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Technical Note

A Quantized Gramian-Based Distributed Observer for Discrete-Time Systems

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Abstract

This paper studies a quantized implementation of a Gramian-based distributed observer for discrete-time systems observed by a network of sensor nodes. The starting point is a distributed observer based on local information matrices obtained from a distributed constructibility Gramian recursion. In order to account for limited communication, the information matrices and information vectors exchanged among neighboring nodes are quantized by means of a uniform quantizer with saturation. The resulting algorithm preserves the original distributed structure and requires only one communication round per sampling instant. A simple perturbation argument shows that the estimation error induced by quantization depends on the perturbation of the local information matrix and information vector, and therefore on the quantization resolution. Numerical results illustrate a sharp practical threshold on the number of communication bits: in the considered example the estimator remains stable for 25 bits, while 23 bits already lead to divergence.

Keywords: distributed observers; quantization; sensor networks; constructibility Gramian; limited communication

1. Introduction

Distributed state estimation has been widely studied in the context of sensor networks, cooperative robotics, and cyber-physical systems; see, for example, the survey in [1] and the application-oriented discussions in [2,3]. Among the most common approaches are distributed Kalman filtering [4–7], diffusion-based estimation [8–10], covariance-intersection and consensus-based filtering [11,12], as well as observer-based distributed designs [13–17]. In many applications, however, communication bandwidth is limited and the exchange of full-precision matrices and vectors is unrealistic.

A Gramian-based distributed observer provides an attractive alternative to diffusion-based filtering because it gives a clear interpretation in terms of local information accumulation and because it allows the use of pre-computed distributed information structures. In the unquantized case, each node combines its local information contribution with information received from its neighbors and computes a local state estimate by solving a linear system involving a local information matrix. This viewpoint is closely related to the distributed constructibility-Gramian observer proposed in [17], and is also connected to earlier distributed observer constructions based on Luenberger-type designs [13–15]. In contrast with consensus-based filtering approaches such as [12], the Gramian-based formulation makes the role of the local information matrix explicit and is therefore particularly convenient for studying the impact of communication quantization.

The purpose of this paper is to introduce quantization in that Gramian-based distributed observer. More precisely, we consider a communication model where the information matrices and information vectors exchanged among agents are quantized by a uniform quantizer with saturation. This yields a simple algorithmic modification, but the numerical effect is nontrivial: quantization perturbs the information matrices directly, which may deteriorate conditioning and eventually compromise the matrix inversion step used to recover the state estimate. The present development is also conceptually

related to quantized distributed estimation schemes such as the distributed quantized Luenberger filter of [18], although here the quantization acts on the Gramian-based information exchange rather than on a consensus recursion over local Luenberger updates.

The contribution of the paper is therefore twofold. First, we formulate a quantized version of the Gramian-based distributed observer that fits naturally within the original recursion of [17]. Second, we provide a simple perturbation analysis that explains the dependence of the estimation error on the quantization resolution and connects the observed performance loss with deterioration of the conditioning of the local information matrix. Numerical results show that, for the example considered, there is a sharp empirical transition between stable and unstable behavior.

1.1. Notation

Throughout the paper, $\otimes$ denotes the Kronecker product, $\rho(\cdot)$ the spectral radius, and I_n the $n \times n$ identity matrix. The notation $\text{col}(X^i)$ stacks the matrices X^i vertically, and $\text{diag}(X^i)$ denotes the block-diagonal matrix with diagonal blocks X^i . When clear from the context, the superscript i refers to node $i \in \mathcal{N} := \{1, \dots, N\}$.

2. Problem Formulation

Consider the discrete-time system

$$x_{t+1} = A_t x_t + w_t, \quad (1)$$

where $x_t \in \mathbb{R}^n$ is the system state, $A_t \in \mathbb{R}^{n \times n}$ is the state transition matrix, and $w_t \in \mathbb{R}^n$ is process noise. Node $i \in \mathcal{N}$ measures

$$y_t^i = C_t^i x_t + v_t^i, \quad (2)$$

where $y_t^i \in \mathbb{R}^{m_i}$, $C_t^i \in \mathbb{R}^{m_i \times n}$, and $v_t^i \in \mathbb{R}^{m_i}$ is measurement noise.

The communication graph is described by $(\mathcal{N}, \mathcal{A})$, where $(j, i) \in \mathcal{A}$ means that node i receives information from node j . Let $\Pi = [\pi^{i,j}] \in \mathbb{R}^{N \times N}$ be a stochastic primitive consensus matrix compatible with the graph.

Assumption 1

The matrix A_t is invertible for all $t \geq 0$.

Assumption 2

The system is collectively observable through the sensor network over a finite information horizon.

The objective is to compute at each node a local estimate $\hat{x}_t^i$ of the global state, using local measurements and one round of communication per sampling time, when inter-agent communication is quantized.

3. Unquantized Gramian-Based Distributed Observer

The starting point is the Gramian-based distributed observer. Let $\bar{\Omega}_t^i \in \mathbb{R}^{n \times n}$ denote the local information matrix at node i , obtained from a distributed constructibility Gramian recursion. Given a local prediction $\hat{x}_t^j$ at each node j , node i computes

$$\Omega_t^i = C_t^{i\top} V_i C_t^i + \sum_{j \in \mathcal{N}} \pi^{i,j} \bar{\Omega}_t^j, \quad (3)$$

$$q_t^i = C_t^{i\top} V_i y_t^i + \sum_{j \in \mathcal{N}} \pi^{i,j} \bar{\Omega}_t^j \hat{x}_t^j, \quad (4)$$

and updates its estimate as

$$\hat{x}_{t+1}^i = A_t (\Omega_t^i)^{-1} q_t^i. \quad (5)$$

Here $V_i \succ 0$ is a local weighting matrix. The matrix $\bar{\Omega}_t^j$ is obtained from the distributed Gramian recursion developed in the unquantized observer design [1] and is not repeated here in full. The only fact needed for the sequel is that each node can compute its own $\bar{\Omega}_t^j$ locally and then transmit it, together with the corresponding information vector contribution $\bar{\Omega}_t^j \hat{x}_t^j$, to neighboring agents.

4. Uniform Quantization with Saturation

To account for communication constraints, we replace the transmission of $\bar{\Omega}_t^j$ and $\bar{\Omega}_t^j \hat{x}_t^j$ by quantized messages. We adopt a uniform quantizer with saturation.

For a scalar variable x , step size $\Delta > 0$, and half-range $X_{\max} > 0$, define

$$Q_{\text{sat}}(x) = \Delta \text{round}\left(\frac{\min\{\max\{x, -X_{\max}\}, X_{\max}\}}{\Delta}\right). \quad (6)$$

For vectors and matrices, the quantizer is applied elementwise. If n_b bits are used, we set

$$\Delta = \frac{2X_{\max}}{2^{n_b}}. \quad (7)$$

In the simulations below, $X_{\max}$ is chosen adaptively at each time instant according to the current magnitude of the information matrix and information vector, namely

$$X_{\max}^{\Omega}(t) = \gamma_{\Omega} \max\left\{\max_j \|\bar{\Omega}_t^j\|_{\infty}, \varepsilon\right\}, \quad (8)$$

$$X_{\max}^q(t) = \gamma_q \max\left\{\max_j \|\bar{\Omega}_t^j \hat{x}_t^j\|_{\infty}, \varepsilon\right\}, \quad (9)$$

where $\gamma_{\Omega}, \gamma_q \geq 1$ and $\varepsilon > 0$ is a small safeguard constant.

5. Quantized Gramian-Based Observer

With quantized communication, node i receives

$$\tilde{\Omega}_t^j = Q_{\text{sat}}(\bar{\Omega}_t^j), \quad \tilde{q}_t^j = Q_{\text{sat}}(\bar{\Omega}_t^j \hat{x}_t^j), \quad (10)$$

and computes

$$\Omega_{t,q}^i = C_t^{i\top} V_i C_t^i + \sum_{j \in \mathcal{N}} \pi^{i,j} \tilde{\Omega}_t^j, \quad (11)$$

$$q_{t,q}^i = C_t^{i\top} V_i y_t^i + \sum_{j \in \mathcal{N}} \pi^{i,j} \tilde{q}_t^j. \quad (12)$$

The local estimate is then updated as

$$\hat{x}_{t+1,q}^i = A_t(\Omega_{t,q}^i)^{-1} q_{t,q}^i. \quad (13)$$

In practice, quantization may destroy the exact symmetry and positive definiteness of the information matrix. In the implementation used in the numerical section, after quantization we enforce symmetry by replacing $\Omega_{t,q}^i$ with $\frac{1}{2}(\Omega_{t,q}^i + \Omega_{t,q}^{i\top})$ and, when needed, add a small diagonal regularization to guarantee invertibility.

The resulting distributed procedure is summarized below.

Algorithm 1 Quantized Gramian-based observer at node i **Input:** Π , local measurement y_t^i , local matrix C_t^i , local Gramian recursion for $\bar{\Omega}_t^i$, number of bits n_b **Output:** $\hat{x}_{t+1,q}^i$

- 1: **for all** $t \geq 0$ **do**
- 2: Compute $\bar{\Omega}_t^i$ from the local Gramian recursion.
- 3: Form the local message $\bar{\Omega}_t^i \hat{x}_t^i$.
- 4: Quantize the transmitted information matrix and information vector with Q_{sat} .
- 5: Receive quantized messages from in-neighbors.
- 6: Build $\Omega_{t,q}^i$ and $q_{t,q}^i$ from Equations (11) and (12).
- 7: Symmetrize and, if necessary, regularize $\Omega_{t,q}^i$.
- 8: Update $\hat{x}_{t+1,q}^i = A_t(\Omega_{t,q}^i)^{-1} q_{t,q}^i$.
- 9: **end for**

6. A Simple Perturbation Analysis

Define the quantization errors

$$\tilde{\Omega}_t^j = \bar{\Omega}_t^j + E_{\Omega,t}^j, \quad \tilde{q}_t^j = \bar{\Omega}_t^j \hat{x}_t^j + e_{q,t}^j. \quad (14)$$

Then the local information matrix and information vector used by node i can be written as

$$\Omega_{t,q}^i = \Omega_t^i + \Delta\Omega_t^i, \quad (15)$$

$$q_{t,q}^i = q_t^i + \Delta q_t^i, \quad (16)$$

with

$$\Delta\Omega_t^i = \sum_{j \in \mathcal{N}} \pi^{ij} E_{\Omega,t}^j, \quad (17)$$

$$\Delta q_t^i = \sum_{j \in \mathcal{N}} \pi^{ij} e_{q,t}^j. \quad (18)$$

The following statement is immediate from standard perturbation arguments.

Proposition 1Assume that Ω_t^i is nonsingular and that $\Omega_{t,q}^i$ is also nonsingular. Then

$$\|\hat{x}_{t+1,q}^i - \hat{x}_{t+1}^i\| \leq \|A_t\| \|(\Omega_{t,q}^i)^{-1}\| \left(\|\Delta q_t^i\| + \|\Delta\Omega_t^i\| \|\hat{x}_{t+1}^i\| \right). \quad (19)$$

In particular, for sufficiently small perturbations the difference between quantized and unquantized estimates is of order $\mathcal{O}(\|\Delta q_t^i\| + \|\Delta\Omega_t^i\|)$.**Proof.** Using Equations (5) and (13),

$$\begin{aligned} \hat{x}_{t+1,q}^i - \hat{x}_{t+1}^i &= A_t \left((\Omega_{t,q}^i)^{-1} (q_t^i + \Delta q_t^i) - (\Omega_t^i)^{-1} q_t^i \right) \\ &= A_t (\Omega_{t,q}^i)^{-1} \left(\Delta q_t^i - \Delta\Omega_t^i \hat{x}_{t+1}^i \right), \end{aligned}$$

from which Equation (19) follows by taking norms. $\square$

Proposition 1 does not by itself guarantee stability of the quantized implementation. Its role is more modest: it shows that, as long as the quantized information matrix remains well conditioned, the effect of quantization scales with the perturbation of the transmitted information. Conversely, when quantization degrades conditioning too severely, the inversion step becomes numerically fragile and may lead to divergence.

7. Numerical Results

We now illustrate the behavior of the quantized observer through numerical simulations. In all cases, the plant and network configuration are the same: the network has 20 agents, the local state dimension is 2, and the communication graph is circular, with each node communicating with one neighbor on each side. The state-transition and measurement matrices are randomly perturbed over time. The observer design parameters are $\beta = 0.5$, $\alpha = 10^{-6}$, and the information horizon is

$$k = \left\lceil \frac{N}{\text{Circ_order}} + \text{dim} \right\rceil.$$

Three cases are considered. In Case 1, the Gramian-based distributed observer is used without quantization. In Case 2, the same observer is implemented with quantized communication and $n_b = 25$ bits. In Case 3, the observer is again quantized, but with $n_b = 23$ bits.

The main empirical observation is that the transition between stable and unstable behavior is sharp. For the considered problem, the quantized observer remained numerically stable for 25 bits, whereas 23 bits led to divergence. This indicates that the effect of quantization is not merely a gradual degradation in estimation quality. Instead, once the quantization-induced perturbation becomes sufficiently large, the conditioning of the local information matrix deteriorates to the point where the inversion step becomes numerically unreliable.

Figure 1 shows the reference behavior in the unquantized case, comparing the centralized observer, the distributed observer, and the Gramian-based distributed observer. Figure 2 shows the corresponding behavior for the quantized Gramian-based observer with 25 bits. To further illustrate the behavior of the adaptive quantizer, Figures 3–6 report the evolution of the quantization step and adaptive range for the two quantized cases.

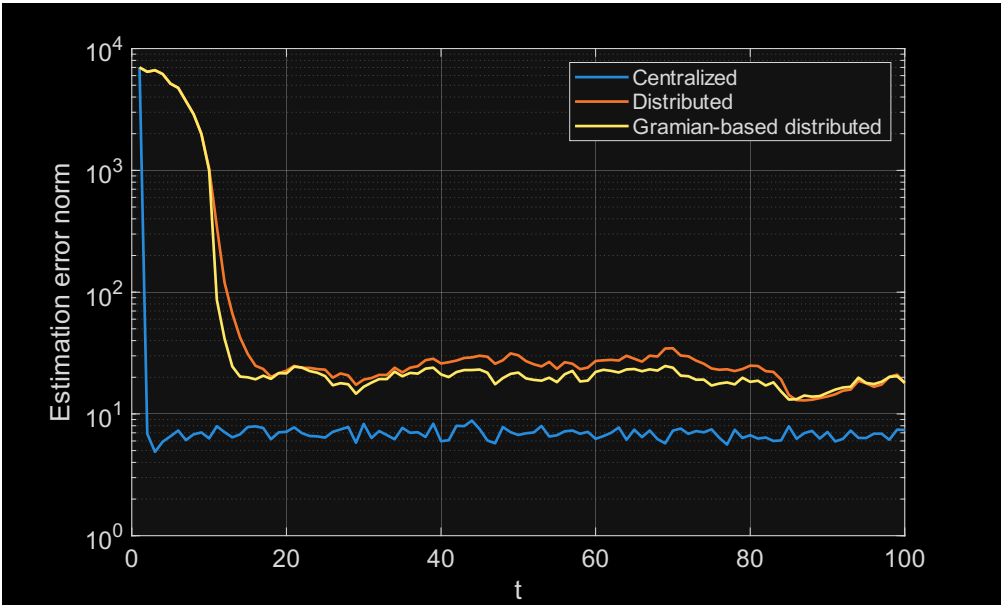


Figure 1. Average estimation error norm in the unquantized case.

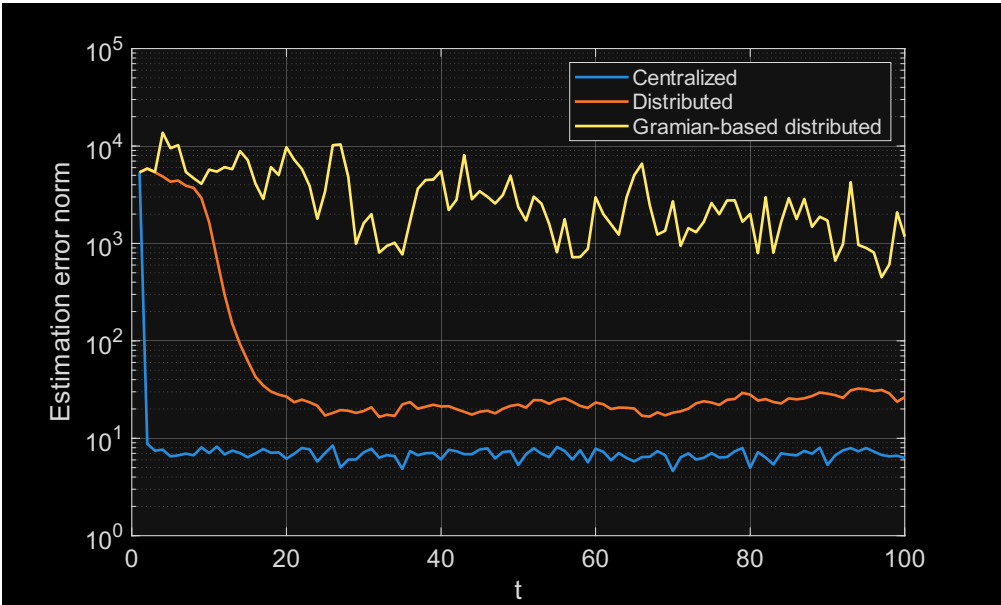


Figure 2. Average estimation error norm in the quantized stable regime with 25 bits.

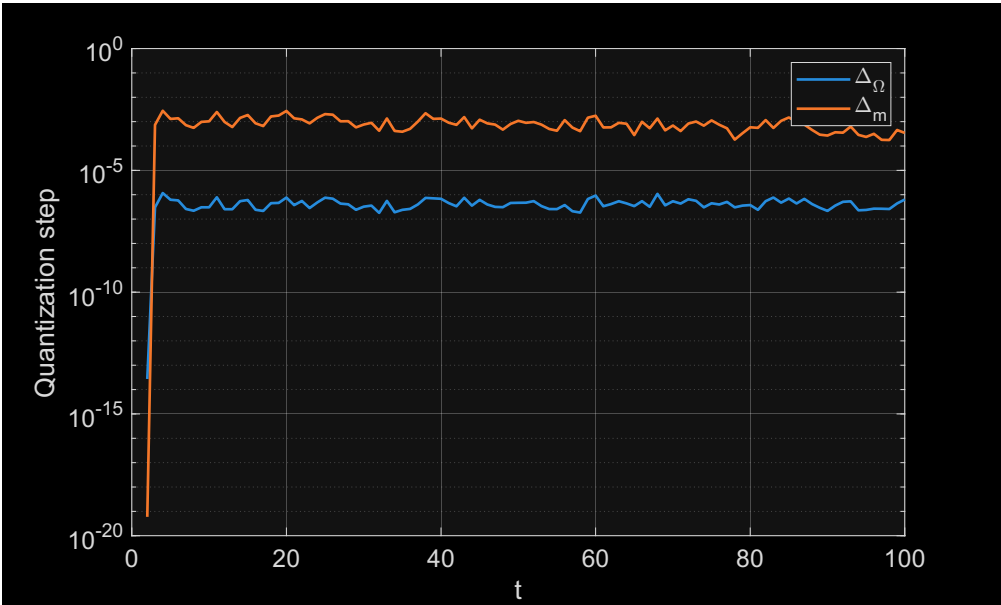


Figure 3. Adaptive quantization steps Δ_Ω and Δ_m in the stable quantized regime with 25 bits.

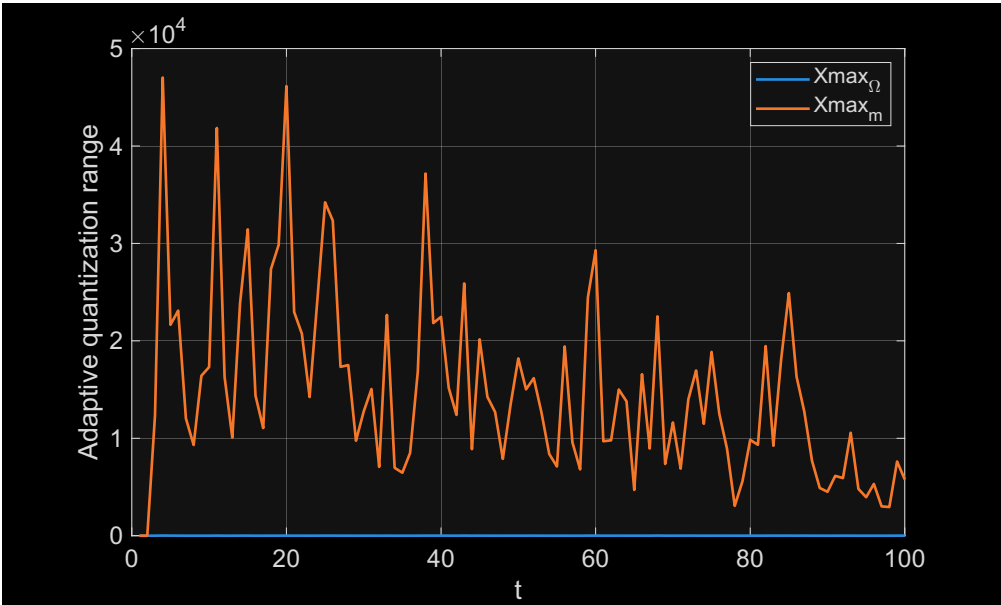


Figure 4. Adaptive quantization ranges $X_{\max_{\Omega}}$ and $X_{\max_m}^m$ in the stable quantized regime with 25 bits.

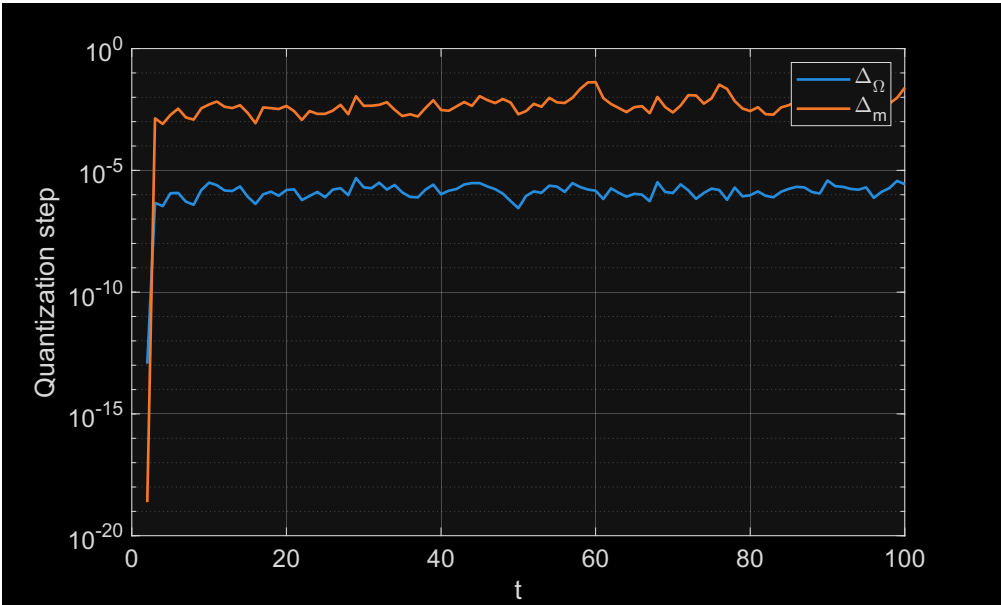


Figure 5. Adaptive quantization steps Δ_{Ω} and Δ_m in the unstable quantized regime with 23 bits.

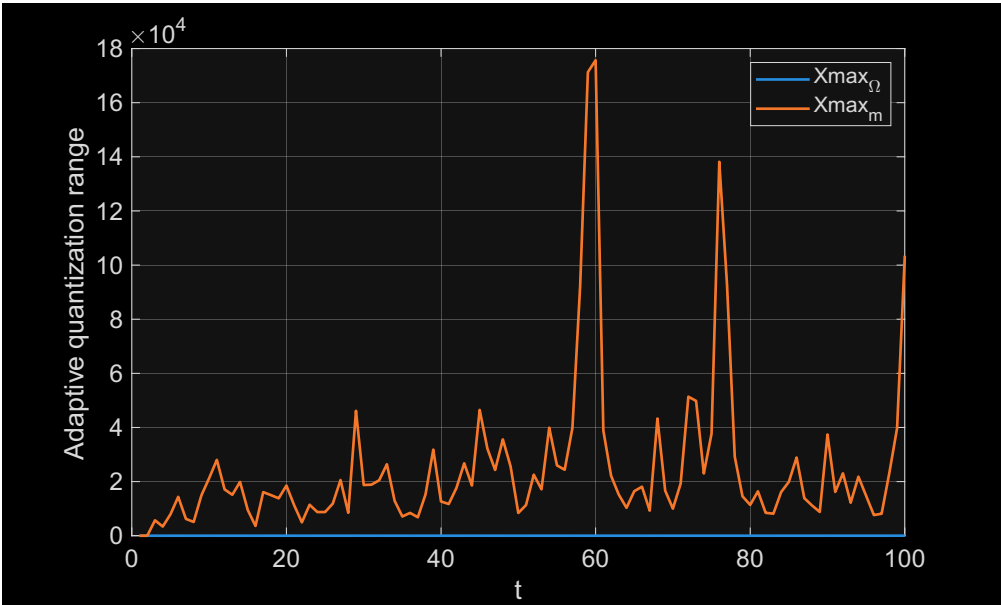


Figure 6. Adaptive quantization ranges $X_{\max}^{\Omega}$ and $X_{\max}^m$ in the unstable quantized regime with 23 bits.

The numerical results support the perturbation-based interpretation of Section 6. In the stable regime, quantization acts as a bounded perturbation of the information recursion and the observer retains good performance. In the unstable regime, the dominant effect is the loss of conditioning of the local information matrix, which makes the inversion step increasingly sensitive and eventually leads to divergence.

8. Conclusions

This paper introduced a quantized version of a Gramian-based distributed observer for discrete-time systems. The proposed method preserves the original distributed recursion and uses a uniform quantizer with saturation for the information matrices and information vectors exchanged among neighboring agents. A simple perturbation argument shows that the effect of quantization is directly linked to the perturbation of the local information matrix and information vector. Numerical results reveal a sharp empirical threshold in the communication resolution: in the example considered, the quantized observer remained stable for 25 bits and diverged for 23 bits.

The results suggest two natural research directions. The first is the derivation of explicit sufficient conditions on the number of bits required to preserve conditioning of the information matrix. The second is the design of structure-preserving quantization schemes that better respect symmetry and positive definiteness of the transmitted information matrices.

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Abbreviations

The following abbreviations are used in this manuscript:

LTI	Linear time-invariant
LTV	Linear time-varying
PSD	Positive semidefinite

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