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Codes with Weighted Poset Metrics Based on the Lattice of Subgroups of Z_m

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Codes with Weighted Poset Metrics Based on the Lattice of Subgroups of $\mathbb{Z}_m$

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Abstract: By combining a lattice subgroup diagram of $\mathbb{Z}_m$ with a weighted poset metric, we introduce a new weighted coordinates poset metric for codes over $\mathbb{Z}_m$, called LS-poset metric. When I is an ideal in a poset, the concept of I-perfect codes with LS-poset metric is investigated. We obtain a Singleton bound for codes with LS-poset metric and define MDS codes. When the poset in poset metric is a chain, we provide sufficient conditions for a code with LS-poset metric to be r-perfect for some $r \in \mathbb{N}$.

Keywords: MDS codes; poset codes; pomset codes; perfect codes

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1. Introduction

The concept of poset metric codes over a finite field $\mathbb{F}_q$ was introduced by Brualdi (see [2]) in 1995. Over the past two decades, coding theory has seen significant developments through the study of codes in the poset metric. This generalization of classical coding metrics has opened up new avenues for research and applications, particularly in scenarios where traditional metrics like the Hamming or Lee distance are not sufficient to model the complexities of error patterns. We refer to [1,3,5–7] for some results on poset metric spaces such as packing radius, the existence of r-error-correcting codes, perfect codes, and group of isometries. In 2018, the pomset metric was introduced by the authors in [8] to accommodate Lee metric for codes over $\mathbb{Z}_m$. This metric is a further generalization of the poset metric and is based on the concept of pomsets, or partially ordered multisets. In both the poset and pomset metrics, the Singleton bound, MDS and I-perfect property for codes are studied (see [4,9]).

Both the poset and pomset metrics are constructed based on the structure of posets. The structure of a poset serves as the foundation for defining these metrics, as it establishes the relationships and dependencies between the elements of the codeword positions. In the present paper, we introduces a weighted poset metric based on subgroups diagram of $\mathbb{Z}_m$. By using the poset of the power set of a multiset, we can effectively visualize the subgroup relationships in $\mathbb{Z}_m$. The poset captures the inclusion relationships between subgroups, while the multiset represents the different ways subgroups can be generated based on the divisors of m. This approach is especially powerful for cyclic groups where the subgroup structure is tightly related to the divisors of the group's order.

1.1. Poset Metrics

Let $P = ([n], \preceq_P)$ be a poset on the set $[n] := \{1, 2, \ldots, n\}$ of coordinates of a vector in $\mathbb{F}_q^n$ (or $\mathbb{Z}_m^n$). For $I \subseteq [n]$, I is called an (*order*) *ideal* of P if $i \in I$, $j \preceq_P i$ imply that $j \in I$. For a subset S of P, we denote $\langle S \rangle$ the smallest ideal containing S. Given a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$, the *support* of $\mathbf{x}$ is $\text{supp}(\mathbf{x}) = \{i \in [n] \ : \ x_i \neq 0\}$. The *poset weight* of $\mathbf{x}$ is defined as

$$w_P(\mathbf{x}) := |\langle \text{supp}(\mathbf{x}) \rangle|.$$

For $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$, the *poset distance* between $\mathbf{x}$ and $\mathbf{y}$ is

$$d_P(\mathbf{x}, \mathbf{y}) := w_P(\mathbf{x} - \mathbf{y}).$$

It was shown in [2] that d_P is a metric on $\mathbb{F}_q^n$. Notice that the poset metric d_P simplifies to the Hamming metric d_H when the poset P is an antichain.

1.2. Multisets and Pomsets

For given a nonempty set X and a map $\mathfrak{c} : X \to \mathbb{N}_0$, an mset M is considered as a pair $M = (X, \mathfrak{c})$. We write $a \in^n M$ (or $n/a \in M$), if $\mathfrak{c}(a) \geq n > 0$, (i.e., a occurs in M at least n times).

An mset $M = (X, \mathfrak{c})$, drawn from $X = \{a_1, a_2, \ldots, a_t\}$, is represented as

$$M = \{k_1/a_1, k_2/a_2, \ldots, k_t/a_t\},$$

where $\mathfrak{c}(a_i) = k_i > 0$ for $i = 1, \ldots, t$. If $k/a \in M$, then $r/a \in M$ for all $1 \leq r \leq k$. The cardinality of an mset $M = (X, \mathfrak{c})$ is defined as $|M| = \sum_{x \in X} \mathfrak{c}(x)$.

For $m \in \mathbb{N}$, we denote $\mathcal{M}^m(X)$ as the (*regular*) mset of height m drawn from the set X such that all elements of X occur with the same multiplicity m, i.e., $|\mathcal{M}^m(X)| = m|X|$. The *mset space* $[X]^m$ is the set of all msets drawn from X such that no element in an mset occurs more than m times.

A *submultiset* (or *submset*) of an mset $M = (X, \mathfrak{c})$ is a multiset $S = (X, \mathfrak{c}_S)$ such that $\mathfrak{c}_S(x) \leq \mathfrak{c}(x)$ for all $x \in X$. For an mset $M = (X, \mathfrak{c})$, the set $M^* = \{x \in X \ : \ \mathfrak{c}(x) > 0\}$ is called the *root set* of M.

For two msets $M_1 = (X, \mathfrak{c}_1)$ and $M_2 = (X, \mathfrak{c}_2)$, we list some definitions of operations in msets [8] as follows:

- The *addition*(*sum*) of M_1 and M_2 is the mset $M_1 \oplus M_2 = (X, \mathfrak{s})$, where $\mathfrak{s}(x) = \mathfrak{c}_1(x) + \mathfrak{c}_2(x)$ for all $x \in X$.
- The *subtraction* (*difference*) of M_1 from M_2 is the mset $M_2 \ominus M_1 = (X, \mathfrak{d})$, where $\mathfrak{d}(x) = \max\{\mathfrak{c}_2(x) - \mathfrak{c}_1(x), 0\}$ for all $x \in X$.
- The *union* of M_1 and M_2 is the mset $M_1 \cup M_2 = (X, \mathfrak{u})$, where $\mathfrak{u}(x) = \max\{\mathfrak{c}_1(x), \mathfrak{c}_2(x)\}$ for all $x \in X$.
- The *intersection* of M_1 and M_2 is the mset $M_1 \cap M_2 = (X, \mathfrak{i})$, where $\mathfrak{i}(x) = \min\{\mathfrak{c}_1(x), \mathfrak{c}_2(x)\}$ for all $x \in X$.

For $M_1, M_2 \in [X]^m$, the *mset sum* $M_1 \oplus M_2 = (X, \mathfrak{s}) \in [X]^m$, where $\mathfrak{s}(x) = \min\{m, \mathfrak{c}_{M_1}(x) + \mathfrak{c}_{M_2}(x)\}$ for all $x \in X$. Given a submset $S = (X, \mathfrak{b})$ of an mset $\mathcal{M}^m(X)$, the *complement* of S is an mset $S^c = (X, \mathfrak{b}_c) \in [X]^m$, where $\mathfrak{b}_c(x) = m - \mathfrak{b}(x)$ for all $x \in X$.

For two msets M_1, M_2 drawn fron a set X, we define the *cartesian product* $M_1 \times M_2$ by $M_1 \times M_2 := \{rs/(r/x, s/y) \ : \ r/x \in M_1, \ s/y \in M_2\}$. A submset $R = (M \times M, \mathfrak{g})$ of $M \times M$ is said to be an *mset relation* on M if $\mathfrak{g}(r/x, s/y) = rs$.

An mset relation R on M is called a *partially ordered mset relation* (or *pomset relation*) R on M if the following properties are all satisfied:

(1) [reflexivity] $\forall \ m/x \in M, \ (m/x, m/x) \in R$,
(2) [antisymmetry] if $(m/x, n/y), (n/y, m/x) \in R \Rightarrow m = n, \ x = y$, and
(3) [transitivity] if $(m/x, n/y), (n/y, k/z) \in R \Rightarrow (m/x, k/z) \in R$.

Notice that if $(m/x, m/x) \in R$, then $(r/x, s/x) \in R$ for all $1 \leq r, s \leq m$.

For given a poset $P = (X, \preceq_P)$, we define the pomset relation $\preccurlyeq_P$ on $\mathcal{M}^m(X)$ having P-shape by

$$\preccurlyeq_P := \{m^2/(m/a, m/a), \ m^2/(m/a, m/b) \ : \ \forall a, b \in X, \ a \preceq_P b\}.$$

The pair $(\mathcal{M}^m(X), \preccurlyeq_P)$ is known as a *partially ordered multiset* (*pomset*), denoted by $\mathbb{P}$.

The *dual pomset* of the pomset $\mathbb{P} = (\mathcal{M}^m(X), \preccurlyeq_P)$, denoted by $\widetilde{\mathbb{P}}$, is the pomset on $M^m(X)$ having $\widetilde{P}$-shape, where $\widetilde{P}$ is the dual poset of P. That is, $m/a \preccurlyeq_P m/b$ in $\mathbb{P}$ if and only if $m/b \preccurlyeq_{\widetilde{P}} m/a$ in $\widetilde{\mathbb{P}}$.

Let S be a submset of $\mathcal{M}^m(X)$ in a pomset $\mathbb{P} = (\mathcal{M}^m(X), \preccurlyeq_P)$. An element $t/a \in S$ is said to be a *maximal element* in S if there is no element $k/c \in S$ $(c \neq a)$ such that $t/a \preccurlyeq_P k/c$. An element $r/b \in S$ is said to be a *minimal element* in S if there is no element $k/c \in S$ $(c \neq b)$ such that $k/c \preccurlyeq_P t/b$.

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Let $\mathbb{P} = (\mathcal{M}^k([n]), \preccurlyeq_P)$ be the pomset of height k having P-shape where the poset $P = ([n], \preceq_P)$. An *ideal* in $\mathbb{P}$ is a submset $I \subseteq \mathcal{M}^k([n])$ with the property that if $j/b \in I$ and $i/a \preccurlyeq_P j/b$ $(a \neq b)$ then $i/a \in I$. Given a submset S of $\mathcal{M}^k([n])$, we denote by $\langle S \rangle$ the smallest ideal containing S.

An ideal I of $\mathcal{M}^k([n])$ is called an *ideal with full count* if $i/a \in I \Rightarrow k/a \in I$; otherwise, it is called an *ideal with partial count*.

Example 1. *From the poset P_1 as in Figure 1, we consider the pomset $\mathbb{P}_1 = (\mathcal{M}^3([4]), \preccurlyeq_{P_1})$. Let $I_1 = \{3/1, 3/4\}$ and $I_2 = \{3/1, 2/2, 1/3\}$ be ideals with full count and partial count in $\mathbb{P}_1$, respectively. Then the complements $I_1^c = \{3/2, 3/3\}$ and $I_2^c = \{1/2, 2/3, 3/4\}$ are ideals with full count and partial count in $\widetilde{\mathbb{P}}_1$, respectively. Observe that $I_1 \cap I_1^c = \emptyset$ whereas $I_2 \cap I_2^c = \{1/2, 1/3\}$.*

Notice that if I is an ideal with full count in $\mathbb{P} = (\mathcal{M}^k([n]), \preccurlyeq_P)$, then $\{I, I^c\}$ is a partition of $\mathcal{M}^k([n])$, that is $I \cap I^c = \emptyset$ and $I \cup I^c = \mathcal{M}^k([n])$. However, for given any submset J of $\mathcal{M}^k([n])$, $J \oplus J^c = \mathcal{M}^k([n])$.

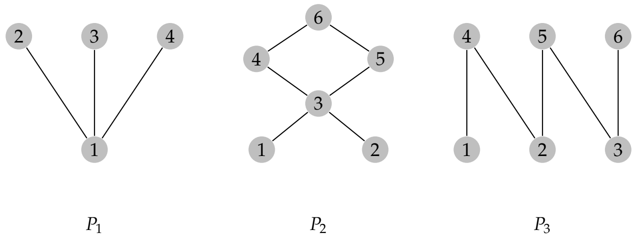

Figure 1. The posets $P_1,\ P_2,\ P_3$.

1.3. Pomset Metrics

In the space $\mathbb{Z}_m^n$ with the pomset $\mathbb{P} = (\mathcal{M}^{\lfloor m/2 \rfloor}([n]), \preccurlyeq_P)$. For a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_m^n$, the *support of* $\mathbf{x}$ *with respect to Lee weight* is defined to be

$$\text{supp}_L(\mathbf{x}) := \{t/i \ : \ x_i \neq 0, \text{and } t = \min\{x_i, m - x_i\}\}.$$

The *pomset weight* of $\mathbf{x} \in \mathbb{Z}_m^n$ is defined to be $w_{\mathbb{P}}(\mathbf{x}) := |\langle \text{supp}_L(\mathbf{x}) \rangle|$, and the *pomset distance* between two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{Z}_m^n$ is defined by $d_{\mathbb{P}}(\mathbf{x}, \mathbf{y}) := w_{\mathbb{P}}(\mathbf{x} - \mathbf{y})$. It is known that the pomset distance is a metric on $\mathbb{Z}_m^n$ (see [8]), and it is called a *pomset metric*. When the pomset is an antichain, there is no hierarchical structure to influence the weight calculation, making the pomset metric equivalent to the Lee metric in terms of how the weight of the codeword is computed.

2. An Ordinal Product

Let $A = \{n_1/a_1, n_2/a_2, \ldots, n_t/a_t\}$ be an mset with $A^* = \{a_1, a_2, \ldots, a_t\}$. Then $|A| = \sum_{i=1}^{t} n_i$. Let $\mathcal{P}(A)$ be the power set of the mset A. With a slight change of notation, we will use $a_{i_1}^{[t_{i_1}]} a_{i_2}^{[t_{i_2}]} \cdots a_{i_s}^{[t_{i_s}]}$ for the mset $\{t_{i_1}/a_{i_1}, t_{i_2}/a_{i_2}, \ldots, t_{i_s}/a_{i_s}\} \in \mathcal{P}(A)$. Here we let $\mathscr{P}(A) := \mathcal{P}(A) \backslash \{\emptyset\}$.

Note that we may write $a_i^{[0]}$ to indicate that a_i does not appear. For $\alpha_1, \alpha_2 \in \mathscr{P}(A)$, which $\alpha_1 = a_1^{[l_1]} a_2^{[l_2]} \cdots a_t^{[l_t]}$ and $\alpha_2 = a_1^{[h_1]} a_2^{[h_2]} \cdots a_t^{[h_t]}$, define the *mset sum* $\alpha_1 \oplus \alpha_2 = a_1^{[\mathfrak{s}_1(a_1)]} a_2^{[\mathfrak{s}_2(a_2)]} \cdots a_t^{[\mathfrak{s}_t(a_t)]} \in \mathscr{P}(A)$, where $\mathfrak{s}_i(a_i) = \min\{n_i, l_i + h_i\}$ for all $i = 1, \ldots, t$.

For each $\alpha = a_{i_1}^{[t_{i_1}]} a_{i_2}^{[t_{i_2}]} \cdots a_{i_s}^{[t_{i_s}]} \in \mathscr{P}(A)$ with $\alpha \neq A$, the *dual* of α is

$$\widehat{\alpha} = a_{i_1}^{[n_{i_1} - t_{i_1}]} a_{i_2}^{[n_{i_2} - t_{i_2}]} \cdots a_{i_s}^{[n_{i_s} - t_{i_s}]} \in \mathscr{P}(A),$$

which $\alpha \oplus \widehat{\alpha} = A$.

Under the submset relation $\subseteq_A$, $(\mathscr{P}(A), \subseteq_A)$ is a partially ordered set, denoted by $\mathscr{P}_A$. For each

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$\alpha \in \mathscr{P}(A)$, we denote $\langle \alpha \rangle$ the ideal in $\mathscr{P}(A)$ having α as its maximum element. It is clear that $\langle A \rangle = \mathscr{P}(A)$. For example, let $a^{[1]}b^{[2]}, b^{[3]} \in \mathscr{P}(a^{[2]}b^{[3]})$. Then $\langle a^{[1]}b^{[2]} \rangle = \{a^{[1]}b^{[2]}, a^{[1]}b^{[1]}, b^{[2]}, a^{[1]}, b^{[1]}\}$, and $\langle b^{[3]} \rangle = \{b^{[3]}, b^{[2]}, b^{[1]}\}$.

Remark 1. *For* $\alpha = a_{i_1}^{[t_{i_1}]} a_{i_2}^{[t_{i_2}]} \cdots a_{i_s}^{[t_{i_s}]} \in \mathscr{P}(A)$, *we have* $|\alpha| = \sum_{j=1}^{s} t_{i_j}$, *and* $|\langle \alpha \rangle| = \prod_{j=1}^{s}(t_{i_j} + 1) - 1$.

Given a poset $P = ([n], \preceq_P)$, we define a relation γ on $[n] \times \mathscr{P}(A)$ by

$$(i, \alpha) \ \gamma \ (j, \beta) \quad \Leftrightarrow \quad \begin{cases} i = j \text{ and } \alpha \subseteq_A \beta \\ i \preceq_P j \text{ where } i \neq j \end{cases}.$$

It is clear that $([n] \times \mathscr{P}(A), \gamma)$ is a poset, denoted by $P \times \mathscr{P}_A$. By the property of any ideal in a poset that contains every element smaller than or equal to some of its elements, we have that if $(i, \alpha) \in I$ and I is an ideal in $P \times \mathscr{P}_A$, then $\{i\} \times \langle \alpha \rangle \subseteq I$.

An ideal I in $P \times \mathscr{P}_A = ([n] \times \mathscr{P}(A), \gamma)$ is called an *ideal with full count* if $(i, \alpha) \in I \Rightarrow (i, A) \in I$; otherwise, it is also called an *ideal with partial count*. Let $\mathcal{I}(P \times \mathscr{P}_A)$ be the set of all ideal in $P \times \mathscr{P}_A$. For $I \in \mathcal{I}(P \times \mathscr{P}_A)$, we denote

$$\omega_{I_f} := \{i \in [n] \ : \ (i, A) \in I\}, \text{ and}$$
$$\omega_{I_p} := \{i \in [n] \ : \ (i, A) \notin I \text{ but } (i, \alpha) \in I \text{ for some } \alpha \in \mathscr{P}(A)\}.$$

Given an ideal with partial count I in $P \times \mathscr{P}_A$ and for $i \in \omega_{I_p}$, we let $\mathcal{A}(I; i) := \{\alpha \in \mathscr{P}(A) \ : (i, \alpha) \in I\}$. An ideal I in $P \times \mathscr{P}_A$ is called *normal* if $\forall i \in \omega_{I_p}$, $\mathcal{A}(I; i) = \langle \alpha_i \rangle$ for some $\alpha_i \in \mathscr{P}(A)$. We denote by $\mathfrak{I}(P \times \mathscr{P}_A)$ the collections of normal ideals in $\mathcal{I}(P \times \mathscr{P}_A)$.

The *dual poset with respect to* P of $P \times \mathscr{P}_A$ is the poset $\widetilde{P} \times \mathscr{P}_A$, where $\widetilde{P}$ is the dual poset of P. Let $I \in \mathfrak{I}(P \times \mathscr{P}_A)$. The *complement* of I, denoted by $I^{\mathfrak{C}}$, is a normal ideal in the dual poset $\widetilde{P} \times \mathscr{P}_A$ which satisfies:

(i) $\omega_{I_p^{\mathfrak{C}}} = \omega_{I_p}$ and $\omega_{I_f^{\mathfrak{C}}} = [n] \backslash (\omega_{I_f} \cup \omega_{I_p})$, and
(ii) for $i \in \omega_{I_p^{\mathfrak{C}}}$, $\mathcal{A}(I^{\mathfrak{C}}; i) = \langle \widehat{\alpha}_i \rangle$, where $\mathcal{A}(I; i) = \langle \alpha_i \rangle$ for some $\alpha_i \in \mathscr{P}(A)$.

Example 2. *Consider the poset* $P_3 = ([6], \preceq_{P_3})$ *as in Figure 1, and the mset* $A = a^{[3]}b^{[2]}$. *Let* $I_1, I_2, I_3 \in \mathcal{I}(P_3 \times \mathscr{P}_A)$ *be defined by*

$$I_1 = \{1, 2\} \times \langle a^{[3]}b^{[2]} \rangle \ \cup \ \{4\} \times \langle a^{[2]}b^{[2]} \rangle,$$
$$I_2 = \{1\} \times \langle a^{[3]}b^{[2]} \rangle, \text{ and } \ I_3 = \{1\} \times \left(\langle a^{[1]}b^{[2]} \rangle \cup \langle a^{[3]} \rangle \right).$$

It can see that $|I_1| = 30$, $|I_2| = 11$, *and* $|I_3| = 7$. *We have* I_1 *and* I_3 *are ideals with partial count such that* $I_1 = \langle (4, a^{[2]}b^{[2]}) \rangle$ *and* $I_3 = \langle \{(1, a^{[1]}b^{[2]}), (1, a^{[3]})\} \rangle$, *and* I_2 *is an ideal with full count such that* $I_2 = \langle (1, a^{[3]}b^{[2]}) \rangle$. *Clearly,* I_1 *and* I_2 *are normal, but* I_3 *is not normal. For the complements of* I_1 *and* I_2, *we have*

$$I_1^{\mathfrak{C}} = \{3, 5, 6\} \times \langle a^{[3]}b^{[2]} \rangle \ \cup \ \{(4, a^{[1]})\}, \text{ and } I_2^{\mathfrak{C}} = \{2, 3, 4, 5, 6\} \times \langle a^{[3]}b^{[2]} \rangle.$$

For $\alpha = a_1^{[h_1]} a_2^{[h_2]} \cdots a_t^{[h_t]} \in \mathscr{P}(A)$, where the mset $A = a_1^{[n_1]} a_2^{[n_2]} \cdots a_t^{[n_t]}$ and $0 \leq h_i \leq n_i$ for $1 \leq i \leq t$, we define

$$\lfloor \alpha \rceil := \begin{cases} |\alpha|, & \text{if } h_j = n_j = 1 \text{ for some } j \in \{1, \dots, t\}, \\ |\alpha| + 1, & \text{otherwise.} \end{cases}$$

The following result is directly obtained.

Proposition 1. *Given an mset $A = (A^*, \mathfrak{c})$ and $\alpha \in \mathscr{P}(A)$, we have that*

1. *If $\lfloor \alpha \rceil = |\alpha| + 1 > 2$, $\exists \alpha' \in \mathscr{P}(A)$ such that $\alpha' \subseteq_A \alpha$ and $\lfloor \alpha' \rceil = |\alpha'| + 1 = |\alpha|$.*
2. *If $\lfloor \alpha \rceil = |\alpha| > 1$, $\exists \alpha' \in \mathscr{P}(A)$ such that $\alpha' \subseteq_A \alpha$ and $\lfloor \alpha' \rceil = |\alpha'| = |\alpha| - 1$.*
3. *For $\alpha_1 \subseteq_A \alpha_2 \in \mathscr{P}(A)$, if $\lfloor \alpha_1 \rceil = |\alpha_1| + 1 = |\alpha_2| = \lfloor \alpha_2 \rceil$, then there is a unique $x \in \alpha_2^* \subseteq A^*$ such that $\mathfrak{c}(x) = 1$.*
4. *If $\mathfrak{c}(x) > 1$ for all $x \in A^*$, then there is no $\alpha \in \mathscr{P}(A)$ such that $\lfloor \alpha \rceil = 1$.*

Define a map $\zeta_A : \mathfrak{I}(P \times \mathscr{P}_A) \to \mathbb{N}$ by

$$\zeta_A(I) = |\omega_{I_f}| \cdot \lfloor A \rceil + \sum_{\substack{i \in \omega_{I_p} \text{ such that} \\ \mathcal{A}(I;i) = \langle \alpha_i \rangle \text{ for some } \alpha_i \in \mathscr{P}(A)}} \lfloor \alpha_i \rceil. \tag{1}$$

Observe that $\zeta_A(I) \le |I|$ for all $I \in \mathfrak{I}(P \times \mathscr{P}_A)$. Moreover, if $|A| = 1$, then $\zeta_A(I) = |I|$ for all $I \in \mathfrak{I}(P \times \mathscr{P}_A)$. For $I \in \mathfrak{I}(P \times \mathscr{P}_A)$, let

$$\Omega_{I_p} := \{\alpha \in \mathscr{P}(A) \ : \ \mathcal{A}(I;i) = \langle \alpha \rangle \text{ and } \lfloor \alpha \rceil = |\alpha| \text{ for } \ i \in \omega_{I_p}\} \text{ and}$$
$$\bar{\Omega}_{I_p} := \{\alpha \in \mathscr{P}(A) \ : \ \mathcal{A}(I;i) = \langle \alpha \rangle \text{ and } \lfloor \alpha \rceil = |\alpha| + 1 \text{ for } \ i \in \omega_{I_p}\}.$$

Then $\zeta_A(I) = |\omega_{I_f}| \cdot \lfloor A \rceil + [\Omega_{I_p}]$, where $[\Omega_{I_p}] = \sum_{\alpha \in \Omega_{I_p} \cup \bar{\Omega}_{I_p}} \lfloor \alpha \rceil$.

With Proposition 1 and by deleting a maximal element of a normal ideal I in $P \times \mathscr{P}_A$, it gives a way to construct a normal ideal $J \subseteq I$. The next result is directly obtained.

Proposition 2. *Consider an mset $A = (A^*, \mathfrak{c})$ and the poset $P \times \mathscr{P}_A = ([n] \times \mathscr{P}(A), \gamma)$.*

1. *For each $0 \le t \le n$, there exists an ideal J with full count such that $\zeta_A(J) = t \cdot \lfloor A \rceil$.*
2. *Let $I \in \mathfrak{I}(P \times \mathscr{P}_A)$ be such that $\omega_{I_p} \neq \emptyset$.*
   (2.1) If $\Omega_{I_p} \neq \emptyset$, then for each $0 \le t \le [\Omega_{I_p}]$ there exists $J \in \mathfrak{I}(P \times \mathscr{P}_A)$ such that $J \subseteq I$ and $\zeta_A(J) = |\omega_{I_f}| \cdot \lfloor A \rceil + t$.
   (2.2) If $\Omega_{I_p} = \emptyset$, then for each $0 \le t \le [\Omega_{I_p}]$ with $t \neq 1$, there exists $J \in \mathfrak{I}(P \times \mathscr{P}_A)$ such that $J \subseteq I$ and $\zeta_A(J) = |\omega_{I_f}| \cdot \lfloor A \rceil + t$.

3. Supports and Weights

Suppose that $m = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, where $p_1, \ldots, p_k$ are distinct prime numbers and $\beta_1, \ldots, \beta_k$ are positive integers. By considering the mset $A = p_1^{[\beta_1]} p_2^{[\beta_2]} \cdots p_k^{[\beta_k]}$, it can see that the lattice of subgroups of $\mathbb{Z}_m$ and the poset structure of $\mathcal{P}(A)$ under $\subseteq_A$ are the same.

Let $\mathrm{Sub}(\mathbb{Z}_m)$ be the set of all subgroups of $\mathbb{Z}_m$. The map $\psi : \mathrm{Sub}(\mathbb{Z}_m) \to \mathcal{P}(A)$ defined by $\psi(H) = \alpha_H = p_1^{[t_1]} p_2^{[t_2]} \cdots p_k^{[t_k]}$, where $|H| = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}$ for $0 \le t_i \le \beta_i$, is an order-isomorphism.

Let $P = ([n], \preceq_P)$ be a poset. Given $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_m^n$, we define the *support of* $\mathbf{x}$ *associated with the lattice of subgroups of* $\mathbb{Z}_m$ as

$$\mathrm{supp}_{LS}(\mathbf{x}) := \{(i, p_1^{[t_1]} p_2^{[t_2]} \cdots p_k^{[t_k]}) \ : \ x_i \neq 0 \ \text{ and } \ |\langle x_i \rangle| = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k}\}$$

a subset of $[n] \times \mathscr{P}(A)$. By considering $\langle \mathrm{supp}_{LS}(\mathbf{x}) \rangle \in \mathfrak{I}(P \times \mathscr{P}_A)$ as the smallest ideal in $P \times \mathscr{P}_A$ containing $\mathrm{supp}_{LS}(\mathbf{x})$, the *LS-poset weight* of $\mathbf{x} \in \mathbb{Z}_m^n$ is defined to be $w_{LS}(\mathbf{x}) := \zeta_A(\langle \mathrm{supp}_{LS}(\mathbf{x}) \rangle)$, and the *LS-poset distance* between $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_m^n$ is $d_{LS}(\mathbf{x}, \mathbf{y}) := w_{LS}(\mathbf{x} - \mathbf{y})$. Now we prove that the *LS*-poset distance is a metric on $\mathbb{Z}_m^n$.

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Theorem 1. *Let $P = ([n], \preceq_P)$ be a poset and $m = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ where p_i are distinct primes. Under the poset $P \times \mathscr{P}_A$ with the mset $A = p_1^{[\beta_1]} p_2^{[\beta_2]} \cdots p_k^{[\beta_k]}$, the LS-poset distance $d_{_{LS}}(\cdot,\cdot)$ is a metric on $\mathbb{Z}_m^n$.*

Proof. It is clear that $d_{_{LS}}(\mathbf{x},\mathbf{y}) \geq 0$. As a group $\mathbb{Z}_m$, we have $d_{_{LS}}(\mathbf{x},\mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y}$. Moreover, for any $x \in \mathbb{Z}_m$, $\langle x \rangle = \langle -x \rangle$, which implies that $d_{_{LS}}(\cdot,\cdot)$ is symmetric. To show that the triangle inequality of $d_{_{LS}}(\cdot,\cdot)$ holds, we let $a,b \in \mathbb{Z}_m$. By applying the fundamental theorem of finite cyclic groups, we assume that $\langle a \rangle = \langle p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k} \rangle$ and $\langle b \rangle = \langle p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \rangle$ for some nonnegative integers t_i, s_i, $(1 \leq i \leq k)$. Then $|\langle a \rangle| = p_1^{\beta_1 - t_1} p_2^{\beta_2 - t_2} \cdots p_k^{\beta_k - t_k}$ and $|\langle b \rangle| = p_1^{\beta_1 - s_1} p_2^{\beta_2 - s_2} \cdots p_k^{\beta_k - s_k}$. Suppose that $\langle a+b \rangle = \langle p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \rangle$ for some $r_i \geq 0$. It is clear that $\langle a+b \rangle \subseteq H = \langle p_1^{\min\{t_1,s_1\}} p_2^{\min\{t_2,s_2\}} \cdots p_k^{\min\{t_k,s_k\}} \rangle$. Then $\alpha_{_{\langle a+b \rangle}} = p_1^{[\beta_1 - r_1]} p_2^{[\beta_2 - r_2]} \cdots p_k^{[\beta_k - r_k]} \subseteq_A \alpha_{_H} = p_1^{[\beta_1 - \min\{t_1,s_1\}]} p_2^{[\beta_2 - \min\{t_2,s_2\}]} \cdots p_k^{[\beta_k - \min\{t_k,s_k\}]}$. Observe that $|\alpha_{_{\langle a+b \rangle}}| = \sum_{i=1}^{k} (\beta_i - r_i) \leq \sum_{i=1}^{k} [\beta_i - \min\{t_i, s_i\}] = |\alpha_{_H}| \leq \sum_{i=1}^{k} [(\beta_i - t_i) + (\beta_i - s_i)] = |\alpha_{_{\langle a \rangle}}| + |\alpha_{_{\langle b \rangle}}|$. If $|\alpha_{_{\langle a+b \rangle}}| < |\alpha_{_H}|$, it follows that $\lfloor \alpha_{_{\langle a+b \rangle}} \rceil \leq \lfloor \alpha_{_{\langle a \rangle}} \rceil + \lfloor \alpha_{_{\langle b \rangle}} \rceil$. Now, we suppose $|\alpha_{_{\langle a+b \rangle}}| = |\alpha_{_H}|$. From $\alpha_{_{\langle a+b \rangle}} \subseteq_A \alpha_{_H}$, this forces $\alpha_{_{\langle a+b \rangle}} = \alpha_{_H}$ which means $r_i = \min\{t_i, s_i\}$ for all i. It is clear for the case $\lfloor \alpha_{_{\langle a+b \rangle}} \rceil = |\alpha_{_{\langle a+b \rangle}}|$. Next, assume that $\lfloor \alpha_{_{\langle a+b \rangle}} \rceil = |\alpha_{_{\langle a+b \rangle}}| + 1$. It follows that if $\beta_j = 1$ for $j \in \{1, \ldots, k\}$, then $1 = r_j = \min\{t_j, s_j\}$ which implies $\lfloor \alpha_{_{\langle a \rangle}} \rceil = |\alpha_{_{\langle a \rangle}}| + 1$ and $\lfloor \alpha_{_{\langle b \rangle}} \rceil = |\alpha_{_{\langle b \rangle}}| + 1$. Consequently, $\lfloor \alpha_{_{\langle a+b \rangle}} \rceil \leq \lfloor \alpha_{_{\langle a \rangle}} \rceil + \lfloor \alpha_{_{\langle b \rangle}} \rceil$. This completes the proof. □

The metric $d_{_{LS}}(\cdot,\cdot)$ on $\mathbb{Z}_m^n$ is called as the *LS-poset metric*. Let C be a submodule of $\mathbb{Z}_m^n$ with the LS-poset metric $d_{_{LS}}$. Then C is called an *LS-poset code* of length n over $\mathbb{Z}_m$. The *minimum LS-poset distance* $d_{_{LS}}(C)$ is the smallest LS-poset distance between two distinct codewords of C. The dual of an LS-poset code C is defined as

$$C^{\perp} = \{\mathbf{v} \in \mathbb{Z}_m^n \mid \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = 0 \ \text{ for all } \mathbf{u} \in C\}.$$

To obtain more information on each element of $\mathbb{Z}_m \backslash \{0\}$ which is placed on the poset structure of $\mathscr{P}(A)$, we let

$$\mathcal{G}_t := \{x \in \mathbb{Z}_m \backslash \{0\} \mid \lfloor \alpha_{\langle x \rangle} \rceil = t\}.$$

Example 3. *Consider $\mathbb{Z}_{20}$ with the mset $A = 2^{[2]} 5^{[1]}$. We have the following table:*

$\mathscr{P}(A)$	$\lfloor \cdot \rceil$	$\mathbb{Z}_{20} \backslash \{0\}$
$5^{[1]}$	1	4, 8, 12, 16
$2^{[1]}$	2	10
$2^{[1]} 5^{[1]}$	2	2, 6, 14, 18
$2^{[2]}$	3	5, 15
$2^{[2]} 5^{[1]}$	3	1, 3, 7, 9, 11, 13, 17, 19

Recall some properties of the Euler ϕ-function as follows:

1. If p is a prime, then $\phi(p) = p - 1$ and $\phi(p^k) = p^k - p^{k-1}$ for all $k \in \mathbb{N}$.
2. For $x, y \in \mathbb{N}$, if $\gcd(x,y) = 1$, then $\phi(xy) = \phi(x)\phi(y)$.

Remark 2. *Suppose $m = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, where p_i are distinct primes. Consider the mset $A = p_1^{[\beta_1]} p_2^{[\beta_2]} \cdots p_k^{[\beta_k]}$. We have that*

- *If $\beta_i > 1$ for all i, then $\mathcal{G}_1 = \emptyset$ and $|\mathcal{G}_{|A|+1}| = \phi(m)$.*
- *If $\beta_i = 1$ for all i, then for $1 \leq t \leq k$,*

$$|\mathcal{G}_t| = \sum_{\substack{(s_1, \ldots, s_k) \text{ where} \\ t = \sum_i^k s_i,\ s_i \in \{0,1\}}} \phi(p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}).$$

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Example 4. *In the space $\mathbb{Z}^4_{180}$, we consider the poset $P_1 \times \mathscr{P}_A = ([4] \times \mathscr{P}(A), \gamma)$ where the mset $A = 2^{[2]}3^{[2]}5^{[1]}$ and the poset $P_1 = ([4], \preceq_{P_1})$ is as shown in Figure 1. Consider the vector $(0, 5, 12, 120) \in \mathbb{Z}^4_{180}$. We have*

- *(Poset weight)* $w_{P_1}((0, 5, 12, 120)) = |\langle\{2, 3, 4\}\rangle| = 4.$
- *(Pomset weight)* $w_{\mathbb{P}_1}((0, 5, 12, 120)) = |\langle\{5/2, 12/3, 60/4\}\rangle| = 77.$
- *(LS-poset weight)*

$$w_{LS}((0, 5, 12, 120)) = \zeta_A \langle\{(2, 2^{[2]}3^{[2]}), (3, 3^{[1]}5^{[1]}), (4, 3^{[1]})\}\rangle = 14.$$

Notice that for a space $\mathbb{Z}^n_p$ with prime p, the poset metric d_P and the LS-poset metric d_{LS} are the same, while the pomset metric $d_{\mathbb{P}}$ and the LS-poset metric d_{LS} are equivalent when $p = 2, 3$. The diagram in Figure 2 illustrates these facts.

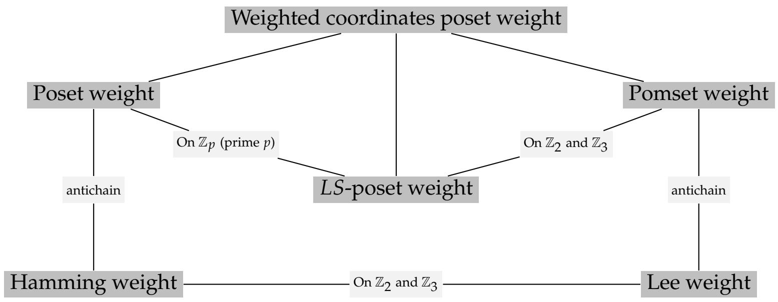

Figure 2. Weight relationship tree

4. r-Balls and I-Balls

Let $\mathbf{u}$ be a vector in the space $\mathbb{Z}^n_m$ with LS-poset metric d_{LS} and $r \in \mathbb{N}_0$. With center at $\mathbf{u}$ and radius r, the *r-ball* and the *r-sphere*, respectively, are as follows:

$$B_{r,LS}(\mathbf{u}) := \{\mathbf{v} \in \mathbb{Z}^n_m \mid d_{LS}(\mathbf{u}, \mathbf{v}) \le r\},$$
$$S_{r,LS}(\mathbf{u}) := \{\mathbf{v} \in \mathbb{Z}^n_m \mid d_{LS}(\mathbf{u}, \mathbf{v}) = r\}.$$

It is clear that $|B_{r,LS}(\mathbf{u})| = 1 + \sum_{i=1}^{r} |S_{i,LS}(\mathbf{u})|$.

Definition 1. *Let C be a code of $\mathbb{Z}^n_m$ with LS-poset metric d_{LS}. Then C is said to be a r-perfect LS-poset code if the r-balls centered at the codewords of C are pairwise disjoint and their union is $\mathbb{Z}^n_m$.*

Let $I^t(s) \in \mathfrak{I}(P \times \mathscr{P}_A)$ be such that $\zeta_A(I^t(s)) = t$ and $I^t(s)$ has exactly s maximal elements. We let $\omega_{I^t(s)} = \{i \in [n] \mid (i, \alpha)$ is a maximal element of $I^t(s)$ for some $\alpha \in \mathscr{P}(A)\}$. Given a vector $\mathbf{v} = (v_1, \ldots, v_n)$ of $\mathbb{Z}^n_m$, we rewrite it as

$$\mathbf{v} := (\mathbf{v_1} : \mathbf{v_2} : \mathbf{0}) = (\{v_i\}_{i \in \omega_{I^t(s)}} : \{v_i\}_{i \in I^t(s)^* \setminus \omega_{I^t(s)}} : \mathbf{0}),$$

where for each $i \in [n]$, v_i is an element in $\mathbb{Z}_m$ satisfying:

(i) If $i \in \omega_{I^t(s)}$, $v_i \in \mathcal{G}_{\lfloor \alpha_i \rceil}$,
where $\mathcal{A}(I^t(s); i) = \langle \alpha_i \rangle$ for some $\alpha_i \in \mathscr{P}(A)$;

(ii) If $i \in I^t(s)^* \setminus \omega_{I^t(s)}$, $v_i \in \mathbb{Z}_m$;

(iii) If $i \notin I^t(s)^*$, $v_i = 0$.

Observe that $\langle \mathrm{supp}_{LS}(\mathbf{v}) \rangle = I^t(s)$. Now, letting $\mathbb{A}_{I^t(s)}$ the collection of all vectors $\mathbf{v}$ in $\mathbb{Z}^n_m$ such that $\langle \mathrm{supp}_{LS}(\mathbf{v}) \rangle = I^t(s)$, we have

$$|\mathbb{A}_{I^t(s)}| = m^{|I^t(s)^* \setminus \omega_{I^t(s)}|} \cdot \prod_{i \in \omega_{I^t(s)}} |\mathcal{G}_{\lfloor \alpha_i \rceil}|.$$

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Obviously, for two distinct ideals I and J in $\mathfrak{I}(P \times \mathscr{P}_A)$, $\mathbb{A}_I \cap \mathbb{A}_J = \emptyset$. Now, we denote by $\mathcal{I}^t(s)$ the set of all ideals $I \in \mathfrak{I}(P \times \mathscr{P}_A)$ such that $\zeta_A(I) = t$, and I has exactly s maximal elements. Then the number of vectors in an r-ball with center $\mathbf{u}$ equals

$$|B_{r,LS}(\mathbf{u})| = 1 + \sum_{i=1}^{r} \sum_{j=1}^{i} \sum_{I \in \mathcal{I}^i(j) \neq \emptyset} |\mathbb{A}_I|. \tag{2}$$

Given an ideal $I \in \mathfrak{I}(P \times \mathscr{P}_A)$, the I-*ball* centered at $\mathbf{u}$ and the I-*sphere* centered at $\mathbf{u}$, respectively, are defined as

$$B_{I,LS}(\mathbf{u}) := \{\mathbf{v} \in \mathbb{Z}_m^n \mid \langle \mathrm{supp}_{LS}(\mathbf{u} - \mathbf{v}) \rangle \subseteq I\},$$
$$S_{I,LS}(\mathbf{u}) := \{\mathbf{v} \in \mathbb{Z}_m^n \mid \langle \mathrm{supp}_{LS}(\mathbf{u} - \mathbf{v}) \rangle = I\}.$$

Definition 2. *Let C be a code of $\mathbb{Z}_m^n$ with LS-poset metric d_{LS} and I be an ideal in $P \times \mathscr{P}_A$. Then C is called an I-perfect LS-poset code if the I-balls centered at the codewords of C are pairwise disjoint and their union is $\mathbb{Z}_m^n$.*

Under pomset metric $d_{\mathbb{P}}$ in $\mathbb{Z}_m^n$, it was shown in [9] that I-balls are no more linear subspaces of $\mathbb{Z}_m^n$ if I is an ideal with partial count in $\mathbb{P} = (\mathcal{M}^{\lfloor m/2 \rfloor}([n]), \preccurlyeq_P)$. On the other hand, with LS-poset metric, the I-ball centered at the zero vector is a submodule of $\mathbb{Z}_m^n$.

Proposition 3. *Let $I \in \mathfrak{I}(P \times \mathscr{P}_A)$. Then $B_{I,LS}(\mathbf{0})$ is a submodule of $\mathbb{Z}_m^n$.*

Proof. Clearly, if I is an ideal with full count in $P \times \mathscr{P}_A$, then $B_{I,LS}(\mathbf{0})$ is a submodule of $\mathbb{Z}_m^n$ with dimension $|\omega_{I_f}|$. Now, suppose that I is an ideal with partial count. Then $|\omega_{I_p}| > 0$. For each $i \in \omega_{I_p}$, let $\mathcal{A}(I;i) = \langle \alpha_i \rangle$ for some $\alpha_i \in \mathscr{P}(A)$. By considering H_i as a subgroup of $\mathbb{Z}_m$ such that $\psi(H_i) = \alpha_i$ for $i \in \omega_{I_p}$, for $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n) \in B_{I,LS}(\mathbf{0})$, we have $u_i, v_i \in H_i$ for all $i \in \omega_{I_p}$. It follows that $\mathbf{u} + \mathbf{v}, c\mathbf{u} \in B_{I,LS}(\mathbf{0})$ for $c \in \mathbb{Z}_m$. Hence, $B_{I,LS}(\mathbf{0})$ is a submodule of $\mathbb{Z}_m^n$. □

For $I \in \mathfrak{I}(P \times \mathscr{P}_A)$, let $B_{I^{\mathfrak{C}},\widetilde{LS}}(\mathbf{0})$ denote the $I^{\mathfrak{C}}$-ball centered at $\mathbf{0}$ under the poset $\widetilde{P} \times \mathscr{P}_A$.

Proposition 4. *Let $I \in \mathfrak{I}(P \times \mathscr{P}_A)$. Then the following statements hold:*

1. *For $\mathbf{u} \in \mathbb{Z}_m^n$, $B_{I,LS}(\mathbf{u}) = \mathbf{u} + B_{I,LS}(\mathbf{0})$.*
2. *For $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_m^n$, $B_{I,LS}(\mathbf{u})$ and $B_{I,LS}(\mathbf{v})$ are either identical or disjoint. Moreover,*
$$B_{I,LS}(\mathbf{u}) = B_{I,LS}(\mathbf{v}) \quad \Leftrightarrow \quad \mathrm{supp}_{LS}(\mathbf{u} - \mathbf{v}) \subseteq I.$$
3. *$B_{I,LS}^{\perp}(\mathbf{0}) = B_{I^{\mathfrak{C}},\widetilde{LS}}(\mathbf{0})$.*

Proof. (1) Let $\mathbf{v} \in B_{I,LS}(\mathbf{u})$. It follows that $\mathbf{u} - \mathbf{v} \in B_{I,LS}(\mathbf{0})$, and $\mathbf{v} = \mathbf{u} + (\mathbf{v} - \mathbf{u}) \in \mathbf{u} + B_{I,\mathbb{P}_C}(\mathbf{0})$. For $\mathbf{w} \in B_{I,\mathbb{P}_C}(\mathbf{0})$, we have $\mathrm{supp}_{LS}(\mathbf{u} - (\mathbf{u} + \mathbf{w})) = \mathrm{supp}_{LS}(-\mathbf{w}) = \mathrm{supp}_{LS}(\mathbf{w}) \subseteq I$. Hence $\mathbf{u} + \mathbf{w} \in B_{I,LS}(\mathbf{u})$.

(2) For each $i \in \omega_{I_f} \cup \omega_{I_p}$, we let H_i be a subgroup of $\mathbb{Z}_m$ such that $\psi(H_i) = \alpha_{H_i} \in \mathscr{P}(A)$, where $\mathcal{A}(I;i) = \langle \alpha_{H_i} \rangle$. For $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Z}_m^n$, suppose $\mathbf{w} = (w_1, \ldots, \omega_n) \in B_{I,LS}(\mathbf{u}) \cap B_{I,LS}(\mathbf{v})$. We have $\mathrm{supp}_{LS}(\mathbf{u} - \mathbf{w}) \subseteq I$ and $\mathrm{supp}_{LS}(\mathbf{v} - \mathbf{w}) \subseteq I$. If $i \notin \omega_{I_f} \cup \omega_{I_p}$, then $u_i - w_i = w_i - v_i = 0$, so $u_i - v_i = 0$. For the case $i \in \omega_{I_f} \cup \omega_{I_p}$, we have $u_i - w_i, w_i - v_i \in H_i$, which implies $u_i - v_i \in H_i$. That is, $(i, \alpha_{\langle u_i - v_i \rangle}) \in I$ for all $i \in \omega_{I_f} \cup \omega_{I_p}$. Consequently, $\mathbf{u} - \mathbf{v} \in B_{I,LS}(\mathbf{0})$, which means that $B_{I,LS}(\mathbf{u}) = B_{I,LS}(\mathbf{v})$.

(3) If I is an ideal with full count in $P \times \mathscr{P}_A$, then $I^{\mathfrak{C}}$ is also an ideal with full count in $\widetilde{P} \times \mathscr{P}_A$. Since $\omega_{I_f} \cap \omega_{I_f^{\mathfrak{C}}} = \emptyset$ and $\omega_{I_f} \cup \omega_{I_f^{\mathfrak{C}}} = [n]$, we derive the result.

Next, suppose that $m = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, where p_i are distinct primes, and the mset $A = p_1^{[\beta_1]} p_2^{[\beta_2]} \cdots p_k^{[\beta_k]}$. Let $I \in \mathfrak{I}(P \times \mathscr{P}_A)$ be an ideal with partial count. From $\omega_{I_p} = \omega_{I_p^{\mathfrak{C}}}$, for each $i \in \omega_{I_p}$, we let

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$\mathcal{A}(I;i) = \langle \alpha_i \rangle$ for some $\alpha_i \in \mathscr{P}(A)$, where $\alpha_i = p_1^{[t_{i_1}]} p_2^{[t_{i_2}]} \cdots p_k^{[t_{i_k}]}$, and let H_i and K_i be subgroups of $\mathbb{Z}_m$ such that $\psi(H_i) = \alpha_i$ and $\psi(K_i) = \widehat{\alpha}_i$. Then $H_i = \langle p_1^{\beta_1 - t_{i_1}} p_2^{\beta_2 - t_{i_2}} \cdots p_k^{\beta_k - t_{i_k}} \rangle$ and $K_i = \langle p_1^{t_{i_1}} p_2^{t_{i_2}} \cdots p_k^{t_{i_k}} \rangle$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in B_{I^{\mathfrak{C}}, \widetilde{LS}}(\mathbf{0})$ and $\mathbf{y} = (y_1, \ldots, y_n) \in B_{I,LS}(\mathbf{0})$. Then $\mathbf{x} \cdot \mathbf{y} = \sum\limits_{i \in \omega_{I_p}} x_i y_i$, where for each i, $x_i \in H_i$ and $y_i \in K_i$. It follows that $\mathbf{x} \cdot \mathbf{y}$ is congruent to 0 modulo m. That is, $B_{I^{\mathfrak{C}}, \widetilde{LS}}(\mathbf{0}) \subseteq B^{\perp}_{I,LS}(\mathbf{0})$. Now, we assume that there is $\mathbf{z} \in B^{\perp}_{I,LS}(\mathbf{0}) \backslash B_{I^{\mathfrak{C}}, \widetilde{LS}}(\mathbf{0})$. Since $\mathbf{z} \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in B_{I,LS}(\mathbf{0})$, we can, without loss of generality, write $\mathbf{z} = (0, \ldots, 0, z_s, 0, \ldots, 0)$, where $s \in \omega_{I_p}$ and $z_s \notin K_s = \langle p_1^{t_{s_1}} p_2^{t_{s_2}} \cdots p_k^{t_{s_k}} \rangle$. Then $z_s = y p_1^{r_{s_1}} p_2^{r_{s_2}} \cdots p_k^{r_{s_k}}$, where $\gcd(y, p_j) = 1$ and $0 \leq r_{s_j} < t_{s_j}$ for some $j \in [k]$. Choosing $\mathbf{w} = (w_1, \ldots, w_n) \in B_{I,LS}(\mathbf{0})$, defined by $w_i = 0$ for all $i \neq s$, and $w_s = p_1^{\beta_1 - t_{s_1}} p_2^{\beta_2 - t_{s_2}} \cdots p_k^{\beta_k - t_{s_k}}$, it follows that $\mathbf{z} \cdot \mathbf{w} \not\equiv 0$ modulo m, which is a contradiction. $\square$

Example 5. *Consider $120 = 2^3 \cdot 3 \cdot 5$ and the mset $A = 2^{[3]}3^{[1]}5^{[1]}$. On $\mathscr{P}(A)$, we choose $\alpha_1 = 2^{[1]}3^{[1]}5^{[1]}$, $\alpha_2 = 2^{[3]}5^{[1]}$, $\alpha_3 = 2^{[2]}3^{[1]}$. The structure of each $\langle \alpha_i \rangle$ when $i = 1, 2, 3$, is demonstrated via the lattice of nontrivial subgroup for $\mathbb{Z}_{120}$ (see in Figure 3) in which $\widehat{\alpha}_1 = 2^{[2]}$, $\widehat{\alpha}_2 = 3^{[1]}$, and $\widehat{\alpha}_3 = 2^{[1]}5^{[1]}$. Now let us consider the poset $P_1 \times \mathscr{P}_A = ([4] \times \mathscr{P}(A), \gamma)$, where the poset $P_1 = ([4], \preceq_{P_1})$ is as shown in Figure 1. Let $I = \langle \{(2, \alpha_1), (3, \alpha_2), (4, \alpha_3)\} \rangle \in \mathfrak{I}(P_1 \times \mathscr{P}_A)$. Then I is an ideal with partial count. It is easy to see that $B_{I,LS}(\mathbf{0}) = \mathbb{Z}_{120} \times \langle 4 \rangle \times \langle 3 \rangle \times \langle 10 \rangle$, whereas $B^{\perp}_{I,LS}(\mathbf{0}) = \langle 0 \rangle \times \langle 30 \rangle \times \langle 40 \rangle \times \langle 12 \rangle$.*

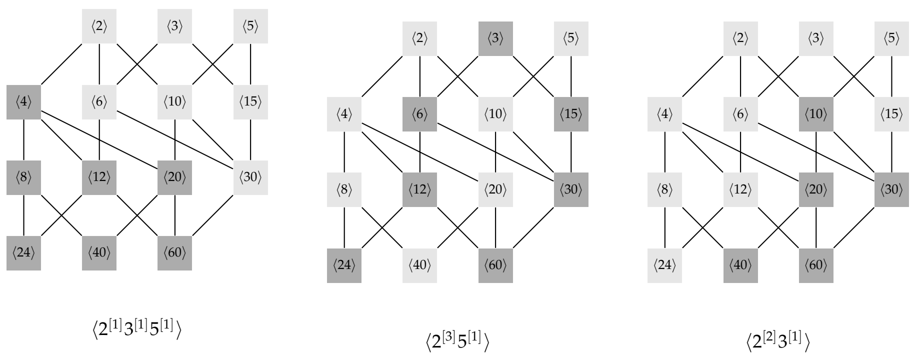

Figure 3. The lattice of nontrivial subgroups for $\mathbb{Z}_{120}$

Observe that the I-ball centered at the zero vector can be considered as a direct product of cyclic subgroups of $\mathbb{Z}_m$. If m is a prime power, the following result is directly obtained.

Proposition 5. *In the space $\mathbb{Z}^n_{q^\beta}$, let $I \in \mathfrak{I}(P \times \mathscr{P}_A)$.Then*

1. *If $\beta = 1$, then $|B_{I,LS}(\mathbf{0})| = q^{|I|}$ and $|B^{\perp}_{I,LS}(\mathbf{0})| = q^{n-|I|}$.*
2. *If $\beta > 1$, then*
   $|B_{I,LS}(\mathbf{0})| = q^{\beta|\omega_{I_f}| + \sum_{i \in \omega_{I_p}} |\alpha_i|}$ *and* $|B^{\perp}_{I,LS}(\mathbf{0})| = q^{\beta(n - |\omega_{I_f} \cup \omega_{I_p}|) + \sum_{i \in \omega_{I_p}} |\widehat{\alpha}_i|}$,
   where for each $i \in \omega_{I_p}$, $\mathcal{A}(I;i) = \langle \alpha_i \rangle$ for some $\alpha_i \in \mathscr{P}(A)$.

From Proposition 3 and 4, the following theorem shows the existence of an I-perfect code with LS-poset metric when I is an ideal with full count.

Theorem 2. *For any ideal I with full count in $P \times \mathscr{P}_A$, we have*

1. *$B_{I^{\mathfrak{C}}, \widetilde{LS}}(\mathbf{0})$ is an I-perfect LS-poset code for the poset $P \times \mathscr{P}_A$.*
2. *$B_{I,LS}(\mathbf{0})$ is an $I^{\mathfrak{C}}$-perfect LS-poset code for the poset $\widetilde{P} \times \mathscr{P}_A$.*

In the case of ideals I with partial count, the I-ball centered at the zero vector is not always I-perfect. The next lemma is a key for the existence of I-perfect code with LS-poset metric.

For each $i \in [n]$, let $\mathbf{e}_i = (e_1, \ldots, e_n) \in \mathbb{Z}_m^n$ be such that $e_j = 0$ if $j \neq i$, and $e_i = 1$.

Lemma 1. *Let $I \in \mathfrak{I}(P \times \mathscr{P}_A)$ with $\omega_{I_p} \neq \emptyset$. For each $i \in \omega_{I_p}$, let H_i be a nontrivial subgroup of $\mathbb{Z}_m$ such that $\mathcal{A}(I;i) = \langle \alpha_{H_i} \rangle$. Then the following statements hold:*

1. *If there is $i \in \omega_{I_p}$ such that $\alpha^*_{H_i} = A^*$, then there is no I-perfect LS-poset code.*
2. *Suppose C is an I-perfect LS-poset code of $\mathbb{Z}_m^n$. Then for each $i \in \omega_{I_p}$, there is a maximal subgroup K_i of $\mathbb{Z}_m$ such that $A^* = \alpha^*_{K_i} \dot{\cup} \alpha^*_{H_i}$. Moreover, $\mathbf{e}_i \cdot C = K_i$ and $K_i + H_i = \mathbb{Z}_m$.*

Proof. (1) Suppose that $\alpha^*_{H_i} = A^*$. Choose $\mathbf{v} = (0, \ldots, 0, v_i, 0, \ldots, 0) \in \mathbb{Z}_m^n$, where $v_i \notin H_i$. Then $\alpha_{\langle v_i \rangle} \notin \langle \alpha_{H_i} \rangle$. It follows that $\mathbf{v} \notin B_{I,LS}(\mathbf{0})$. Suppose there is an I-perfect LS-poset code C of $\mathbb{Z}_m^n$. Then $\mathbf{v} \in B_{I,LS}(\mathbf{c})$ for some $\mathbf{c} = (c_1, \ldots, c_n) \in C$. That is, $\text{supp}_{LS}(\mathbf{c} - \mathbf{v}) \subseteq I$. This implies that $\alpha_{\langle c_i - v_i \rangle} \in \langle \alpha_{H_i} \rangle$ which means $0 \neq c_i \in v_i + H_i$. From $\alpha^*_{H_i} = A^*$, there is $N_0 \in \mathbb{N}$ such that $N_0 c_i \neq 0$ and $\alpha_{\langle N_0 c_i \rangle} \in \langle \alpha_{H_i} \rangle$. Consequently, $\text{supp}_{LS}(N_0 \mathbf{c}) \subseteq I$. As a submodule $B_{I,LS}(\mathbf{0})$ of $\mathbb{Z}_m^n$, we have $\mathbf{0} \neq N_0 \mathbf{c} \in B_{I,LS}(\mathbf{0})$, which is a contradiction to the I-perfect of C.

(2) Suppose C is an I-perfect LS-poset code of $\mathbb{Z}_m^n$. Let $i \in \omega_{I_p}$. From (1), $\alpha^*_{H_i} \neq A^*$. Then there is a maximal subgroup K_i of $\mathbb{Z}_m$ such that $\alpha^*_{K_i} \cap \alpha^*_{H_i} = \emptyset$. Let $0 \neq x \in K_i$. Consider $\mathbf{v} = (0, \ldots, 0, v_i = x, 0, \ldots, 0) \in \mathbb{Z}_m^n$. Then $\mathbf{v} \notin B_{I,LS}(\mathbf{0})$. We choose $\mathbf{c} = (c_1, \ldots, c_n) \in C$ such that $\mathbf{v} \in B_{I,LS}(\mathbf{c})$. Since $B_{I,LS}(\mathbf{0}) \cap B_{I,LS}(\mathbf{c}) = \emptyset$, it follows that $c_i \neq 0$ and $\alpha_{\langle c_i \rangle} \notin \langle \alpha_{H_i} \rangle$. Indeed, by proceeding as before, we have $\alpha^*_{\langle c_i \rangle} \cap \alpha^*_{H_i} = \emptyset$. That is, $\langle c_i \rangle \cap H_i = \{0\}$. Since $\text{supp}_{LS}(\mathbf{c} - \mathbf{v}) \subseteq I$, we have $\alpha_{\langle c_i - x \rangle} \in \langle \alpha_{H_i} \rangle$ which means $\langle c_i - x \rangle \subseteq H_i$. Then $c_i - x = y$ for some $y \in H_i$. From $\alpha^*_{K_i} \cap \alpha^*_{H_i} = \emptyset$, we have $K_i \cap H_i = \{0\}$. Then there is $N \in \mathbb{N}$ such that $N \nmid |H_i|$ and $Nx \equiv 0$ modulo m. Thus, $Nc_i \equiv Ny$ modulo m. These force $y = 0$. Hence, $c_i = x$. That is, $\mathbf{e}_i \cdot C = K_i$. Since C is I-perfect, by a similar technique, it can be shown that $K_i + H_i = \mathbb{Z}_m$. $\square$

Given an mset A, let $\mathbb{E}_A := \{\alpha \in \mathscr{P}(A) \mid \alpha^* \cap \widehat{\alpha}^* = \emptyset\}$.

Theorem 3. *Let $I \in \mathfrak{I}(P \times \mathscr{P}_A)$ with $\omega_{I_p} \neq \emptyset$. Then $B^{\perp}_{I,LS}(\mathbf{0})$ is an I-perfect LS-poset code of $\mathbb{Z}_m^n$ if and only if for each $i \in \omega_{I_p}$, $\mathcal{A}(I;i) = \langle \alpha_i \rangle$ for some $\alpha_i \in \mathbb{E}_A$.*

Proof. Suppose $B^{\perp}_{I,LS}(\mathbf{0})$ is I-perfect. By Proposition 4(3), $B^{\perp}_{I,LS}(\mathbf{0}) = B_{I^{\mathfrak{c}},\widehat{LS}}(\mathbf{0})$. For the necessary condition, let $i \in \omega_{I_p}$. We have $\mathcal{A}(I;i) = \langle \alpha_{H_i} \rangle$ and $\mathcal{A}(I^{\mathfrak{c}};i) = \langle \widehat{\alpha}_{H_i} \rangle = \langle \alpha_{K_i} \rangle$, where two subgroups H_i, K_i are as in Lemma 1. Then $\alpha_{H_i} \in \mathbb{E}_A$.

For each $\alpha \in \mathbb{E}_A$, we have that $\alpha^* \cap \widehat{\alpha}^* = \emptyset$ and $\alpha \oplus \widehat{\alpha} = A$. These imply that $\psi^{-1}(\alpha) \cap \psi^{-1}(\widehat{\alpha}) = \{0\}$ and $|\psi^{-1}(\alpha)| \cdot |\psi^{-1}(\widehat{\alpha})| = m$. Hence, the converse is proved. $\square$

Corollary 1. *There is no I-perfect LS-poset code of $\mathbb{Z}^n_{q^\beta}$ if I is an ideal with partial count in $P \times \mathscr{P}_A$.*

Example 6. *In Example 5, we have $\alpha_2 = 2^{[3]}5^{[1]} \in \mathbb{E}_A$ with $\widehat{\alpha}_2 = 3^{[1]}$, but $\alpha_1, \alpha_3 \notin \mathbb{E}_A$. Consider the poset $P_2 = ([6], \preceq_{P_2})$ as in Figure 1. In the space $\mathbb{Z}^6_{120}$ with LS-poset metric, we let $I = \langle (4, 2^{[3]}5^{[1]}) \rangle \in \mathfrak{I}(P_2 \times \mathscr{P}_A)$ as an ideal with partial count. By Theorem 3, we have $B^{\perp}_{I,LS}(\mathbf{0}) = \{(0,0,0,x,y,z) \mid x \in \langle 40 \rangle = \psi^{-1}(3^{[1]})$, and $y, z \in \mathbb{Z}_{120}\}$ is an I-perfect LS-poset code of $\mathbb{Z}^6_{120}$.*

5. MDS LS-Poset Codes and Codes in Chain Poset Structure

Theorem 4. (Singleton Bound) *Let $P \times \mathscr{P}_A$ be the poset on $[n] \times \mathscr{P}(A)$ and $C \subseteq \mathbb{Z}_m^n$ be an LS-poset code. Then*

$$\log_m |C| \leq n - \left\lfloor \frac{d_{LS}(C) - 1}{\lfloor A \rceil} \right\rfloor \tag{3}$$

Proof. Choose $\mathbf{x}, \mathbf{y} \in C$ such that $d_{LS}(\mathbf{x}, \mathbf{y}) = d_{LS}(C)$. Consider the ideal I generated by $\mathrm{supp}_{LS}(\mathbf{x} - \mathbf{y})$. We have $d_{LS}(C) - 1 < \zeta_A(I) \leq \lfloor A \rceil \cdot |\omega_{I_f} \cup \omega_{I_p}|$. From Proposition 2, there is a normal ideal J with full count of $P \times \mathscr{P}_A$ such that $|\omega_{J_f}| = \left\lfloor \frac{d_{LS}(C) - 1}{\lfloor A \rceil} \right\rfloor$. Then $\zeta_A(J) = \left\lfloor \frac{d_{LS}(C) - 1}{\lfloor A \rceil} \right\rfloor \lfloor A \rceil \leq d_{LS}(C) - 1$. That is, there is no codeword $\mathbf{c}$ in C such that $\mathrm{supp}_{LS}(\mathbf{c}) \subseteq J$, and any two distinct codewords of C will not coincide in all position $j \in [n] \backslash \omega_{J_f}$. These imply that $|C| \leq m^{n - |\omega_{J_f}|}$. So we have $\log_m |C| \leq n - \left\lfloor \frac{d_{LS}(C) - 1}{\lfloor A \rceil} \right\rfloor$. □

Definition 3. *An LS-poset code C of length n over $\mathbb{Z}_m$ is said to be a* maximum distance separable *LS*-poset code *(or simply MDS LS-poset code) if it attains the Singleton bound as in* (3).

Theorem 5. *In the space $\mathbb{Z}_m^n$ with the poset $P \times \mathscr{P}_A$, let $C \subseteq \mathbb{Z}_m^n$ be an LS-poset code such that $|C| = m^t$. Then C is an MDS LS-poset code if and only if C is an I-perfect LS-poset code for all ideals $I \in \mathcal{I}(P \times \mathscr{P}_A)$ with full count such that $|\omega_{I_f}| = n - t$.*

Proof. Let I be an ideal with full count in $P \times \mathscr{P}_A$ such that $|\omega_{I_f}| = n - t$. It is clear that $|B_{I,LS}(\mathbf{0})| = m^{n-t}$. Suppose that C is an MDS LS-poset code. From (3), it follows that $\left\lfloor \frac{d_{LS}(C) - 1}{\lfloor A \rceil} \right\rfloor = n - t$. Then $d_{LS}(C) > \zeta_A(I)$, that is, $B_{I,LS}(\mathbf{x}) \cap B_{I,LS}(\mathbf{y}) = \emptyset$ for two distinct elements $\mathbf{x}, \mathbf{y} \in C$. By $|C| = m^t$, we have C is I-perfect.

To show that C is MDS, we choose $\mathbf{c} \in C \backslash \{\mathbf{0}\}$ such that $w_{LS}(\mathbf{c}) = d_{LS}(C)$. Let $J = \langle \mathrm{supp}_{LS}(\mathbf{c}) \rangle$. Suppose $\zeta_A(J) \leq \lfloor A \rceil (n - t)$. By Proposition 2, we can construct an ideal I with full count in $P \times \mathscr{P}_A$ containing J such that $\zeta_A(I) = \lfloor A \rceil (n - t)$. But this would imply that $\mathbf{c} \in B_{I,LS}(\mathbf{0})$, which is impossible since C is I-perfect. This forces that $d_{LS}(C) = \zeta_A(J) > \lfloor A \rceil (n - t)$. Then $\left\lfloor \frac{d_{LS}(C) - 1}{\lfloor A \rceil} \right\rfloor \geq n - t$. By Theorem 4, we have C is MDS. □

Example 7. *Let $A = 2^{[3]} 3^{[1]} 5^{[1]}$ with $\lfloor A \rceil = 5$. Consider $C_1 = \{(0, 0, x, y, z, w) \mid x, y, z, w \in \mathbb{Z}_{120}\}$, and $C_2 = \{(0, 0, 0, x, y, z) \mid x, y, z \in \mathbb{Z}_{120}\}$.*

- *Under the poset $P_2 \times \mathscr{P}_A$, where $P_2 = ([6], \preceq_{P_2})$ as in Figure 1, it is clear that $d_{LS}(C_1) = 11$ and $d_{LS}(C_2) = 16$. Then C_1 and C_2 are MDS. Moreover, the poset $P_2 \times \mathscr{P}_A$ has exactly one ideal $I = \langle \{(1, A), (2, A)\} \rangle$ such that $\zeta_A(I) = 10$, and has exactly one ideal $J = \langle (3, A) \rangle$ such that $\zeta_A(J) = 15$. Clearly, C_1 and C_2 are I-perfect and J-perfect, respectively.*
- *Under the poset $P_3 \times \mathscr{P}_A$, where $P_3 = ([6], \preceq_{P_3})$ as in Figure 1, it is clear that $d_{LS}(C_1) = 1$ and $d_{LS}(C_2) = 6$. Then C_1 and C_2 are not MDS. In addition, there are two ideals $I_1 = \langle (6, A) \rangle$ and $I_2 = \langle (4, A) \rangle$ such that $\zeta_A(I_1) = 10$ and $\zeta_A(I_2) = 15$ in which C_1 is not I_1-perfect, and C_2 is not I_2-perfect.*

Observe that with the full count property of ideals in $P \times \mathscr{P}_A$, it was a main tool to study the MDS LS-poset code $C \subseteq \mathbb{Z}_m^n$ where $|C| = m^t, 0 \leq t \leq n$.

Next, we denote by $\mathcal{C}_n$ the chain poset $P = ([n], \preceq_P)$ with $\min P = 1$. Observe that every ideal I with full count in $\mathcal{C}_n \times \mathscr{P}_A$ has a unique maximal element. Suppose $r = t \lfloor A \rceil \leq n \lfloor A \rceil$ for some $t \in \mathbb{N}$. There is only one ideal I with full count such that $\zeta_A(I) = t \lfloor A \rceil$. It follows that $B_{r,LS}(\mathbf{u}) = B_{I,LS}(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{Z}_m^n$.

The following results are some immediate consequences.

Proposition 6. *In the space $\mathbb{Z}_m^n$ with the poset $\mathcal{C}_n \times \mathscr{P}_A$, given an ideal I with full count in $\mathcal{C}_n \times \mathscr{P}_A$, let C be an I-perfect LS-poset code of $\mathbb{Z}_m^n$. Then*

1. *C is an $\zeta_A(I)$-perfect LS-poset code.*

2. *C is an MDS LS-poset code.*

Recall that the cardinality of an r-ball with center $\mathbf{u} \in \mathbb{Z}_m^n$ as in (2). By considering m as a prime power, we obtain the following result.

Proposition 7. *In the space $\mathbb{Z}_{q^\beta}^n$ with the poset $\mathcal{C}_n \times \mathscr{P}_A$, if $\beta > 1$, the cardinality of $B_{r,LS}(\mathbf{0})$ is q^{r-t}, where $(t-1)(\beta+1) < r \leq t(\beta+1)$ for $0 < t \leq n$.*

Proof. As $A = q^{[\beta]}$, we have the poset $(\mathscr{P}(A), \subseteq_A)$ becomes a chain. This implies that every ideal I in $\mathcal{C}_n \times \mathscr{P}_A$ contains a unique maximal element. Since $\beta > 1$, $\lfloor A \rceil = \beta + 1$. We write $r = (t-1)(\beta+1) + s + 1$, where $0 \leq s \leq \beta$. From (2), we have

$$\begin{aligned}
|B_{r,LS}(\mathbf{0})| &= 1 + \sum_{i=1}^{r} |\mathbb{A}_{I^i(1)}| \\
&= 1 + \sum_{j=0}^{t-2} q^{j\beta}[\phi(q) + \phi(q^2) + \cdots + \phi(q^\beta)] + q^{(t-1)\beta}[\phi(q) + \phi(q^2) + \cdots + \phi(q^s)] \\
&= 1 + \sum_{j=0}^{t-2} q^{j\beta}[q^\beta - 1] + q^{(t-1)\beta}[q^s - 1] \\
&= 1 + [q^\beta - 1]\left(\frac{q^{(t-1)\beta} - 1}{q^\beta - 1}\right) + q^{(t-1)\beta}[q^s - 1] = q^{r-t}.
\end{aligned}$$

□

From above Proposition, although $|B_{r,LS}(\mathbf{0})|$ divides $q^{\beta n}$, by Corollary 1, there is no r-perfect LS-poset code of $\mathbb{Z}_{q^\beta}^n$ if $\beta + 1$ does not divide r.

Next, we have thus established the sufficient condition of ideals with partial count in $\mathcal{C}_n \times \mathscr{P}_A$ for an LS-poset code of $\mathbb{Z}_m^n$ to be an r-perfect.

Theorem 6. *For $m = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ with $k > 1$, in the space $\mathbb{Z}_m^n$ with the poset $\mathcal{C}_n \times \mathscr{P}_A$, if there is a unique p_i such that $\beta_i = 1$, then there is an $(t\lfloor A \rceil + 1)$-perfect LS-poset code of $\mathbb{Z}_m^n$ for $0 \leq t < n$.*

Proof. By the assumption, we have $\lfloor A \rceil = |A|$, and $p_i^{[1]}$ is the unique element of $\mathscr{P}(A)$ such that $\lfloor p_i^{[1]} \rceil = 1$. Let $0 \leq t < n$. By applying Proposition 2 (2.1), there is a unique ideal I with partial count in $\mathcal{C}_n \times \mathscr{P}_A$ such that $\zeta_A(I) = t\lfloor A \rceil + 1$. Since $p_i^{[1]} \in \mathbb{E}_A$, by Theorem 3, it follows that $B_{I,LS}^{\perp}(\mathbf{0})$ becomes an $(t\lfloor A \rceil + 1)$-perfect LS-poset code. □

Example 8. *In the space $\mathbb{Z}_{12}^3$ with the poset $\mathcal{C}_3 \times \mathscr{P}_A$ and the mset $A = 2^{[2]}3^{[1]}$, we consider $C = \{000, 003, 006, 009\}$. Observe that $d_{LS}(C) = w_{LS}(006) = 8$. Then we have $B_{7,LS}(000) = \mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \langle 4 \rangle$, $B_{7,LS}(003) = \mathbb{Z}_{12} \times \mathbb{Z}_{12} \times (3 + \langle 4 \rangle)$, $B_{7,LS}(006) = \mathbb{Z}_{12} \times \mathbb{Z}_{12} \times (6 + \langle 4 \rangle)$, and $B_{7,LS}(009) = \mathbb{Z}_{12} \times \mathbb{Z}_{12} \times (9 + \langle 4 \rangle)$. It can see that C is an 7-perfect LS-poset code. Moreover, C is also an I-perfect LS-poset code when the ideal $I = \langle (3, 3^{[1]}) \rangle$ with $\zeta_A(I) = 7$.*

In the space $\mathbb{Z}_{60}^3$ with the poset $\mathcal{C}_3 \times \mathscr{P}_A$ and the mset $A = 2^{[2]}3^{[1]}5^{[1]}$, let $D = \{(0,0,0), (0,0,15), (0,0,30), (0,0,45)\}$. By Theorem 3, it can see that D is an J-perfect LS-poset code, where the ideal $J = \langle (3, 3^{[1]}5^{[1]}) \rangle$ such that $\zeta_A(J) = 10$. Observe that $d_{LS}(D) = w_{LS}(0,0,30) = 10$. However, the space $\mathbb{Z}_{60}^3$ is not covered by the union of r-balls centered at the codewords of D for any $r < 10$.

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