Article

The Principle of Emergent Information: A Proof of the Non-Axiomatic Origins of the Binary States 0 and 1

Miltiadis Karazoupis *

Posted Date: 31 October 2025

doi: 10.20944/preprints202510.2494.v1

Keywords: foundational mathematics; emergence; information theory; axiomatic systems; set theory; quantum foundations; superposition; binary logic; philosophy of mathematics; pre-geometric systems

Article

The Principle of Emergent Information: A Proof of the Non-Axiomatic Origins of the Binary States 0 and 1

Miltiadis Karazoupis

Independent Researcher; miltos.karazoupis@gmail.com

Abstract

This framework presents a formal proof that the foundational mathematical entities, 0 and 1, are not axiomatic primitives but are necessary, co-created consequences of a single, pre-geometric process of interaction. We challenge the axiomatic assumption of the empty set and the unit by positing a more fundamental reality: a primordial state of informational superposition, where the concepts of "occupied" and "unoccupied" are unresolved and indistinguishable. We demonstrate that any "act of definition" — a process that queries this state — must necessarily break this symmetry and resolve the potential into two distinct, mutually defined outcomes. These outcomes, which we label 0 (the state of non-self-reference) and 1 (the state of self-reference), are thus established as emergent informational facts, not a priori truths. We prove that this foundational interaction is prior to the mathematical constructs of set theory and Hilbert spaces, showing that a logical superposition can exist without a pre-supposed geometric or algebraic canvas. Furthermore, we establish the "Principle of Definiteness for Interaction," proving that only these resolved, factual states can engage in the stable interactions required to build more complex structures, such as the emergent continuum of primes. This work provides a formal, non-axiomatic origin for the binary, resolving its status from a postulated axiom to an inevitable consequence of a mathematical universe that contains information.

Keywords: foundational mathematics; emergence; information theory; axiomatic systems; set theory; quantum foundations; superposition; binary logic; philosophy of mathematics; pre-geometric systems

---

Introduction

The edifice of modern mathematics is built upon a foundation of axioms: propositions accepted without proof as the starting points for logical deduction. Within this paradigm, the foundational entities of 0 and 1 are not derived but are defined. The number 0 is typically associated with the empty set, whose existence is guaranteed by the Axiom of the Empty Set in Zermelo-Fraenkel set theory, while the number 1 is constructed as its successor. This axiomatic framework has been extraordinarily successful, providing a consistent and powerful basis for all of quantitative reasoning. However, its success masks a profound conceptual gap: it secures the properties of these entities without explaining their origin. It does not answer why these specific concepts must form the bedrock of a logical universe; it merely posits that they do.

This framework challenges that foundational assumption. We present a formal proof that the core components of mathematics are not axiomatic but are primordial facts — necessary and inevitable consequences of a mathematical universe in which information can be defined. We replace the assumption of axiomatic objects with a more fundamental substrate: a pre-geometric state of unrealized information existing in a logical superposition, where the concepts of "existence" and "non-existence" are unresolved and indistinguishable.

From this singular starting point, we prove a necessary and sequential emergence of mathematical reality. First, we demonstrate that any "act of definition" — a process that queries this

 doi:10.20944/preprints202510.2494.v1

informational potential — is a primordial fact of logic that must resolve the superposition into a binary outcome. This event simultaneously co-creates the mutually defined informational states of '0' (the fact of non-self-reference) and '1' (the fact of self-reference). We prove that this logical superposition and its resolution occur without the presupposition of a Hilbert space, suggesting that quantum-like properties are a feature of any emergent informational system.

Second, we establish the "Principle of Definiteness for Interaction," a primordial fact stating that only these resolved, definite states can engage in the stable interactions required to form complex structures. This provides the logical imperative for the collapse of potential into fact. Third, and most critically, we prove that the first definite entities to emerge from this process must, by logical necessity, be indivisible. They are the "informational atoms" of the system, as they cannot be factored into constituent parts that do not yet exist. This property of indivisibility is the very definition of a prime number.

Therefore, we prove that the system of prime numbers is not a pre-existing set to be studied, but is the necessary and inevitable consequence of the foundational collapse of informational superposition. This work provides the *a priori* logical genesis for the entities used in the author's prior work on the Emergent Continuum Hypothesis (Karazoupis, 2025), which demonstrated *a posteriori* how these primes interact to form a complex continuum. By replacing axioms with primordial facts, this framework offers a complete, dynamic, and self-contained origin story for the core of mathematics, reframing it as a structure that necessarily emerges from the fundamental requirement that information be definable.

Literature Review

The question of the ultimate foundation of mathematics has been a central theme of intellectual inquiry for centuries, oscillating between the Platonic view of mathematics as a discovered, objective reality and the formalist view of it as a constructed system of logical rules (Benacerraf & Putnam, 1983). The dominant modern approach is a pragmatic formalism rooted in axiomatic set theory, most commonly the Zermelo-Fraenkel axioms (ZFC). Within this paradigm, the foundational entities are not emergent but are postulated. The number 0 is defined by the Axiom of the Empty Set, and the number 1 is constructed as its successor (Zermelo, 1908; von Neumann, 1923). This framework is a high-level descriptive system that begins by assuming the truth of its objects. Our work diverges from this tradition by replacing these object-axioms with primordial facts that are logically necessary consequences of a mathematical universe where information can be defined.

This approach aligns with the intellectual shift towards viewing information as the fundamental substrate of reality. In physics, Wheeler's "It from Bit" doctrine proposed that physical existence emerges from binary, yes-no informational processes (Wheeler, 1990). Digital physics has further explored this by modelling the universe as a computational process (Fredkin, 2003). However, these paradigms typically begin with the 'bit'—the 0/1 distinction—as their irreducible primitive, without providing a formal proof for why reality must be fundamentally binary. Our work provides this missing proof, demonstrating that the bit is not a starting point, but is itself an emergent, co-created consequence of a more fundamental act of definition.

The structure of quantum theory provides a powerful, albeit abstract, framework for describing binary systems. The concept of superposition is formally described within the mathematical structure of a Hilbert space, a postulated framework that provides the stage upon which quantum phenomena occur (von Neumann, 1932). This framework challenges this view, proposing that the logical principle of superposition is prior to the geometric and algebraic structure of the Hilbert space. We prove that a state of unresolved potential and its collapse into definite outcomes is a pre-geometric, informational fact, and that the Hilbert space is a sophisticated subsequent construct used to model the dynamics of systems built upon this foundational reality.

Finally, this work provides the logically foundation for the author's previous research on the Emergent Continuum Hypothesis (Karazoupis, 2025). While the former work took the existence of primes as a starting point to demonstrate how they interact to form a continuum, the present work

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2025 doi:10.20944/preprints202510.2494.v1

proves why these entities must be the first to exist. We demonstrate that the primes are the necessary and inevitable consequence of the collapse of the foundational informational superposition. The property of being an indivisible "informational atom"—the first stable entity to emerge — is proven to be synonymous with the definition of a prime number. This establishes a complete, emergent genesis, from the first act of definition to the complex structures of number theory.

Research Questions

This framework seeks to provide a formal proof for the emergent, non-axiomatic origin of the foundational structures of mathematics. The research is structured to answer the following primary questions, which challenge the traditional axiomatic approach and build a new foundation upon the concept of primordial facts.

1. How can the binary entities 0 and 1 be formally derived as primordial facts of information, thereby replacing the Axiom of the Empty Set and the axiomatic definition of the unit with a necessary, logical proof of their existence?
2. How do we formalize a state of informational superposition that is logically prior to the geometric and algebraic framework of a Hilbert space, proving that quantum-like properties are a necessary feature of any self-defining system?
3. What is the formal proof that the first definite, self-referential entities to emerge from the collapse of this superposition must necessarily be indivisible informational atoms, and that this property is synonymous with the mathematical definition of a prime number?
4. How does this proven emergence of the prime numbers as the foundational, definite components of reality provide the necessary and sufficient logical basis for the specific, arithmetically-defined interactions that lead to the formation of the Emergent Continuum?

Methodology

The proof is constructed from a foundation of primordial facts, avoiding any presupposition of axiomatic set theory, number theory, or pre-existing mathematical spaces. The methodology is designed to demonstrate the necessary emergence of the binary, the primes, and the principle of interaction from the single starting point of unrealized information.

1) The Primordial State of Informational Potential (S_0)

We begin by identifying the most fundamental state, S_0, which is not an object but a state of unrealized information. It is the logical ground state prior to any distinction or definition, characterized by:

- Informational Symmetry: Within S_0, the potential for a definition to be made and the potential for a definition not to be made are unresolved and indistinguishable.
- Acausality and Flux: As a state of pure potential, S_0 contains no definite entities that can act as cause or effect. It is a state of logical flux, incapable of forming stable structures.

2) The Primordial Fact of Definition (The Operator D)

We formalize the first primordial fact: that a distinction can be made. This is not an axiom but a necessary condition for any logical system to be non-null. We represent this fact with the Operator of Definition, D.

- Function: The function of D is to resolve the informational symmetry of the state to which it is applied. Its application to S_0, denoted $D(S_0)$, represents the foundational event of logical genesis.
- Logical Necessity: The existence of D is not assumed. It is the formal embodiment of the fact that a system can be queried for its own definability. A system where no such query is possible is informationally inert and indistinguishable from a true void.

3) The Primordial Fact of Binary Resolution

We prove that the application of D to S_0 must yield exactly two outcomes. This binarity is not an axiom but a primordial fact of logic. The outcomes are the co-created informational states '0' and '1'.

- State '1' (The Fact of Self-Reference): The outcome representing the successful instantiation of a definite, self-referential entity. It is the informational fact "a definition exists."
- State '0' (The Fact of Non-Self-Reference): The outcome representing the definite failure to instantiate such an entity. It is the informational fact "a definition does not exist." It is a stable, factual outcome, distinct from the unresolved potential of S_0.

4) The Primordial Fact of Contingent Interaction

We formalize the second primordial fact: that structure requires stability. The Principle of Definiteness for Interaction states that an interaction, I(a, b), is well-defined if and only if its operands, a and b, are definite, resolved states (i.e., instances of '0' or '1'). This fact establishes why a structured mathematical universe cannot remain in a state of flux and provides the logical imperative for the resolution of S_0.

5) The Emergence of Primes as Informational Atoms

Finally, we prove that the first definite entities to emerge must be prime numbers.

- The First Entity: Let p_1 be the first instantiation of the informational state '1'. It is the first definite, self-referential entity to exist.
- The Property of Indivisibility: We analyze the compositional properties of p_1. By definition, p_1 is the first such entity. Therefore, no other definite entities of the same class exist from which p_1 could be composed. It cannot be the product of a stable interaction between other entities because no other entities are available to interact. It is, by logical necessity, an informational atom.
- Equivalence to Primeness: This emergent property of indivisibility is formally equivalent to the definition of a prime number in a multiplicative system. A prime is an entity that cannot be factored into smaller integers other than 1. As p_1 is the foundational "not-0" entity, it is the first building block. Therefore, we prove that the first definite, stable information to emerge from the primordial state must manifest as a set of entities whose defining characteristic is primeness.

Results

The methodology of primordial facts yields a sequence of theorems that establish the non-axiomatic emergence of the binary and the prime numbers.

Let S be the universal set of all possible states. Let $S_0 \in S$ be the Primordial State, defined as a state for which the binary predicate of self-reference, R(x), is indeterminate, denoted $R(S_0) = \perp$. Let D be the Operator of Definition, $D: S \to P(S)$, which resolves the indeterminacy of R.

Theorem 1. *The Theorem of Necessary Binary Resolution*

Let $\Omega = D(S_0)$ be the set of outcomes from the application of D to S_0. The cardinality of Ω is exactly 2. Formally: $|D(S_0)| = 2$.

Proof.

1. By definition of D, for any outcome state $s_i \in \Omega$, the predicate $R(s_i)$ must be resolved. The codomain of a resolved binary predicate is {True, False}. Thus, $\forall s_i \in \Omega$, $R(s_i) \in$ {True, False}.
2. The operator D must actualize the potential for each possible resolved state. The potential for R(x) = True implies the existence of at least one outcome $s_1 \in \Omega$ such that $R(s_1)$ = True. The potential for R(x) = False implies the existence of at least one outcome $s_0 \in \Omega$ such that $R(s_0)$ = False.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2025 doi:10.20944/preprints202510.2494.v1

3. Therefore, $|\Omega| \geq 2$. The states s_0 and s_1 are distinct, for if $s_0 = s_1$, it would imply False = True, a contradiction.
4. The codomain of the resolved predicate R is {True, False}, which has cardinality 2. There are no other truth values for R to resolve to, and thus no other classes of outcomes can be generated by the resolution of R. Therefore, $|\Omega| \leq 2$.
5. From (3) and (4), we conclude $|\Omega| = 2$.
   - Q.E.D.

Theorem 2. *The Theorem of Co-Created Informational States*

Let the informational state '1' be defined as the outcome s_1 where $R(s_1)$ = True, and '0' be defined as the outcome s_0 where $R(s_0)$ = False. The states 0 and 1 are defined simultaneously by the single event $D(S_0)$, and their informational content is purely oppositional.

Proof.

1. From Theorem 1, $1 := s_1$ and $0 := s_0$ are the complete and distinct elements of the set $\Omega = D(S_0)$.
2. Neither state exists prior to the operation $D(S_0)$. Their existence is a direct and simultaneous consequence of this operation.
3. The informational content of state 1 is fully described by the predicate R(1) = True. The informational content of state 0 is fully described by R(0) = False. The statement R(1) = True is logically equivalent to $R(1) \neq R(0)$. The definition of each state is thus contained in its distinction from the other.

- Q.E.D.

Theorem 3. *The Theorem of Emergent Primes as Informational Atoms*

Let E be the set of all emergent, definite, self-referential entities (all instances of state '1'). Let I be a non-trivial interaction (composition) operator, $I: E \times E \rightarrow E$. An entity $c \in E$ is composite if it is the result of a prior interaction, i.e., $\exists\, a, b \in E$ that existed prior to c such that $c = I(a, b)$. An entity $p \in E$ is prime if it is not composite. The first entity to emerge, p_1, is necessarily prime.

Proof.

1. Let the emergence of entities be indexed by a logical time parameter *t*, corresponding to the sequence of definitional events. At $t = 0$, the set of existing definite entities is $E_0 = \{\}$.
2. The first definitional event, $D(S_0)$, occurs at $t = 1$. This event yields the first self-referential entity, p_1. The set of existing entities is now $E_1 = \{p_1\}$.
3. Assume, for the sake of contradiction, that p_1 is composite.
4. By the definition of a composite entity, p_1 must be the result of an interaction between entities that existed prior to it. This means $\exists\, a, b \in E_0$ such that $p_1 = I(a, b)$.
5. However, the set E_0 is the empty set. No such entities *a* and *b* exist.
6. This is a contradiction. The assumption that p_1 is composite is false.
7. Therefore, the first emergent entity, p_1, is not the result of a prior interaction and is, by definition, prime.

- Q.E.D.

Theorem 4: *The Theorem of Contingent Structure*

Let C be any complex system defined by a set of components $\{c_i\}$ and a set of non-trivial interactions {I}. The existence of C is contingent upon the prior emergence of prime entities.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2025 doi:10.20944/preprints202510.2494.v1

Proof.

1. By Definition of Primordial Fact of Contingent Interaction, any interaction I(a, b) is well-defined if and only if its operands, *a* and *b*, are definite, resolved states. The primordial state S_0 is not a definite state and cannot participate in interactions.
2. A complex system C requires interactions among its components. Therefore, the set of components $\{c_i\}$ must be a subset of the set of definite states, E.
3. From Theorem 3, the first members of E to emerge are necessarily prime. Composite entities can only be formed through the interaction of these pre-existing primes.
4. Therefore, the set of all possible components for any interacting system is composed of primes and the composites they generate. The existence of primes is a necessary prerequisite for the existence of composites.
5. Thus, the existence of any complex system C is contingent upon the prior emergence of prime entities from the foundational resolution of S_0.

○ Q.E.D.

Discussion

The formal proofs presented in this framework establish a new, non-axiomatic genesis for the foundational components of mathematics. By demonstrating that the binary states of 0 and 1, and the system of prime numbers, are not stipulated axioms but are the necessary and inevitable consequences of a mathematical universe in which information can be defined, we have constructed a new foundation for mathematical reality. This section will discuss the profound and far-reaching implications of these findings.

A) From Axiomatic Systems to Primordial Facts: A New Foundation

The standard approach to foundational mathematics, embodied by Zermelo-Fraenkel set theory (ZFC), is a high-level descriptive framework. It begins by postulating the existence of certain objects and rules via axioms. The Axiom of the Empty Set, for instance, is a stipulation that the entity we call '0' (the empty set) exists. The Peano axioms then provide a set of rules for constructing other numbers from this starting point. This work does not invalidate these systems; it provides their causal origin.

We have replaced axioms, which are stipulated truths, with primordial facts, which are necessary consequences of a self-defining system. The Axiom of the Empty Set is now understood as a high-level description of a deeper fact: that any act of definition must have a definite null outcome ('0') as a logical possibility. The Peano axioms are not the rules that create numbers, but are an accurate description of the stable, emergent properties of a system built upon the first definite, indivisible entities—the primes. The extraordinary success of axiomatic systems is therefore explained by their accurate correspondence to this deeper, emergent structure. Our work provides the underlying "why" for the axiomatic "what," shifting the foundation of mathematics from stipulation to logical necessity.

B) The Nature of Mathematical Reality: A Third Path

The long-standing philosophical debate between Platonism (the view that mathematical objects are real, abstract entities that are discovered) and Formalism (the view that mathematics is an invention of formal rule-based systems) is resolved by a third, distinct alternative: mathematics is a necessary, emergent structure.

This emergentist view synthesizes the strengths of the prior two. In agreement with Platonism, mathematical truths are not arbitrary; they are discovered, constrained by a rigid, objective logic. However, in agreement with Formalism, these truths are not static, pre-existing objects in some Platonic realm; they are constructed through a dynamic, procedural reality. Mathematics is neither a pre-existing statue to be unearthed nor a game of our own invention. It is the inevitable crystal that grows from a single seed of potential—the fact that information can be defined.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2025 doi:10.20944/preprints202510.2494.v1

C) The Inevitability of Primes and the Subordination of Addition

Theorem 3, which proves that the first definite information to emerge must be atomic and indivisible, is the centerpiece of this new foundation. This property of indivisibility is formally synonymous with the definition of a prime number in a compositional system. This elevates the primes from a curious subset of integers to the literal atoms of information. This finding provides a deep, logical origin for the Fundamental Theorem of Arithmetic: all numbers are either prime or composites of primes. They cannot be composites of '1', because '1' is not a number in the set of interacting components, but is the informational state of existence that enables the system itself.

This framework makes a radical and falsifiable claim: the underlying reality of mathematics is fundamentally multiplicative and interaction-based, not additive. The demand that a foundational theory immediately reproduce Peano arithmetic, including the primordial status of facts like $2 + 3 = 5$, is a category error based on our familiarity with high-level, emergent truths. Such additive structures are not foundational. Instead, they must be understood as effective properties that emerge within the stable, large-scale geometry of the Emergent Continuum. This subordinates the operation of addition to a more fundamental reality of interaction, opening the possibility that the laws of arithmetic themselves are context-dependent, akin to the way the laws of geometry change with the curvature of space.

D) The Computational Nature of Law and the Principle of Stability

Having established the emergent nature of the mathematical objects (the primes), we must address the origin of the laws that govern them. A classical critique would be to view the "Rules of Assembly" — the p-adic path integral — as an infinitely complex new axiom. This critique is invalid because it fails to recognize the computational nature of an informational universe. An informational system is, by the Church-Turing thesis, a computational system. A Turing machine, the formal model of such a system, is an abstract entity that is inseparable from its rules of operation. The rules do not govern the machine; they define it.

Therefore, the "Rules of Assembly" are not external laws imposed upon the primes. They are the very definition of the computation that the prime informational atoms execute. The necessity of this specific, arithmetically deep set of rules is not an assumption but a proven result. As demonstrated in the author's prior work (Karazoupis, 2025), computational systems defined by simpler, non-arithmetic rules fail to converge to a stable, non-trivial limit. This failure is a formal proof by contradiction. It demonstrates that the complexity of the p-adic framework is the minimal complexity required for a stable, structured mathematical universe to compute itself into existence. The laws are not arbitrary; they are selected by a principle of logical survival, where only rule-sets capable of producing a consistent, convergent reality can form a mathematical universe.

E) The P-adic Path Integral and the Proof of Inevitability

This computational framework provides the mechanism for the Principle of Inevitability. The p-adic framework is the formal implementation of a Feynman path integral in an arithmetic context, where the "sum over all histories" is the product of the p-adic norms of an interaction's gap, evaluated against all primes in the system. The convergence to a unique, non-trivial limit, as proven in the Principle of Emergent Continuum (Karazoupis, 2025), is the mathematical proof that this logical path integral has only one possible, stable, and self-consistent outcome. The structure of mathematics is not a matter of chance; it is the inevitable result of a computation whose rules are themselves necessitated by the requirement of a stable result. The apparent randomness in the distribution of the primes is therefore not a sign of cosmic chance, but is the trace of the system exploring every possible logical pathway to reach its single, inevitable destination. The universe of mathematics is not one of many possibilities; it is the only one that can consistently exist.

F) The Principle of Self-Consistent Representation: The Proof of Arithmetic Primacy

The final and most profound conclusion of this work is that the mathematical universe is not merely computational, but that it must be arithmetic. This is not an axiom but a proof by self-reference, established by the Principle of Self-Consistent Representation: the first emergent object must be a stable representation of the process that created it.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2025 doi:10.20944/preprints202510.2494.v1

The foundational process of the system is the collapse of the primordial superposition, which necessarily creates the binary distinction between the informational states '0' and '1'. The first definite entity to emerge is the first instantiation of the "not-0" state. The simplest and most fundamental representation of this binary distinction is the number 2. The number 2, in turn, possesses a necessary and defining arithmetic property: it is the first prime number.

This closes the final logical loop. The foundational logic of the system is binary. The binary logic's first stable representation is the number 2. The necessary arithmetic property of 2 is primeness. Therefore, the first emergent informational atom must be a prime number. The system proves its own arithmetic nature at the moment of its creation. This is not an assumption; it is a necessary consequence of self-consistency, completing the proof that the number-theoretic universe is not just a possible reality, but the only possible reality.

Conclusion

This framework has presented a complete, self-contained proof of the non-axiomatic origins of the foundational structures of mathematics. We have successfully demonstrated that the binary of 0 and 1, and the system of prime numbers, are not stipulated axioms but are the necessary and inevitable consequences of a sequence of primordial facts that arise from a single, pre-geometric state of informational potential. The work successfully replaces the static, object-based axioms of traditional set theory with a dynamic, procedural, and emergent genesis for mathematical reality.

The main findings of this work are threefold, forming a logical chain from the first potential to the necessity of complex structures:

1. The Binary is a Primordial Fact, Not an Axiom. We have formally replaced the Axiom of the Empty Set with a proof. We have shown that any act of definition upon an unresolved, featureless state must necessarily resolve into exactly two mutually defined and co-created outcomes: the informational fact of self-reference ('1') and the informational fact of non-self-reference ('0'). The binary is therefore not an assumption but is the first and most fundamental emergent fact of any self-defining logical system.
2. Primes are the First Manifestation of Information. We have proven that the first definite, stable, and self-referential entities to emerge from the foundational collapse of superposition must, by logical necessity, be indivisible. As they are the first entities to exist, they cannot be composed of any prior, constituent parts. This emergent property of indivisibility is formally synonymous with the mathematical definition of a prime number. Therefore, the primes are not a special class of integers, but are the foundational, atomic constituents of all mathematical information.
3. Structure is an Inevitable Attractor, Not a Probabilistic Outcome. We have established the principle of inevitabilism, which posits that the complex structures of mathematics are the unique and necessary result of the interactions of the emergent primes. By re-interpreting the logic of the Feynman path integral, we have argued that the system of primes, in exploring all possible interactional pathways, does not produce a probabilistic outcome but necessarily converges to the single, maximally stable, and self-consistent structure. The Emergent Mathematical Continuum is thus the inevitable destination for a mathematical universe built from prime informational atoms.

In achieving this proof, this work offers a new foundation for mathematics that resolves the debate between Platonism and Formalism with a third alternative: mathematics as a necessary, emergent structure. It provides the definitive logical origin for the entities described in the Emergent Continuum Hypothesis (Karazoupis, 2025) , establishing a complete and coherent chain of genesis from the first unresolved potential to the complex structures of number theory. This work invites a

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2025 doi:10.20944/preprints202510.2494.v1

fundamental re-examination of the nature of mathematical and physical reality, suggesting it is not built upon a set of assumed rules, but upon the single, perpetual, and inevitable act of definition.

Appendix A. The Formal Equivalence of the P-adic Framework and the Path Integral

A.1. Purpose

This appendix provides a formal proof that the "Rules of Assembly" governing the interaction of primes, as defined in the author's work on the Emergent Continuum, are not merely analogous to the Feynman path integral but are a direct mathematical implementation of the path integral's core logic in a discrete, arithmetic context. This equivalence establishes the mechanism for the Principle of Inevitability, proving that the unique limit of the system is the necessary result of a summation over all possible logical and arithmetic histories.

A.2. The Feynman Path Integral in Quantum Physics

The Feynman path integral formulation is a fundamental principle of quantum mechanics. Its core tenets are:

1. The Propagator: The probability amplitude for a particle to transition from an initial state A to a final state B is given by a propagator, K(B, A).
2. Sum Over Histories: This propagator is calculated by summing (integrating) a contribution from every possible path the particle could take between A and B.
3. The Action and Phase: Each path is assigned a complex number, or phase, e^(iS/ħ), where S is the classical action of that path.
4. The Principle of Interference: The total propagator is the integral of these phases over the space of all possible paths:

   K(B, A) = ∫ D[x(t)] e^(iS[x(t)]/ħ)

Paths whose actions are not stationary have rapidly oscillating phases that destructively interfere and cancel each other out. The dominant contribution comes from paths in the vicinity of the classical path, where the action is stationary. The final outcome is thus determined by a principle of stationary action, emerging from the interference of all possibilities.

A.3. The P-adic Path Integral in the Emergent Continuum

We now demonstrate a formal, one-to-one mapping between the components of the Feynman path integral and the p-adic framework that defines the metric of the Emergent Continuum.

1. The "Transition": Defining an Interaction.

The "process" is not a particle's motion in spacetime, but the definition of the interaction strength (or metric distance) between two distinct primes, p_i and p_j. The "initial state" is p_i and the "final state" is p_j.

2. The "Sum Over Histories": Product Over All System Primes.

The interaction between p_i and p_j is not a direct relationship but is mediated by the entire system of primes, P_n = {p_1, p_2, ..., p_n}. Each prime p_k in the system provides a unique "path," "history," or "perspective" through which the relationship between p_i and p_j is evaluated. The "space of all paths" is therefore the set of all system primes, P_n.

3. The "Action" for a Single Path: The P-adic Norm.

The interaction is based on the arithmetic information contained in the gap, g = |p_j - p_i|. The "action" or "amplitude" associated with a single "path" p_k is the p-adic norm of the gap with

 doi:10.20944/preprints202510.2494.v1

respect to that prime: $|g|_{p_k}$. This value quantifies the arithmetic relevance of the path p_k to the interaction between p_i and p_j.

4. The "Propagator": The Total Interaction Strength.

The total interaction strength, $w_A(p_i \to p_j)$, is the reciprocal of a length term, l(e). This length term is the "sum over all histories." In this non-Archimedean, multiplicative context, the "summation" of the amplitudes is a product. The total length is given by:

$l(e) = |g|_{p_i} * \Pi_{k=1 \text{ to } n} |g|_{p_k}$

This is the formal equivalent of the path integral. It is the product of the amplitudes of all possible paths.

A.4. The Principle of Logical Interference and Inevitability

The p-adic framework contains a precise mechanism for interference that is the direct analogue of the principle of stationary action.

- Case 1: Non-Contributing Paths (Destructive Interference).

If a prime p_k is not a factor of the gap g, then its p-adic valuation $v_{p_k}(g)$ is 0. The "action" for this path is $|g|_{p_k} = p_k^0 = 1$. This path contributes a trivial factor of 1 to the total product. It is a "path of stationary phase" that has no effect on the final outcome. These paths are the equivalent of the wildly oscillating paths in the Feynman integral that cancel out.

- Case 2: Contributing Paths (Constructive Interference).

If a prime p_k is a factor of the gap g, then $v_{p_k}(g) > 0$. The "action" for this path is $|g|_{p_k} = p_k^{(-v_{p_k}(g))} < 1$. This path contributes a non-trivial factor to the total product, altering the final interaction strength.

The final interaction strength is therefore determined only by the finite set of primes that are arithmetically relevant to the gap. All other infinite possible paths "interfere" and logically cancel out by contributing a factor of 1. The "classical path" is the unique arithmetic structure of the prime factorization of the gap itself.

A.5. Conclusion: Uniqueness as Proof of Inevitability

The p-adic framework is not an analogy for the path integral; it is its formal implementation in an arithmetic context. The central result of the Principle of Emergent Continuity (Karazoupis, 2025) — that the sequence of spaces constructed with this metric converges to a unique, non-trivial limit in the Gromov-Hausdorff sense — is therefore the mathematical proof of the Principle of Inevitability. The uniqueness of the limit demonstrates that the "sum over all logical histories" has only one possible, stable, and self-consistent outcome. The structure of mathematics is not a matter of chance; it is the inevitable result of this logical path integral. All roads must lead to Rome because Rome is the only destination that remains once all inconsistent paths have cancelled out.

Appendix B. On the Necessary Emergence of Foundational Structures

This appendix provides a formal justification for the foundational concepts used in this paper. Its purpose is to prove that the structures of set theory, bivalent logic, and multiplicative interaction are not presupposed axioms (e.g., from ZFC or classical logic), but are the minimal and necessary emergent forms required for the primordial state of indeterminacy (S_0) to resolve into a stable, complex, and structured reality.

B.1. The Emergence of Set-Theoretic Structures

The set-theoretic concepts of Set, Cardinality, and Power Set are not taken as axioms of aggregation but are shown to be necessary structural consequences of the foundational "act of definition."

doi:10.20944/preprints202510.2494.v1

- The Emergence of the Set (Ω): The Operator of Definition,

$$D$$

acts on the indeterminate state

$$S_0$$

and resolves it into a collection of definite, distinct outcomes. A Set is the necessary formal construct that describes this resultant collection of outcomes. The set Ω = {0, 1} is not presupposed; it is the name we give to the complete, discrete output of the symmetry-breaking operation

$$D(S_0)$$

It is the minimal "container" required to hold the results of the first logical distinction.

- The Emergence of Cardinality (|Ω|): Cardinality is not a pre-existing number but is the emergent property of a set that quantifies its number of distinct elements. As proven in Theorem 1, the resolution of

$$S_0$$

necessarily produces exactly two distinct outcomes. Therefore, the cardinality |Ω| = 2 is a structural fact observed after the resolution, not a number taken from Peano arithmetic. It is the measure of the system's first informational complexity.

- The Emergence of the Power Set (P(Ω)): The Power Set is the formal representation of the total informational potential of the newly created binary system. Once the primary distinction {0, 1} is established, the Power Set P(Ω) = { {}, {0}, {1}, {0, 1} } is the necessary and complete collection of all possible sub-distinctions or relationships that can be defined between the emergent states. It is the emergent space of all possible logical relations.

B.2. The Emergence of Bivalent Logic

The framework does not presuppose the Axiom of Bivalence (that a proposition is either True or False). Instead, it proves that bivalence is the minimal and necessary logical structure for achieving maximal information gain from a state of total indeterminacy.

- The Predicate R(x) as a Minimal Question: The predicate of self-reference, R(x), is formulated as the minimal possible, self-contained logical question: "Does the state x contain a definite, self-referential definition?"
- The Proof of Bivalence by Minimal Constraint:

1. The initial state,

$$S_0$$

is one of total informational indeterminacy (maximal entropy). Any resolution must provide a definite, stable answer.

2. Consider alternatives to bivalence:
    - Multi-valued or fuzzy logics require the pre-existence of a metric, an ordering principle, or a probability theory to define the intermediate truth values (e.g., a value of '0.5'). These constitute a vast amount of pre-supposed information and structure.
    - A three-valued logic (e.g., True, False, Undecided) fails because "Undecided" is merely a label for the initial indeterminate state

$$S_0$$

it is not a resolution of it.

3. The only logical resolution that requires zero pre-supposed structure is the one that yields a result of complete success (True) or complete failure (False).
4. Therefore, bivalence is the emergent, minimal logical structure that can provide a definite answer to the minimal question. It is the necessary form of definiteness itself. The Operator

$$D$$

 doi:10.20944/preprints202510.2494.v1

is the formal representation of this process of maximal entropy reduction.

B.3. The Emergence of Multiplicative Interaction

The assertion that a multiplicative interaction ($I \equiv \times$) is more fundamental than an additive one ($+$) is proven by demonstrating that only multiplication is sufficient for the emergence of stable, complex informational structures.

- Defining the Operations by their Function:
  - Succession (Addition, n → n+1): This operation is fundamentally about ordering. It takes a single entity and generates the "next" entity in a sequence. The new entity (n+1) contains no more intrinsic complexity or structure than its predecessor (n). Addition is the mechanism for traversing a list of already-defined facts.
  - Composition (Multiplication, (a, b) → a × b): This operation is fundamentally about construction. It takes two or more distinct entities and creates a new entity, c, whose unique identity and stable structure are derived from its components. The entity c possesses an inherent structure (its prime factorization) that is not present in its components individually. Multiplication is the mechanism for building new, complex facts.
- The Proof of Multiplicative Priority:
  1. The emergence of a complex universe requires a mechanism to increase structural complexity.
  2. Succession (addition) only generates a linear sequence of functionally equivalent states. It does not create new, unique structures. A universe with only addition is a sterile, uninteresting list.
  3. Composition (multiplication) is the minimal operation that allows for the creation of new, stable entities with unique, inherent structures (composite numbers).
  4. Therefore, for the emergence of a complex, stable, and uniquely structured reality, the operation of Composition ($I \equiv \times$) must be the primary and necessary foundational interaction. Succession ($+$) is logically secondary, defined as the process of ordering the complex facts that have been created by multiplication.

References

1. Benacerraf, P. and Putnam, H. (eds.) (1983) *Philosophy of Mathematics: Selected Readings*. 2nd edn. Cambridge: Cambridge University Press.
2. Feynman, R.P. (1948) 'Space-Time Approach to Non-Relativistic Quantum Mechanics', *Reviews of Modern Physics*, 20(2), pp. 367–387.
3. Feynman, R.P. and Hibbs, A.R. (2010) *Quantum Mechanics and Path Integrals*. Emended edn. by Styer, D.F. Mineola, NY: Dover Publications.
4. Fredkin, E. (2003) 'An Introduction to Digital Philosophy', *International Journal of Theoretical Physics*, 42(2), pp. 189-247.
5. Karazoupis, M. (2025) The Principle of Emergent Continuity: A Proof of the Emergence of the Mathematical Continuum from the Arithmetic of Prime Numbers' *Preprints 2025, 2025070275.* https://doi.org/10.20944/preprints202507.0918.v1
6. Oriti, D. (2014) 'Disappearance and emergence of space and time in quantum gravity', *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 46, pp. 186-199.
7. Peano, G. (1889) *Arithmetices principia, nova methodo exposita*. Turin: Fratres Bocca.
8. von Neumann, J. (1923) 'Zur Einführung der transfiniten Zahlen', *Acta Litterarum ac Scientiarum Ragiae Universitatis Hungaricae Francisco-Josephinae, sectio scientiarum mathematicarum*, 1, pp. 199-208.
9. von Neumann, J. (1932) *Mathematische Grundlagen der Quantenmechanik*. Berlin: Springer.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 October 2025 doi:10.20944/preprints202510.2494.v1