Article

Not peer-reviewed version

Comparison Criteria for First Order Polynomial Differential Equations

Gevorg Grigorian *

Posted Date: 2 January 2025

doi: 10.20944/preprints202412.2567.v1

Keywords: comparison criteria; global solvability; Hilbert's 16th problem; the Riccati equation; differential inequalities; sub solution; super solution; usable sequence; global solvability; closed solutions; planar autonomous systems; separator polynomials; generalized Prufer transformation; periodic orbits; limit cycles

Article

Comparison Criteria for First Order Polynomial Differential Equations

Gevorg Grigorian

0019 Armenia c. Yerevan, str. M. Bagramian 24/5, Institute of Mathematics of NAS of Armenia; mathphys2@instmath.sci.am, phone: 098 62 03 05; 010 35 48 61

Abstract: In this paper we use the comparison method for investigation of first order polynomial differential equations. We prove two comparison criteria for these equations. The proved criteria we use to obtain some global solvability criteria for first order polynomial differential equations. On the basis od these criteria we prove some criteria for existence of a closed solution (of closed solutions) for first order polynomial differential equations. The results obtained we compare with some known results. Based on obtained results some criteria of existence of periodic orbits or limit cycles for planar autonomous systems are proved.

Keywords: comparison criteria; global solvability; Hilbert's 16th problem; the Riccati equation; differential inequalities; sub solution; super solution; usable sequence; global solvability; closed solutions; planar autonomous systems; separator polynomials; generalized Prufer transformation; periodic orbits; limit cycles

MSC: 34D20

---

1. Introduction

Let $a_k(t),\ \ k = \overline{1,n}$ be real-valued continuous functions on $[t_0, \tau_0)\ \ (t_0 < \tau_0 \le \infty)$. Consider the first order polynomial differential equation

$$y' + \sum_{n=0}^{n} a_k(t)y^k = 0, \qquad t_0 \le t \le \tau_0. \tag{1.1}$$

According to the general theory of normal systems of differential equations for every $t_1 \ge t_0,\ \ \gamma \in \mathbb{R}$ and for any solution $y(t)$ of Eq. (1.1) with $y(t_1) = \gamma$ there exists $t_2 > t_2$ such that $y(t)$ is continuable on $[t_1, t_2)$. From the point of view of qualitative theory of differential equations an important interest represents the case $t_2 = \infty$. One of effective ways to study the conditions, under which the case $t_2 = \infty$ holds, is the comparison method. This method has been used in [10,11] to obtain some comparison criteria for Eq. (1.1) in the case $n = 2$ (the case of Riccati equations), which were used for qualitative study of different types of equations (see e. g. [11–23]). In the general case Eq. (1.1) attracts the attention of mathematicians in the connection with a relation of the problem of existence of closed solutions of Eq. (1.1) with the problem of determination of the upper bound for the number of limit cycles in two-dimensional polynomial vector fields of degree n. (see [1,2,7,24] the 16th problem of Hilbert [recall that a solution $y(t)$ of Eq. (1.1), existing on any interval $[t_0, T]$, is called closed on that interval, if $y(t_0) = y(T)$]) and many works are devoted to it (see [4,8,9] and cited works therein). Significant results in this direction have been obtained in [25]. Among them we point out the following result.

Theorem 1.1. ([25, p.3, Theorem 1]). *Let us assume that* $a_0(t) \equiv 0$, $\int\limits_{t_0}^{T} a_1(t)dt > 0$. *Let us assume that there exists some* $j = 2, \dots, n$ *such that* $a_k(t) \le 0$ *and* $\sum\limits_{k=j}^{n} a_k(t) < 0$ *for all* $k = j, \dots, n$ *and* $t \in [t_0, T]$. *Then there exists a positive isolated closed solution of Eq. (1.1) on* $[t_0, T]$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

■

This and other theorems of work [25] were obtained by the use of a perturbation method and the contracting mapping principle. Note that an interpretation of Theorem 1.1 is the following

Theorem 1.1*. *Let us assume that* $a_0(t) \equiv 0$, $\int\limits_{t_0}^{T} a_1(t)dt > 0$. *Let us assume that there exists some* $j = 2, \ldots, n$ *such that* $(-1)^k a_k(t) \geq 0$ *and* $\sum\limits_{k=j}^{n} (-1)^k a_k(t) > 0$ *for all* $k = j, \ldots, n$ *and* $t \in [t_0, T]$. *Then there exists a negative isolated closed solution of Eq. (1.1) on* $[t_0, T]$.

■

Note also that the class of equations, described by conditions of Theorem 1.1 (Theorem 1.1*) is not so wide, whereas the classes of equations described by other theorems of work [25] are very wide, but unlike of Theorem 1.1 the other theorems of work [25] are conditional (the conditions of these theorems contain an undetermined parameter λ_0, depending (may be) on the coefficients of Eq. (1.1)).

In this paper we use the comparison method for investigation of Eq. (1.1) for the case $n \geq 3$. In section 3 we prove two comparison criteria for Eq. (1.1). These criteria we use in section 4 to obtain some global solvability criteria for Eq. (1.1). On the basis od these criteria in section 5 we prove some criteria for existence of a closed solution (of closed solutions) of Eq. (1.1), essentially extending the class of equations, described by conditions of Theorem 1.1 (of Theorem 1.1*). The results obtained we compare with results of work [25]. In section 6 we use some results of section 5 to prove criteria of existence of periodic orbits or limit cycles for planar autonomous systems.

2. Auxiliary Propositions

Denote

$$D(t, u, v) \equiv \sum_{k=1}^{n} a_k(t) S_k(u, v),$$

where $S_k(u, v) \equiv \sum\limits_{j=0}^{k-1} u^j v^{k-j-1}$, $u, v \in \mathbb{R}$, $k = \overline{1, n}$, $t \geq t_0$. Let $b_k(t)$, $k = \overline{0, n}$ be real-valued continuous functions on $[t_0, \infty)$. Consider the equation

$$y' + \sum_{k=0}^{n} b_k(t) y^k = 0, \qquad t \geq t_0. \tag{2.1}$$

Let $y_0(t)$ and $y_1(t)$ be solutions of the equations (1.1) and (2.1) respectively on $[t_1, t_2) \subset [t_0, \infty)$. Then

$$[y_0(t) - y_1(t)]' + \sum_{k=0}^{n} a_k(t)[y_0^k(t) - y_1^k(t)] + \sum_{k=0}^{n} [a_k(t) - b_k(t)] y_1^k(t) = 0, \ \ t \in [t_1, t_2).$$

It follows from here and from the obvious equalities $y_0^k(t) - y_1^k(t) = [y_0(t) - y_1(t)] S_k(y_0(t), y_1(t))$, $k = \overline{1, n}$, that

$$[y_0(t) - y_1(t)]' + D(t, y_0(t), y_1(t))[y_0(t) - y_1(t)] + \sum_{k=0}^{n} [a_k(t) - b_k(t)] y_1^k(t) = 0, \ \ t \in [t_1, t_2).$$

It is clear from here that $y_0(t) - y_1(t)$ is a solution of the linear equation

$$x' + D(t, y_0(t), y_1(t))x + \sum_{k=0}^{n} [a_k(t) - b_k(t)] y_1^k(t) = 0, \ \ t \in [t_1, t_2).$$

Then by the Cauchy formula we have

$$y_0(t) - y_1(t) = \exp\left\{-\int_{t_1}^{t} D(\tau, y_0(\tau), y_1(\tau)) d\tau\right\} \Big[y_0(t_1) - y_1(t_1) -$$

 doi:10.20944/preprints202412.2567.v1

$$-\int_{t_1}^{t} \exp\Big\{\int_{t_1}^{\tau} D(s, y_0(s), y_1(s))ds\Big\}\Big(\sum_{k=0}^{n}[a_k(\tau) - b_k(\tau)]\Big)d\tau\Big], \qquad t \in [t_1, t_2). \tag{2.2}$$

Consider the differential inequality

$$\eta' + \sum_{k=0}^{n} a_k(t)\eta^k \geq 0, \qquad t_0 \leq t < \tau_0. \tag{2.3}$$

Definition 2.1. *A continuous on $[t_0, \tau_0)$ $(\tau_0 \leq \infty)$ function $\eta^*(t)$ is called a sub solution of the inequality (2.3) on $[t_0, \tau_0)$ if for every $t_1 \in [t_0, \tau_0)$ there exists a solution $\eta_{t_1}(t)$ of the inequality (2.3) on $[t_0, t_1]$ such that $\eta_{t_1}(t_0) \geq \eta^*(t_0)$, $\eta_{t_1}(t_1) = \eta^*(t_1)$.*

Consider the differential inequality

$$\zeta' + \sum_{k=0}^{n} a_k(t)\zeta^k \leq 0, \qquad t_0 \leq t < \tau_0. \tag{2.4}$$

Definition 2.2. *A continuous on $[t_0, \tau_0)$ $(\tau_0 \leq \infty)$ function $\zeta^*(t)$ is called a super solution of the inequality (2.4) on $[t_0, \tau_0)$ if for every $t_1 \in [t_0, \tau_0)$ there exists a solution $\zeta_{t_1}(t)$ of the inequality (2.4) on $[t_0, t_1]$ such that $\zeta_{t_1}(t_0) \leq \zeta^*(t_0)$, $\zeta_{t_1}(t_1) = \zeta^*(t_1)$.*

Obviously any solution $\eta(t)$ $(\zeta(t))$ of the inequality (2.3) ((2.4)) on $[t_0, \tau_0)$ is also a sub (super) solution of that inequality on $[t_0, \tau_0)$.

Lemma 2.1. *Let $y(t)$ be a solution of Eq. (1.1) on $[t_0, \tau_0)$ and $\eta^*(t)$ be a sub solution of the inequality (2.3) on $[t_0, \tau_0)$ such that $y(t_0) \leq \eta^*(t_0)$. Then $y(t) \leq \eta^*(t)$, $t \in [t_0, \tau_0)$, and if $y(t_0) < \eta^*(t_0)$, then $y(t) < \eta^*(t)$, $t \in [t_0, t_1)$.*

Proof. It is enough to show that if $\eta(t)$ is a solution of the inequality (2.3) on $[t_0, \tau_0)$ with $y(t_0) \leq \eta(t_0)$, then

$$y(t) \leq \eta(t), \;\; t \in [t_9, \tau_0). \tag{2.5}$$

and if $y(t_0) < \eta(t_0)$, then

$$y(t) < \eta(t), \;\; t \in [t_0, \tau_0). \tag{2.6}$$

We set $\widetilde{a}_0(t) \equiv -\eta'(t) - \sum\limits_{k=1}^{n} a_k(t)\eta^k(t)$, $t \in [t_0, \tau_0)$. By (2.3) we have

$$\widetilde{a}_0(t) \leq a_0(t), \;\; t \in [t_0, \tau_0). \tag{2.7}$$

Obviously $\eta(t)$ is a solution of the equation

$$y' + \sum_{k=1}^{n} a_k(t)y^k + \widetilde{a}_0(t) = 0, \;\; t \in [t_0, \tau_0)$$

on $[t_0, \tau_0)$. Then in virtue of (2.2) we have

$$y_0(t) - \eta(t) = \exp\Big\{-\int_{t_0}^{t} D(\tau, y(\tau), \eta(\tau))d\tau\Big\}\Big[y(t_0) - \eta(t_0) +$$

$$+\int_{t_1}^{t} \exp\Big\{\int_{t_1}^{\tau} D(s, y(s), \eta(s))ds\Big\}\Big(\widetilde{a}_0(\tau) - a_0(\tau)\Big)d\tau\Big], \qquad t \in [t_0, \tau_0).$$

This together with (2.7) implies that if $y(t_0) \leq \eta(t_0)$ $(y(t_0) < \eta(t_0))$, then (2.6) ((2.7)) is valid. The lemma is proved.

By analogy with the proof of Lemma 2.1 one can prove the following lemma

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

Lemma 2.2. *Let $y(t)$ be a solution of Eq. (1.1) on $[t_0, \tau_0)$ and $\zeta^*(t)$ be a super solution of the inequality (2.4) on $[t_0, \tau_0)$ such that $\zeta^*(t_0) \le y(t_0)$. Then $\zeta^*(t) \le y(t)$, $t \in [t_0, \tau_0)$, and if $\zeta(t_0) < y(t_0)$, then $\zeta(t) < y(t)$, $t \in [t_0, \tau_0)$.*

■

Remark 2.1. *It is clear that Lemma 2.1 (Lemma 2.2) remains valid if in the case $\tau_0 < +\infty$ the interval $[t_0, \tau_0)$ is replaced by $[t_0, \tau_0]$ in it.*

Let us introduce some denotations

1) $\Omega_\nu \equiv \{P(x)|P(x) \ge 0,\ x \ge \nu\}$, $\nu \in \mathbb{R}$, where $P(x) \equiv \sum\limits_{k=0}^{n} p_k x^k$, $x \in \mathbb{R}$ is any polynomial with real coefficients $p_k \in \mathbb{R}$, $k = \overline{0, n}$.

2) By $\Omega_{-\infty}$ we denote the set $\bigcap\limits_{\nu \in \mathbb{R}} \Omega_\nu$.

3) $\Omega_0^* \equiv \{P(x)|$ if $x \ge 0$, then $P(x) \ge 0$, if $x < 0$, then $P(x) \le 0$, where $P(x) \equiv \sum\limits_{k=0}^{n} p_k x^k$, $x \in \mathbb{R}$ is any polynomial with real coefficients $p_k \in \mathbb{R}$, $k = \overline{0, n}\}$.

It is clear, that if $\nu_1 < \nu_2$, then $\Omega_{\nu_1} \subset \Omega_{\nu_2}$ and if $P_1(x) \in \Omega_{-\infty}$, $P_2(x)$ is any polynomial, then $P_1(P_2(x)) \in \Omega_{-\infty}$. If $P_1(x) \in \Omega_0$, $P_2(x) \in \Omega_\nu$, then $P_1(P_2(x)) \in \Omega_\nu$, $\nu \in \mathbb{R} \bigcup \{-\infty\}$. If $P_j(x) \in \Omega_\nu$, $\lambda_j > 0$, $j = \overline{1, N}$, then $\sum\limits_{j=1}^{N} \lambda_j P_J(x) \in \Omega_\nu$, $\nu \in \mathbb{R} \bigcup \{-\infty\}$. Obviously, if $P_1(x) \in \Omega_0^*$, $P_2(x) \in \Omega_{-\infty}$, then $P_1(x)P_2(x) \in \Omega_0^*$. If $P_j(x) \in \Omega_0^*$, $\lambda_j > 0$, $j = \overline{1, N}$, then $\sum\limits_{j=1}^{N} \lambda_j P_J(x) \in \Omega_0^*$. Obviously $\Omega_0^* \subset \Omega_0$. If $P_j(x) \in \Omega_0^*$, $\lambda_j > 0$, $j = \overline{1, N}$, then $\sum\limits_{j=1}^{N} \lambda_j P_J(x) \in \Omega_0^*$.

Assume $a_k(t) = p_k(t) + r_k(t)$, $k = \overline{0, n}$, where $p_k(t)$ and $r_k(t)$, $k = \overline{0, n}$ are real-valued continuous functions on $[t_0, \infty)$. For any $T > t_0$ and $j = 2, \dots, n$ we set

$$M_{T,j} \equiv \max\left\{1, \max_{\tau \in [t_0, T]} \left\{\sum_{k=0}^{j-1} |r_k(\tau)| \Big/ \sum_{k=j}^{n} r_k(\tau)\right\}\right\}, \qquad M^*_{T,j}(t) \equiv \begin{cases} M_{T,j}, & t \in [t_0, T], \\ M_{t,j}, & t > T. \end{cases}$$

Lemma 2.3. *Let for some $j = 2, \dots, n$ the inequalities $r_k(t) \ge 0$, $k = \overline{j, n}$, $\sum\limits_{k=j}^{n} r_k(t) > 0$, $t \ge t_0$ be satisfied and let $\sum\limits_{k=0}^{n} p_k(t)x^k \in \Omega_0$, $t \ge t_0$. Then $M^*_{T,j}(t)$ is a sub solution of the inequality (2.3) on $[t_0, \infty)$.*

Proof. It is obvious that $M^*_{T,j}(t)$ is a nondecreasing and continuous function on $[t_0, \infty)$. Let $t_1 > t_0$ be fixed. To prove the lemma it is enough to show that $\eta_{t_1}(t) \equiv M_{t,j}$, $t \in [t_0, t_1]$ is a solution of the inequality (2.3) on $[t_0, t_1]$. Since $r_k(t) \ge 0$, $k = \overline{j, n}$, $t \ge t_0$ we have $\sum\limits_{k=j}^{n} r_k(t)\eta^k \ge \left[\sum\limits_{k=j}^{n} a_k(t)\right]\eta^j$ for all $\eta \ge 1$, and $t \ge t_0$. Then under the restriction $R_j(t) \equiv \sum\limits_{k=j}^{n} r_k(t) > 0$, $t \ge t_0$ we get

$$\sum_{k=0}^{n} r_k(t)\eta^k \ge R_j(t)\eta^j\left[1 - \sum_{k=0}^{j-1} |r_k(t)| \Big/ R_j(t)\eta\right]$$

for all $\eta \ge 1$, $t \ge t_0$. It follows from here that $\eta_{t_1}(t)$ is a solution of the inequality

$$\eta' + \sum_{k=0}^{n} r_k(t)\eta^k \ge 0$$

on $[t_0, t_1]$. This together with the condition $\sum\limits_{k=0}^{n} p_k(t)x^k \in \Omega_0$ of the theorem implies that $\eta_{t_1}(t)$ is a solution of the inequality (2.3) on $[t_0, t_1]$. The lemma is proved.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

For any $\gamma \in \mathbb{R}$ and $t_1 \geq t_0$ we set

$$\eta_{\gamma,t_1}(t) \equiv \gamma + \exp\Big\{-\int\limits_{t_0}^{t} a_1(\tau)d\tau\Big\}\Big[c(t_1) - \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\}a_0(\tau)d\tau\Big], \qquad t \in [t_0, t_1],$$

where $c_1(t_1) \equiv \max\limits_{\xi\in[t_0,t_1]} \int\limits_{t_0}^{\xi} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\}a_0(\tau)d\tau.$

Lemma 2.4. *Let the following conditions be satisfied.*

(1) $a_n(t) \geq 0, \ \ t \geq t_0.$

(2) $a_k(t) = a_n(t)c_k(t) + p_k(t), \ \ k = \overline{2, n-1}, \ \ t \geq t_0$, *where* $c_k(t), \ \ k = \overline{2, n-1}$ *are bounded functions on* $[t_0, t_1]$ *for every* $t_1 \geq t_0$ *and*

(3) $\sum\limits_{k=2}^{n-1} p_k(t)x^k \in \Omega_{N_T}, \ \ t \geq t_0$, *where*

$$N_{t_1} \equiv \max\Big\{1, \sup_{t\in[t_0,t_1]} \sum_{k=2}^{n} |c_k(t)|\Big\}, \ \ t_1 \geq T, \ \ \textit{for some} \ \ T \geq t_0.$$

Then

$$\eta^*_T(t) \equiv \begin{cases} \eta_{N_T,T}(t), \ \ t \in [t_0, T], \\ \eta_{N_t,t}(t), \ \ t \geq T \end{cases}$$

is a sub solution of the inequality (2.3) on $[t_0, \infty)$.

Proof. Obviously, $\eta^*_T(t) \in C([t_0, \infty))$. Therefore, to prove the lemma it is enough to show that for every $t_1 \geq T$ the function $\eta_{N_{t_1},t_1}(t)$ is a solution of the inequality (2.3) on $[t_0, t_1]$ and $\eta_{N_{t_1},t_1}(t_0) \geq \eta^*_T(t_0)$. The last inequality follows immediately from the definition of $\eta_{N_{t_1},t_1}(t)$. Consider the function

$$F(t,u) \equiv 1 + \frac{c_{n-1}(t)}{u} + \ldots + \frac{c_2(t)}{u^{n-2}}, \qquad t \in [t_0, t_1], \ \ u \geq 1 \qquad (t_1 \geq T).$$

Obviously $F(t,u) \geq 1 - \frac{\sum\limits_{k=2}^{n-1} |c_k(t)|}{u} \geq 0$ for all $t \in [t_0, t_1]$ and for all $u \geq N_{t_1}$. Moreover, $\eta_{N_{t_1},t_1}(t) \geq N_{t_1}, \quad t \in [t_0, t_1]$. Hence,

$$F(t, \eta_{N_{t_1},t_1}(t)) \geq 0, \qquad t \in [t_0, t_1]. \tag{2.8}$$

It is clear that $\eta_{N_{t_1},t_1}(t) \in C^1([t_0, t_1])$ and $\eta'_{N_{t_1},t_1}(t) + a_1(t)\eta_{N_{t_1},t_1}(t) + a_0(t) = 0, \ \ t \in [t_0, t_1]$. It follows from here and the condition (2) that

$$\eta'_{N_{t_1},t_1}(t) + \sum_{k=0}^{n} a_k(t)\eta^k_{N_{t_1},t_1}(t) = a_n(t)\eta^n_{N_{t_1},t_1}(t)F(t, \eta_{N_{t_1},t_1}(t)) + \sum_{k=2}^{n-1} p_k(t)\eta^k_{N_{t_1},t_1}(t),$$

$t \in [t_0, t_1]$. This together with the conditions (1), (2) and the inequality (2.8) implies that $\eta_{N_{t_1},t_1}(t)$ is a solution of the inequality (2.3) on $[t_0, t_1]$. The lemma is proved.

We set

$$\eta_c(t) \equiv \exp\Big\{-\int\limits_{t_0}^{t} a_1(\tau)d\tau\Big\}\Big[c - \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\}a_0(\tau)d\tau\Big], \qquad t \geq t_), \qquad c \in \mathbb{R}.$$

Lemma 2.5. *Let the following conditions be satisfied.*

(4) $a_2(t) > 0, \ \ t \in [t_0, T],$

 doi:10.20944/preprints202412.2567.v1

(5) for some $c \ge \max_{t\in[t_0,T]} \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\} a_0(\tau)d\tau$ *the inequality*

$\sum\limits_{k=3}^{n} |a_k(t)|\eta_c^{k-2}(t) \le a_2(t), \ t \in [t_0, T]$ *is valid.*

Then the function $\eta_c(t)$ *is a nonnegative solution of the inequality (2.3) on* $[t_0, T]$.

Proof. Obviously

$$\eta_c(t) \ge 0, \qquad t \in [t_0, T] \tag{2.9}$$

and

$$\eta_c'(t) + a_1(t)\eta_c(t) + a_0(t) = 0, \qquad t \in [t_0, T]. \tag{2.10}$$

It follows from the conditions (4), (5) and the inequality (2.9) that

$$\sum_{k=2}^{n} a_k(t)\eta_c^k(t) = a_2(t)\eta_c^2(t)\left[1 + \frac{\sum\limits_{k=3}^{n} a_k(t)\eta_c^{k-2}(t)}{a_2(t)}\right] \ge a_2(t)\eta_c^2(t)\left[1 - \frac{\sum\limits_{k=3}^{n} a_k(t)\eta_c^{k-2}(t)}{a_2(t)}\right] \ge 0,$$

$t \in [t_0, T]$. This together with (2.9) and (2.10) implies that $\eta_c(t)$ is a nonnegative solution of the inequality (2.3) on $[t_0, T]$. The lemma is proved.

We set

$$\alpha(t) \equiv \sum_{k=2}^{n} |a_k(t)| - a_1(t), \ \ \theta_c(t) \equiv \exp\Big\{\int\limits_{t_0}^{t} \alpha(\tau)d\tau\Big\}\Big[c - \int\limits_{t_0}^{t} \exp\Big\{-\int\limits_{t_0}^{\tau} \alpha(s)ds\Big\} a_0(\tau)d\tau\Big],$$

$t \ge t_0, \ \ c \in \mathbb{R}$.

Lemma 2.6. *Let for some* $c \ge \max_{t\in[t_0,T]} \int\limits_{t_0}^{t} \exp\Big\{-\int\limits_{t_0}^{\tau} \alpha(s)ds\Big\} a_0(\tau)d\tau$ *the inequality* $\theta_c(t) \le 1, \ t \in [t_0, T]$ *be satisfied. Then* $\theta_c(t)$ *is a nonnegative solution of the inequality (2.3) on* $[t_0, T]$.

Proof. It is obvious that

$$\theta_c(t) \ge 0, \qquad t \in [t_0, T]. \tag{2.11}$$

Show that $\theta_c(t)$ satisfies (2.3) on $[t_0, T]$. We have

$$\sum_{k=0}^{n} a_k(t)\theta_c^k(t) = \Big(\sum_{k=2}^{n} |a_k(t)|\Big)\theta_c(t) + \sum_{k=2}^{n} a_k(t)\theta_c^k(t) + a_0(t) - \alpha(t)\theta_c(t), \ \ t \in [t_0, T]. \tag{2.12}$$

Obviously,

$$\theta_c'(t) + a_0(t) - \alpha(t)\theta_c(t) = 0, \qquad t \in [t_0, T]. \tag{2.13}$$

It follows from here and (2.12) that if $\sum\limits_{k=2}^{n} |a_k(t)| = 0$ for some fixed $t \in [t_0, T]$, then $\theta_c(t)$ satisfies (2.3) in t. Assume $\sum\limits_{k=2}^{n} |a_k(t)| \ne 0$ for a fixed $t \in [t_0, T]$. Then it follows from the condition $\theta_c(t) \le 1, \ t \in [t_0, T]$ of the lemma and (2.11) that

$$\Big(\sum_{k=2}^{n} |a_k(t)|\Big)\theta_c(t) + \sum_{k=2}^{n} a_k(t)\theta_c^k(t) \ge \Big(\sum_{k=2}^{n} |a_k(t)|\Big)\theta_c(t)\left[1 - \frac{\sum\limits_{k=2}^{n} |a_k(t)|\theta_c^2(t)}{\sum\limits_{k=2}^{n} |a_k(t)|\theta_c(t)}\right] \ge 0$$

for that fixed t. This together with (2.12) and (2.13) implies that $\theta_c(t)$ satisfies (2.3) in that fixed t. Hence, $\theta_c(t)$ satisfies (2.3) for all $t \in [t_0, T]$. The lemma is proved.

Let $F(t, Y)$ be a continuous in t and continuously differentiable in Y vector function on $[t_0, \infty) \times \mathbb{R}^m$. Consider the nonlinear system

$$Y' = F(t, Y), \qquad t \ge t_0. \tag{2.14}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

Every solution $Y(t) = Y(t, t_0, Y_0)$ of this system exists either only a finite interval $[t_0, T)$ or is continuable on $[t_0, \infty)$

Lemma 2.7([5, p. 204, Lemma]). *If a solution $Y(t)$ of the system (2.14) exists only on a finite interval $[t_0, T)$, then*

$$||Y(t)|| \to \infty \ \ as \ \ t \to T - 0,$$

where $||Y(t)||$ is any euclidian norm of $Y(t)$ for every fixed $t \in [t_0, T)$.

■

Lemma 2.8. *For k odd the inequality*

$$S_k(u, v) \geq 0, \ \ u, v \in \mathbb{R} \ \ is\ valid.$$

Proof. If $u = 0$, then

$$S_k(u, v) = v^{k-1} = v^{2m} \geq 0, \qquad v \in \mathbb{R}, \ \ (m \in \mathbb{Z}_+). \tag{2.15}$$

For $u \neq 0$ we have

$$S_k(u, v) = u^{2m} P_k(x), \qquad x \equiv \frac{v}{u}, \qquad P_k(x) \equiv \sum_{j=0}^{k-1} x^j, \ \ x \in \mathbb{R}. \tag{2.16}$$

Since $k - 1$ is even all roots of $P_k(x)$ are complex (not real). Besides $P_k(0) = 1 > 0$. Hence, $P_k(x) > 0, \ \ x \in \mathbb{R}$. This together with (2.15) and (2.16) implies that $S_k(u, v) \geq 0$ for all $u, v \in \mathbb{R}$.. The lemma is proved.

Lemma 2.9 *For k even the inequality*

$$\frac{\partial S_k(u, v)}{\partial u} \geq 0, \qquad u, v \in \mathbb{R} \qquad is\ valid.$$

Proof. Since $\frac{\partial S_k(u,v)}{\partial u} = (k-1)u^{k-2} + \ldots + 2uv^{k-3} + v^{k-2}, \ \ u, v \in R$ and k is even, we have

$$\frac{\partial S_k(u, v)}{\partial u}\Big|_{u=0} = (k-1)u^{2m} \geq 0, \qquad u \in \mathbb{R}, \qquad (m \in \mathbb{Z}_+). \tag{2.17}$$

For $u \neq 0$ the following equality is valid

$$\frac{\partial S_k(u, v)}{\partial u} = u^{2m} Q_k(x), \qquad x \equiv \frac{v}{u}, \qquad Q_k(x) \equiv \sum_{j=0}^{k-1} (j+1) x^j, \qquad x \in \mathbb{R}. \tag{2.18}$$

Consider the polynomials $q_j(x) = (j+1)x^{2j}(1 + x^2), \ \ j = 0, 1, \ldots$. Obviously,

$$q_j(x) \geq 0, \qquad x \in \mathbb{R}, \qquad j = 0, 1, 2, \ldots. \tag{2.19}$$

and

$$q_0(x) + (k-1)x^{2m} \geq 0, \qquad x \in \mathbb{R}, \qquad m \in \mathbb{Z}_+. \tag{2.20}$$

It is not difficult to verify that

$$Q_k(x) = q_0(x) + \ldots + q_{2(m-1)}(x) + (k-1)x^{2m}, \qquad x \in \mathbb{R}, \qquad m \in \mathbb{Z}_+.$$

This together with (2.17)-(2.20) implies that $\frac{\partial S_k(u,v)}{\partial u} \geq 0, \ \ u, v \in \mathbb{R}$. The lemma is proved.

Let $f(t, u)$ be a real-valued continuous function on $[t_0, T] \times \mathbb{R}$. Consider the first order differential equation

$$y' = f(t, y), \qquad t \in [t_0, T] \tag{2.21}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

and the differential inequalities

$$\zeta' \leq f(t,\zeta), \quad t \in [t_0, T], \tag{2.22}$$

$$\eta' \geq f(t,\eta), \quad t \in [t_0, T]. \tag{2.23}$$

Theorem 2.1 ([3, Theorem 2.1]) *Let $\zeta(t)$ and $\eta(t)$ be solutions of the inequalities (2.22) and (2.23) respectively on $[t_0, T]$ such that $\zeta(t) \leq \eta(t),\ t \in [t_0, T],\ \zeta(t_0) \leq \zeta(T)$, $\eta(t_0) \geq \eta(T)$. If any solution $y(t)$ of the Cauchy problem $y = f(t,u),\ y(t_0) = y_0 \in [\zeta(t_0), \eta(t_0)]$ is unique, then Eq. (2.1) has a solution $y_*(t)$ on $[t_0, T]$ such that $y_*(t_0) = y_*(T)$, $\zeta(t) \leq y_*(t) \leq \eta(t),\ t \in [t_0, T]$*

Corollary 2.1. *Let $\zeta(t)$ and $\eta(t)$ be solutions of the inequalities (2.22) and (2.23) respectively on $[t_0, T]$ such that $\eta(t) \leq \zeta(t),\ t \in [t_0, T],\ \zeta(t_0) \leq \zeta(T),\ \eta(t_0) \geq \eta(T)$. If any solution $y(t)$ of the Cauchy problem $y = f(t,u),\ y(T) = y_0 \in [\eta(T), \zeta(T)]$ is unique, then Eq. (2.1) has a solution $y_*(t)$ on $[t_0, T]$ such that $y_*(t_0) = y_*(T)$, $\eta(t) \leq y_*(t) \leq \zeta(t),\ t \in [t_0, T]$*

Proof. In Eq. (2.21) and inequalities (2.22), (2.23) we substitute respectively $t \to -t,,\ \zeta \to \widetilde{\eta},\ \eta \to \widetilde{\zeta}$. We obtain respectively

$$y' = f_1(t,y), \quad t \in [-T, -t_0], \tag{2.24}$$

$$\widetilde{\eta}' \geq f_1(t,\widetilde{\eta}), \quad t \in [-T, -t_0],$$

$$\widetilde{\zeta}' \leq f_1(t,\widetilde{\zeta}), \quad t \in [-T, -t_0],$$

where $f_1(t,u) \equiv -f(t,u),\ t \in [-T, -t_0],\ u \in \mathbb{R}$. Obviously it follows from the conditions of the corollary that all the conditions of Theorem 2.1 for the last equation with $f(t,u) \equiv f_1(t,u),\ \zeta(t) \equiv \widetilde{\zeta}(t),\ \eta(t) \equiv \widetilde{\eta}(t)$ are satisfied. Hence, Eq. (2.24) has a closed solution $y_*(t)$ on $[-T, -t_0]$ such that $y_*(t_0) = y_*(T),\ \widetilde{\zeta}(t) \leq y_*(t) \leq \widetilde{\eta}(t),\ t \in [t_0, T]$. It follows from here that $y(t) \equiv y_*(-t)$ is the required closed solution of Eq. (2.21) on $[t_0, T]$. The corollary is proved.

Let $t_0 < t_1 < \ldots$ be a finite or infinite sequence such that $t_k \in [t_0, \tau_0],\ k = 0, 1, ...$

Definition 2.3. *The sequence $\{t_k\}$ we will call an usable sequence for the interval $[t_0, \tau_0)$, if the maximum of the numbers t_k coincides with τ_0 for finite $\{t_k\}$, and $\lim\limits_{k\to\infty} t_k = \tau_0$ for infinite $\{t_k\}$.*

Let $a(t),\ b(t)$ and $c(t)$ be real valued continuous functions on $[t_0, \tau_0)\ (\tau \leq \infty)$. Consider the Riccati equation

$$y' + a(t)y^2 + b(t)y + c(t) = 0,\ t \in [t_0, \tau_0). \tag{2.25}$$

Definition 2.4. *A solution of Eq. (2.25) is called t_1-regular, if it exists on $[t_1, \infty)$ (here $t_0 \leq t_1 < \tau_0 = \infty$).*

Definition 2.5 *A t_1-regular solution $y_0(t)$ is called t_1-normal, if there exists $\delta > 0$ such that every solution $y(t)$ od Eq. (2.25) with $|y(t_1) - y_0(t_1)| < \delta$ is t_1-regular, otherwise it is called t_1-extremal.*

Lemma 2.10 ([23, Theorem 2.3, II^0]). *Let $a(t) \geq 0,\ c(t) \leq 0,\ t \geq t_0$, and let $a(t)$ and $c(t)$ have unbounded supports. Then the unique t_0-extremal solution of Eq. (2.25) is negative.*

Denote by $reg(t_1)$ the set of initial values $\gamma \in \mathbb{R}$ for which the solution $y(t)$ of Eq. (2.25) with $y(t_1) = \gamma$ exists on $[t_1, \infty)$.

Lemma 2.11 ([22, Lemma 2.1]). *Let Eq. (2.25) has a t_1-regular solution. If $a(t) \geq 0,\ \ t \geq t_0$ and has an unbounded support, then $reg(t_1) = [y_*(t_1), \infty)$, where $y_*(t)$ is the unique t_1-extremal solution of Eq. (2.25).*

Theorem 2.2 ([11, Theorem 4.1]). *Assume $a(t) \geq 0,\ \ t \in [t_0, \tau_0)$ and*

$$\int\limits_{t_k}^{t} \exp\biggl\{ \int\limits_{t_k}^{\tau} \Bigl[b(s) - a(s) \Bigl(\int\limits_{t_k}^{s} \exp\Bigl\{ -\int\limits_{\xi}^{s} b(\zeta)d\zeta \Bigr\} c(\xi)d\xi \Bigr) \Bigr] ds \biggr\} c(\tau)d\tau \leq 0, \quad t \in [t_k, t_{k+1}),$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

$k = 1, 2, \ldots$, where $\{t_k\}$ is an usable sequence for $[t_0, \tau_0)$. Then for every $\gamma \geq 0$ Eq. (2.25) has a solution $y_0(t)$ on $[t_0, \tau_0)$, satisfying the initial condition $y_0(t_0) = \gamma$, and $y_0(t) \geq 0$, $t \in [t_0, \tau_0)$.

Remark 2.1. *Theorem 2.2 remains valid if for $\tau_0 < \infty$ we replace $[t_0, \tau_0)$ by $[t_0, \tau_0]$ in it.*

Lemma 2.12. *Let $a(t) \geq 0$, $t \geq t_0$ has an unbounded support and let Eq. (2.25) has a negative t_0-regular solution. If $a(t) > 0$, $t \in [t_0, T]$, then Eq. (2.25) has a negative solution $y_-(t)$ on $[t_0, T]$ such that $y_-(t_0) \geq y_-(T)$.*

Proof. By Lemma 2.11 it follows from the conditions of the lemma that Eq. (2.25) has the unique t_0-extremal solution $y_*(t) < 0$, $t \geq t_0$. Let γ_- be the lower bound of the initial values γ such that the solutions of Eq. (2.25) with $y(t_0) = \gamma$ exists on $[t_0, T]$. Obviously, $\gamma_- < y_*(t_0) < 0$. Assume $\gamma_- > -\infty$. Then since the solutions of Eq. (2.25) continuously depend on their initial values the solution $y_{\gamma_-}(t)$ with $y_{\gamma_-}(t_0) = \gamma_-$ exists on $[t_0, T)$ and $\liminf\limits_{t \to T-0} y_{\gamma_-}(t) = -\infty$. We claim that there exists a solution $y_-(t)$ of Eq. (2.25) with $y_(t_0) \in (\gamma_-, y_*(t_0)]$ such that $y_-(t_0) \geq y_-(T)$ (obviously by the uniqueness theorem $y_-(t) < 0$, $t \in [t_0, T]$). Suppose this is not true. Then for every solution $y(t)$ of Eq. (2.25) with $y_(t_0) \in (\gamma_-, y_*(t_0)]$ the inequality

$$y(T) > y(t_0) \tag{2.26}$$

is valid. Let $t_k < T$, $k = 1, 2, \ldots$ be a infinite sequence such that $\lim\limits_{k \to \infty} t_k = T$, $\lim\limits_{k \to \infty} y_{\gamma_-}(t_k) = -\infty$. Since the solutions of Eq. (2.25) continuously depend on their initial values for every $k = 1, 2, \ldots$ we chose γ_k, $k = 1, 2, \ldots$ such that for the solutions $y_k(t)$ of Eq. (2.25) with $y_k(t_k) = \gamma_k$, $k = 1, 2, \ldots$ the inequalities $|y_k(t_k) - y_{\gamma_-}(t_k)| < 1$. $k = 1, 2, \ldots$ are valid. Therefore,

$$\lim_{k \to \infty} y_k(t_k) = -\infty. \tag{2.27}$$

We set $m_k \equiv \min\limits_{t \in [t_0, T]} y_k(t)$, $k = 1, 2, \ldots$ and assume $y_k(\tau_k) = m_k$, $k = 1, 2, \ldots$. Then it follows from (2.26), (2.27) and the inequalities $y_k(t_0) > \gamma_-$, $k = 1, 2, \ldots$ that $y_k'(\tau_k) = 0$ for all enough large k and $\lim\limits_{k \to \infty} y_k(\tau_k) = -\infty$ then since $a(t) > 0$, $t \in [t_0, T]$ we get $y_k'(\tau_k) + a(\tau_k) y_k^2(\tau_k) + b(\tau_k) y_k(\tau_k) + c(\tau_k) > 0$ for all enough large k. We obtain a contradiction. Hence, the claim for the case $\gamma_- > -\infty$. To complete the proof of the lemma it is enough to show that the supposition $\gamma_- = -\infty$ leads to a contradiction. Assume $\gamma_- = -\infty$. Let then $y_k(t)$, $k = 1, 2, \ldots$ are the solutions of Eq. (2.25) with $y_k(t_0) = -k$, $k = 1, 2, \ldots$. Then $y_k(t)$, $k = 1, 2, \ldots$ exist on $[t_0, T]$. Let $m_k \equiv \min\limits_{t \in [t_0, T]} y_k(t)$, $k = 1, 2, \ldots$ and $y_k(\tau_k) = m_k$, $k = 1, 2, \ldots$. Obviously,if $\tau_k = t_0$, then $y_k'(\tau_k) \geq 0$, otherwise $y_k'(\tau_k) = 0$ (since according to assumption (2.26) $y_k(t_0) < y_k(T)$). Hence, since $a(t) > 0$, $t \in [t_0, T]$, we have

$$y_k'(\tau_k) + a(\tau_k) y_k^2(\tau_k) + b(\tau_k) y_k(\tau_k) + c(\tau_k) \geq a(\tau_k) k^2 - |b(\tau_k)| k - |c(\tau_k)| > 0$$

for all enough large k. We obtain a contradiction, completing the proof of the lemma.

For any real-valued continuous functions $r_k(t)$, $k = 0, 1, 2$ on $[t_0, \infty)$ we set

$$I_\gamma(t) \equiv \gamma \exp\left\{-\int\limits_{t_0}^{t} r_1(\tau) d\tau\right\} + \int\limits_{t_0}^{t} \exp\left\{-\int\limits_{\tau}^{t} r_1(s) ds\right\} |r_0(\tau)| d\tau, \qquad t \geq t_0.$$

Theorem 2.3. *Assume $a_k(t) = p_k(t) + r_k(t)$, $k = \overline{0,2}$, $t \geq t_0$, where $p_k(t)$, $r_k(t)$, $k = \overline{0,2}$ are real-valued continuous functions on $[t_0, \tau_0)$, $r_2(t) \geq 0$, $\sum\limits_{k=0}^{2} p_k(t) x^k + \sum\limits_{k=3}^{n} a_k(t) x^k \in \Omega_0$, $t \in [t_0, \tau_0)$ and*

$$\int\limits_{t_k}^{t} \exp\left\{\int\limits_{t_k}^{\tau} \left[r_1(s) - r_2(s)\left(\int\limits_{t_k}^{s} \exp\left\{-\int\limits_{\xi}^{s} r_1(\zeta) d\zeta\right\} r_0(\xi) d\xi\right)\right] ds\right\} r_0(\tau) d\tau \leq 0, \tag{2.28}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

$t \in [t_k, t_{k+1}),\ k = 1, 2, ...$ where $\{t_k\}$ is an usable sequence for $[t_0, \tau_0)$. Then for every $\gamma \geq 0$ the inequality (2.3) has a solution $\eta^0_\gamma(t)$ on $[t_0, \tau_0)$, satisfying the initial condition $\eta^0_\gamma(t_0) = \gamma$, and $0 \leq \eta^0_\gamma(t) \leq I_\gamma(t),\ t \in [t_0, \tau_0)$.

Proof. By Theorem 2.2 it follows from the conditions $r_2(t) \geq 0,\ t \in [t_0, \tau_0)$ and (2.28) that for every $\gamma \geq 0$ any solution $y_\gamma(t)$ of the Riccati equation

$$y' + r_2(t)y^2 + r_1(t)y + r_0(t) = 0, \qquad t \in [t_0, \tau_0)$$

with $y_\gamma(t_0) = \gamma$ exists on $[t_0, \tau_0)$ and is nonnegative. It follows from here and from the condition $\sum\limits_{k=0}^{w} p_k(t)x^k + \sum\limits_{k=3}^{n} a_k(t)x^k \in \Omega_0,\ t \in [t_0, \tau_0)$ of the theorem that $\eta^0_\gamma(t) \equiv y_\gamma(t)$ is a nonnegative solution of the inequality (2.3) on $[t_0, \tau_0)$ for every $\gamma \geq 0$. Note that we can interpret $y_\gamma(t)$ as a solution of the linear equation

$$x' + [r_2(t)y_\gamma(t) + r_1(t)]x + r_0(t) = 0,\ t \geq t_0.$$

Then by the Cauchy formula we have

$$y_\gamma(t) = \gamma \exp\Big\{-\int\limits_{t_0}^{t} [r_2(\tau)y_\gamma(\tau) + r_1(\tau)]d\tau\Big\} - \int\limits_{t_0}^{t} \exp\Big\{-\int\limits_{\tau}^{t} [r_2(s)y_\gamma(s) + r_1(s)]ds\Big\} r_0(\tau)d\tau,$$

$t \geq t_0$. Hence, $0 \leq \eta^0_\gamma(t) = y_\gamma(t) \leq I_\gamma(t),\ t \geq t_0$. The theorem is proved.

Consider the differential inequalities

$$\zeta' < f(t, \zeta), \qquad t \in [t_0, T], \tag{2.29}$$

$$\eta' > f(t, \eta), \qquad t \in [t_0, T]. \tag{2.30}$$

Lemma 2.13([25. Lemma A2]) *Let us assume that f is continuous in t and analytic in y. If there exist solutions $\zeta(t)$ and $\eta(t)$ of the inequalities (2.29) and (2.30) respectively on $[t_0, T]$ such that $\zeta(t_0) \leq \zeta(T),\ \eta(t_0) \geq \eta(T),\ \zeta(t) < \eta(t),\ t \in [t_0, T]$ (or $\zeta(t_0) \geq \zeta(T),\ \eta(t_0) \leq \eta(T),\ \zeta(t) > \eta(t),\ t \in [t_0, T]$), then Eq. (2.21) has a isolated closed solution $y(t)$ on $[t_0, T]$ such that $\zeta(t) < y(t) < \eta(t),\ t \in [t_0, T]$ (respectively $\eta(t) < y(t) < \zeta(t),\ t \in [t_0, T]$).*

Lemma 2.14 *Assume for a $j = 0, 1, \ldots, n-1$ the inequalities $a_k(t) \leq 0,\ \sum\limits_{k=0}^{j} a_k(t) < 0,\ t \in [t_0, T]$ are satisfied. Then there exists $\rho \in (0, 1)$ such that $\zeta(t) \equiv \rho$ is a solution of the inequality (2,29) on $[t_0, T]$.*

Proof. For any $\rho \in (0, 1)$ we have $a_0(t) + a_1(t)\rho + \ldots + a_j(t)\rho^j + a_{j+1}(t)\rho^{j+1} + \ldots a_n(t)\rho^n \leq [a_0(t) + a_1(t) + \ldots a_j(t)]\rho^j + [|a_{j+1}(t)| + \ldots + |a_n(t)|]\rho^{j+1} \leq$ $\leq A^0_j(t)\rho^j \Bigg[1 - \Bigg(\max\limits_{t \in [t_0, T]} \frac{\sum\limits_{k=j+1}^{n} |a_k(t)|}{|A^0_j(t)|}\Bigg)\rho\Bigg]$, where $A^0_j(t) \equiv a_0(t) + \ldots + a_j(t),\ t \in [t_0, T]$. Hence, for $0 < \rho < \min\Bigg\{1, \Bigg(\max\limits_{t \in [t_0, T]} \frac{\sum\limits_{k=j+1}^{n} |a_k(t)|}{|A^0_j(t)|}\Bigg)^{-1}\Bigg\}$ (it is assumed that the trivial case $\sum\limits_{k=j+1}^{n} |a_k(t)| \equiv 0$, for which the lemma is obvious, is excluded) $\zeta(t) \equiv 0,\ t \in [t_0, T]$ is a solution of the inequality (2.29). The lemma is proved.

Lemma 2.15. *Assume $a_k(t) = p_k(t) + r_k(t),\ k = \overline{0, n},\ t \in [t_0, T]$, where $p_k(t), r_k(t)$, $k = \overline{0, n}$ are real-valued continuous functions on $[t_0, T]$ such that for some $j = 2, \ldots, n$, $r_k(t) \geq 0,\ k = \overline{j, n},\ R_j(t) \equiv \sum\limits_{k=j}^{n} r_k(t) > 0,\ \sum\limits_{k=0}^{n} p_k(t)x^k \in \Omega_0,\ t \in [t_0, T]$. Then there exists $M > 1$ such that $\eta(t) \equiv M,\ t \in [t_0, T]$ is a solution of the inequality (2.30) on $[t_0, T]$.*

Proof. It is clear from the proof of Lemma 2,3 that for $M > \max\{1, M_{T,j}\}$ the inequality $\sum\limits_{k=0}^{n} r_k(t) > 0, \ t \in [t_0, T]$ is satisfied. Then since $\sum\limits_{k=0}^{n} p_k(t)M^k \geq 0, \ t \in [t_0, T]$ (as for as $\sum\limits_{k=0}^{n} p_k(t)x^k \in \Omega_0, \ t \in [t_0, T]$) we have $\sum\limits_{k=0}^{n} a_k(t)M^k > 0, \ t \in [t_0, T]$. Therefore $\eta(t) \equiv M, \ t \in [t_0, T]$ is a solution of the inequality (2,30) on $[t_0, T]$. The lemma is proved.

Lemma 2.16. *Let the inequalities* $\sum\limits_{k=2}^{n} |a_k(t)| > 0, \ \theta_c(t) < 1, \ t \in [t_0, T]$ *for some* $c > \max\limits_{t\in[t_0,T]} \int\limits_{t_0}^{T} \exp\Big\{-\int\limits_{t_0}^{\tau} \alpha(s)ds\Big\} a_0(\tau)d\tau$. *Then* $\theta_c(t)$ *is a solution of the inequality (2.30).*

Proof. Obviously

$$\theta_c(t) > 0, \ t \in [t_0, T]. \tag{2.31}$$

It was shown in the proof of Lemma 2.6 that

$$\Big(\sum_{k=2}^{n} |a_k(t)|\Big)\theta_c(t) + \sum_{k=2}^{n} a_k(t)\theta_c^k(t) \geq \Big(\sum_{k=2}^{n} |a_k(t)|\Big)\theta_c(t)\left[1 - \frac{\sum\limits_{k=2}^{n} |a_k(t)|\theta_c^2(t)}{\sum\limits_{k=2}^{n} |a_k(t)|\theta_c(t)}\right], \ t \in [t_0, T].$$

This together with (2.13), (2.31) and the conditions $\sum\limits_{k=2}^{n} |a_k(t)| > 0, \ \theta_c(t) < 1, \ t \in [t_0, T]$ of the lemma implies that $\theta_c(t)$ is a solution of the inequality (2.30) on $[t_0, T]$. The lemma is proved.

3. Comparison Criteria

In this section we prove two comparison criteria for Eq. (1.1). These criteria with the aid of section 2 we use in section 4 to obtain some global solvability criteria for Eq. (1.1).

Theorem 3.1. *Let* $y_1(t)$ *be a solution of Eq. (2.1) on* $[t_0, \infty)$ *and* $\eta^*(t)$ *be a sub solution of the inequality (2.3) on* $[t_0, \infty)$ *such that* $y_1(t_0) < \eta^*(t_0)$. *Moreover, let the following conditions be satisfied*
(I) $D(t, u, y_1(t)) \leq D_1(t, u, y_1(t)), \ u \geq y_1(t), \ t \geq t_0$, *where* $D_1(t, u, y_1(t))$ *is a nondecreasing in* $u \geq y_1(t)$ *function for every* $t \geq t_0$.
(II) $\gamma - y_1(t) + \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} D_1(s, \eta^*(s), y_1(s))ds\Big\}\Big(\sum\limits_{k=0}^{n} [b_k(\tau) - a_k(\tau)]y_1^k(\tau)\Big)d\tau \geq 0, \quad t \geq t_0$ *for some* $\gamma \in [y_1(t_0), \eta^*(t_0)]$.
Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \in [\gamma, \eta^*(t_0)]$ *exists on* $[t_0, \infty)$ *and*

$$y_1(t) \leq y(t) \leq \eta^*(t), \quad t \geq t_0.$$

Furthermore, if $y_1(t_0) < y(t_0) \ \ (y(t_0) < \eta^*(t_0))$, *then*

$$y_1(t) < y(t) \ \ (y(t) < \eta^*(t)), \ t \geq t_0.$$

Proof. Let $y(t)$ be a solution of Eq. (1.1) with $y(t_0) \in [\gamma, \eta^*(t_0)]$ and $[t_0, t_1)$ be its maximum existence interval. Then by Lemma 2.1 we have

$$y(t) \leq \eta^*(t), \ t \in [t_0, t_1), \tag{3.1}$$

and if $y(t_0) < \eta^*(t_0)$, then

$$y(t) < \eta^*(t), \quad t \in [t_), t_1). \tag{3.2}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

In virtue of (2.2) we have

$$y(t) - y_1(t) = \exp\Big\{-\int\limits_{t_0}^{t} D(\tau, y(\tau), y_1(\tau))d\tau\Big\}\Big[y(t_0) - y_1(t_0) -$$

$$-\int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} D(s, y(s), y_1(s))ds\Big\}\Big(\sum_{k=0}^{n}[a_k(\tau) - b_k(\tau)]y_1^k(\tau)\Big)d\tau\Big], \ t \in [t_0, t_1). \tag{3.3}$$

Let us show that

$$y_1(t) \le y(t), \ t \in [t_0, t_1). \tag{3.4}$$

At first we consider the case $y(t_0) > y_1(t_0)$. Show that in this case

$$y_1(t) < y(t), \qquad t \in [t_0, t_1). \tag{3.5}$$

Suppose it is not true. Then there exists $t_2 \in (t_0, t_1)$ such that

$$y_1(t) < y(t), \qquad t \in [t_0, t_2).$$

$$y_1(t_2) = y(t_2). \tag{3.6}$$

It follows from here, (3.1) and the condition (I) that

$$D(t, y(t), y_1(t)) \le D_1(t, \eta^*(t), y_1(t)), \qquad t \in [t_0, t_2).$$

Hence, the function

$$H(\tau) \equiv \exp\Big\{\int\limits_{t_0}^{\tau}\Big[D(s, y(s), y_1(s)) - D_1(s, \eta^*(s), y_1(s))\Big]ds\Big\}, \qquad \tau \in [t_0, t_2)$$

is positive and non increasing on $[t_0, t_2)$. By mean value theorem for integrals (see [6, p. 869]) it follows from here that

$$\int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} D(s, y(s), y_1(s))ds\Big\}\Big(\sum_{k=0}^{n}[a_k(\tau) - b_k(\tau)]y_1^k(\tau)\Big)d\tau =$$

$$= \int\limits_{t_0}^{\kappa(t)} \exp\Big\{\int\limits_{t_0}^{\tau} D_1(s, \eta^*(s), y_1(s))ds\Big\}\Big(\sum_{k=0}^{n}[a_k(\tau) - b_k(\tau)]y_1^k(\tau)\Big)d\tau$$

for some $\kappa(t) \in [t_0, t)$, $t \in [t_0, t_2)$. This together with (3.3) and the condition (II) implies that $y_1(t_2) < y(t_2)$, which contradicts (3.6). The obtained contradiction proves (3.5), hence proves (3.4). Let us show that (3.4) is also valid for the case $y(t_0) = y_1(t_0)$. Suppose, for some $t_3 \in (t_0, t_1)$

$$y(t_3) < y_1(t_3). \tag{3.7}$$

Let $\widetilde{y}_\delta(t)$ be a solution of Eq. (1.1) with $\widetilde{y}_\delta(t_0) > y_1(t_0)$. Then by already proven (3.5) we have $\widetilde{y}_\delta(t_3) > y_1(t_3)$. As for as the solutions of Eq. (1.1) continuously depend on their initial values we chose $\delta > 0$ enough small such that $\widetilde{y}_\delta(t_3) - y_1(t_3) < \frac{y_1(t_3) - y(t_3)}{2}$. Since, $\widetilde{y}_\delta(t_3) > y_1(t_3)$ it follows from the last inequality that

$$y_1(t_0) - y(t_3) < \widetilde{y}_\delta(t_3) - y_1(t_3) < \frac{y_1(t_3) - y(t_3)}{2},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

which contradicts (3.7). The obtained contradiction proves that (3.4) is also valid for $y(t_0) = y_1(t_0)$. Note that the proof of (3.3) and (3.4) in the general case $y(t_0) \geq y_1(t_0)$ repeats the proof of them for the case $y(t_0) > y_1(t_0)$. Therefore, due to (3.1), (3.2), (3.4) and (3.5) to complete the proof of the theorem it remains to show that

$$t_1 = \infty. \tag{3.8}$$

Suppose $t_1 < \infty$. Then it follows from (3.1) and (3.4) that $y(t)$ is bounded on $[t_0, t_1)$. By Lemma 2.7 it follows from here that $[t_0, t_1)$ is not the maximum existence interval for $y(t)$, which contradicts our supposition. The obtained contradiction proves (3.8). The proof of the theorem is completed.

Note that every function $y_1(t) \equiv \zeta(t) \in C^1([t_0,\infty))$ is a solution of Eq. (2.1) with $b_0(t) = -\zeta'(t),\ b_1(t) = \ldots = b_n(t) \equiv 0,\ t \geq t_0$. Then

$$\sum_{k=0}^{n}[b_k(t) - a_k(t)]y_1^k(t) = -\Big[\zeta'(t) + \sum_{k=0}^{n} a_k(t)\zeta^k(t)\Big] \quad t \geq t_0.$$

From here and Theorem 3.1 we obtain immediately

Corollary 3.1. *Let $\eta^*(t)$ be a sub solution of the inequality (2.3) on $[t_0,\infty)$ and for some $y_1(t) \equiv \zeta(t) \in C^1([t_0,\infty))$ with $\zeta(t_0) < \eta^*(t_0)$ the condition (I) of Theorem 3.1 and the following condition be satisfied*

$$(II^0)\ \ \zeta(t_0) - \gamma + \int_{t_0}^{t} \exp\Big\{\int_{t_0}^{\tau} D_1(s,\eta^*(s),\zeta(s))ds\Big\}\Big(\zeta'(\tau) + \sum_{k=0}^{n} a_k(\tau)\zeta^k(\tau)\Big)d\tau \leq 0,\ \ t \geq t_0,$$

for some $\gamma \in [\zeta(t_0), \eta^*(t_0)]$.
Then every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\gamma, \eta^*(t_0)]$ exists on $[t_0,\infty)$ and

$$\zeta(t) \leq y(t) \leq \eta^*(t), \qquad t \geq tt_0.$$

Furthermore, if $\zeta(t_0) < y(t_0)$ $(y(t_0) < \eta^*(t_0))$, then

$$\zeta(t) < y(t)\ \ (y(t) < \eta^*(t)), \qquad t \geq t_0.$$

■

Remark 3.1. *It is clear form the proofs of Theorem 3.1 and Corollary 3.1 that we can replace $\eta^*(t)$ in the conditions (II) and (II^0) respectively of Theorem 3.1 and Corollary 3.1 by a continuous function $\widetilde{\eta}^*(t) \geq \eta^*(t),\ t \in [t_0,\infty)$.*

Let $e_k(t),\ k = \overline{0,n}$ be real-valued continuous functions on $[t_0,\infty)$. Consider the equation

$$y' + \sum_{k=0}^{n} e_k(t)y^k = 0, \qquad t \geq t_0. \tag{3.9}$$

Theorem 3.2 *Let $y_1(t)$ and $y_2(t)$ be solutions of the equations (2.1) and (3.9) respectively on $[t_0,\infty)$ such that $y_1(t_0) \leq y_2(t_0)$ and the following conditions be satisfied.*
(III) $\sum_{k=0}^{n}(b_k(t) - a_k(t))y_1^k(t) \geq 0,\ \ t \geq t_0,$
(IV) $\sum_{k=0}^{n}(e_k(t) - a_k(t))y_2^k(t) \leq 0,\ \ t \geq t_0.$
Then every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [y_1(t_0), y_2(t_0)]$ exists on $[t_0,\infty)$ and

$$y_1(t) \leq y(t) \leq y_2(t), \qquad t \geq t_0.$$

Furthermore, if $y_1(t_0) < y(t_0)$ $(y(t_0) < y_2(t_0))$, then

$$y_1(t) < y(t) \quad (y(t) < y_2(t)), \qquad t \geq t_0.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

Proof. Let $y(t)$ be a solution of Eq. (1.1) with $y(t_0) \in [y_1(t_0), y_2(t_0)]$ and $[t_0, t_1)$ be its maximum existence interval. Then by (2.2) the following equations are valid

$$y(t) - y_1(t) = \exp\Big\{-\int\limits_{t_0}^{t} D(\tau, y(\tau), y_1(\tau))d\tau\Big\}\Big[y(t_0) - y_1(t_0) -$$

$$-\int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} D(s, y(s), y_1(s))ds\Big\}\Big(\sum_{k=0}^{n}[a_k(\tau) - b_k(\tau)]y_1^k(\tau)\Big)d\tau, \qquad t \in [t_0, t_1),$$

$$y(t) - y_2(t) = \exp\Big\{-\int\limits_{t_0}^{t} D(\tau, y(\tau), y_2(\tau))d\tau\Big\}\Big[y(t_0) - y_2(t_0) -$$

$$-\int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} D(s, y(s), y_2(s))ds\Big\}\Big(\sum_{k=0}^{n}[a_k(\tau) - e_k(\tau)]y_1^k(\tau)\Big)d\tau, \qquad t \in [t_0, t_1).$$

It follows from here and the conditions (III) and (IV) of the theorem that

$$y_1(t) \le y(t) \le y_2(t), \qquad t \in [t_0, t_1). \tag{3.10}$$

and if $y_1(t_0) < y(t_0)$ $(y(t_0) < y_2(t_0))$, then

$$y_1(t) < y(t) \ (y(t) < y_2(t)), \qquad t \in [t_0, t_1).$$

Therefore, to complete the proof of the theorem it remains to show that

$$t_1 = \infty. \tag{3.11}$$

Suppose $t_1 < \infty$. Then by Lemma 2.7 it follows from (3.10) that $[t_0, t_1)$ is not the maximum existence interval for $y(t)$, which contradicts our supposition. The obtained contradiction proves (3.11). The proof of the theorem is completed.

Corollary 3.2. *Let $\eta^*(t)$ and $\zeta^*(t)$ be sub and super solutions of the inequalities (2.3) and (2.4) respectively on $[t_0, \infty)$ such that $\zeta^*(t) \le \eta^*(t)$. Then every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\zeta^*(t_0), \eta^*(t_0)]$ exists on $[t_0, \infty)$ and*

$$\zeta^*(t) \le y(t) \le \eta^*(t), \qquad t \ge t_0.$$

Furthermore, if $\zeta^*(t_0) < y(t_0)$ $(y(t_0) < \eta^*(t_0))$, then

$$\zeta^*(t) < y(t) \qquad (y(t) < \eta^*(t)), \qquad t \ge t_0.$$

Proof. To prove the corollary it is enough to show that for every $\tau_0 > t_0$ and solutions $\zeta(t)$ and $\eta(t)$ of the inequalities (2.4) and (2.3) respectively on $[t_0, \tau_9]$ with $\zeta(t_0) \le \eta(t_0)$ any solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\zeta(t_0), \eta(t_0)]$ exists on $[t_0, \tau_0]$ and

$$\zeta(t) \le y(t) \le \eta(t), \qquad t \in [t_0, \tau_0], \tag{3.12}$$

and if $\zeta(t_0) < y(t_0)$ $(y(t_0) < \eta(t_0))$, that

$$\zeta(t) < y(t) \qquad (y(t) < \eta(t)), \qquad [t_0, \tau_0]. \tag{3.13}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

The function $y_1(t) \equiv \zeta(t)$ is a solution of Eq. (2.1) on $[t_0, \tau_0]$ for $b_0(t) \equiv -\zeta'(t),\ b_1(t) =, \ldots = b_n(t) \equiv 0$, and $y_2(t) \equiv \eta(t)$ is a solution of the equation (3.9) on $[t_0, \tau_0]$ for $e_0(t) \equiv -\eta'(t),\ e_1(t) = \ldots = e_n(t) \equiv 0$. Then the condition (III) gives us

$$\zeta'(t) + \sum_{k=0}^{n} a_k(t)\zeta^k(t) \leq 0, \qquad t \geq t_0$$

and the condition (IV) gives us

$$\eta'(t) + \sum_{k=0}^{n} a_k(t)\eta^k(t) \geq 0, \qquad t \geq t_0.$$

Therefore by Theorem 3.2 (note that Theorem 3.2 remains valid if we replace $[t_0, +\infty)$ by $[t_0, \tau_0]$ in it) the inequalities (3.12) and (3.13) are valid. The corollary is proved.

4. Global Solvability Criteria

For any continuous on $[t_0, \infty)$ function $f(t)$ denote $f^+(t) \equiv \max\{0, f(t)\},\ t \geq t_0$.

Theorem 4.1. *Let the condition of Lemma 2.3 and the following condition be satisfied*
(A) *for a nonnegative* $\zeta(t) \in C^1([t_0, \infty))$ *with* $\zeta(t_0) < M^*_{\gamma,T}(t_0)$ *and for some* $\nu \in [\zeta(t_0), M^*_{\gamma,T}(t_0)]$

$$\zeta(t_0) - \nu + \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\Big[\sum_{k=2}^{n} a_k^+(s)S_k(M^*_{\gamma,T}(s), \zeta(s)) + a_1(s)\Big]ds\Big\} \times$$

$$\times \Big(\zeta'(\tau) + \sum_{k=0}^{n} a_k(\tau)\zeta^k(\tau)\Big)d\tau \leq 0, \qquad t \geq t_0.$$

Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \in [\nu, M^*_{\gamma,T}(t_0)]$ *exists on* $[t_0, \infty)$ *and*

$$\zeta(t) \leq y(t) \leq M^*_{\gamma,T}(t), \qquad t \geq t_0. \tag{4.1}$$

Furthermore, if $\zeta(t_0) < y(t_0)$ $(y(t_0) < M^*_{\gamma,T}(t_0))$, *then*

$$\zeta(t) < y(t) \ \ (y(t) < M^*_{\gamma,T}(t)), \qquad t \geq t_0. \tag{4.2}$$

Proof. By Lemma 2.3 $M^*_{\gamma,T}(t)$ is a sub solution of the inequality (2.3) on $[t_0, \infty)$. Note that $y_1(t) \equiv \zeta(t)$ is a solution of Eq. (2.1) on $[t_0, \infty)$ for $b_0(t) \equiv -\zeta'(t),\ b_1(t) = \ldots = b_n(t) \equiv 0$. Then since $\zeta(t)$ is nonnegative we have

$$D(t, u, v) \leq \sum_{k=2}^{n} a_k^+(t)S_k(u, \zeta(t)) + a_1(t), \ \text{ for all } \ u \geq \zeta(t),\ t \geq t_0.$$

Moreover, $\sum\limits_{k=2}^{n} a_k^+(t)S_k(u, \zeta(t))$ is nondecreasing in $u \geq \zeta(t) \geq 0$, for all $t \geq t_0$. It follows from here and (A) that the conditions of Corollary 3.1 are satisfied. Hence, every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\nu, M^*_{\gamma,T}(t_0)]$ exists on $[t_0, \infty)$ and the inequalities (4.1) and (4.2) are valid. The theorem is proved.

By analogy with the proof of Theorem 4.1 it can be proved the following theorem

Theorem 4.2. *Let the conditions of Lemma 2.4 and the condition*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

(B) for a nonnegative $\zeta(t) \in C^1([t_0,\infty))$ with $\zeta(t_0) < \eta^*_T(t_0)$ and for some $\nu \in [\zeta(t_0), \eta^*_T(t_0)]$

$$\zeta(t_0) - \nu + \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\Big[\sum_{k=2}^{n} a_k^+(s) S_k(\eta^*_T(s),\zeta(s)) + a_1(s)\Big] ds\Big\} \times$$

$$\times\Big(\zeta'(\tau) + \sum_{k=0}^{n} a_k(\tau)\zeta^k(\tau)\Big) d\tau \le 0, \qquad t \ge t_0.$$

be satisfied.
Then every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\nu, \eta^*_T(t_0)]$ exists on $[t_0,\infty)$ and

$$\zeta(t) \le y(t) \le \eta^*_T(t), \qquad t \ge t_0.$$

Furthermore, if $\zeta(t_0) < y(t_0)$ $(y(t_0) < \eta^*_T(t_0))$, then

$$\zeta(t) < y(t) \ \ (y(t) < \eta^*_T(t)), \qquad t \ge t_0.$$

■

Corollary 4.1. *Let the conditions of Lemma 2.3 or Lemma 2.4 be satisfied. If $a_0(t) \le 0$, $t \ge t_0$, then every solution $y(t)$ of Eq. (1.1) with $y(t_0) \ge 0$ exists on $[t_0,\infty)$ and is nonnegative*

Proof. Let $y(t)$ be a solution of Eq. (1.1) with $y(t_0) \ge 0$. Under the conditions of Lemma 2.3 (of Lemma 2.4) we can take $M^*_T(t)$ $(\eta^*_T(t))$ so that $y(t_0) \le M^*_T(t)$ $(y(t_0) \le \eta^*_T(t))$. Then the condition $r_0(t) \le 0$, $t \ge t_0$ provides the satisfiability of the condition (A) of Theorem 4.1 (of the condition (B) of Theorem 4.2) for $\nu = 0$, $\zeta(t) \equiv 0$. Hence, the assertion of the corollary is valid. The corollary is proved.

Theorem 4.3. *Let $a_k(t) = p_k(t) + r_k(t)$, $k = \overline{0,2}$, $t \ge t_0$, where $p_k(t)$, $r_k(t)$, $k = \overline{0,2}$ are real-valued continuous functions on $[t_0,\infty)$, $\sum\limits_{k=0}^{2} p_k(t)x^k + \sum\limits_{k=3}^{n} a_k(t)x^k \in \Omega_0$, $t \ge t_0$ and the following conditions be satisfied.*
(C) $r_2(t) \ge 0$, $t \ge t_0$.
(D) $\int\limits_{t_l}^{t} \exp\Big\{\int\limits_{t_l}^{\tau}\Big[r_1(s) - r_2(s)\Big(\int\limits_{t_l}^{s}\exp\Big\{-\int\limits_{\xi}^{s} r_1(\zeta)d\zeta\Big\} r_0(\xi)d\xi\Big)\Big] ds\Big\} r_0(\tau)d\tau \le 0,$
$t \in [t_l, t_{l+1})$, $l = 1,2,\dots$, where $\{t_l\}$ is an usable sequence for $[t_0,\infty)$..
(E) $\int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\Big[\sum\limits_{k=2}^{n} a_k^+(s) I_\gamma^{k-1}(s) + a_1(s)\Big] ds\Big\} a_0(\tau)d\tau \le 0, \qquad t \ge t_0.$
Then every solution $y(t)$ of Eq. (1.1) with $y(t_0) = \gamma \ge 0$ exists on $[t_0,\infty)$ and

$$0 \le y(t) \le I_\gamma(t), \ \ t \ge t_0.$$

Proof. By Theorem 2.3 it follows from the conditions (C) and (D) of the theorem that for every $\gamma > 0$ the inequality (2.3) has a nonnegative solution $\eta^0_\gamma(t)$ with $\eta^0_\gamma(t_0) = \gamma$. It is clear that $D(t,u,0) \le D_1(t,u,0) \equiv \sum\limits_{k=1}^{n} a_k^+(t)u^{k-1}$, $u \ge 0$ and $D_1(t,u,0)$ is a nondecreasing function for $u \ge 0$. Then (taking into account Remark 2.1) it follows from (E) that the conditions of Corollary 3.1 (for $\zeta(t) \equiv 0$) are satisfied. Therefore, every solution $y(t)$ of Eq. (1.1) with $y(t_0) = \gamma \ge 0$ exists on $[t_0,\infty)$ and $0 \le y(t) \le I_\gamma(t)$, $t \ge t_0$. The theorem is proved.

We set $\sigma_k^\pm \equiv \frac{1\pm(-1)^k}{2}$, $k = 0,1,2,\dots$. Obviously,

$$\sigma_k^+ = \begin{cases} 1, & \text{for } k \text{ even}, \\ 0, & \text{for } k \text{ odd}, \end{cases} \qquad \sigma_k^- = \begin{cases} 0, & \text{for } k \text{ even}, \\ 1, & \text{for } k \text{ odd}, \end{cases} \qquad k = 0,1,\dots.$$

Theorem 4.4. *Let the conditions of Lemma 2.4 and the following conditions be satisfied.*

 doi:10.20944/preprints202412.2567.v1

(F) $(-1)^k a_k(t) \geq 0,\ \ k = \overline{2,n},\qquad t \geq t_0,$
(G) *for some* $\zeta(t) \in C^1([t_0,\infty))$ *with* $\zeta(t_0) < \eta^*_T(t_0)$ *and for some* $\nu \in [\zeta(t_0), \eta^*_T(t_0)]$

$$\zeta(t_0) - \nu + \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\Big[\sum_{k=2}^{n}\sigma_k^+ a_k^+(s) S_k(\eta^*_T(s),\zeta(s)) + a_1(s)\Big]ds\Big\} \times$$
$$\times \Big(\zeta'(\tau) + \sum_{k=0}^{n} a_k(\tau)\zeta^k(\tau)\Big)d\tau \leq 0, \qquad t \geq t_0.$$

Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \in [\nu, \eta^*_T(t_0)]$ *exists on* $[t_0,\infty)$ *and*

$$\zeta(t) \leq y(t) \leq \eta^*_T(t), \qquad t \geq t_0. \tag{4.3}$$

Furthermore, if $\zeta(t_0) < y(t_0)\ \ (y(t_0) < \eta^*_T(t_0))$, *then*

$$\zeta(t) < y(t)\ \ (y(t) < \eta^*_T(t)), \qquad t \geq t_0. \tag{4.4}$$

Proof. By virtue of Lemma 2.4 $\eta^*_T(t)$ is a sub solution of the inequality (2.3) on $[t_0,\infty)$. Since

$$D(t,u,v) = \sum_{k=2}^{n}\sigma_k^+ a_k(t)S_k(u,v) + \sum_{k=2}^{n}\sigma_k^- a_k(t)S_k(u,v) + a_1(t), \qquad u,v \in \mathbb{R},\ \ t \geq t_0$$

By Lemmas 2.8 and 2.9 it follows from (F) that $\sum\limits_{k=2}^{n}\sigma_k^+ a_k(t)S_k(u,v) + a_1(t)$ is nondecreasing in $u \geq \zeta(t)$ for all t and $\sum\limits_{k=2}^{n}\sigma_k^- a_k(t)S_k(u,v) \leq 0,\ \ t \geq t_0$. Hence, $D(t,u,\zeta(t)) \leq \sum\limits_{k=2}^{n}\sigma_k^+ a_k(t)S_k(u,\zeta(t)) + a_1(t),\ \ u \geq \zeta(t),\ \ t \geq t_0$. It follows from here and (G) that the condition (II) of Theorem 3.1 is satisfied for the case $b_0(t) = -\zeta'(t),\ \ b_1(t) = \ldots = b_n(t) \equiv 0$. Thus, all conditions of Theorem 3.1 are satisfied. Therefore, every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\nu, \eta^*_T(t_0)]$ exists on $[t_0,\infty)$ and the inequalities (4.3) and (4.4) are satisfied. The theorem is proved.

Theorem 4.5. *Let the conditions of Lemma 2.5 and the following condition be satisfied.*
(H) *for a nonnegative* $\zeta(t) \in C^1([t_0,T])$ *with* $\zeta(t_0) < \eta_c(t_0)$ *and for some* $\nu \in [\zeta(t_0), \eta_c(t)]$

$$\zeta(t_0) - \nu + \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\Big[\sum_{k=2}^{n} a_k^+(s) S_k(\eta_c(s),\zeta(s)) + a_1(s)\Big]ds\Big\} \times$$
$$\times \Big(\zeta'(\tau) + \sum_{k=0}^{n} a_k(\tau)\zeta^k(\tau)\Big)d\tau \leq 0, \qquad t \in [t_0,T].$$

Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \in [\nu, \eta_c(t_0)]$ *exists on* $[t_0,T]$ *and*

$$\zeta(t) \leq y(t) \leq \eta_c(t), \qquad t \in [t_0,T]. \tag{4.5}$$

Furthermore, if $\zeta(t_0) < y(t_0)\ \ (y(t_0) < \eta_c(t_0))$, *then*

$$\zeta(t) < y(t)\ \ (y(t) < \eta_c(t)), \qquad t \in [t_0,T]. \tag{4.6}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

Proof. By virtue of Lemma 2.5 $\eta_c(t)$ is a solution of the inequality (2.3) on $[t_0, T]$. Since $\zeta(t)$ is nonnegative we have

$$D(t, u, \zeta(t)) \le \sum_{k=2}^{n} a_k^+(t) S_k(u, \zeta(t)) + a_1(t), \qquad t \in [t_0, T].$$

It follows from here and from the condition (H) that the condition (II) of Theorem 3.1 for $[t_0, T]$ and for the case $b_0(t) \equiv -\zeta'(t),\ b_1(t) = \ldots = b_n(t) \equiv 0,\ t \in [t_0, T]$ is satisfied. Thus all conditions of Theorem 3.1 for $[t_0, T]$ are satisfied. Therefore, every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\nu, \eta_c(t_0)]$ exists on $[t_0, T]$ and the inequalities (4.5) and (4.6) are satisfied. The theorem is proved.

By analogy with the proof of Theorem 4.5 it can be proved the following theorem

Theorem 4.6. *Let the condition of Lemma 2.6 and the following condition be satisfied for a nonnegative $\zeta(t) \in C^1([t_0, T])$ with $\zeta(t_0) < \theta_c(t_0)$ and for some $\nu \in [\zeta(t_0), \theta_c(t_0))$*

$$\zeta(t_0) - \nu + \int_{t_0}^{t} \exp\Big\{\int_{t_0}^{\tau} \Big[\sum_{k=2}^{n} a_k^+(s) S_k(\theta_c(s), \zeta(s)) + a_1(s)\Big] ds\Big\} \times$$

$$\times \Big(\zeta'(\tau) + \sum_{k=0}^{n} a_k(\tau) \zeta^k(\tau)\Big) d\tau \le 0, \ \ t \in [t_0, T].$$

Then every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\nu, \theta_c(t_0)]$ exists on $[t_0, T]$ and

$$\zeta(t) \le y(t) \le \theta_c(t), \qquad t \in [t_0, T].$$

Furthermore, if $\zeta(t_0) < y(t_0),\ \ (y(t_0) < \theta_c(t_0))$, then

$$\zeta(t) < y(t),\ \ (y(t) < \theta_c(t)), \qquad t \in [t_0, T].$$

■

Corollary 4.2. *Let the conditions of Lemma 2.5 and the following conditions be satisfied*

(I) $a_1(t) < 0,\ \ t \in [t_0, T]$,

(J) for some $\zeta_0 \in (0, \eta_c(t_0))$ with $\sum_{k=2}^{n} |a_k(t)| \zeta_0^{k-1} \le |a_1(t)|,\ \ t \in [t_0, T]$ and for some $\nu \in [\zeta_0, \eta_c(t_0)]$

$$\zeta_0 - \nu + \int_{t_0}^{t} \exp\Big\{\int_{t_0}^{\tau} \Big[\sum_{k=2}^{n} a_k^+(s) S_k(\eta_c(s), \zeta_0) + a_1(s)\Big] ds\Big\} a_0(\tau) d\tau \le 0, \qquad t \in [t_0, T].$$

Then every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\nu, \eta_c(t_0)]$ exists on $[t_0, T]$ and

$$\zeta_0 \le y(t) \le \eta_c(t), \qquad t \in [t_0, T]. \tag{4.7}$$

Furthermore, if $\zeta_0 < y(t_0),\ \ (y(t_0) < \eta_c(t_0))$, then

$$\zeta_0 < y(t),\ \ (y(t) < \eta_c(t)), \qquad t \in [t_0, T]. \tag{4.8}$$

Proof. It follows from the condition (I) that for some (enough small) $\zeta_0 \in (0, \eta_c(t_0))$ with $\sum_{k=2}^{n} |a_k(t)| \zeta_0^{k-1} \le |a_1(t)|,\ \ t \in [t_0, T]$ the inequality $\sum_{k=2}^{n} a_k(t) \zeta_0^k \le 0,\ \ t \in [t_0, T]$ is satisfied. This together with the condition (J) implies the condition (H) of Theorem 4.5. Thus all conditions of Theorem 4.5 are satisfied. Therefore, every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\nu, \eta_c(t_0)]$ exists on $[t_0, T]$ and the inequalities (4.7) and (4.8) are satisfied. The corollary is proved.

By analogy with the proof of Corollary 4.2 one can prove the following assertion.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

Corollary 4.3. *Let the conditions of Lemma 2.5 and the following conditions be satisfied*
$a_1(t) > 0, \ \ t \in [t_0, T],$
for some $\zeta_0 < 0$ *with* $\sum\limits_{k=2}^{n} |a_k(t)||\zeta_0|^{k-1} \le a_1(t), \ \ t \in [t_0, T]$ *and for some* $\nu \in [\zeta_0, \eta_c(t_0)]$

$$\zeta_0 - \nu + \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} \Big[\sum_{k=2}^{n} a_k^{+}(s) S_k(\eta_c(s), \zeta_0) + a_1(s)\Big] ds\Big\} a_0(\tau) d\tau \le 0, \qquad t \in [t_0, T].$$

Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \in [\nu, \eta_c(t_0)]$ *exists on* $[t_0, T]$ *and*

$$\zeta_0 \le y(t) \le \eta_c(t), \qquad t \in [t_0, T].$$

Furthermore, if $\zeta_0 < y(t_0), \ \ (y(t_0) < \eta_c(t_0))$*, then*

$$\zeta_0 < y(t), \ \ (y(t) < \eta_c(t)), \qquad t \in [t_0, T].$$

■

For any $\gamma \in \mathbb{R}, \ \ t_1 \ge t_0$ we set

$$\zeta_{\gamma, t_1}(t) \equiv -\gamma - \exp\Big\{-\int\limits_{t_0}^{t} a_1(\tau) d\tau\Big\} \Big[c(t_1) + \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s) ds\Big\} a_0(\tau) d\tau\Big], \ \ t \in [t_0, t_1),$$

where $c(t_1) \equiv \max\limits_{\xi \in [t_0, t_1]} \Big(-\int\limits_{t_0}^{\xi} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s) ds\Big\} a_0(\tau) d\tau\Big).$

Theorem 4.7. *Let the conditions of Lemma 2.4 and the following conditions be satisfied.*
(K) $\sum\limits_{k=2}^{n-1} (-1)^{k+1} p_k(t) x^k \in \Omega_{N_T}, \ \ t \ge t_0.$
(L) *n is odd.*
Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \in [\zeta_T^*(t_0), \eta_T^*(t_0)]$ *exists on* $[t_0, \infty)$ *and*

$$\zeta_T^*(t) \le y(t) \le \eta_T^*(t), \qquad t \ge t_0, \tag{4.9}$$

where $\eta_T^*(t)$ *is defined in Lemma 2.4 and* $\zeta_T^*(t) \equiv \begin{cases} \zeta_{N_T, T}(t), \ \ t \in [t_0, T], \\ \zeta_{N_t, t}(t), \ \ t \ge T, \end{cases}$ *such that* $\zeta_T^*(t_0) \le \eta_T^*(t_0).$
furthermore, if $\zeta_T^*(t_0) < y(t_0) \ \ (y(t_0) < \eta_T^*(t_0))$*, then*

$$\zeta_T^*(t) < y(t) \ \ (y(t) < \eta_T^*(t)), \qquad t \ge t_0. \tag{4.10}$$

Proof. By Lemma 2.4 $\eta_T^*(t)$ is a sub solution of the inequality (2.3) on $[t_0, +\infty)$. Show that $\zeta_T^*(t)$ is a super solution of the inequality (2.4) on $[t_0, +\infty)$. Consider the differential inequality

$$\eta' + \sum_{k=0}^{n} \widetilde{a}_k(t) \eta^k \ge 0, \qquad t \ge t_0, \tag{4.11}$$

where $\widetilde{a}_k(t) = (-1)^{k+1} a_k(t), \ \ k = \overline{0, n}, \ \ t \ge t_0$ It follows from (K) and the condition (1) of Lemma 2.4 that
$\widetilde{(1)}$ $\widetilde{a}_n(t) \ge 0, \ \ t \ge t_0.$
it follows from the condition (2) of Lemma 2.4 that

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

$\widetilde{(2)}$ $\widetilde{a}_k(t) = \widetilde{a}_n(t)\widetilde{c}_k(t) + \widetilde{d}_k(t),\ k = \overline{2, n-1},\ t \geq t_0$, where $\widetilde{c}_k(t) = (-1)^{k+1}c_k(t),\ k = \overline{2, n-1},\ t \geq t_0$ are bounded function on $[t_0, t_1]$ for every $t_1 \geq t_0,\ \widetilde{d}_k(t) = (-1)^{k+1}d_k(t)$, $k = \overline{2, n-1},\ t \geq t_0$.
It follows from the condition (K), that
$\widetilde{(3)}$ $\sum\limits_{k=2}^{n-1} \widetilde{d}_k(t)u^k \geq 0$ for all $u \geq N_T,\ t \geq t_0$.
We see that all conditions of Lemma 2.4 for the inequality (4.11) are satisfied. Hence, by Lemma 2.4 $\widetilde{\eta}_T^*(t) \equiv -\zeta_T^*(t)$ is a sub solution of the inequality (4.11) on $[t_0, \infty)$. Then $\zeta_T^*(t)$ is a super solution of the inequality (2.4) on $[t_0, \infty)$. By Corollary 3.2 it follows from here that every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\zeta_T^*(t_0), \eta_T^*(t_0)]$ (note that always $\zeta_T^*(t_0) \leq \eta_T^*(t_0)]$) exists on $[t_0, \infty)$ and the inequalities (4.9) and (4.10) are satisfied. The theorem is proved.

We set

$$\theta_c^-(t) \equiv -\exp\left\{\int\limits_{t_0}^{t} \alpha(\tau)d\tau\right\}\left[c + \int\limits_{t_0}^{t} \exp\left\{-\int\limits_{t_0}^{\tau} \alpha(s)ds\right\}a_0(\tau)d\tau\right], \quad t \geq t_0, \quad t \in \mathbb{R}.$$

Theorem 4.8. *Assume for some* $c^+ \geq \max\limits_{t\in[t_0,T]} \int\limits_{t_0}^{t} \exp\left\{-\int\limits_{t_0}^{\tau} \alpha(s)ds\right\}a_0(\tau)d\tau$, $c^- \geq -\min\limits_{t\in[t_0,T]} \int\limits_{t_0}^{t} \exp\left\{-\int\limits_{t_0}^{\tau} \alpha(s)ds\right\}a_0(\tau)d\tau$ *the inequalities*

$$\theta_{c^+}(t) \leq 1, \quad |\theta_{c^-}^-(t)| \leq 1, \quad t \in [t_0, T]$$

are valid. Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \in [\theta_{c^-}^-(t), \theta_{c^+}(t)]$ *exists on* $[t_0, T]$ *and*

$$\theta_{c^-}^-(t) \leq y(t) \leq \theta_{c^+}(t), \quad t \in [t_0, T]. \tag{4.12}$$

Furthermore, if $\theta_{c^-}^-(t_0) < y(t_0)\ \ (y(t_0) < \theta_{c^+}(t_0))$*, then*

$$\theta_{c^-}^-(t) < y(t)\ \ (y(t) < \theta_{c^+}(t)), \quad t \in [t_0, T]. \tag{4.13}$$

Proof. We have
$\theta_{c^-}^-(t_0) \leq \min\limits_{t\in[t_0,T]} \int\limits_{t_0}^{t} \exp\left\{-\int\limits_{t_0}^{\tau} \alpha(s)ds\right\}a_0(\tau)d\tau \leq \max\limits_{t\in[t_0,T]} \int\limits_{t_0}^{t} \exp\left\{-\int\limits_{t_0}^{\tau} \alpha(s)ds\right\}a_0(\tau)d\tau \leq \theta_{c^+}(t_0)$.
Therefore, the relation $y(t_0) \in [\theta_{c^-}^-(t), \theta_{c^+}(t)]$ is correct. By Lemma 2.6 $\theta_{c^+}(t)$ is a solution of the inequality (2.3) on $[t_0, T]$, and $-\theta_{c^-}^-(t)$ is a solution of the inequality (4.11) on $[t_0, T]$. Then, since $\theta_{c^-}^-(t) \leq \theta_{c^+}(t)]$, by Corollary 3.3 every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [\theta_{c^-}^-(t), \theta_{c^+}(t)]$ exists on $[t_0, T]$ and the inequalities (4.12) and (4.13) are valid. The theorem is proved.

Theorem 4.9. *Assume* $a_k(t) = p_k(t) + r_k(t),\ k = \overline{3, n}$*, where* $p_k(t)$ *and* $r_k(t)$ *are real-valued continuous function on* $[t_0, \infty)$*. If*
M) $r_k(t) \geq 0,\ k = \overline{3, n},\ \sum\limits_{k=3}^{n} r_k(t) > 0,\ t \geq t_0,$
N) $\sum\limits_{k=3}^{n} p_k(t)x^k \in \Omega_0^*,$
O) $a_0(t) \leq 0,\ a_2(t) \geq 0,\ t \geq t_0$ *and* $a_0(t),\ a_2(t)$ *have unbounded supports.*
Then every solution $y(t)$ *of Eq. (1.1) with* $y(t_0) \geq y_*(t_0)$ *exists on* $[t_0, \infty)$ *and* $y(t) \geq y_*(t),\ t \geq t_0$*, where* $y_*(t)$ *is the unique* t_0*-extremal solution of Eq. (2.26) (here* $\tau_0 = \infty$*).*

Proof. Since $\Omega_0^* \subset \Omega_0$ by Lemma 2.3 it follows from the conditions $M),\ N)$ that for every $\gamma > 0$ the inequality (2.3) has a sub solution $\eta^*(t)$ on $[t_0, \infty)$ with $\eta^*(t_0) = \gamma$. By Lemma 2.10 it follows from the conditions $O)$ that Eq. (2.26) has the unique t_0-extremal solution $y_*(y) < 0,\ t \geq t_0$. Then it follows from the condition $N)$ that $y_*(t)$ is a solution of the inequality (2.4) on $[t_0, \infty)$. Hence, by

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

virtue of Corollary 3.2 every solution $y(t)$ of Eq. (1.1) with $y(t_0) \in [y_*(t_0), \gamma]$ exists on $[t_0, \infty)$ and $y(t) \geq y_*(t), \ t \geq t_0$. Since $\gamma > 0$ can be arbitrarily large the proof of the theorem is completed.

5. Closed Solutions

Theorem 5.1. *Assume* $a_k(t) = p_k(t) + r_k(t), \ k = \overline{0,n}, \ t \in [t_0, T]$, *where* $p_k(t)$ *and* $r_k(t), \ k = \overline{0,n}$ *are real-valued continuous functions on* $[t_0, T]$ *such that* $\sum\limits_{k=0}^{n} p_k(t)x^k \in \Omega_0$, $t \in [t_0, T]$, *and let the following conditions be satisfied.*

$1^0)$ *for some* $j = 2, \dots, n$ *the inequalities* $r_k(t) \geq 0, \ k = \overline{j,n}, \ \sum\limits_{k=j}^{n} r_k(t) > 0, \ t \in [t_0, T]$ *are valid,*

$2^0)$ $\int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\Big[\sum\limits_{k=2}^{n} a_k^+(s)M_{T+\gamma}^{k-1} + a_1(s)\Big]ds\Big\}a_0(\tau)d\tau \leq 0, \ t \in [t_0, T]$ *for some* $\gamma \geq 0$.

Then the following statements are valid.

$\alpha)$ *Eq. (1.1) has a nonnegative closed solution* $y_*(t)$ *on* $[t_0, T]$,

$\beta)$ *In particular, if* $a_0(t) \not\equiv 0$ *and* $a_0(t) \leq 0, \ t \in [t_0, T]$, *then* $y_*(t)$ *is positive,*

$\gamma)$ *In particular, if* $j = 2$ *and* $\int\limits_{t_0}^{T} a_1(\tau)d\tau > 0$, *then* $y_*(t)$ *is isolated.*

Proof. Let us prove α). It follows from the conditions of the theorem that for $\zeta(t) \equiv 0$ the conditions of Theorem 4.1 are satisfied. Then by Theorem 4.1 the solutions $y_1(t)$ and $y_2(t)$ of Eq. (1.1) with $y_1(t_0) = 0$ and $y_2(t_0) = M_{T+\gamma}$ exist on $[t_0, T]$ and $y_1(T) \geq 0, \ y_2(T) \leq M^*_{\gamma,T}(T) = M^*_{\gamma,T}(t_0)$. By Theorem 2.1 it follows from here that Eq. (1.1) has a nonnegative closed solution $y_*(t)$ on $[t_0, T]$. The statement α) is proved. Let us prove β). If $a_0(t) \not\equiv 0$ and $a_0(t) \geq 0, \ t \in [t_0, T]$, then $y_1(t) \not\equiv 0$. Hence,

$$y_1(t_1) > 0 \ \text{ for sone } \ t_1 \in [t_0, T] \tag{5.1}$$

(since $y_1(t) \geq 0, \ t \in [t_0, T]$). Consider the equation

$$y' + \sum_{k=1}^{n} a_k(t)y^k = 0, \ t \in [t_0, T].$$

Since $y_0(t) \equiv 0$ is a solution of this equation by (2.2) we have

$$y_1(T) = \exp\Big\{-\int\limits_{t_1}^{T} D(\tau, 0, y_1(\tau))d\tau\Big\}\Big[y_1(t_1) - \int\limits_{t_1}^{T} \exp\Big\{\int\limits_{t_1}^{\tau} D(s, 0, y_1(s))ds\Big\}a_0(\tau)d\tau\Big].$$

It follows form here, (5.1) and the conditions of β) that $y_1(T) > 0 = y_1(t_0)$. Therefore, $y_1(t)$ is not a closed solution of Eq. (1.1) on $[t_0, T]$. By the uniqueness theorem it follows from here and the statement α) that $y_*(t)$ is positive. The statement β) is proved. It remains to prove γ). Let us show that $y_*(t)$ is isolated. Suppose $y_*(t)$ is not isolated. Then there exists a sequence $\{y_m(t)\}_{m+1}^{\infty}$ of closed solutions of Eq. (1.1) on $[t_0, T]$ such that $y_m(t_0) \to y_*(t_0)$ for $m \to \infty$. By (2.2) we have

$$y_*(T) - y_m(T) = \exp\Big\{-\int\limits_{t_0}^{T}\Big(\sum_{k=2}^{n} a_k(\tau)S_k(y_*(\tau), y_m(\tau)) + a_1(\tau)\Big)d\tau\Big\}[y_*(t_0) - y_m(t_0)], \tag{5.2}$$

$m = 1, 2, \dots$. Since $j = 2$ it follows from the conditions 1^0) that $\int\limits_{t_0}^{T}\Big(\sum\limits_{k=2}^{n} a_k(\tau)S_k(y_*(\tau), y_*(\tau))\Big)d\tau \geq 0$. Then since the solutions of Eq. (1.1) continuously depend on their initial values and $\int\limits_{t_0}^{T} a_1(\tau)d\tau > 0$ we can chose $m = m_0$ enough large such that

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

$\int\limits_{t_0}^{T}\Big(\sum\limits_{k=2}^{n} a_k(\tau)S_k(y_*(\tau), y_m(\tau)) + a_1(\tau)\Big)d\tau > 0$. It follows from here and (5.2) that $y_{m_0}(t)$ is not closed. We obtain a contradiction, proving $\gamma)$. The proof of the theorem is completed.

Corollary 5.1. *Assume $a_k(t) = p_k(t) + r_k(t)$, $t \in [t_0, T]$, where $p_k(t)$ and $r_k(t)$ are real-valued continuous functions on $[t_0, T]$, $k = \overline{0, n}$, such that $\sum\limits_{k=0}^{n}(-1)^k p_k(t)x^k \in \Omega_0$, $t \in [t_0, T]$, and let the following conditions be satisfied.*
for some $j = 2, \ldots, n$ the inequalities $(-1)^k r_k(t) \geq 0$, $k = \overline{j, n}$, $\sum\limits_{k=j}^{n}(-1)^k r_k(t) > 0$, $r_0(t) \leq 0$, $t \in [t_0, T]$ hold.
Then the following statements are valid
$\alpha^0)$ Eq. (1.1) has a non positive closed solution $y_*(t)$ on $[t_0, T]$.
$\beta^0)$ In particular, if $r_0(t) \not\equiv 0$ and $r_0(t) \leq 0$, $t \in [t_0, T]$, then $y_*(t)$ is negative,
$\gamma^0)$ In particular, if $j = 2$ and $\int\limits_{t_0}^{T} r_1(\tau)d\tau > 0$, then $y_*(t)$ is isolated.

Proof. In Eq. (1.1) we substitute

$$y = -z, \qquad t \to -t. \tag{5.3}$$

We obtain

$$z' + \sum_{k=0}^{n}(-1)^k a_k(-t)z^k = 0, \;\; t \leq -t_0.$$

Then by Theorem 5.1 it follows from the conditions of the corollary that the transformed (last) equation has a nonnegative closed solution $z_*(t)$ on $[-T, -t_0]$, for which the statements $\alpha) - \gamma)$ of Theorem 5.1 are valid. It follows from here and (5.3) that $y_*(t) \equiv -z_*(-t)$ is a nonnegative closed solution of Eq. (1.1) on $[t_0, T]$, for which the statements $\alpha^0) - \gamma^0)$ are valid. The corollary is proved.

Note that in the statement $\alpha)$ of Corollary 5.1 the condition $a_0(t) \equiv 0$ of Theorem 1.1 is weakened up to $a_0(t) \leq 0$, $t \in [t_0, T]$ and the condition $\int\limits_{t_0}^{T} a_1(\tau)d\tau > 0$ is omitted. Therefore, Corollary 5.1 is a complement of Theorem 1.1.

The inequality $\sum\limits_{k=j}^{n} a_k(t) > 0$, $t \in [t_0, T]$ in conditions of Theorem 5.1 looks like a strict limitation. The next theorem attempts to partially weaken it.

Theorem 5.2. *Let the conditions of Theorem 4.3 be satisfied. If $a_k(t) \geq 0$, $k = \overline{2.n}$, $\sum\limits_{k=2}^{n} a_k(t) \not\equiv 0$ or $\int\limits_{t_0}^{T} a_1(\tau)d\tau > 0$, then Eq. (1.1) has a nonnegative closed solution on $[t_0, T]$. In the case $\int\limits_{t_0}^{T} a_1(\tau)d\tau > 0$ it is isolated.*

Proof. By Theorem 4.3 for every $\gamma \geq 0$ Eq. (1.1) has a nonnegative solution $y_\gamma(t)$ on $[t_0, T]$ with $y_\gamma(t_0) = \gamma$. Let us show that there exists $\gamma > 0$ such that

$$y_\gamma(t_0) \geq y_\gamma(T). \tag{5.4}$$

First we show that if $\sum\limits_{k=2}^{n} a_k(t) \not\equiv 0$, $t \in [t_0, T]$, then

$$\lim_{\gamma \to +\infty} \int_{t_0}^{T} \Big[\sum_{k=2}^{n} a_k(t) y_\gamma^{k-1}(t)\Big] dt = \infty. \tag{5.5}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

By (1.1) we can interpret $y_\gamma(t)$ as a solution of the linear equation

$$x' + \Big[\sum_{k=1}^{n} a_k(t) y_\gamma^{k-1}(t)\Big] x + a_0(t) = 0, \qquad t \in [t_0, T].$$

Then by the Cauchy formula we have

$$y_\gamma(t) = \gamma \exp\Big\{-\int_{t_0}^{t}\Big[\sum_{k=1}^{n} a_k(\tau) y_\gamma^{k-1}(\tau)\Big] d\tau\Big\} -$$

$$-\int_{t_0}^{t} \exp\Big\{-\int_{\tau}^{t}\Big[\sum_{k=1}^{n} a_k(s) y_\gamma^{k-1}(s)\Big] ds\Big\} a_0(\tau) d\tau, \ \ t \in [t_0, T]. \tag{5.6}$$

Multiplying both sides of this equality by $\Big[\sum\limits_{k=2}^{n} a_k(t) y_\gamma^{k-2}(t)\Big] \exp\Big\{-\int\limits_{t_0}^{t}\Big[\sum\limits_{k=2}^{n} a_k(\tau) y_\gamma^{k-1}(\tau)\Big] d\tau\Big\}$ and integrating over $[t_0, T]$ we obtain

$$\exp\Big\{\int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(\tau) y_\gamma^{k-1}(\tau)\Big] d\tau\Big\} = 1 + \gamma \int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(t) y_\gamma^{k-2}(t)\Big] \exp\Big\{-\int_{t_0}^{t} a_1(\tau) d\tau\Big\} -$$

$$-\int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(t) y_\gamma^{k-2}(t)\Big] dt \int_{t_0}^{t} \exp\Big\{\int_{t_0}^{\tau}\Big[\sum_{k=2}^{n} a_k(s) y_\gamma^{k-1}(s)\Big] ds - \int_{\tau}^{t} a_1(s) ds\Big\} a_0(\tau) d\tau$$

From here we obtain

$$\exp\Big\{\int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(\tau) y_\gamma^{k-1}(\tau)\Big] d\tau\Big\} \geq 1 + \int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(t) y_\gamma^{k-2}(t)\Big] \exp\Big\{-\int_{t_0}^{t} a_1(\tau) d\tau\Big\} dt \times$$

$$\times \Big[\gamma - \int_{t_0}^{T} \exp\Big\{\int_{t_0}^{\tau}\Big[\sum_{k=2}^{n} a_k(s) y_\gamma^{k-1}(s)\Big] ds + \int_{t_0}^{\tau} a_1(s) ds\Big\} |a_0(\tau)| d\tau\Big]. \tag{5.7}$$

Suppose

$$\int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(t) y_\gamma^{k-2}(t)\Big] dt \leq M, \ \ \gamma > 0. \tag{5.8}$$

Then (5.7) implies

$$\exp\Big\{\int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(\tau) y_\gamma^{k-1}(\tau)\Big] d\tau\Big\} \geq 1 + \int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(t) y_\gamma^{k-2}(t)\Big] \exp\Big\{-\int_{t_0}^{t} a_1(\tau) d\tau\Big\} dt \times$$

$$\times \Big[\gamma - \exp\Big\{M\Big\} \int_{t_0}^{T} \exp\Big\{\int_{t_0}^{\tau} a_1(s) ds\Big\} |a_0(\tau)| d\tau\Big]. \tag{5.9}$$

By the uniqueness theorem $y_\gamma(t) > 0, \ t \in [t_0, T]$ for all $\gamma > 0$. Therefore (since $a_k(t) \geq 0$, $k = \overline{2, n}, \ \ t \in [t_0, T]$ and $\sum\limits_{k=2}^{n} a_k(t) \not\equiv 0$)

$$\int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(t) y_\gamma^{k-2}(t)\Big] \exp\Big\{-\int_{t_0}^{t} a_1(\tau) d\tau\Big\} dt \geq \int_{t_0}^{T}\Big[\sum_{k=2}^{n} a_k(t) y_{\gamma_0}^{k-2}(t)\Big] \exp\Big\{-\int_{t_0}^{t} a_1(\tau) d\tau\Big\} dt > 0$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

for all $\gamma \ge \gamma_0 > 0$. It follows from here that the right part of the inequality (5.9) tends to ∞ as $\gamma \to \infty$, whereas, according to (5.8) its left part is bounded. We obtain a contradiction, proving (5.5). It follows from (5.6) that

$$y_\gamma(T) = \gamma \exp\Big\{-\int\limits_{t_0}^{T}\Big[\sum_{k=1}^{n} a_k(\tau) y_\gamma^{k-1}(\tau)\Big]d\tau\Big\} -$$

$$-\int\limits_{t_0}^{T} \exp\Big\{-\int\limits_{\tau}^{t}\Big[\sum_{k=1}^{n} a_k(s) y_\gamma^{k-1}(s)\Big]ds\Big\} a_0(\tau)d\tau, \ \ t \in [t_0, T], \ \ \gamma \ge 0.$$

Therefore $y_\gamma(T) \le y_\gamma(t_0) = \gamma$, provided

$$\gamma\Big(1 - \exp\Big\{\int\limits_{t_0}^{T}\Big(\sum_{k=1}^{n} a_k(\tau) y_\gamma^{k-1}(\tau)\Big)d\tau\Big\}\Big) \ge -\int\limits_{t_0}^{T} \exp\Big\{-\int\limits_{\tau}^{t}\Big[\sum_{k=1}^{n} a_k(s) y_\gamma^{k-1}(s)\Big]ds\Big\} a_0(\tau)d\tau,$$

which will be fulfilled if by virtue of (5.5) we chose $\gamma \ge 2\int\limits_{t_0}^{T} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\}|a_0(\tau)|d\tau$ enough large such that $\int\limits_{t_0}^{T} \sum\limits_{k=1}^{n} a_k(\tau) y_\gamma(\tau) d\tau \ge \ln 2$. Therefore (5.4) is proved for the case $\sum\limits_{k=2}^{n} a_k(t) \not\equiv 0$. If $\sum\limits_{k=2}^{n} a_k(t) \equiv 0$ and $\int\limits_{t_0}^{T} a_1(\tau)d\tau > 0$, then from the obvious equality $y_\gamma(T) = \gamma \exp\Big\{-\int\limits_{t_0}^{T} a_(\tau)d\tau\Big\} - \int\limits_{t_0}^{T} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\} a_0(\tau)d\tau$ we derive that for

$$\gamma \ge \int\limits_{t_0}^{T} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\}|a_0(\tau)|d\tau \Big/ \Big(1 - \exp\Big\{-\int\limits_{t_0}^{T} a_1(\tau)d\tau\Big\}\Big)$$

the inequality (5.4) is fulfilled. Thus, under the restriction $\sum\limits_{k=2}^{n} a_k(t) \not\equiv 0$ or $\int\limits_{t_0}^{T} a_1(\tau)d\tau > 0$ of the theorem the inequality (5.4) is valid. Then since $y_0(t_0) = 0 \le y_0(T)$, by Theorem 2.1 Eq. (1.1) has a nonnegative closed solution $y_*(t)$ on $[t_0, T]$. To complete the proof of the theorem it remains to show that if $\int\limits_{t_0}^{T} a_1(\tau)d\tau > 0$, then $y_*(t)$ is isolated. The proof of this fact is similar to the proof of the assertion γ) of Theorem 5.1. Therefore we omit it. The proof of the theorem is completed.

Theorem 5.3. *Let the following conditions be satisfied.*

3^0) $a_n(t) \ge 0, \ \ t \in [t_0, T]$,

4^0) $a_k(t) = a_n(t) c_k(t) + p_k(t), \ \ k = \overline{2, n-1}$, *where* $c_k(t), \ \ k = \overline{2, n-1}$ *are bounded functions on* $[t_0, T]$,

5^0) $\sum\limits_{k=2}^{n-1} p_k(t) x^k \in \Omega_{N_T}, \ \ t \in [t_0, T]$,

6^0) $\max\limits_{t \in [t_0, T]} \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\} a_0(\tau)d\tau \Big[1 - \exp\Big\{\int\limits_{t_0}^{T} a_1(\tau)d\tau\Big\}\Big] \le \int\limits_{t_0}^{T} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\} a_0(\tau)d\tau.$

7^0) $\int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\Big[\sum\limits_{k=2}^{n} a_k^+(s) \eta_{\gamma,T}^{k-1}(s) + a_1(s)\Big]ds\Big\} a_0(\tau)d\tau \le 0, \ \ t \in [t_0, T]$. *for some* $\gamma \ge 0$,

Then Eq. (1.1) has a nonnegative closed solution on $[t_0, T]$.

Proof. By virtue of Lemma 2.4 it follows from the conditions $3^0) - 5^0)$ that $\eta_{\gamma,T}(t)$ is a solution of the inequality (2.3) on $[t_0, T]$. Then it follows from the condition 7^0) that the conditions of Theorem 4.2 with $\zeta(t) \equiv 0$ are satisfied. Hence, according to Theorem 4.2 the solutions $y_1(t)$ and $y_2(t)$ of Eq. (1.1) with $y_1(t_0) = 0, \ \ y_2(t_0) = \eta_{\gamma,T}(t_0) = c(T)$ exist on $[t_0, T]$ and $y_1(T) \ge 0, \ \ y_2(T) \le \eta_{\gamma,T}(T)$. It follows

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

from the condition 6^0) that $\eta_{\gamma,T}(T) \le \eta_{\gamma,T}(t_0)$. Therefore $y_2(T) \le y_2(t_0)$. By Theorem 2.1 it follows from here that Eq. (1.1) has a nonnegative closed solution on $[t_0, T]$. The theorem is proved.

Example 5.1. *Consider the equation*

$$y' + \sum_{k=0}^{6} a_k(t) y^k = 0, \qquad t \ge t_0, \tag{5.10}$$

where, $a_0(t) \equiv -\sin 10t$, $a_1(t)$ *is any continuous function,* $a_2(t) = \cos^4 t$, $a_3(t) = -2|sint \cos^3 t|$, $a_4(t) = \sin^2 t \cos^2 t$, $a_5(t) = -\sin^2 t|cos \pi t|$, $a_6(t) = \sin^2 t$, $[t_0, T]$. *Obviously, the conditions of Corollary 5.1 for Eq. (5.10) are satisfied. It is not difficult to verify that the conditions of Theorem 5.3 with* $c_2(t) = c_3(t) = c_4(t) \equiv 0$, $c_5(t) = -|\cos \pi t|$, $p_2(t) = \cos^4 t$, $p_3(t) = -2|\sin t \cos^3 t|$, $p_4(t) = \sin^2 t \cos^2 t$, $p_5(t) \equiv 0$, $t \in [t_0, T]$ *for Eq. (5.10) are satisfied. Therefore Eq. (5.10) has at least a nonnegative closed solution* $y_+(t)$ *on* $[t_0, T]$ *and at least a non positive closed solution* $y_-(t)$ *on* $[t_0, T]$ *(for every* $T > t_0$*). Since* $a_0(t) \not\equiv 0$ *we have* $y_+(t) \ne y_-(t)$, $t \in [t_0, T]$.

Theorem 5.4. *Let the following conditions be satisfied.*

8^0) $a_2(t) > 0$, $t \in [t_0, T]$,

9^0) *for some* $c \ge \max\limits_{t\in[t_0,T]} \int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\} a_0(\tau)d\tau$, *the inequality* $\sum\limits_{k=3}^{n} |a_k(t)|\eta_c^{k-2}(t) \le a_2(t)$, $t \in [t_0, T]$ *is valid and*

10^0) $\int\limits_{t_0}^{t} \exp\Big\{\int\limits_{t_0}^{\tau}\Big[\sum\limits_{k=2}^{n} a_k^+(s)\eta_c^{k-1}(s) + a_1(s)\Big]ds\Big\} a_0(\tau)d\tau \le 0$, $t \in [t_0, T]$,

11^0) $c\Big(1 - \exp\Big\{\int\limits_{t_0}^{T} a_1(\tau)d\tau\Big\}\Big) \le \int\limits_{t_0}^{T} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\} a_0(\tau)d\tau.$

Then Eq. (1.1) has a nonnegative closed solution on $[t_0, T]$.

Proof. By Lemma 2.5 it follows from the conditions 8^0) and 9^0) that $\eta_c(t)$ is a solution of the inequality (2.3) on $[t_0, T]$. It follows from the condition 10^0) that the condition (E) with $\zeta(t) \equiv 0$ of Theorem 4.5 is satisfied. It follows from the condition 11^0) that $\eta_c(t_0) \ge \eta_c(T)$. Then by Theorems 2.1 and 4.5 Eq. (1.1) has a nonnegative closed solution on $[t_0, T]$. The theorem is proved.

Let us write $a_2(t) = \lambda p(t)$, $p(t) > 0$, $t \in [t_0, T]$. Then for all $\lambda \ge \lambda_0 \equiv \equiv \max\limits_{t\in[t_o,T]} \{(\sum\limits_{k=3}^{n} |a_k(t)|\eta_c^{k-2}(t))/p(t)\}$ the condition 9^0) of Theorem 5.4 will be satisfied. If we write $a_2(t) = \lambda + p(t)$, $p(t) \in C([t_0, T])$, then for all $\lambda \ge \lambda_0 \equiv \max\limits_{t\in[t_o,T]} \{(\sum\limits_{k=3}^{n} |a_k(t)|\eta_c^{k-2}(t)) - p(t)\}$ the condition 9^0) of the Theorem 5.4 will be satisfied as well. Unlike of this in Theorems 2 and 3 of work [25] the parameter λ_0 is undetermined. Moreover, for $a_0(t) \equiv 0$, $c = 0$ the conditions 10^0) and 11^0) of Theorem 5.4 are satisfied. Therefore, Theorem 5.4 is a complement of both mentioned above Theorems 2 and 3.

Theorem 5.5. *Let the following conditions be satisfied.*

12^0) $a_n(t) \ge 0$, $t \in [t_0, T]$,

13^0) $a_k(t) = a_n(t)c_k(t) + p_k(t)$, $k = \overline{2, n-1}$, $t \in [t_0, T]$, *where* $c_k(t)$, $k = \overline{2, n-1}$ *are bounded functions on* $[t_0, T]$ *and*

14^0) $\sum\limits_{k=2}^{n-1} p_k(t)x^k \in \Omega_{N_T}$, $t \in [t_0, T]$,

15^0) $\sum\limits_{k=2}^{n-1} (-1)^{k+1} p_k(t)x^k \in \Omega_{N_T}$, $t \in [t_0, T]$,

16^0) *n is odd,*

17^0)
$$\max_{\xi\in[t_0,T]} \Big(\int\limits_{t_0}^{\xi} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\} a_0(\tau)d\tau\Big)\Big[1 - \exp\Big\{\int\limits_{t_0}^{T} a_1(\tau)d\tau\Big\}\Big] \le$$
$$\le \int\limits_{t_0}^{T} \exp\Big\{\int\limits_{t_0}^{\tau} a_1(s)ds\Big\} a_0(\tau)d\tau,$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

$18^0)$ $\min\limits_{\xi\in[t_0,T]}\left(\int\limits_{t_0}^{\xi}\exp\left\{\int\limits_{t_0}^{\tau}a_1(s)ds\right\}a_0(\tau)d\tau\right)\left[1-\exp\left\{\int\limits_{t_0}^{T}a_1(\tau)d\tau\right\}\right]\ge$

$$\ge\int\limits_{t_0}^{T}\exp\left\{\int\limits_{t_0}^{\tau}a_1(s)ds\right\}a_0(\tau)d\tau.$$

Then Eq. (1.1) has a closed solution on $[t_0, T]$.

Proof. By Lemma 2.4 it follows from $12^0) - 14^0)$ that $\eta_{N_T,T}(t),\ t\in[t_0,T]$ is a solution of the inequality (2.3) on $[t_0,T]$ and it follows from the conditions, $12^0),\ 13^0),\ 15^0),\ 16^0)$ that $\zeta_{N_T,T}(t),\ t\in[t_0,T]$ is a solution of the inequality (2.4) on $[t_0,T]$. It follows from the condition $17^0)$ that $\eta_{N_T,T}(t_0)\ge\eta_{N_T,T}(T)$, and it follows form the condition $18^0)$ that $\zeta_{N_T,T}(t_0)\le\zeta_{N_T,T}(T)$. Therefore, by virtue of Theorem 2.1 Eq. (1.1) has a closed solution on $[t_0,T]$. The theorem is proved.

Remark 5.1. *The conditions $14^0),\ 15^0)$ of Theorem 5.5 for n odd are satisfied if, in particular, $p_2(t)=p_{n-1}(t)\equiv 0,\ p_3(t)>0,\ p_{n-2}(t)>0,\ p_k(t)=\alpha_k(t)+\beta_k(t),\alpha_k(t)>0,\ \beta_k(t)>0,\ k=5,\ 7,\dots,n-4,$*

$$p_4^2(t)-4\alpha_5(t)p_3(t)\le 0,$$

$$p_6^2(t)-4\alpha_7(t)\beta_5(t)\le 0,$$

$$p_8^2(t)-4\alpha_9(t)\beta_7(t)\le 0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$p_{n-5}^2(t)-4\alpha_{n-4}(t)\beta_{n-6}(t)\le 0,$$

$$p_{n-3}^2(t)-4d_{n-2}(t)\beta_{n-4}(t)\le 0,\ \ t\in[t_0,T]\ \ (n\ge 7)$$

(since under the above restrictions the "square trinomials" $\alpha_5(t)x^2\pm p_4(t)x+p_3(t),\dots,$ $p_{n-2}(t)x^2\pm p_{n-3}(t)x+\beta_{n-4}(t)$ are nonnegative for all $t\in[t_0,T],\ \ u\in\mathbb{R}$). Note that the conditions $17^0)$ and $18^0)$ are satisfied if, in particular, $\int\limits_{t_0}^{T}\exp\left\{\int\limits_{t_0}^{\tau}a_1(s)ds\right\}a_0(\tau)d\tau=0$, $\int\limits_{t_0}^{T}a_1(t)dt\ge 0$. Indeed, under these restrictions the left part of $17^0)$ is non positive and the left part of $18^0)$ is nonnegative.

Example 5.2 *For $n=7,\ a_7(t)=\sin^2 t,\ a_6(t)=\sin^2 t\cos t,\ a_5(t)=7\sin^2 t\cos 3t+2,$ $a_4(t)=4\sin^2 t\arctan t+\sin(\cos t),\ a_3(t)=10\sin^4 t\cos e^t+2,\ a_2(t)=\sin^8 t\cos^9 t,\ t\ge t_0,$ $\int\limits_{t_0}^{T}a_1(t)dt\ge 0,\ \int\limits_{t_0}^{T}\exp\left\{\int\limits_{t_0}^{\tau}a_1(s)ds\right\}a_0(\tau)d\tau=0$ the conditions of Theorem 5.5 are satisfied. Here we take $c_2(t)=\sin^6 t\cos^9 t,\ c_3(t)=10\sin^2 t\cos e^t,\ c_4(t)=4\arctan t,\ c_5(t)=7\cos 3t,\ c_6(t)=$ $\cos t,\ p_2(t)\equiv 0,\ p_3(t)=2,\ p_4(t)=\sin(\cos t),\ p_5(t)=2,\ p_6(t)\equiv 0,\ t\in[t_0,T].$*

Theorem 5.6. *Let the following conditions be satisfied*

$19^0)$ for some

$$c^+\ge\max_{t\in[t_0,T]}\int\limits_{t_0}^{t}\exp\left\{-\int\limits_{t_0}^{\tau}\alpha(s)ds\right\}a_0(\tau)d\tau,\ \ c^-\ge-\min_{t\in[t_0,T]}\int\limits_{t_0}^{t}\exp\left\{-\int\limits_{t_0}^{\tau}\alpha(s)ds\right\}a_0(\tau)d\tau$$

the inequalities

$$\theta_{c^+}(t)\le 1,\qquad |\theta_{c^-}^-(t)|\le 1,\qquad t\in[t_0,T]$$

are valid,

$20^0)$ $c^+\left(1-\exp\left\{-\int\limits_{t_0}^{T}\alpha(\tau)d\tau\right\}\right)\le\int\limits_{t_0}^{T}\exp\left\{-\int\limits_{t_0}^{\tau}\alpha(s)ds\right\}a_0(\tau)d\tau,$

$$c^-\left(1-\exp\left\{-\int\limits_{t_0}^{T}\alpha(\tau)d\tau\right\}\right)\ge\int\limits_{t_0}^{T}\exp\left\{-\int\limits_{t_0}^{\tau}\alpha(s)ds\right\}a_0(\tau)d\tau,$$

Then Eq. (1.1) has a closed solution $y_*(t)$ on $[t_0,T]$ such that

$$\theta_{c^-}^-(t)\le y_*(t)\le\theta_{c^+}(t),\qquad t\in[t_0,T],$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

and if $\theta^-_{c^-}(t_0) < y_*(t_0)$ $(y_*(t_0) < \theta_{c^+}(t_0))$, *then*

$$\theta^-_{c^-}(t) < y_*(t) \ \ (y_*(t) < \theta_{c^+}(t)), \qquad t \in [t_0, T].$$

Proof. By Lemma 2.6 it follows from the condition $19^0)$ that $\theta_{c^+}(t)$ and $\theta^-_{c^-}(t)$ are solutions of the inequalities (2.3) and (2.4) respectively on $[t_0, T]$. It is not difficult to verify that the conditions $20^0)$ imply that

$$\theta_{c^+}(t_0) \ge \theta_{c^+}(T), \qquad \theta^-_{c^-}(t_0) \le \theta^-_{c^-}(T)$$

By Lemmas 2.1 and 2.2 it follows from here that the solutions $y_1(t)$ and $y_2(t)$ of Eq. (1.1) with $y_1(t_0) = \theta^-_{c^-}(t_0)$, $y_2(t_0) = \theta_{c^+}(t_0)$ exist on $[t_0, T]$ and

$$y_1(t_0) \le y_1(T), \qquad y_2(t_0) \ge y_2(T).$$

By Theorem 2.1 it follows from here that Eq. (1.1) has a closed solution $y_*(t)$ on $[t_0, T]$ such that

$$\theta^-_{c^-}(t) \le y_*(t) \le \theta_{c^+}(t), \qquad t \in [t_0, T],$$

and if $\theta^-_{c^-}(t_0) < y_*(t_0)$ $(y_*(t_0) < \theta_{c^+}(t_0))$, then

$$\theta^-_{c^-}(t) < y_*(t) \ \ (y_*(t) < \theta_{c^+}(t)), \qquad t \in [t_0, T].$$

The theorem is proved.

Remark 5.2. *The conditions* $20^0)$ *of Theorem 5.6 are satisfied, if in particular,* $\int\limits_{t_0}^{T} \exp\Big\{-\int\limits_{t_0}^{\tau} \alpha(s)ds\Big\} a_0(\tau)d\tau = 0$, $\int\limits_{t_0}^{T} \alpha(\tau)d\tau \le 0$. *Indeed, note that* $c^+ \ge 0$, $c^- \le 0$. *Therefore, if* $\int\limits_{t_0}^{T} \alpha(\tau)d\tau \le 0$, *then the left part of the first inequality of* $20^0)$ *is non positive and the left part of the second inequality of* $20^0)$ *is nonnegative.*

Example 5.3. *Assume* $a_0(t) = -\lambda\alpha(t)$, $\lambda = const > 0$, $c^- = \lambda\Big[1 - \exp\Big\{-\max\limits_{\xi\in[t_0,T]} \int\limits_{t_0}^{\xi} \alpha(s)ds\Big\}\Big]$, $c^+ = \lambda\Big[\exp\Big\{-\min\limits_{\xi\in[t_0,T]} \int\limits_{t_0}^{\xi} \alpha(s)ds\Big\} - 1\Big]$. *Then it is not difficult to verify that*

$$\theta_{c^+}(t) = \lambda\Bigg[\exp\Bigg\{\int\limits_{t_0}^{t} \alpha(s)ds - \min_{\xi\in[t_0,T]} \int\limits_{t_0}^{\xi} \alpha(s)ds\Bigg\} - 1\Bigg] \ge 0, \ \ t \in [t_0, T],$$

$$\theta^-_{c^-}(t) = \lambda\Bigg[\exp\Bigg\{\int\limits_{t_0}^{t} \alpha(s)ds - \max_{\xi\in[t_0,T]} \int\limits_{t_0}^{\xi} \alpha(s)ds\Bigg\} - 1\Bigg] \le 0, \ \ t \in [t_0, T].$$

Therefore if $\alpha(t) \not\equiv 0$, $\lambda \le \min\Bigg\{\max\limits_{t\in[t_0,T]}\Bigg[\exp\Bigg\{\int\limits_{t_0}^{t} \alpha(s)ds - \min\limits_{\xi\in[t_0,T]} \int\limits_{t_0}^{\xi} \alpha(s)ds\Bigg\} - 1\Bigg]^{-1}, \max\limits_{t\in[t_0,T]}\Bigg[1 - \exp\Bigg\{\int\limits_{t_0}^{t} \alpha(s)ds - \max\limits_{\xi\in[t_0,T]} \int\limits_{t_0}^{\xi} \alpha(s)ds\Bigg\}\Bigg]^{-1}\Bigg\}$, *then* $\theta_{c^+}(t) \le 1$, $|\theta^-_{c^-}(t)| \le 1$, $t \in [t_0, T]$. *Due to Remark 5.1 it follows from here that if* $\int\limits_{t_0}^{T} \alpha(\tau)d\tau = 0$. *Then all the conditions of Theorem 5.6 are satisfied. Hence, under the above conditions Eq. (1.1) has a closed solution on* $[t_0, T]$.

Using Lemmas 2.13 and 2.16 instead of Lemma 2.1, 2.2, 2.6 and Theorem 2.1 by analogy with the proof of Theorem 5.6 one can prove the following theorem

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

Theorem 5.7. *Let for some*

$$c^+ > \max_{t\in[t_0,T]} \int_{t_0}^{t} \exp\Big\{-\int_{t_0}^{\tau} \alpha(s)ds\Big\} a_0(\tau)d\tau, \;\; c^- > -\min_{t\in[t_0,T]} \int_{t_0}^{t} \exp\Big\{-\int_{t_0}^{\tau} \alpha(s)ds\Big\} a_0(\tau)d\tau$$

the inequalities

$$\theta_{c^+}(t) < 1, \qquad |\theta^-_{c^-}(t)| < 1, \qquad t \in [t_0, T]$$

and the conditions $20^9)$ *of Theorem 5.6 be satisfied.*
Then Eq. (1.1) has an isolated closed solution.

Theorem 5.8. *Let* $a_k(t) = p_k(t) + r_k(t), \;\; k = 1, 2,$ *where* $p_k(t)$ *and* $r_k(t), \;\; k = 1, 2$ *are real-valued continuous functions on* $[t_0, T]$ *and the following conditions be satisfied*
$21^0)$ $r_2(t) \geq 0, \;\; a_0(t) \leq 0, \;\; t \in [t_0, T],$
$22^0)$ $-\sum\limits_{k=3}^{n} a_k(t)x^k - \sum\limits_{k=1}^{2} p_k(t)x^k \in \Omega^*_0, \;\; t \in [t_0, T].$
Then Eq. (1.1) has a closed solution on $[t_0, T]$.

Proof. Consider the Riccati equation

$$y' + r_2(t)y^2 + r_1(t)y + a_0(t) = 0, \qquad t \in [t_0, T]. \tag{5.11}$$

without loss of generality taking into account the conditions $21^0)$ we can take that $r_k(t) \geq 0, \;\; a_0(t) \leq 0, \;\; t \geq t_0$ and have unbounded supports. Then by virtue of Theorem 2.2 the solution $y_+(t)$ of Eq. (5.11) with $y_+(t_0) = 0$ exists on $[t_0, T]$ and $y_+(T) \geq 0$. Hence,

$$y_+(t_0) \leq y_+(T). \tag{5.12}$$

By Lemma 2.10 it follows from the conditions $21^0)$ that Eq. (5.11) has a negative t_0-regular solution. Then by Lemma 2.12 Eq. (5.11) has a negative solution $y_-(t)$ on $[t_0, T]$ such that

$$y_-(t_0) \geq y_-(T). \tag{5.13}$$

It follows from $22^0)$ that $\zeta(t) \equiv y_+(t)$ is a solution of the inequality (2.4) on $[t_0, T]$ and $\eta(t) \equiv y_-(t)$ is a solution of the inequality (2.3) on $[t_0, T]$ (since $\eta(t) < 0, \;\; \zeta(t) \geq 0, \;\; t \in [t_0, T]$). Moreover, according to (5.12) and (5.13)

$$\zeta(t_0) \leq \zeta(t), \;\; \eta(t_0) \geq \eta(T), \tag{5.14}$$

Obviously, $\eta(t) \leq \zeta(t), \;\; t \in [t_0, T]$. Then by Corollary 2.1 it follows from (5.14) that Eq. (1.1) has a closed solution on $[t_0, T]$. The theorem is proved.

Theorem 5.9. *Assume* $a_k(t) = p_k(t) + r_k(t), \;\; k = \overline{2, n},$ *where* $p_k(t)$ *and* $r_k(t), \;\; k = \overline{2, n}$ *real-valued continuous functions on* $[t_0, T]$ *such that*
a) for some $j = 2, 3, \ldots n, \;\; r_k(t) \geq 0, \;\; k = \overline{j, n}, \;\; \sum\limits_{k=j}^{n} r_k(t) > 0, \;\; t \in [t_0, T],$
$\sum\limits_{k=2}^{n} p_k(t)x^k \in \Omega_0, \;\; t \in [t_0, T]$
b) for some $j_0 = 0, 1, \ldots, j-1 \;\; a_k(t) \leq 0, \;\; k = \overline{0, j_0}, \;\; \sum\limits_{k=0}^{j_0} a_k(t) < 0, \;\; t \in [t_0, T]$
Then Eq. (1.1) has a positive isolated closed solution on $[t_0, T]$.

Proof. By Lemma 2.15 it follows from the conditions a) that for enough large $M > 1$ the function $\eta(t) \equiv M, \;\; t \in [t_0, T]$ is a solution of the inequality (2.30) on $[t_0, T]$. By lemma 2.14 it follows from the conditions b) that for enough small $\rho > 0 \;\; (\rho < 1)$ the function $\zeta(t) \equiv \rho, \;\; t \in [t_0, T]$ is a solution of the inequality (2.19) on $[t_0, T]$. Then by Lemma 2.13 Eq. (1.1) has a positive isolated closed solution on $[t_0, T]$. The theorem is proved.

 doi:10.20944/preprints202412.2567.v1

6. Some Applications to Planar Autonomous Systems

Let $\mathcal{P}(x,y)$ be a polynomial. Consider the function

$$I_{\mathcal{P}}(\theta) \equiv \sin\theta\big[\mathcal{P}'_x(\cos\theta,\sin\theta)\mathcal{P}(\sin\theta,\cos\theta) - \mathcal{P}'_y(\sin\theta, cos\theta)\mathcal{P}(\cos\theta,\sin\theta)\big] +$$

$$+\cos\theta\big[\mathcal{P}'_x(\sin\theta,\cos\theta)P(\cos\theta,\sin\theta) - \mathcal{P}'_y(\cos\theta,\sin\theta)\mathcal{P}(\sin\theta,\cos\theta)\big], \ \ t \in R.$$

Definition 6.1. *A polynomial $\mathcal{P}(x,y)$ is called a separator polynomial ar,simply, a separator if $I_{\mathcal{P}}(\theta) \neq 0, \ \ t \in R$.*

Hereafter for any polynomial $\mathcal{P}(x,y)$ the function $I_{\mathcal{P}}(\theta)$ we will call the indicator of separation of $\mathcal{P}(x,y)$ or simply the indicator of $\mathcal{P}(x,y)$. Indicate some polynomials with their indicators.

1) $\mathcal{P}(x,y) = x, \ \ I_{\mathcal{P}}(\theta) \equiv 1,$
2) $\mathcal{P}(x,y) = x + \lambda x^3, \ \ \lambda \in R, \ \ I_{\mathcal{P}}(\theta) = 1 + \lambda\big[\sin^4\theta + \cos^4\theta\big] + \big[6\lambda + 3\lambda^2\big]\sin^2\theta\cos^2\theta,$
3) $\mathcal{P}(x,y) = x^3, \ \ I_{\mathcal{P}}(\theta) = 3\sin^2\theta\cos^2\theta,$
4) $\mathcal{P}(x,y) = x + x^5, \ \ I_{\mathcal{P}}(\theta) = 1 + \sin^6\theta + \cos^6\theta + 5\sin^4\theta\cos^4\theta + 5\sin^2\theta\cos^2\theta,$
5) $\mathcal{P}(x,y) = x + \lambda y, \ \ \lambda \in R, \ \ I_{\mathcal{P}}(\theta) \equiv 1 - \lambda^2.$

Problem. *Describe all separator polynomials.*

Definition 6.2. *The transformation*

$$\phi = r\mathcal{P}(\cos\theta,\sin\theta), \qquad \psi = r\mathcal{P}(\sin\theta,\cos\theta), \qquad r,t \in \mathbb{R} \tag{6.1}$$

with any separator $\mathcal{P}(x,y)$ is called a generalized Prufer transformation.

Next we will see that a generalized Prufer transformation allows to extend the classes of systems of planar autonomous systems, studied in [25], to which Eq. (1.1) is applicable.

Consider the autonomous system

$$\begin{cases} \phi' = \sum\limits_{k=1}^{n} P_k(\phi,\psi), \\ \psi' = \sum\limits_{k=1}^{n} Q_k(\phi,\psi). \end{cases} \tag{6.2}$$

where $P_k, \ \ Q_k, \ \ k = \overline{1,n}$ are homogeneous polynomials of degree k. In this section we use some results of previous sections to establish some sufficient conditions for existence of a periodic solution or a limit cycle of the last system.

The substitution (6.1) reduces (6.2) to the system

$$\begin{cases} r'\mathcal{P}(\cos\theta,\sin\theta) + r\theta' A_{\mathcal{P}}(\theta) = \sum\limits_{k=1}^{n} P_k(\mathcal{P}(\cos\theta, sin\theta),\mathcal{P}(\sin\theta,\cos\theta))r^k, \\ r'\mathcal{P}(\sin\theta,\cos\theta) + r\theta' B_{\mathcal{P}}(\theta) = \sum\limits_{k=1}^{n} Q_k(\mathcal{P}(\cos\theta,\sin\theta),\mathcal{P}(\sin\theta,\cos\theta))r^k, \end{cases} \tag{6.3}$$

where $A_{\mathcal{P}}(\theta) \equiv -\mathcal{P}'_x(\cos\theta,\sin\theta)\sin\theta + \mathcal{P}'_y(\cos\theta,\sin\theta)\cos\theta, \ \ B_{\mathcal{P}}(\theta) \equiv \mathcal{P}'_x(\sin\theta,\cos\theta)\cos\theta - \mathcal{P}'_y(\sin\theta,\cos\theta)\sin\theta, \ \ \theta \in \mathbb{R}$. Multiplying both sides of the firs equation of the obtained system by $B_{\mathcal{P}}(\theta)$ and the second equation of that system by $A_{\mathcal{P}}(\theta)$ and subtracting from the first obtained equation the second one we get

$$I_{\mathcal{P}}(\theta)r' = \sum_{k=1}^{n}\big[B_{\mathcal{P}}(\theta)P_k(\mathcal{P}(\cos\theta,\sin\theta),\mathcal{P}(\sin\theta,\cos\theta)) -$$

$$- A_{\mathcal{P}}(\theta)Q_k(\mathcal{P}(\cos\theta,\sin\theta),\mathcal{P}(\sin\theta,\cos\theta))\big]r^k.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

Similarly, multiplying both sides of the first equation of the system (6.2) by $\mathcal{P}(\sin\theta, \cos\theta)$ and both sides of the second equation of that system by $\mathcal{P}(\cos\theta, \sin\theta)$ and subtraction from the second obtained equation the first obtained one we get

$$I_{\mathcal{P}}(\theta) r\theta' = \sum_{k=1}^{n} \big[\mathcal{P}(\cos\theta, \sin\theta) Q_k(\mathcal{P}(\cos\theta, \sin\theta), \mathcal{P}(\sin\theta, \cos\theta)) -$$

$$- \mathcal{P}(\sin\theta, \cos\theta) P_k(\mathcal{P}(\cos\theta, \sin\theta), \mathcal{P}(\sin\theta, \cos\theta),)\big] r^k$$

Therefore (6.2) is reduced to the system

$$\begin{cases} r' = \sum\limits_{k=1}^{n} f_k(\theta) r^k, \\ \\ \theta' = \sum\limits_{k=1}^{n} g_k(\theta) r^{k-1}, \end{cases} \tag{6.4}$$

where

$$f_k(\theta) = \big[B_{\mathcal{P}}(\theta) P_k(\mathcal{P}(\cos\theta, \sin\theta), \mathcal{P}(\sin\theta, \cos\theta)) -$$

$$- A_{\mathcal{P}}(\theta) Q_k(\mathcal{P}(\cos\theta, \sin\theta), \mathcal{P}(\sin\theta, \cos\theta))\big] / I_{\mathcal{P}}(\theta), \tag{6.5}$$

$$g_k(\theta) = \big[\mathcal{P}(\cos\theta, \sin\theta) Q_k(\mathcal{P}(\cos\theta, \sin\theta), \mathcal{P}(\sin\theta, \cos\theta)) -$$

$$- \mathcal{P}(\sin\theta, \cos\theta) P_k(\mathcal{P}(\cos\theta, \sin\theta), \mathcal{P}(\sin\theta, \cos\theta))\big] / I_P(\theta). \tag{6.6}$$

In some cases the system (6.2) is reducible to a single equation like Eq. (1.1). Then a closed solution of the obtained single equation will be represent a periodic orbit for the system (6.2), moreover if the closed solution is isolated, then it corresponds to a limit cycle for that system (see [25]). First we consider the system

$$\begin{cases} \phi' = a\phi + b\psi + \sum\limits_{k=1}^{n-1} \Big[\phi F_k(\phi, \psi) + (\phi^2 + \alpha_k \phi\psi + \beta_0 \psi^2) G_k(\phi, \psi)\Big], \\ \\ \psi' = c\phi + d\psi + \sum\limits_{k=1}^{n-1} \Big[\psi F(\phi, \psi) + (\gamma_0 \phi^2 + \lambda_0 \phi\psi + \mu_k \psi^2) G_k(\phi, \psi)\Big], \end{cases} \tag{6.7}$$

where F_k and G_k are homogeneous polynomials of degrees k and $k-1$ respectively $k = \overline{1, n-1}$, a, b, c, d, α_k, β_k, γ_k $\overline{1, n-1}$ are some real constants.

Let us assume that for some $\beta \neq \pm 1$ the following equalities hold

$$A) \quad \begin{cases} \gamma_0 + \lambda_0 \beta + (\mu_k - \alpha_k)\beta^2 - \beta_0 \beta^3 = \beta, \\ 3\gamma_0 \beta + \lambda_0 (1 + 2\beta^2) + (\mu_k - \alpha_k)(2\beta + \beta^3) - 3\beta_0 \beta^2 = 1 + 2\beta^2, \\ 3\gamma_0 \beta^2 + \lambda_0 (2\beta + \beta^3) + (\mu_k - \alpha_k)(1 + 2\beta^2) - 3\beta_0 \beta = 2\beta + \beta^3, \\ \gamma_0 \beta + \lambda_0 \beta^2 + (\mu_k - \alpha_k)\beta - \beta_0 = \beta^2, \ \ k = \overline{1, n-1}. \end{cases}$$

Then the substitution

$$\phi = r(\cos\theta + \beta \sin\theta), \qquad \psi = r(\sin\theta + \beta\cos\theta) \tag{6.8}$$

reduces (6.7) to the system

$$\begin{cases} r' = \sum\limits_{k=1}^{n} \mathfrak{f}_k(\theta) r^k, \\ \\ \theta' = \mathfrak{g}_1(\theta) r, \end{cases} \tag{6.9}$$

where according to formulae (6.5) and (6.6)

$$\mathfrak{f}_1(\theta) = \frac{1}{1-\beta^2}\Big\{[a + (b+c)\beta + a\beta^2]\cos^2\theta - [d + (b+c)\beta + a\beta^2]\sin^2\theta +$$
$$+ [b - c + d - d\beta + (c-b)\beta^2]\sin\theta\cos\theta\Big\},$$

$$\mathfrak{f}_{k+1}(\theta) = F_k(\cos\theta + \beta\sin\theta, \sin\theta + \beta\cos\theta) +$$
$$+\Big[a_k\cos^3\theta + b_k\cos^2\theta\sin\theta + c_k\cos\theta\sin^2\theta + d_k\sin^3\theta\Big]G_k(\cos\theta + \beta\sin\theta, \sin\theta + \beta\cos\theta),$$

$$a_k = \tfrac{1}{1-\beta^2}\left[1 + \alpha_k\beta + \beta_0\beta^2 - \gamma_0\beta - \lambda_0\beta^2 - \mu_k\beta^3\right],$$
$$b_k = \tfrac{1}{1-\beta^2}\left[\beta + \alpha_k + \beta_0(2\beta - \beta^3) - \gamma_0(2\beta^2 - 1) - \lambda_0\beta^3 - \mu_k\beta^2\right],$$
$$c_k = \tfrac{1}{1-\beta^2}\left[-\beta^2 - \alpha_k\beta^3 + \beta_0(1 - 2\beta^2) - \gamma_0(\beta^3 - 2\beta) + \lambda_0 + \mu_k\beta\right],$$
$$d_k = \tfrac{1}{1-\beta^2}\left[-\beta^3 - \alpha_k\beta^2 - \beta_0\beta + \gamma_0\beta^2 + \lambda_0\beta + \mu_k\right],\ k = \overline{1, n-1},$$

$$\mathfrak{g}_1(\theta) = \frac{1}{1-\beta^2}\Big\{[c + (d-a)\beta - b\beta^2]\cos^2\theta + [d - b - a\beta + c\beta^2]\sin^2\theta +$$
$$+ [d - a + 2(c-b)\beta + (d-a)\beta^2]\sin\theta\cos\theta\Big\}.$$

Assume $\mathfrak{g}_1(\theta) \neq 0,\ \ \theta \in \mathbb{R}$. Then by considering r as a function of θ from (6.9) we derive the equation

$$\frac{dr}{d\theta} = \sum_{k=1}^{n}\frac{\mathfrak{f}_k(\theta)}{\mathfrak{g}_1(\theta)}r^k, \qquad \theta \in \mathbb{R}. \tag{6.10}$$

Theorem 6.1. *Let us assume that for some $\beta \neq \pm 1$ the conditions $A)$ and the following conditions be satisfied.*
$\mathfrak{g}_1(\theta) \neq 0,\ \ \theta \in [0, 2\pi],\ \ \frac{\mathfrak{f}_k(\theta)}{\mathfrak{g}_1(\theta)} = p_k(\theta) + r_k(\theta),\ \ k = \overline{1, n}$, *where* $p_k(\theta),\ \ r_k(\theta),\ \ k = \overline{1, n}$ *are real-valued continuous functions on* $[0, 2\pi]$, *such that* $-\sum_{k=1}^{n} p_k(\theta)x^k \in \Omega_0$, *for some* $j = 2, \dots, n,\ \ r_k(\theta) \leq 0,\ \ k = \overline{j, n},\ \ \theta \in [0, 2\pi]$ *and for some* $j_0 = 0, \dots, j-1,\ \ r_k(\theta) \geq 0,\ \ k = \overline{0, j_0},\ \ \theta \in [0, 2\pi]$. *Moreover,* $\sum_{k=j}^{n} r_k(\theta) < 0,\ \ \sum_{k=0}^{j_0} r_k(\theta) > 0$. *Then the system (6.7) has a limit cycle.*

Proof. One can verify that the conditions $A)$, the condition $\mathfrak{g}_1(\theta) \neq 0,\ \ \theta \in [0, 2\pi]$ and the transformation (6.8) imply the reduction of the system (6.7) to the single equation (6.10). Then it follows from the remaining conditions of the theorem, that all conditions of Theorem 5.9 foe Eq. (6.10) are satisfied. Then the assertion of the theorem is a direct consequence of Theorem 5.9. The theorem is proved.

Consider the system

$$\begin{cases} \phi' = a\phi + b\psi + P_{m+1}(\phi, \psi) + \sum\limits_{k=1}^{n-1}\Big[\phi F_{mk}(\phi, \psi) + (\phi^2 + \alpha_k\phi\psi + \beta_0\psi^2)G_{mk}(\phi, \psi)\Big], \\ \psi' = c\phi + d\psi + Q_{m+1}(\phi, \psi) + \sum\limits_{k=1}^{n-1}\Big[\psi F_{mk}(\phi, \psi) + (\gamma_0\phi^2 + \lambda_0\phi\psi + \mu_k\psi^2)G_{mk}(\phi, \psi)\Big], \end{cases} \tag{6.11}$$

where $P_{m+1}(x,y)$ and $Q_{m+1}(x,y)$ are homogeneous polynomials of degree $m+1$, F_{mk} and G_{mk} are homogeneous polynomials of degrees mk and $mk-1$ respectively, $m \in \mathbb{N}$, $k = \overline{2.n-1}$. Assume the conditions $A)$ hold. Then the substitution (6.8) reduces (6.11) to the system

$$\begin{cases} r' = \mathfrak{f}_1(\theta) + \mathfrak{f}^0_{m+1}(\theta) r^{m+1} + \sum_{k=2}^{n} \mathfrak{f}^0_{mk}(\theta) r^{mk+1}, \\ \theta' = \mathfrak{g}^0_{m+1}(\theta) r^m, \end{cases} \tag{6.12}$$

where

$$\mathfrak{f}^0_{m(k+1)}(\theta) \equiv F_{mk}(\cos\theta + \beta\sin\theta, \sin\theta + \beta\cos\theta) +$$
$$+\Big[a_k\cos^3\theta + b_k\cos^2\theta\sin\theta + c_k\cos\theta\sin^2\theta + d_k\sin^3\theta\Big] G_{mk}(\cos\theta + \beta\sin\theta, \sin\theta + \beta\cos\theta),$$

$k = \overline{1, n-1}$,

$$\mathfrak{g}^0_{m+1}(\theta) \equiv \frac{1}{1-\beta^2}\Big[(\cos\theta + \beta\sin\theta) Q_{m+1}(\cos\theta + \beta\sin\theta, \sin\theta + \beta\cos\theta) -$$
$$-(\sin\theta + \beta\cos\theta) P_{m+1}(\cos\theta + \beta\sin\theta, \sin\theta + \beta\cos\theta)\Big].$$

Assume $\mathfrak{g}^0_{m+1}(\theta) \neq 0,\ t \in \mathbb{R}$. Then by considering r as a function of θ, from (6.12) we derive the single equation

$$\frac{dr}{d\theta} = \frac{1}{\mathfrak{g}^0_{m+1}(\theta)}\Big[\mathfrak{f}_1(\theta) r + \mathfrak{f}^0_{m+1}(\theta) r^{m+1} + \sum_{k=2}^{n} \mathfrak{f}^0_{mk}(\theta) r^{mk+1}\Big].$$

After the change of variables $R = r^m$ from the last equation we get the following equation of type (1.1)

$$\frac{dR}{d\theta} = \frac{m}{\mathfrak{g}^0_{m+1}(\theta)}\Big[\mathfrak{f}_1(\theta) + \mathfrak{f}^0_{m+1}(\theta) R + \sum_{k=2}^{n} \mathfrak{f}^0_{mk}(\theta) R^k\Big]. \tag{6.13}$$

Theorem 6.2 *Assume the conditions $A)$ and the following conditions be satisfied* $\mathfrak{g}^0_{m+1}(\theta) \neq 0,\ \frac{m\mathfrak{f}^0_{m+1}(\theta)}{\mathfrak{g}^0_{m+1}(\theta)} = p_1(\theta) + r_1(\theta),\ \frac{m\mathfrak{f}^0_{2m}(\theta)}{\mathfrak{g}^0_{m+1}(\theta)} = p_2(\theta) + r_2(\theta),\ \theta \in [0, 2\pi]$, *where* $p_k(t),\ r_k(t),\ k = 1, 2$ *are real-valued continuous functions on* $[0, 2\pi]$ *such that* $r_2(\theta) \leq 0,\ \frac{\mathfrak{f}_1(\theta)}{\mathfrak{g}^0_{m+1}(\theta)} \geq 0,\ \sum_{k=0}^{2} p_k(t) x^k + \sum_{k=3}^{n} \frac{m\mathfrak{f}^0_{mk}(\theta)}{\mathfrak{g}^0_{m+1}(\theta)} x^k \in \Omega^*_0,\ \theta \in [0, 2\pi]$.
Then the system (6.12) has a periodic orbit.

Proof. Under the restrictions $A)$ and $\mathfrak{g}^0_{m+1}(\theta) \neq 0,\ \theta \in [0, 2\pi]$ the system (6.12) is reducible to Eq. (6.13). It is clear that the conditions of Theorem 5.8 for Eq. (6.13) are satisfied. Then the assertion of the theorem is a direct consequence of Theorem 5.8. The theorem is proved.

Theorem 6.3. *Assume the conditions $A)$ and the following conditions be satisfied* $\mathfrak{g}^0_{m+1}(\theta) \neq 0,\ \frac{m\mathfrak{f}_1(\theta)}{\mathfrak{g}^0_{m+1}(\theta)} = p_0(\theta) + r_0(\theta),\ \frac{m\mathfrak{f}^0_{m+1}(\theta)}{\mathfrak{g}^0_{m+1}(\theta)} = p_1(\theta) + r_1(\theta),\ \frac{m\mathfrak{f}^0_{mk}(\theta)}{\mathfrak{g}^0_{m+1}(\theta)} = p_k(\theta),\ k = \overline{2, n}\ \theta \in [0, 2\pi]$,*where* $p_k(t),\ r_k(t),\ k = \overline{0, v}$ *are real-valued continuous functions on* $[0, 2\pi]$ *such that* $-\sum_{k=0}^{n} p_k(\theta) x^k \in \Omega_0,\ \theta \in [0, 2\pi]$, *for some* $j = 2, \ldots, , n$ *the inequalities* $(-1)^k r_k(\theta) \leq 0, k = \overline{j, n},\ \sum_{k=j}^{n} r_k(\theta) < 0,\ r_0(\theta) \geq 0,\ r_0(\theta) \not\equiv 0,\ \theta \in [0, 2\pi]$ *hold. Then the system (6.12) has a periodic orbit* $(\phi_*(t), \psi_*(t))$*. In particular if* $j = 2$ *and* $\int_0^{2\pi} r_1(\theta) d\theta < 0$*, then* $(\phi_*(t), \psi_*(t))$ *is a limit cycle.*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 January 2025 doi:10.20944/preprints202412.2567.v1

Proof. As in the case of previous theorem under the restrictions $A)$ and $\mathfrak{g}^0_{m+1}(\theta) \neq 0,\ \ \theta \in [0, 2\pi]$ the system (6.12) is reducible to Eq. (6.13). It is not difficult to verify that the conditions of Corollary 5.1 for Eq. (6.13) are satisfied. Then the assertion of the theorem immediately follows from Corollary 5.1. The theorem is proved.

Denote

$$\mathfrak{F}(\theta) \equiv \frac{m}{\mathfrak{g}^0_{m+1}(\theta)} \sum_{k=2}^{n} |\mathfrak{f}^0_{mk}(\theta)| + \frac{m\mathfrak{f}^0_{m+1}(\theta)}{\mathfrak{g}^0_{m+1}(\theta)},$$

$$\mathfrak{d}_C(\theta) \equiv \exp\left\{\int_0^\theta \mathfrak{F}(\tau)d\tau\right\}\left[C + \int_0^\theta \exp\left\{-\int_0^\tau \mathfrak{F}(s)ds\right\}\frac{m\mathfrak{f}_1(\tau)}{\mathfrak{g}^0_{m+1}(\tau)}d\tau\right],$$

$$\mathfrak{d}^-_C(\theta) \equiv \exp\left\{\int_0^\theta \mathfrak{F}(\tau)d\tau\right\}\left[\int_0^\theta \exp\left\{-\int_0^\tau \mathfrak{F}(s)ds\right\}\frac{m\mathfrak{f}_1(\tau)}{\mathfrak{g}^0_{m+1}(\tau)}d\tau - C\right],\ \ \theta, c \in \mathbb{R}.$$

Theorem 6.4. *Let the conditions $A)$ and the following conditions be satisfied*
$\mathfrak{g}^0_{m+1}(\theta) \neq 0$
for some

$$C^+ \geq -\min_{\theta\in[0,2\pi]} \int_0^\theta \exp\left\{-\int_0^\tau \mathfrak{F}(s)ds\right\}\frac{m\mathfrak{f}_1(\tau)}{\mathfrak{g}^0_{m+1}(\tau)}d\tau, \tag{6.14}$$

$$C^- \geq \max_{\theta\in[0,2\pi]} \int_0^\theta \exp\left\{-\int_0^\tau \mathfrak{F}(s)ds\right\}\frac{m\mathfrak{f}_1(\tau)}{\mathfrak{g}^0_{m+1}(\tau)}d\tau, \tag{6.15}$$

the inequalities

$$\mathfrak{d}_{C^+}(\theta) \leq 1, \qquad |\mathfrak{d}^-_{C^-}(\theta)| \leq 1, \qquad \theta \in [0, 2\pi], \tag{6.16}$$

$$C^+\left(\exp\left\{-\int_0^{2\pi} \mathfrak{F}(\tau)d\tau\right\} - 1\right) \geq \int_0^{2\pi} \exp\left\{-\int_0^\tau \mathfrak{F}(s)ds\right\}\frac{m\mathfrak{f}_1(\tau)}{\mathfrak{g}^0_{m+1}(\tau)}d\tau,$$

$$C^-\left(\exp\left\{-\int_0^{2\pi} \mathfrak{F}(\tau)d\tau\right\} - 1\right) \leq \int_0^{2\pi} \exp\left\{-\int_0^\tau \mathfrak{F}(s)ds\right\}\frac{m\mathfrak{f}_1(\tau)}{\mathfrak{g}^0_{m+1}(\tau)}d\tau$$

are valid.
Then the system (6.12) has a periodic orbit. If the inequalities (6.14) - (6.16) are strict, then the system (6.12) has a limit cycle.

Proof. The conditions $A)$ imply that the system (6.12) is reducible to Eq. (6.13). It follows from the conditions of the theorem that the conditions of Theorem 5.6 for Eq. (6.13) are satisfied. Moreover, If the inequalities (6.14) - (6.16) are strict then the conditions of Theorem 5.7 for Eq. (6.13) are satisfied. Then the assertion of the theorem is a direct consequence of Theorems 5.6 and 5.7. The theorem is proved.

References

1. S. Akram, A. Nawaz, N. Yasmin, H. Kalsoom, Yu-M. Chu, Periodic solutions for first-order cubic non-autonomous differential equation with bifurcation analysis. Journal of Taibah University for Science Volume 14, 2020 - Issue 1.
2. M. J. Alvarez, J. L. Bravo, M. Fernadez, Uniqueness of limit cycles for polynomial first-order differential equations. J. Math. Anal. Appll. 360 (2009) 168–189.

3. S. E. Ariaku, E. C. Mbuh, C. C. Asogva, P. U. Nwokoro. Lower and Upper Solutions of First Order Non-Linear Ordinary Differential Equations. International Journal of Scientific Engineering and Science, Vol. 3, Issue 11, 2019, pp. 59–61.
4. M. Briskin and Y. Yumdin, Tangential version of Hilbert 16th problem for the Abel equation. Moscow Mathematical Journal, vol. 5, No. 1, 2005, 23–53.
5. B. P. Demidovich, Lectures on the mathematical theory of stability. Moskow, "Nauka", 1967.
6. R. E. Edwards, A formal background to mathematics (Springer-Verlag, New York, Haidelberg, Berlin, 1980).
7. M. Calanchi, B. Ruf, On the number of closed solutions for polynomial ODE's and a special case of Hilbert's 16th problem. Adv, Differential equations, 7 (2002), no. 2, 197–216.
8. A. Gassull, J. Libre, Limit cycles for a class of Abel equations. SIAM J. Math. Anal. 21 (1990), 1235–1244.
9. A. Gassull, R. Prohens, J. Torregrossa, Limit cycles for rigid cubic systems. J. Math. Anal. Appl. 303 (2005), 391–404.
10. G. A. Grigorian, On two comparison tests for second-order linear ordinary differential equations, Diff. Urav., vol 47, (2011), 1225–1240 (in Russian), Diff. Eq., vol. 47, (2011), 1237–1252 (in English).
11. G. A. Grigorian, Two comparison criteria for scalar Riccati equations and some of their applications Izv. Vissh. Uchebn. Zaved. Mat. (2012), no 11, 20-25. Russian Math. (Iz. VUZ) 56 (2.12), no. 11, 17-30.
12. G. A. Grigorian, Interval oscillation criteria for linear matrix Hamiltonian systems. Rocky Mountain J. Math. 50 (2000), no. 6, 2047-2057.
19. G. A. Grigorian, New reducibility criteria for systems of two linear first-order ordinary differential equations. Monatsh. Math. 198 (2022), no. 2. 311-322.
14. G. A. Grigorian, Oscillation and non-oscillation criteria for linear nonhomogeneous systems of two first-order ordinary differential equations. J. Math. Anal. Appl. 507 (2022), no. 1, Paper No. 125734, 10 pp.
21. G. A. Grigorian, Oscillatory and non-oscillatory criteria for linear four-dimensional Hamiltonian systems Math. Bohem. 146 (2021), no. 3, 289-304.
16. G. A. Grigorian, On the reducibility of systems of two linear first-order ordinary differential equations. Monatsh. Math. 195 (2021), no. 1, 107-117.
17. G. A. Grigorian, Oscillation criteria for linear matrix Hamiltonian systems. Proc. Amer. Math. Soc. 148 (2020), no. 8, 3407-3415.
18. G. A. Grigorian, Oscillatory and non oscillatory criteria for the systems of two linear first order two by two dimensional matrix ordinary differential equations. Arch. Math. (Brno) 54 (2018), no. 4, 169-203.
19. G. A. Grigorian, Stability criterion for systems of two first-order linear ordinary differential equations. Mat. Zametki 103 (2018), no. 6, 831-840. Math. Notes 103 (2018), no. 5-6, 892-900.
20. G. A. Grigorian, Oscillatory criteria for the systems of two first-order linear differential equations. Rocky Mountain J. Math. 47 (2017), no. 5. 1497-1524.
21. G. A. Grigorian, Some properties of the solutions of third order linear ordinary differential equations Rocky Mountain J. Math. 46 (2016), no. 1, 147-168.
22. G. A. Grigorian, Stability criteria for systems of two first-order linear ordinary differential equations. Math. Slovaca 72 (2022), no. 4, 935-944.
23. G. A. Grigorian, The behavior of solutions of the system of two first order linear ordinary differential equations. Mathematica Pannonica 26/1 (2017-2018), pp. 67–11.
24. Yu. Ilyashenko, Centennial History of Hilbert's 16th Problem. Bul. (New Series) of the AMS, Vol. 39, Num. 3, pp. 301–354.
25. P. J. Torres, Existence of closed solutions for a polynomial first order differential equation. J. Math. Anal. Appll. 328 (2007) 1108–1116.

doi:10.20944/preprints202412.2567.v1