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Article

Noncommutative Heisenberg-Robertson-Schrodinger Uncertainty Principles

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Abstract: Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ be possibly unbounded self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we show that

$$(1) \quad \Delta_x(B)^2 d_x(A)^2 + \Delta_x(A)^2 d_x(B)^2 \geq \frac{(\langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\})^2 - (\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])^2}{2}$$

and

$$(2) \quad \Delta_x(A)\Delta_x(B) \geq \frac{\sqrt{\|(\langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\})^2 - (\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])^2\|}}{2},$$

where

$$\Delta_x(A) := \|Ax - \langle Ax, x \rangle x\|, \quad d_x(A) := \sqrt{\langle Ax, Ax \rangle - \langle Ax, x \rangle^2},$$

$$[A, B] := AB - BA, \quad \{A, B\} := AB + BA,$$

$$\{\langle Ax, x \rangle, \langle Bx, x \rangle\} := \langle Ax, x \rangle \langle Bx, x \rangle + \langle Bx, x \rangle \langle Ax, x \rangle,$$

$$[\langle Ax, x \rangle, \langle Bx, x \rangle] := \langle Ax, x \rangle \langle Bx, x \rangle - \langle Bx, x \rangle \langle Ax, x \rangle.$$

We call Inequalities (1) and (2) as noncommutative Heisenberg-Robertson-Schrodinger uncertainty principles. They reduce to the Heisenberg-Robertson-Schrodinger uncertainty principle (derived by Schrodinger in 1930) whenever $\mathcal{A} = \mathbb{C}$.

Keywords: Uncertainty Principle; Hilbert C^* -module

MSC: 46L08

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and A be a possibly unbounded self-adjoint operator defined on the domain $\mathcal{D}(A) \subseteq \mathcal{H}$. For $h \in \mathcal{D}(A)$ with $\|h\| = 1$, define the **uncertainty** of A at the point h as

$$\Delta_h(A) := \|Ah - \langle Ah, h \rangle h\| = \sqrt{\|Ah\|^2 - \langle Ah, h \rangle^2}.$$

In 1929, Robertson [1] derived the following mathematical form of the uncertainty principle (also known as uncertainty relation) by Heisenberg in 1927 [2]. Recall that for two linear operators $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$, we define the commutator $[A, B] := AB - BA$ and the anti-commutator $\{A, B\} := AB + BA$.

Theorem 1. [1–4] (*Heisenberg-Robertson Uncertainty Principle*) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$\frac{1}{2}(\Delta_h(A)^2 + \Delta_h(B)^2) \geq \frac{1}{4}(\Delta_h(A) + \Delta_h(B))^2 \geq \Delta_h(A)\Delta_h(B) \geq \frac{1}{2}|\langle [A, B]h, h \rangle|. \quad (1)$$

In 1930, Schrodinger improved Inequality (1) [5].

Theorem 2. [5,6] (*Heisenberg-Robertson-Schrodinger Uncertainty Principle*) Let $A : \mathcal{D}(A) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ and $B : \mathcal{D}(B) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be self-adjoint operators. Then for all $h \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\|h\| = 1$, we have

$$\begin{aligned}\Delta_h(A)\Delta_h(B) &\geq |\langle Ah, Bh \rangle - \langle Ah, h \rangle \langle Bh, h \rangle| = \frac{\sqrt{|\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle|^2 + |\langle [A, B]h, h \rangle|^2}}{2} \\ &= \frac{\sqrt{(\langle \{A, B\}h, h \rangle - 2\langle Ah, h \rangle \langle Bh, h \rangle)^2 - \langle [A, B]h, h \rangle^2}}{2}.\end{aligned}$$

We are fundamentally concerned with the question: What is the noncommutative analogues of Theorem 2? We then naturally look to the notion of Hilbert C^* -modules which are first introduced by Kaplansky [7] for modules over commutative C^* -algebras and later developed for modules over arbitrary C^* -algebras by Paschke [8] and Rieffel [9].

Definition 1. [7–9] Let $\mathcal{A}$ be a unital C^* -algebra. A left module $\mathcal{E}$ over $\mathcal{A}$ is said to be a (left) **Hilbert C^* -module** if there exists a map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$ such that the following hold.

- (i) $\langle x, x \rangle \geq 0, \forall x \in \mathcal{E}$. If $x \in \mathcal{E}$ satisfies $\langle x, x \rangle = 0$, then $x = 0$.
- (ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \forall x, y, z \in \mathcal{E}$.
- (iii) $\langle ax, y \rangle = a\langle x, y \rangle, \forall x, y \in \mathcal{E}, \forall a \in \mathcal{A}$.
- (iv) $\langle x, y \rangle = \langle y, x \rangle^*, \forall x, y \in \mathcal{E}$.
- (v) $\mathcal{E}$ is complete w.r.t. the norm $\|x\| := \sqrt{\|\langle x, x \rangle\|}, \forall x \in \mathcal{E}$.

We are going to use the following noncommutative Cauchy-Schwarz inequality.

Theorem 3. [8,9] Let $\mathcal{E}$ be a Hilbert C^* -module. Then

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle y, y \rangle\| \|\langle x, x \rangle\|, \quad \forall x, y \in \mathcal{E}.$$

We derive noncommutative analogues of Theorem 2 in Theorem 4.

2. Noncommutative Heisenberg-Robertson-Schrodinger Uncertainty Principles

Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$ and A be a possibly unbounded self-adjoint morphism defined on domain $\mathcal{D}(A) \subseteq \mathcal{E}$. For $x \in \mathcal{D}(A)$ with $\langle x, x \rangle = 1$, we define the **noncommutative norm-uncertainty** of A at the point x as

$$\Delta_x(A) := \|Ax - \langle Ax, x \rangle x\| = \sqrt{\|\langle Ax, Ax \rangle - \langle Ax, x \rangle^2\|}.$$

We define the **noncommutative modular-uncertainty** of A at the point x as

$$d_x(A) := \sqrt{\langle Ax, Ax \rangle - \langle Ax, x \rangle^2} = \sqrt{\langle Ax - \langle Ax, x \rangle x, Ax - \langle Ax, x \rangle x}.$$

Then $d_x(A) \in \mathcal{A}$ and

$$\Delta_x(A) = \|d_x(A)\|.$$

Given two elements $a, b \in \mathcal{A}$, we define $[a, b] := ab - ba$ and $\{a, b\} := ab + ba$. Similarly, for two morphisms $A : \mathcal{D}(A) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \rightarrow \mathcal{E}$, we define $[A, B] := AB - BA$ and $\{A, B\} := AB + BA$.

Theorem 4. (*Noncommutative Heisenberg-Robertson-Schrodinger Uncertainty Principles*) Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we have

(i)

$$\begin{aligned}\Delta_x(B)^2 d_x(A)^2 &\geq (\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)(\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle) \\ &\geq -\frac{(\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])^2}{4} \\ &= \frac{(\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])(\langle x, [A, B]x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])}{4}.\end{aligned}$$

(ii)

$$\begin{aligned}\Delta_x(B) d_x(A) &\geq \sqrt{(\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)(\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle)} \\ &\geq \frac{\sqrt{(\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])(\langle x, [A, B]x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])}}{2}.\end{aligned}$$

(iii)

$$\begin{aligned}\Delta_x(A)^2 d_x(B)^2 &\geq (\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle)(\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle) \\ &\geq -\frac{(\langle [B, A]x, x \rangle + [\langle Bx, x \rangle, \langle Ax, x \rangle])^2}{4} \\ &= \frac{(\langle [B, A]x, x \rangle + [\langle Bx, x \rangle, \langle Ax, x \rangle])(\langle x, [B, A]x \rangle + [\langle Bx, x \rangle, \langle Ax, x \rangle])}{4}.\end{aligned}$$

(iv)

$$\begin{aligned}\Delta_x(A) d_x(B) &\geq \sqrt{(\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle)(\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)} \\ &\geq \frac{\sqrt{(\langle [B, A]x, x \rangle + [\langle Bx, x \rangle, \langle Ax, x \rangle])(\langle x, [B, A]x \rangle + [\langle Bx, x \rangle, \langle Ax, x \rangle])}}{2}.\end{aligned}$$

(v)

$$\begin{aligned}\Delta_x(A) \Delta_x(B) &\geq \|\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle\| \\ &\geq \frac{\sqrt{\|(\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])(\langle x, [A, B]x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])\|}}{2} \\ &= \frac{\|(\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])\|}{2}.\end{aligned}$$

(vi)

$$\begin{aligned}\Delta_x(B)^2 d_x(A)^2 + \Delta_x(A)^2 d_x(B)^2 &\geq \frac{(\langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\})^2 - (\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])^2}{2} \\ &= \frac{(\langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\})^2 + (\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])(\langle x, [A, B]x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])}{2}.\end{aligned}$$

(vii)

$$\begin{aligned}\Delta_x(A) \Delta_x(B) &\geq \frac{\sqrt{\|(\langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\})^2 - (\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])^2\|}}{2} \\ &= \frac{\sqrt{\|(\langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\})^2 + (\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])(\langle x, [A, B]x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])\|}}{2}.\end{aligned}$$

(viii) If $\mathcal{A}$ is commutative, then

$$\begin{aligned}\Delta_x(B)^2 d_x(A)^2 + \Delta_x(A)^2 d_x(B)^2 &\geq \frac{(\langle \{A, B\}x, x \rangle - 2\langle Ax, x \rangle \langle Bx, x \rangle)^2 - \langle [A, B]x, x \rangle^2}{2} \\ &= \frac{(\langle \{A, B\}x, x \rangle - 2\langle Ax, x \rangle \langle Bx, x \rangle)^2 + \langle [A, B]x, x \rangle \langle x, [A, B]x \rangle}{2}.\end{aligned}$$

(ix) If $\mathcal{A}$ is commutative, then

$$\begin{aligned}\Delta_x(A)\Delta_x(B) &\geq \frac{\sqrt{\|(\langle \{A, B\}x, x \rangle - 2\langle Ax, x \rangle \langle Bx, x \rangle)^2 - \langle [A, B]x, x \rangle^2\|}}{2} \\ &= \frac{\sqrt{\|(\langle \{A, B\}x, x \rangle - 2\langle Ax, x \rangle \langle Bx, x \rangle)^2 + \langle [A, B]x, x \rangle \langle x, [A, B]x \rangle\|}}{2}.\end{aligned}$$

Proof. Let $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ be such that $\langle x, x \rangle = 1$.

(i) Using noncommutative Cauchy-Schwarz inequality,

$$\begin{aligned}(\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)(\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle) \\ = \langle Ax - \langle Ax, x \rangle x, Bx - \langle Bx, x \rangle x \rangle \langle Bx - \langle Bx, x \rangle x, Ax - \langle Ax, x \rangle x \rangle \\ \leq \|Bx - \langle Bx, x \rangle x\| \|Ax - \langle Ax, x \rangle x\| \\ = \|Bx - \langle Bx, x \rangle x\|^2 (\langle Ax, Ax \rangle - \langle Ax, x \rangle^2) = \Delta_x(B)^2 d_x(A)^2.\end{aligned}$$

We now note that

$$\begin{aligned}(\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle)(\langle Bx, Ax \rangle - \langle Bx, x \rangle \langle Ax, x \rangle) \\ \geq (\operatorname{Im}(\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle))^2 \\ = \left(\frac{\langle Ax, Bx \rangle - \langle Ax, x \rangle \langle Bx, x \rangle - \langle Bx, Ax \rangle + \langle Bx, x \rangle \langle Ax, x \rangle}{2i} \right)^2 \\ = -\frac{(\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])^2}{4}.\end{aligned}$$

Now we see that

$$-(\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])^2 = (\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])(\langle x, [A, B]x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle]).$$

(ii) Follows from (i) by noting that square root respects the order of positive elements.

(iii) Similar to (i).

(iv) Follows from (iii).

(v) We get by taking norm on (i) (norm respects the order of positive elements).

(vi) Define

$$z := Ax - \langle Ax, x \rangle x, \quad w := Bx - \langle Bx, x \rangle x.$$

Then

$$\begin{aligned}\langle z, w \rangle + \langle w, z \rangle &= \langle Ax - \langle Ax, x \rangle x, Bx - \langle Bx, x \rangle x \rangle + \langle Bx - \langle Bx, x \rangle x, Ax - \langle Ax, x \rangle x \rangle \\ &= \langle BAx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle + \langle ABx, x \rangle - \langle Bx, x \rangle \langle Ax, x \rangle \\ &= \langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\}\end{aligned}$$

and

$$\begin{aligned}\langle z, w \rangle - \langle w, z \rangle &= \langle Ax - \langle Ax, x \rangle x, Bx - \langle Bx, x \rangle x \rangle - \langle Bx - \langle Bx, x \rangle x, Ax - \langle Ax, x \rangle x \rangle \\ &= \langle BAx, x \rangle - \langle Ax, x \rangle \langle Bx, x \rangle - \langle ABx, x \rangle + \langle Bx, x \rangle \langle Ax, x \rangle \\ &= -\langle [A, B]x, x \rangle - [\langle Ax, x \rangle, \langle Bx, x \rangle].\end{aligned}$$

Now adding (i) and (iii) gives

$$\begin{aligned}\Delta_x(B)^2 d_x(A)^2 + \Delta_x(A)^2 d_x(B)^2 &\geq \langle z, w \rangle \langle w, z \rangle + \langle w, z \rangle \langle z, w \rangle \\ &= \frac{(\langle z, w \rangle + \langle w, z \rangle)^2 - (\langle z, w \rangle - \langle w, z \rangle)^2}{2} \\ &= \frac{(\langle \{A, B\}x, x \rangle - \{\langle Ax, x \rangle, \langle Bx, x \rangle\})^2 - (\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])^2}{2}.\end{aligned}$$

(vii) We get by taking norm on (vi).

(viii) Follows from (vi). Note that

$$-\langle [A, B]x, x \rangle^2 = \langle [A, B]x, x \rangle \langle x, [A, B]x \rangle.$$

(ix) Follows from (vii).

□

Corollary 1. (Noncommutative Heisenberg-Robertson Uncertainty Principles) Let $\mathcal{E}$ be a Hilbert C^* -module over a unital C^* -algebra $\mathcal{A}$. Let $A : \mathcal{D}(A) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ and $B : \mathcal{D}(B) \subseteq \mathcal{E} \rightarrow \mathcal{E}$ be self-adjoint morphisms. Then for all $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ with $\langle x, x \rangle = 1$, we have

(i)

$$\Delta_x(B)^2 d_x(A)^2 + \Delta_x(A)^2 d_x(B)^2 \geq \frac{(\langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])(\langle x, [A, B]x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle])}{2}.$$

(ii)

$$\Delta_x(A) \Delta_x(B) \geq \frac{\| \langle [A, B]x, x \rangle + [\langle Ax, x \rangle, \langle Bx, x \rangle] \|}{2}.$$

(iii) If $\mathcal{A}$ is commutative, then

$$\Delta_x(B)^2 d_x(A)^2 + \Delta_x(A)^2 d_x(B)^2 \geq \frac{\langle [A, B]x, x \rangle \langle x, [A, B]x \rangle}{2}.$$

(iv) If $\mathcal{A}$ is commutative, then

$$\Delta_x(A) \Delta_x(B) \geq \frac{\| \langle [A, B]x, x \rangle \|}{2}.$$

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