i , under the 2D linear constraint, is:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det \exp \left(-\frac{1}{2} \theta \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \right)}}_{\text{Spin(2) Invariant Ensemble}} \underbrace{\det \exp \left(-\frac{1}{2} \theta \begin{bmatrix} 0 & -b_i \\ b_i & 0 \end{bmatrix} \right)}_{\text{Spin(2) Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \quad (84)$$

In 2D, the Lagrange multiplier θ correspond to an angle of rotation. For comparison, in 1+1D it would correspond to the rapidity ζ :

$$2D : \quad \exp \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \theta \text{ is the angle of rotation} \quad (85)$$

$$1 + 1D : \quad \exp \zeta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cosh \zeta & \sinh \zeta \\ \sinh \zeta & \cosh \zeta \end{bmatrix} \quad \zeta \text{ is the rapidity} \quad (86)$$

Definition 7 (Spin(2)-valued Wavefunction).

$$|\psi\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1} R_1 \\ \vdots \\ \sqrt{\rho_2} R_2 \end{bmatrix} \quad (87)$$

where $\sqrt{\rho_i} = e^{\frac{1}{2}a_i}$ representing the square root of the probability and $R_i = e^{\frac{1}{2}\mathbf{b}_i}$ representing a rotor in 2D (or boost in 1+1D).

The partition function of the probability measure can be expressed using the bilinear form applied to the Spin(2)-valued Wavefunction:

Theorem 4 (Partition Function). $Z = \langle\langle \psi | \psi \rangle\rangle$

Proof.

$$\langle\langle \psi | \psi \rangle\rangle = \sum_i \psi_i^\dagger \psi_i = \sum_i \rho_i R_i^\dagger R_i = \sum_i \rho_i = Z \quad (88)$$

□

Definition 8 (Spin(2)-valued Evolution Operator).

$$T = \begin{bmatrix} e^{-\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} \quad (89)$$

Theorem 5. The partition function is invariant with respect to the Spin(2)-valued evolution operator.

Proof. We note that:

$$\langle\langle T \mathbf{v} | T \mathbf{v} \rangle\rangle = \langle\langle \mathbf{v} | \mathbf{v} \rangle\rangle = \mathbf{v}^\dagger T^\dagger T \mathbf{v} \implies T^\dagger T = I \quad (90)$$

then, since $\begin{bmatrix} e^{\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} \begin{bmatrix} e^{-\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} = I$, the relation $T^\dagger T = I$ is satisfied. □

We note that the even sub-algebra of $GA(2)$, being closed under addition and multiplication and constituting an inner product through its bilinear form, allows for the construction of a Hilbert space. In this context, the Hilbert space is $Spin(2)$ -valued. The primary distinction between a wavefunction in a complex Hilbert space and one in a $Spin(2)$ -valued Hilbert space lies in the subject matter of the theory. Specifically, in the latter, the construction governs the change in orientation experienced by an observer, which in turn dictates the measurement basis used in the experiment, consistently with the rotational symmetry and freedom of the system.

The dynamics of observer orientation transformations are described by a variant of the Schrödinger equation, which is derived by taking the derivative of the wavefunction with respect to the Lagrange multiplier, θ :

Definition 9 ($Spin(2)$ -valued Schrödinger Equation).

$$\frac{d}{d\theta} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mathbf{b}_1 & & \\ & \ddots & \\ & & -\frac{1}{2}\mathbf{b}_n \end{bmatrix} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} \quad (91)$$

Here, θ represents a global one-parameter evolution parameter akin to time, which is able to transform the wavefunction under the $Spin(2)$, locally across the states of the Hilbert space. This is an extremely general equation that captures all transformations that can be done consistently with the symmetries of the wavefunction for the $Spin(2)$ group.

Definition 10 (David Hestenes' Formulation). *In 3+1D, the David Hestenes' formulation [7] of the wavefunction is $\psi = \sqrt{\rho}Re^{ib/2}$, where $R = e^{\mathbf{f}/2}$ is a Lorentz boost or rotation and where $e^{ib/2}$ is a phase. In 2D, as the algebra only admits a bivector, his formulation would reduce to $\psi = \sqrt{\rho}R$, which is the form we have recovered.*

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

Definition 11 (Dirac Current). *Given the basis σ_x and σ_y , the Dirac current for the 2D theory is defined as:*

$$J_x \equiv \psi^\dagger \sigma_x \psi = \rho \underbrace{R^\dagger \sigma_x R}_{SO(2)} = \rho \tilde{\sigma}_x \quad (92)$$

$$J_y \equiv \psi^\dagger \sigma_y \psi = \rho \underbrace{R^\dagger \sigma_y R}_{SO(2)} = \rho \tilde{\sigma}_y \quad (93)$$

where $\tilde{\sigma}_x$ and $\tilde{\sigma}_y$ are a $SO(2)$ rotated basis vectors.

2.2.1. 1+1D Obstruction

As stated in the introduction, of the dimensional cases, only 2D and 3+1D are free of obstructions. For instance, the 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\mathbf{e}_0 \wedge \mathbf{e}_1)(a + b\mathbf{e}_0 \wedge \mathbf{e}_1)$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. This is why we had to use 2D instead of 1+1D in this two-dimensional introduction. In the following section, we will investigate the 3+1D case, then we will show why all other dimensional cases are obstructed.

2.3. RQM in 3+1D

In this section, we extend the concepts and techniques developed for multivector amplitudes in 2D to the more physically relevant case of 3+1D dimensions. The Lagrange multiplier equation is as follows:

$$\mathcal{L} = - \sum_i \rho_i \ln \frac{\rho_i}{p_i} + \lambda \left(1 - \sum_i \rho_i \right) + \zeta \text{tr} \left(\bar{\mathbf{M}} - \frac{1}{2} \sum_i \rho_i \mathbf{M}_i \right) \quad (94)$$

The solution (proof in Annex B) is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det \exp(-\zeta \frac{1}{2} \mathbf{M}_i)}}_{\text{Spin}^c(3,1) \text{ Invariant Ensemble}} \underbrace{\det \exp(-\zeta \frac{1}{2} \mathbf{M}_i)}_{\text{Spin}^c(3,1) \text{ Born Rule}} \underbrace{p_i}_{\text{Initial Preparation}} \quad (95)$$

where ζ is a "twisted-phase" rapidity. (If the invariance group was $\text{Spin}(3,1)$ instead of $\text{Spin}^c(3,1)$, obtainable by posing $\mathbf{b} \rightarrow 0$, then it would simply be the rapidity).

2.3.1. Preliminaries

Our initial goal will be to express the partition function as a self-product of elements of the vector space. As such, we begin by defining a general multivector in the geometric algebra $\text{GA}(3,1)$.

Definition 12 (Multivector). Let $\mathbf{u}$ be a multivector of $\text{GA}(3,1)$. Its general form is:

$$\mathbf{u} = a \quad (96)$$

$$+ t\gamma_0 + x\gamma_1 + y\gamma_2 + z\gamma_3 \quad (97)$$

$$+ f_{01}\gamma_0 \wedge \gamma_1 + f_{02}\gamma_0 \wedge \gamma_2 + f_{03}\gamma_0 \wedge \gamma_3 + f_{12}\gamma_1 \wedge \gamma_2 + f_{13}\gamma_1 \wedge \gamma_3 + f_{23}\gamma_2 \wedge \gamma_3 \quad (98)$$

$$+ p\gamma_1 \wedge \gamma_2 \wedge \gamma_3 + q\gamma_0 \wedge \gamma_2 \wedge \gamma_3 + v\gamma_0 \wedge \gamma_1 \wedge \gamma_3 + w\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \quad (99)$$

$$+ b\gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 \quad (100)$$

where $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are the basis vectors in the real Majorana representation.

A more compact notation for $\mathbf{u}$ is

$$\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b} \quad (101)$$

where a is a scalar, $\mathbf{x}$ a vector, $\mathbf{f}$ a bivector, $\mathbf{v}$ is pseudo-vector and $\mathbf{b}$ a pseudo-scalar.

This general multivector can be represented by a 4×4 real matrix using the real Majorana representation:

Definition 13 (Matrix Representation of $\mathbf{u}$).

$$\mathbf{M} = \begin{bmatrix} a + f_{02} - q - z & b - f_{13} + w - x & -f_{01} + f_{12} - p + v & f_{03} + f_{23} + t + y \\ -b + f_{13} + w - x & a + f_{02} + q + z & f_{03} + f_{23} - t - y & f_{01} - f_{12} - p + v \\ -f_{01} - f_{12} + p + v & f_{03} - f_{23} + t - y & a - f_{02} + q - z & -b - f_{13} - w - x \\ f_{03} - f_{23} - t + y & f_{01} + f_{12} + p + v & b + f_{13} - w - x & a - f_{02} - q + z \end{bmatrix} \quad (102)$$

To manipulate and analyze multivectors in $\text{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 14 (Multivector Conjugate (in 4D)).

$$\mathbf{u}^\dagger = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \quad (103)$$

Definition 15 (3,4 Blade Conjugate). *The 3,4 blade conjugate of $\mathbf{u}$ is*

$$[\mathbf{u}]_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \quad (104)$$

The results of Lundholm[8], demonstrates that the multivector norms in the following definition, are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{u}\mathbf{v}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

Definition 16. *The self-products associated with low-dimensional geometric algebras are:*

$$\text{GA}(0,1) : \quad \varphi^\dagger \varphi \quad (105)$$

$$\text{GA}(2,0) : \quad \varphi^\dagger \varphi \quad (106)$$

$$\text{GA}(3,0) : \quad [\varphi^\dagger \varphi]_3 \varphi^\dagger \varphi \quad (107)$$

$$\text{GA}(3,1) : \quad [\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi \quad (108)$$

$$\text{GA}(4,1) : \quad ([\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi)^\dagger ([\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi) \quad (109)$$

We can now express the determinant of the matrix representation of a multivector via the self-product $[\varphi^\dagger \varphi]_{3,4} \varphi^\dagger \varphi$. Again, this choice is not arbitrary, but the unique choice with allows us to represent the determinant of the matrix representation of a multivector within $\text{GA}(3,1)$:

Theorem 6 (Determinant as a Multivector Self-Product).

$$[\mathbf{u}^\dagger \mathbf{u}]_{3,4} \mathbf{u}^\dagger \mathbf{u} = \det \mathbf{M} \quad (110)$$

Proof. Please find a computer assisted proof of this equality in Annex C. $\square$

Definition 17 ($\text{GA}(3,1)$ -valued Vector).

$$|V\rangle\rangle = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{f}_1 + \mathbf{v}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{f}_n + \mathbf{v}_n + \mathbf{b}_n \end{bmatrix} \quad (111)$$

These constructions allow us to express the partition function in terms of the multivector self-product:

Definition 18 (Double-Copy Product). *Instead of an inner product, we obtain what we call a double-copy product:*

$$\langle\langle V|V|V|V \rangle\rangle = \sum_i \underbrace{[\psi_i^\dagger \psi_i]_{3,4}}_{\text{copy 1}} \underbrace{\psi_i^\dagger \psi_i}_{\text{copy 2}} \quad (112)$$

$$= \underbrace{[\mathbf{u}_1^\dagger \quad \dots \quad \mathbf{u}_n] \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix}}_{\text{copy 1}} \underbrace{\begin{bmatrix} \mathbf{u}_1^\dagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\dagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix}}_{\text{copy 2}} \quad (113)$$

Theorem 7 (Partition Function). $Z = \langle\langle V|V|V|V \rangle\rangle$

Proof.

$$\langle\langle V|V|V|V \rangle\rangle \quad (114)$$

$$= \mathbb{L} \begin{bmatrix} \mathbf{u}_1^\dagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \mathbb{J}_{3,4} \begin{bmatrix} \mathbf{u}_1^\dagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\dagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \quad (115)$$

$$= \mathbb{L} \begin{bmatrix} \mathbf{u}_1^\dagger \mathbf{u}_1 & \dots & \mathbf{u}_n \mathbf{u}_n \end{bmatrix} \mathbb{J}_{3,4} \begin{bmatrix} \mathbf{u}_1^\dagger \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n^\dagger \mathbf{u}_n \end{bmatrix} \quad (116)$$

$$= \mathbb{L} \mathbf{u}_1^\dagger \mathbf{u}_1 \mathbb{J}_{3,4} \mathbf{u}_1^\dagger \mathbf{u}_1 + \dots + \mathbb{L} \mathbf{u}_n^\dagger \mathbf{u}_n \mathbb{J}_{3,4} \mathbf{u}_n^\dagger \mathbf{u}_n \quad (117)$$

$$= \sum_{i=1}^n \det \mathbf{M}_{\mathbf{u}_i} \quad (118)$$

$$= Z \quad (119)$$

□

Desirable properties for the double-copy product are introduced by addressing the issue of non-positivity. First, we establish non-negativity:

Theorem 8 (Non-negativity). *The double-copy product, applied to the even subalgebra of $\text{GA}(3,1)$, is always non-negative.*

Proof. Let $|V\rangle\rangle = \begin{bmatrix} a_1 + \mathbf{f}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{f}_n + \mathbf{b}_n \end{bmatrix}$. Then,

$$\langle\langle V|V|V|V \rangle\rangle \quad (120)$$

$$= \mathbb{L} \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\dagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \mathbb{J}_{3,4} \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\dagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \quad (121)$$

$$= \mathbb{L} \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \mathbb{J}_{3,4} \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} \quad (122)$$

$$= \mathbb{L} \begin{bmatrix} a_1^2 + a_1 \mathbf{f}_1 + a_1 \mathbf{b}_1 - \mathbf{f}_1 a_1 - \mathbf{f}_1^2 - \mathbf{f}_1 \mathbf{b}_1 + \mathbf{b}_1 a_1 + \mathbf{b}_1 \mathbf{f}_1 + \mathbf{b}_1^2 & \dots \end{bmatrix} \mathbb{J}_{3,4} \dots \quad (123)$$

$$= \mathbb{L} \begin{bmatrix} a_1^2 - \mathbf{f}_1^2 + \mathbf{b}_1^2 & \dots \end{bmatrix} \mathbb{J}_{3,4} \dots \quad (124)$$

We note 1) $\mathbf{b}^2 = (bI)^2 = -b^2$ and 2) $\mathbf{f}^2 = -E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2 + 4e_0 e_1 e_2 e_3 (E_1 B_1 + E_2 B_2 + E_3 B_3)$

$$= \mathbb{L} \begin{bmatrix} a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2 - 4e_0 e_1 e_2 e_3 (E_1 B_1 + E_2 B_2 + E_3 B_3) & \dots \end{bmatrix} \mathbb{J}_{3,4} \dots \quad (125)$$

We note that the terms are now complex numbers, which we rewrite as $\Re(z) = a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2$ and $\Im(z) = -4(E_1 B_1 + E_2 B_2 + E_3 B_3)$

$$= \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}_{3,4} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \quad (126)$$

$$= \begin{bmatrix} z_1^\dagger & \dots & z_n^\dagger \end{bmatrix} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \quad (127)$$

$$= z_1^\dagger z_1 + \dots + z_n^\dagger z_n \quad (128)$$

which is always non-negative. $\square$

To achieve *positive-definiteness* of the double-copy product, we must address the issue of non-zero vectors that have zero norm due to the indefinite metric signature of spacetime in $\text{GA}(3,1)$. In such algebras, null vectors (vectors with zero norm) can be non-zero, which prevents the inner product from being strictly positive-definite.

To resolve this, we introduce an equivalence relation that identifies all non-zero vectors of zero norm with the zero vector. Formally, we define the equivalence relation:

Definition 19 (Equivalence Relation for Null Vectors). *For vectors v, w in the even subalgebra of $\text{GA}(3,1)$, we say:*

$$v \sim w \quad \text{if and only if} \quad v - w = 0 \quad \text{or} \quad \|v - w\| = 0.$$

This means that in our quotient space $\mathcal{V}/\sim$, the only vector of zero norm is the zero vector itself. Consequently, vectors that were previously non-zero but had zero norm are now identified with the zero vector, ensuring that the inner product is positive-definite on the quotient space.

To implement this equivalence relation and to represent physical states appropriately, we define the $\text{Spin}^c(3,1)$ -valued wavefunction, which takes values in the even subalgebra of $\text{GA}(3,1)$:

Definition 20 ($\text{Spin}^c(3,1)$ -Valued Wavefunction).

$$|\psi\rangle\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{f}_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1} R_1 B_1 \\ \vdots \\ \sqrt{\rho_n} R_n B_n \end{bmatrix},$$

where:

- $\sqrt{\rho_i} = e^{\frac{1}{2}a_i} \geq 0$ is a positive scalar factor ensuring non-negativity.
- $R_i = e^{\frac{1}{2}\mathbf{f}_i}$ is a rotor representing Lorentz transformations (rotations and boosts in spacetime).
- $B_i = e^{\frac{1}{2}\mathbf{b}_i}$ is a complex phase factor, as $\mathbf{b}_i = b_i I$ and $e^{\frac{1}{2}\mathbf{b}_i} = e^{\frac{1}{2}b_i I} = \cosh\left(\frac{b_i}{2}\right) + I \sinh\left(\frac{b_i}{2}\right)$.

In this representation:

- The exponential map $e^{\frac{1}{2}(a_i + \mathbf{f}_i + \mathbf{b}_i)}$ maps elements of the algebra to the connected component of the identity in the spin group $\text{Spin}^c(3,1)$, except at the zero vector, where the map is not injective.
- The wavefunction $|\psi\rangle\rangle$ captures both the amplitude (through $\sqrt{\rho_i}$) and the phase (through R_i and B_i) of the quantum state.

By considering the quotient space under the equivalence relation $\sim$, the double-copy product $\langle\langle\psi|\psi|\psi|\psi\rangle\rangle$ becomes positive-definite. This is because:

- The double-copy product yields zero if and only if $|\psi\rangle\rangle$ is the zero vector in the quotient space.
- All non-zero vectors $|\psi\rangle\rangle$ in the quotient space have a strictly positive norm.
- The identification of null vectors with the zero vector removes the degeneracy caused by the indefinite metric signature.

Now, let us turn our attention to the evolution operator, which leaves the partition function invariant:

Definition 21 ($\text{Spin}^c(3,1)$ Evolution Operator).

$$T = \begin{bmatrix} e^{-\frac{1}{2}\zeta(\mathbf{f}_1 + \mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} \quad (129)$$

In turn, this leads to a variant of the Schrödinger equation obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier ζ :

Definition 22 ($\text{Spin}^c(3,1)$ -valued Schrödinger equation).

$$\frac{d}{d\zeta} \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(\mathbf{f}_1 + \mathbf{b}_1) & & \\ & \ddots & \\ & & -\frac{1}{2}(\mathbf{f}_n + \mathbf{b}_n) \end{bmatrix} \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} \quad (130)$$

In this case ζ represents a one-parameter evolution parameter akin to time, which is able to transform the measurement basis under action of the $\text{Spin}^c(3,1)$ group. This is an extremely general equation that captures all transformations that can be done consistently with the symmetries of the wavefunction.

Theorem 9 ($\text{Spin}^c(3,1)$ invariance). *Let $e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}$ be a general element of $\text{Spin}^c(3,1)$. Then, the equality:*

$$[\psi^\dagger\psi]_{3,4}\psi^\dagger\psi = [(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}})^\dagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi]_{3,4}(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\dagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \quad (131)$$

is always satisfied.

Proof.

$$[(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\dagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi]_{3,4}(e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi)^\dagger e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \quad (132)$$

$$= [\psi^\dagger e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi]_{3,4}\psi^\dagger e^{-\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\psi \quad (133)$$

$$= [\psi^\dagger e^{\mathbf{b}}\psi]_{3,4}\psi^\dagger e^{\mathbf{b}}\psi \quad (134)$$

$$= [\psi^\dagger\psi]_{3,4}e^{-\mathbf{b}}e^{\mathbf{b}}\psi^\dagger\psi \quad (135)$$

$$= [\psi^\dagger\psi]_{3,4}\psi^\dagger\psi \quad (136)$$

□

2.3.2. RQM

Definition 23 (David Hestenes' Wavefunction). *The $\text{Spin}^c(3,1)$ -valued wavefunction we have recovered is formulated identically to David Hestenes'[7] formulation of the wavefunction within $GA(3,1)$.*

$$\psi = e^{\frac{1}{2}(a+\mathbf{f}+\mathbf{b})} = \sqrt{\rho}Re^{-ib/2} \quad (137)$$

where $e^{\frac{1}{2}a} = \sqrt{\rho}$, $e^{\frac{1}{2}\mathbf{f}} = R$ and $e^{\frac{1}{2}\mathbf{b}} = e^{-ib/2}$.

Before we continue the RQM investigation, let us note that the double-copy product contains two copies of a bilinear form $\psi^\dagger\psi$:

$$\underbrace{[\psi^\dagger\psi]_{3,4}}_{\text{copy 1}} \underbrace{\psi^\dagger\psi}_{\text{copy 2}} \quad (138)$$

In the present section, we will investigate the properties of each copy individually, leaving the properties specific to the double-copy for the section on quantum gravity.

Taking a single copy, the Dirac current is obtained directly from the gamma matrices, as follows:

Definition 24 (Dirac Current). *The definition of the Dirac current is the same as Hestenes':*

$$J \equiv \psi^\dagger\gamma_\mu\psi = \rho R^\dagger B^\dagger \gamma_\mu B R = \rho R^\dagger \gamma_\mu B^{-1} B R = \rho \underbrace{R^\dagger \gamma_\mu R}_{SO(3,1)} = \rho \tilde{\gamma}_\mu \quad (139)$$

where $\tilde{\gamma}_\mu$ is a $SO(3,1)$ rotated basis vector.

2.3.3. Standard Model Gauge Symmetries

We will now demonstrate that the double-copy product is automatically invariant under transformations corresponding to the $U(1)$, $SU(2)$, and $SU(3)$ symmetries, as well as under unitary transformations satisfying $U^\dagger U = I$, all of which play fundamental roles in the Standard Model of particle physics. These symmetries constitute the set of transformations that leave the Dirac current invariant, i.e., $(T\psi)^\dagger\gamma_0 T\psi = \psi^\dagger\gamma_0\psi$ with T valued in $GA(3,1)$.

Theorem 10 ($U(1)$ Invariance). *Let $e^{\frac{1}{2}\mathbf{b}}$ be a general element of $U(1)$. Then, the equality*

$$[\psi^\dagger\gamma_0\psi]_{3,4}\psi^\dagger\gamma_0\psi = \underbrace{[(e^{\frac{1}{2}\mathbf{b}}\psi)^\dagger\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi]_{3,4}}_{\text{copy 1}} \underbrace{(e^{\frac{1}{2}\mathbf{b}}\psi)^\dagger\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi}_{\text{copy 2}} \quad (140)$$

is satisfied, yielding a $U(1)$ symmetry for each copied bilinear form.

Proof. Equation (140) is invariant if this expression is satisfied:

$$e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}} = \gamma_0 \quad (141)$$

This is always satisfied simply because $e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}} = \gamma_0 e^{-\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{b}} = \gamma_0$ $\square$

Theorem 11 ($SU(2)$ Invariance). *Let $e^{\frac{1}{2}\mathbf{f}}$ be a general element of $Spin(3,1)$. Then, the equality:*

$$[\psi^\dagger\gamma_0\psi]_{3,4}\psi^\dagger\gamma_0\psi = \underbrace{[(e^{\frac{1}{2}\mathbf{f}}\psi)^\dagger\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi]_{3,4}}_{\text{copy 1}} \underbrace{(e^{\frac{1}{2}\mathbf{f}}\psi)^\dagger\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi}_{\text{copy 2}} \quad (142)$$

is satisfied for if $\mathbf{f} = \theta_1\gamma_2\gamma_3 + \theta_2\gamma_1\gamma_3 + \theta_3\gamma_1\gamma_2$ (which generates $SU(2)$), yielding a $SU(2)$ symmetry for each copied bilinear form.

Proof. Equation (142) is invariant if this expression is satisfied[9]:

$$e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}} = \gamma_0 \quad (143)$$

We now note that moving the left-most term to the right of the gamma matrix yields:

$$e^{-E_1\gamma_0\gamma_1-E_2\gamma_0\gamma_2-E_3\gamma_0\gamma_3-\theta_1\gamma_2\gamma_3-\theta_2\gamma_1\gamma_3-\theta_3\gamma_1\gamma_2}\gamma_0e^{\frac{1}{2}\mathbf{f}} \quad (144)$$

$$= \gamma_0e^{E_1\gamma_0\gamma_1+E_2\gamma_0\gamma_2+E_3\gamma_0\gamma_3-\theta_1\gamma_2\gamma_3-\theta_2\gamma_1\gamma_3-\theta_3\gamma_1\gamma_2}e^{\frac{1}{2}\mathbf{f}} \quad (145)$$

Therefore, the product $e^{-\frac{1}{2}\mathbf{f}}\gamma_0e^{\frac{1}{2}\mathbf{f}}$ reduces to γ_0 if and only if $E_1 = E_2 = E_3 = 0$, leaving $\mathbf{f} = \theta_1\gamma_2\gamma_3 + \theta_2\gamma_1\gamma_3 + \theta_3\gamma_1\gamma_2$:

Finally, we note that $e^{\theta_1\gamma_2\gamma_3+\theta_2\gamma_1\gamma_3+\theta_3\gamma_1\gamma_2}$ generates $SU(2)$. $\square$

Theorem 12 ($SU(3)$). *The generators of $SU(3)$ in $GA(3,1)$ are given by Anthony Lesenby in [10] and are as follows:*

$$\hat{E}_{ij} = \hat{e}_i\hat{e}_j - \hat{f}_i\hat{f}_j \quad \text{where } i < j \quad (146)$$

$$\hat{F}_{ij} = \hat{e}_i\hat{f}_j + \hat{e}_j\hat{f}_i \quad \text{where } i < j \quad (147)$$

$$\hat{J} = \hat{e}_i\hat{f}_i \quad \text{where } i = 1, 2, 3 \quad (148)$$

where

$$\hat{e}_i = \text{multiplication on the left by } \sigma_i, \text{ so that } \hat{e}_i(F) = \sigma_i F \quad (149)$$

$$\hat{f}_i = \text{multiplication on the right by } I\sigma_i, \text{ so that } \hat{f}_i(F) = I\sigma_i F \quad (150)$$

This defines the 9 generators of $U(3)$.

With the additional restriction on $\hat{J}$

$$\alpha_1\hat{J}_1 + \alpha_2\hat{J}_2 + \alpha_3\hat{J}_3, \text{ with } \alpha_1 + \alpha_2 + \alpha_3 = 0 \quad (151)$$

the number generators is reduced to 8, consistently with $SU(3)$.

We now must show that the following equation is satisfied for all 8 generators:

$$[\psi^\dagger\gamma_0\psi]_{3,4}\psi^\dagger\gamma_0\psi = \underbrace{[(e^{\theta^i\lambda_i}\psi)^\dagger\gamma_0e^{\theta^i\lambda_i}\psi]_{3,4}}_{\text{copy 1}} \underbrace{(e^{\theta^i\lambda_i}\psi)^\dagger\gamma_0e^{\theta^i\lambda_i}\psi}_{\text{copy 2}} \quad (152)$$

Proof. First, we note the following action:

$$-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0 \quad (153)$$

which we can rewrite as follows:

$$-(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2)\gamma_0\mathbf{f} \quad (154)$$

The first three terms anticommute with γ_0 , while the last three commute with γ_0 :

$$= \gamma_0(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 - B_1\gamma_2\gamma_3 - B_2(q)\gamma_1\gamma_3 - B_3(q)\gamma_1\gamma_2)\mathbf{f}(q) \quad (155)$$

This can be written as:

$$\gamma_0(\mathbf{E} - \mathbf{B})(\mathbf{E} + \mathbf{B}) \quad (156)$$

$$= \gamma_0(\mathbf{E}^2 + \mathbf{E}\mathbf{B} - \mathbf{B}\mathbf{E} - \mathbf{B}^2) \quad (157)$$

where $\mathbf{E} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3$ and $\mathbf{B} = B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$.

Thus, for $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$, we require: 1) $\mathbf{E}^2 - \mathbf{B}^2 = 1$ and 2) $\mathbf{E}\mathbf{B} = \mathbf{B}\mathbf{E}$. The first requirement expands as follows:

$$\mathbf{E}^2 - \mathbf{B}^2 = (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) = 1 \quad (158)$$

which is the defining conditions for the SU(3) symmetry group.

Finally, as the SU(3) norm is a consequence of preserving the Dirac current, it follows that the SU(3) generators provided by Lasenby, acting on $\mathbf{f}$, cannot change the SU(3) norm, hence must also preserve the Dirac current. $\square$

Theorem 13 (Unitary invariance). *Let U be $n \times n$ unitary matrices. Then unitary invariance:*

$$\langle\langle \psi | \gamma_\mu \psi | \psi | \gamma_\nu \psi \rangle\rangle = \langle\langle U\psi | \gamma_\mu U\psi | U\psi | \gamma_\nu U\psi \rangle\rangle \implies U^\dagger U = I \quad (159)$$

is individually satisfied for each copied bilinear form.

Proof. Equation (159) is satisfied if $U^\dagger \gamma_\mu U = \gamma_\mu$. Since U is valued in complex numbers, then $U^\dagger = U^T$, and since $\gamma_\mu \gamma_0 \gamma_1 \gamma_2 \gamma_3 = -\gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_\mu$, it follows that:

$$\gamma_\mu U^\dagger U = \gamma_\mu \quad (160)$$

which is satisfied when $U^\dagger U = I$. $\square$

The invariances SU(3), SU(2) and U(1) discussed above can be promoted to local symmetries using the usual gauge symmetry construction techniques, along with the Dirac equation or field Lagrangian.

In conventional QM, the Born rule naturally leads to a U(1)-valued gauge theory due to the following symmetry:

$$(e^{-i\theta(x)}\psi(x))^\dagger e^{-i\theta(x)}\psi(x) = \psi(x)^\dagger \psi(x) \quad (161)$$

However, the SU(3) and SU(2) symmetries do not emerge from the probability measure in the same straightforward manner and are typically introduced manually, justified by experimental observations. This raises the question: why these specific symmetries and not others? In contrast, within the double-copy product framework, all three symmetry groups—U(1), SU(2), and SU(3)—as well as the Spin(3,1) and unitary symmetries, follow naturally from the invariance of the probability measure, in the same way that the U(1) symmetry follows from the Born rule. This suggests a deeper underlying principle governing the symmetries in fundamental physics.

2.3.4. A Starting Point for a Theory of Quantum Gravity

In the previous section, we developed a quantum theory valued in $\text{Spin}^c(3,1)$, which served as the arena for RQM. We then demonstrated how a single copy of this theory leads to the gauge symmetries of the standard model, the Dirac current and other features of RQM. The goal of this section is to extend this methodology to arbitrary basis vectors, in which the metric tensor emerges as an observable. To achieve this, we will utilize both copies.

We recall the definition of the metric tensor in terms of basis vectors of geometric algebra, as follows:

$$g_{\mu\nu} = \frac{1}{2}(\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu) \quad (162)$$

Then, we note that the double-copy product acts on a pair of basis element $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$, as follows:

$$\frac{1}{2} \left(\underbrace{[\psi^\dagger \mathbf{e}_\mu \psi]_{3,4}}_{\text{copy 1}} \underbrace{\psi^\dagger \mathbf{e}_\nu \psi}_{\text{copy 2}} + \underbrace{[\psi^\dagger \mathbf{e}_\nu \psi]_{3,4}}_{\text{copy 2}} \underbrace{\psi^\dagger \mathbf{e}_\mu \psi}_{\text{copy 1}} \right) \quad (163)$$

$$= \frac{1}{2} \left(\underbrace{\tilde{R} \rho e^{ib/2} e^{-ib/2}}_{\text{Born rule copy 1}} \mathbf{e}_\mu R \tilde{R} \underbrace{\rho e^{-ib/2} e^{ib/2}}_{\text{Born rule copy 2}} \mathbf{e}_\nu R + \underbrace{\tilde{R} \rho e^{ib/2} e^{-ib/2}}_{\text{Born rule copy 2}} \mathbf{e}_\nu R \tilde{R} \underbrace{\rho e^{-ib/2} e^{ib/2}}_{\text{Born rule copy 1}} \mathbf{e}_\mu R \right) \quad (164)$$

$$= \frac{1}{2} \rho^2 (\tilde{R} \mathbf{e}_\mu R \tilde{R} \mathbf{e}_\nu R + \tilde{R} \mathbf{e}_\nu R \tilde{R} \mathbf{e}_\mu R) \quad (165)$$

$$= \underbrace{\rho^2}_{\text{probability}} \underbrace{\frac{1}{2} (\tilde{\mathbf{e}}_\mu \tilde{\mathbf{e}}_\nu + \tilde{\mathbf{e}}_\nu \tilde{\mathbf{e}}_\mu)}_{\text{metric tensor}} \quad (166)$$

where $\tilde{\mathbf{e}}_\mu$ and $\tilde{\mathbf{e}}_\nu$ are SO(3,1) rotated basis vectors, and where ρ^2 is a probability measure.

As one can swap $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ and obtain the same metric tensor, the double-copy product guarantees that $g_{\mu\nu}$ is symmetric.

Furthermore, since $\mathbf{e}_\mu^\dagger = -\mathbf{e}_\mu$, we get:

$$[(\mathbf{e}_\mu \psi)^\dagger \psi]_{3,4} (\mathbf{e}_\nu \psi)^\dagger \psi \quad (167)$$

$$= [\psi^\dagger (-1) \mathbf{e}_\mu^\dagger \psi]_{3,4} \psi^\dagger (-1) \mathbf{e}_\nu^\dagger \psi \quad (168)$$

$$= [\psi^\dagger \mathbf{e}_\mu \psi]_{3,4} \psi^\dagger \mathbf{e}_\nu \psi \quad (169)$$

which allows us to conclude that $\mathbf{e}_\mu$ and $\mathbf{e}_\nu$ are self-adjoint within the double-copy product, entailing the interpretation of $g_{\mu\nu}$ as an observable.

In the double-copy product, the metric tensor emerges as a double copy of Dirac currents. This formulation suggests that the metric tensor encodes the probabilistic structure of a quantum theory of gravity *in the form of a rank-2 tensor*, analogous to how the Dirac current encodes the probabilistic structure of a special relativistic quantum theory *in the form of a 4-vector*.

Let us now investigate the dynamics. We recall that the evolution operator (Definition 21) is:

$$T = \begin{bmatrix} e^{-\frac{1}{2}\zeta(\mathbf{f}_1 + \mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} \quad (170)$$

Acting on the wavefunction, the effect of this operator cascades down to the basis vectors via the double-copy product:

$$\underbrace{[\psi^\dagger T^\dagger \mathbf{e}_\mu T \psi]_{3,4}}_{\text{copy 1}} \underbrace{\psi^\dagger T^\dagger \mathbf{e}_\nu T \psi}_{\text{copy 2}} + \underbrace{[\psi^\dagger T^\dagger \mathbf{e}_\nu T \psi]_{3,4}}_{\text{copy 2}} \underbrace{\psi^\dagger T^\dagger \mathbf{e}_\mu T \psi}_{\text{copy 1}} \quad (171)$$

which realizes an SO(3,1) transformation of the metric tensor via action of the exponential of a bivector, and a double-copy unitary invariant transformation via action of the exponential of a pseudo-scalar:

$$\underbrace{[\psi^\dagger \underbrace{e^{\frac{1}{2}\zeta \mathbf{f}} \mathbf{e}_\mu e^{-\frac{1}{2}\zeta \mathbf{f}}}_{\text{SO(3,1) evolution}} \underbrace{e^{\frac{1}{2}\zeta \mathbf{b}} e^{-\frac{1}{2}\zeta \mathbf{b}}}_{\text{unitary evolution}} \psi]_{3,4}}_{\text{copy 1}} \psi^\dagger \underbrace{e^{\frac{1}{2}\zeta \mathbf{f}} \mathbf{e}_\nu e^{-\frac{1}{2}\zeta \mathbf{f}}}_{\text{SO(3,1) evolution}} \underbrace{e^{\frac{1}{2}\zeta \mathbf{b}} e^{-\frac{1}{2}\zeta \mathbf{b}}}_{\text{unitary evolution}} \psi + \dots \quad (172)$$

In summary, this initial investigation has identified a scenario in which the metric tensor is measured using basis vectors. The evolution operator, governed by the Schrödinger equation, dynamically realizes SO(3,1) transformations on the metric tensor. Furthermore, the amplitudes associated with

possible metric tensors are derived from a double-copy of unitary quantum theories acting on the basis vectors. This formulation simultaneously preserves the $SO(3,1)$ symmetry, essential for describing spacetime structure, and the unitary symmetry, fundamental to quantum mechanics. It describes all changes of basis transformations that an observer in 3+1D spacetime can perform prior to measuring (in the quantum sense) a basis system in spacetime, and attributes a probability to the outcome (the outcome being the metric tensor).

2.4. Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to resolve the entropy maximization problem for other dimensional configurations. We found that all geometric configurations except those we have explored here (e.g. $GA(0) \cong \mathbb{R}$, $GA(0,1) \cong \mathbb{C}$, $GA(2,0)$ and $GA(3,1)$) are obstructed. By obstructed, we mean that the solution to the entropy maximization problem, ρ , does not satisfy all axioms of probability theory.

Dimensions	Obstruction	
$GA(0)$	Unobstructed $\implies$ statistical mechanics	(173)
$GA(0,1)$	Unobstructed $\implies$ quantum mechanics	(174)
$GA(1,0)$	Negative probabilities in the RQM	(175)
$GA(2,0)$	Unobstructed $\implies$ toy model	(176)
$GA(1,1)$	Negative probabilities in the RQM	(177)
$GA(0,2)$	Not isomorphic to a real matrix algebra	(178)
$GA(3,0)$	Not isomorphic to a real matrix algebra	(179)
$GA(2,1)$	Not isomorphic to a real matrix algebra	(180)
$GA(1,2)$	Not isomorphic to a real matrix algebra	(181)
$GA(0,3)$	Not isomorphic to a real matrix algebra	(182)
$GA(4,0)$	Not isomorphic to a real matrix algebra	(183)
$GA(3,1)$	Unobstructed $\implies$ quantum gravity $\wedge SU(3) \times SU(2) \times U(1)$	(184)
$GA(2,2)$	Negative probabilities in the RQM	(185)
$GA(1,3)$	Not isomorphic to a real matrix algebra	(186)
$GA(0,4)$	Not isomorphic to a real matrix algebra	(187)
$GA(5,0)$	Not isomorphic to a real matrix algebra	(188)
$\vdots$	$\vdots$	
$GA(6,0)$	No probability measure as a self-product	(189)
$\vdots$	$\vdots$	
∞		(190)

Let us now demonstrate the obstructions mentioned above.

Theorem 14 (Not isomorphic to a real matrix algebra). *The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.*

Proof. These geometric algebras are classified as follows:

$$\text{GA}(0, 2) \cong \mathbb{H} \quad (191)$$

$$\text{GA}(3, 0) \cong \mathbb{M}_2(\mathbb{C}) \quad (192)$$

$$\text{GA}(2, 1) \cong \mathbb{M}_2^2(\mathbb{R}) \quad (193)$$

$$\text{GA}(1, 2) \cong \mathbb{M}_2(\mathbb{C}) \quad (194)$$

$$\text{GA}(0, 3) \cong \mathbb{H}^2 \quad (195)$$

$$\text{GA}(4, 0) \cong \mathbb{M}_2(\mathbb{H}) \quad (196)$$

$$\text{GA}(1, 3) \cong \mathbb{M}_2(\mathbb{H}) \quad (197)$$

$$\text{GA}(0, 4) \cong \mathbb{M}_2(\mathbb{H}) \quad (198)$$

$$\text{GA}(5, 0) \cong \mathbb{M}_2^2(\mathbb{H}) \quad (199)$$

The determinant of these objects is valued in $\mathbb{C}$ or in $\mathbb{H}$, where $\mathbb{C}$ are the complex numbers, and where $\mathbb{H}$ are the quaternions. $\square$

Theorem 15 (Negative Probabilities in the RQM). *The even sub-algebra, which associates to the RQM part of the theory, of these dimensional configurations allows for negative probabilities, making them unsuitable as a RQM.*

Proof. This category contains three dimensional configurations:

GA(1,0): Let $\psi = a + be_1$, then:

$$(a + be_1)^\dagger(a + be_1) = (a - be_1)(a + be_1) = a^2 - b^2e_1e_1 = a^2 - b^2 \quad (200)$$

which is valued in $\mathbb{R}$.

GA(1,1): Let $\psi = a + be_0e_1$, then:

$$(a + be_0e_1)^\dagger(a + be_0e_1) = (a - be_0e_1)(a + be_0e_1) = a^2 - b^2e_0e_1e_0e_1 = a^2 - b^2 \quad (201)$$

which is valued in $\mathbb{R}$.

GA(2,2): Let $\psi = a + be_0e_1e_2$, where $e_0^2 = -1, e_1^2 = -1, e_2^2 = 1$, then:

$$[(a + \mathbf{b})^\dagger(a + \mathbf{b})]_{3,4}(a + \mathbf{b})^\dagger(a + \mathbf{b}) \quad (202)$$

$$= [a^2 + 2a\mathbf{b} + \mathbf{b}^2]_{3,4}(a^2 + 2a\mathbf{b} + \mathbf{b}^2) \quad (203)$$

We note that $\mathbf{b}^2 = b^2e_0e_1e_2e_0e_1e_2 = b^2$, therefore:

$$= (a^2 + b^2 - 2a\mathbf{b})(a^2 + b^2 + 2a\mathbf{b}) \quad (204)$$

$$= (a^2 + b^2)^2 - 4a^2\mathbf{b}^2 \quad (205)$$

$$= (a^2 + b^2)^2 - 4a^2b^2 \quad (206)$$

which is valued in $\mathbb{R}$.

In all of these cases the RQM probability can be negative. $\square$

Conjecture 1 (No probability measures as a self-product (in 6D)). *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 16 collapse. The research by Acus et al.[11] in 6D geometric algebra demonstrates that the determinant, so far

defined through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) \quad (207)$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[11].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2 a_{47}^2 + b_2 a_0^2 a_{47}^2 p_{412} p_{422} + \langle 72 \text{ monomials} \rangle = 0 \quad (208)$$

$$b_1 a_0^3 a_{52} + 2b_2 a_0 a_{47}^2 a_{52} p_{412} p_{422} p_{432} p_{442} p_{452} + \langle 72 \text{ monomials} \rangle = 0 \quad (209)$$

$$\langle 74 \text{ monomials} \rangle = 0 \quad (210)$$

$$\langle 74 \text{ monomials} \rangle = 0 \quad (211)$$

From Equation (207), it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1 \mathbf{O} B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 B g_5(g_4(B) g_3(g_2(B) g_1(B))) = b_1 B f_5(f_4(B) f_3(f_2(B) f_1(B))) + b_2 \mathbf{O} B g_5(g_4(B) g_3(g_2(B) g_1(B))) \quad (212)$$

Any equality of the above type between $b_1 \mathbf{O}$ and $b_2 \mathbf{O}$ is frustrated by the factors b_1 and b_2 , forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory theory (with non-trivial observables) be constructible in 6D, using our method. $\square$

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a relevant quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2 (No probability measures as a self-product (above 6D)). *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out $GA(1,3)$ even if in relativity, the signature of the metric $(+, -, -, -)$ versus $(-, -, -, +)$ does not influence the physics. However, in geometric algebra, $GA(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $GA(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

Consequently, the only dimensional configuration (other than the purely scalar configurations of $GA(0) \cong \mathbb{R}$ and $GA(0,1) \cong \mathbb{C}$) in which a 'least biased' solution to the problem of maximizing the Shannon entropy of quantum measurements relative to an initial preparation exists is 3+1D.

3. Conclusions

In this work, we have presented a novel framework that derives fundamental physical laws from a single, unifying principle: the maximization of the Shannon entropy of all possible measurements relative to a system's initial state, subject to specific linear constraints. By carefully choosing the nature of these constraints, we have demonstrated how this principle recovers various pillars of modern physics within a unified and parsimonious approach.

Specifically, imposing a vanishing $U(1)$ group constraint leads directly to the emergence of quantum mechanics, encapsulating complex probability amplitudes and interference effects intrinsic

to quantum phenomena. Extending this framework by adopting a $\text{Spin}^c(3,1)$ group constraint, we naturally obtain a relativistic quantum theory. Remarkably, in 3+1 dimensions, this approach uniquely results in the metric tensor of general relativity via a double-copy mechanism applied to the Dirac current. Furthermore, it inherently incorporates the $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ gauge symmetries of the Standard Model, providing a unified description of fundamental interactions.

Our findings suggest that the foundational structures of physics—quantum mechanics, special relativity, general relativity, and the gauge symmetries governing particle interactions—can be understood as natural consequences of an underlying principle of maximum entropy under linear constraints. This principle offers a least-biased theory with respect to arbitrary measurements, implying that the laws of physics are not arbitrary but are instead determined by the requirement of maximal informational entropy consistent with the imposed constraints.

Moreover, the emergence of a 3+1-dimensional spacetime within this framework is particularly noteworthy. The dimensionality arises not from empirical input but as a unique solution to the entropy maximization problem under the specified constraints. This provides a potential explanation for why our universe exhibits precisely four spacetime dimensions and suggests a deep connection between information theory and the fundamental structure of reality.

By reducing the complexity of fundamental physics to a single, parsimonious principle, this work opens new avenues for understanding the interconnectedness of physical laws. It highlights the power of entropy maximization as a foundational tool in theoretical physics and invites further exploration into how other physical phenomena might be derived from similar informational principles.

Data Availability Statement: No datasets were generated or analyzed during the current study. During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

Conflicts of Interest: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

Appendix A. SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L} = \underbrace{-k_B \sum_i \rho_i \ln \rho_i}_{\text{Boltzmann Entropy}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\bar{E} - \sum_i \rho_i E_i\right)}_{\text{Average Energy Constraint}} \quad (\text{A1})$$

We solve the maximization problem as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_i, \dots, \rho_n)}{\partial \rho_i} \quad (\text{A2})$$

$$= -\ln \rho_i - 1 - \lambda - \beta E_i \quad (\text{A3})$$

$$= \ln \rho_i + 1 + \lambda + \beta E_i \quad (\text{A4})$$

$$\Rightarrow \ln \rho_i = -1 - \lambda - \beta E_i \quad (\text{A5})$$

$$\Rightarrow \rho_i = \exp(-1 - \lambda) \exp(-\beta E_i) \quad (\text{A6})$$

$$= \frac{1}{Z(\tau)} \exp(-\beta E_i) \quad (\text{A7})$$

The partition function, is obtained as follows:

$$1 = \sum_i \exp(-1 - \lambda) \exp(-\beta E_i) \quad (\text{A8})$$

$$\Rightarrow (\exp(-1 - \lambda))^{-1} = \sum_i \exp(-\beta E_i) \quad (\text{A9})$$

$$Z(\tau) := \sum_i \exp(-\beta E_i) \quad (\text{A10})$$

Finally, the probability measure is:

$$\rho_i = \frac{1}{\sum_i \exp(-\beta E_i)} \exp(-\beta E_i) \quad (\text{A11})$$

Appendix B. RQM in 3+1D

$$\mathcal{L} = \underbrace{-\sum_i \rho_i \ln \frac{\rho_i}{p_i}}_{\text{Relative Entropy}} + \underbrace{\lambda \left(1 - \sum_i \rho_i\right)}_{\text{Normalization Constraint}} + \underbrace{\zeta \left(-\text{tr} \frac{1}{2} \sum_i \rho_i \mathbf{M}_i\right)}_{\substack{\text{Vanishing Anti-Constraint} \\ \text{Relativistic-Phase}}}} \quad (\text{A12})$$

The solution is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho_i = \underbrace{\frac{1}{\sum_i p_i \det \exp(-\zeta \frac{1}{2} \mathbf{M}_i)}}_{\substack{\text{Spin}^c(3,1) \text{ Invariant Ensemble}}} \underbrace{\det \exp(-\zeta \frac{1}{2} \mathbf{M}_i)}_{\substack{\text{Spin}^c(3,1) \text{ Born Rule}}} \underbrace{p_i}_{\text{Initial Preparation}} \quad (\text{A13})$$

Proof. The Lagrange multiplier equation can be solved as follows:

$$0 = \frac{\partial \mathcal{L}(\rho_1, \dots, \rho_n)}{\partial \rho_i} \quad (\text{A14})$$

$$= -\ln \frac{\rho_i}{p_i} - p_i - \lambda - \zeta \text{tr} \frac{1}{2} \mathbf{M}_i \quad (\text{A15})$$

$$= \ln \frac{\rho_i}{p_i} + p_i + \lambda + \zeta \text{tr} \frac{1}{2} \mathbf{M}_i \quad (\text{A16})$$

$$\Rightarrow \ln \frac{\rho_i}{p_i} = -p_i - \lambda - \zeta \text{tr} \frac{1}{2} \mathbf{M}_i \quad (\text{A17})$$

$$\Rightarrow \rho_i = p_i \exp(-p_i - \lambda) \exp\left(-\zeta \text{tr} \frac{1}{2} \mathbf{M}_i\right) \quad (\text{A18})$$

$$= \frac{1}{Z(\zeta)} p_i \exp\left(-\zeta \text{tr} \frac{1}{2} \mathbf{M}_i\right) \quad (\text{A19})$$

The partition function $Z(\zeta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_i p_i \exp(-p_i - \lambda) \exp\left(-\zeta \text{tr} \frac{1}{2} \mathbf{M}_i\right) \quad (\text{A20})$$

$$\Rightarrow (\exp(-p_i - \lambda))^{-1} = \sum_i p_i \exp\left(-\zeta \text{tr} \frac{1}{2} \mathbf{M}_i\right) \quad (\text{A21})$$

$$Z(\zeta) := \sum_i p_i \exp\left(-\zeta \text{tr} \frac{1}{2} \mathbf{M}_i\right) \quad (\text{A22})$$

□

Appendix C. SageMath program showing $[u^\dagger u]_{3,4} u^\dagger u = \det M_u$

```

from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix

# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])

# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)

# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()

# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')

# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3
udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4

u2 = u.clifford_conjugate()*u

u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4

I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

#MAJORANA MATRICES
y0 = Matrix(SR, [[0, 0, 0, 1],
                [0, 0, -1, 0],
                [0, 1, 0, 0],

```

```

        [-1, 0, 0, 0]])

y1 = Matrix(SR, [[0, -1, 0, 0],
                 [-1, 0, 0, 0],
                 [0, 0, 0, -1],
                 [0, 0, -1, 0]])

y2 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, -1, 0, 0],
                 [1, 0, 0, 0]])

y3 = Matrix(SR, [[-1, 0, 0, 0],
                 [0, 1, 0, 0],
                 [0, 0, -1, 0],
                 [0, 0, 0, 1]])

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())

```

The program outputs

True

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the double-copy form: $\det \mathbf{M}_u = [\mathbf{u}^\dagger \mathbf{u}]_{3,4} \mathbf{u}^\dagger \mathbf{u}$.

1. Dirac, P.A.M. *The principles of quantum mechanics*; Number 27, Oxford university press, 1981.
2. Von Neumann, J. *Mathematical foundations of quantum mechanics: New edition*; Vol. 53, Princeton university press, 2018.
3. Jaynes, E.T. Information theory and statistical mechanics. *Physical review* **1957**, 106, 620.
4. Jaynes, E.T. Information theory and statistical mechanics. II. *Physical review* **1957**, 108, 171.
5. Kullback, S.; Leibler, R.A. On information and sufficiency. *The annals of mathematical statistics* **1951**, 22, 79–86.
6. Shannon, C.E. A mathematical theory of communication. *Bell system technical journal* **1948**, 27, 379–423.
7. Hestenes, D. Spacetime physics with geometric algebra. *American Journal of Physics* **2003**, 71, 691–714.
8. Lundholm, D. Geometric (Clifford) algebra and its applications. *arXiv preprint math/0605280* **2006**.
9. Hestenes, D. Space-time structure of weak and electromagnetic interactions. *Foundations of Physics* **1982**, 12, 153–168.
10. Lasenby, A. Some recent results for $SU(3)$ and Octonions within the Geometric Algebra approach to the fundamental forces of nature. *arXiv preprint arXiv:2202.06733* **2022**.
11. Acus, A.; Dargys, A. Inverse of multivector: Beyond $p+q=5$ threshold. *arXiv preprint arXiv:1712.05204* **2017**.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.