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Article

Electromagnetic Dynamics: Equilibrium Solutions for the Electric Field and Charge Density of a Continuously Distributed Charge

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Abstract: An electrostatic problem for stationary equilibrium of a continuously distributed charge is formulated in the form of finding both the electric field as well as the charge distribution. This is distinct from typical corresponding problems dealing with finding either the electric field or the charge distribution given one of them as input data. Maxwell and mass continuity equations representing the conservation of charge and mass, as well as the evolution of the electromagnetic fields are kinematic equations to be complemented by a momentum equation governing the dynamics of motion of the distributed charge. It is shown that while two types of possible stationary equilibria, one trivial (i.e. zero value of charge density and electric field) and the other one non-trivial (i.e. non-zero values of charge density and electric field) are possible, only the non-trivial one can materialize in reality. Oscillations may and do occur around the non-trivial equilibrium, but yield non-realistic results if they occur around the trivial equilibrium. The latter is the major reason for rejecting the trivial equilibrium and adopting the non-trivial one.

Keywords: electric field; charge density; lorenz force; charge conjugation; stationary equilibrium

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1. Introduction

A typical electrostatic problem for stationary equilibrium of a continuously distributed charge is formulated in the form of (Jackson 1999, Reitz *et al.* 1993, Jefimenko 1989, Landau *et al.* 1984) finding the electric field $\boldsymbol{E}(\boldsymbol{x})$ given a known charge density distribution $r_q(\boldsymbol{x})$, or the inverse problem of finding $r_q(\boldsymbol{x})$ given $\boldsymbol{E}(\boldsymbol{x})$, where $\boldsymbol{x} = x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z$ is the position vector in Cartesian coordinates, and $\hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y, \text{and}\, \hat{\boldsymbol{e}}_z$ are the unit vectors in the x, y, and z directions, respectively. However, this is an artificial problem as it assumes that either the charge density distribution or the electric field are imposed upfront. In reality, the charge density distribution is a result just the same way as the electric field is. Both $\boldsymbol{E}(\boldsymbol{x})$ as well as $r_q(\boldsymbol{x})$ should be part of the solution. Maxwell and mass continuity equations representing the conservation of charge and mass, as well as the evolution of the electromagnetic fields are kinematic equations to be complemented by a momentum equation governing the dynamics of motion of the distributed charge. This problem is formulated and solved in the current paper identifying two stationary equilibrium solutions, one trivial (i.e. zero value of charge density and electric field, $r_q = \boldsymbol{E} = 0$) and the second one non-trivial (i.e. non-zero values of charge density and electric field, $r_q(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x})^1\ 0$). It is shown that oscillations may and do occur around the non-trivial equilibrium, but yield non-realistic results if they are set around the trivial equilibrium, consequently the latter is rejected as a non-realistic solution. The former becomes the only possible stationary equilibrium should this solution be unique (excluding the trivial one).

2. Problem Formulation

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An electric charge q having a mass m_q is continuously distributed within a volume $\tilde{V}_o$, leading to a charge density distribution $r_q(t,\boldsymbol{x})$, a mass density distribution of $r(t,\boldsymbol{x})$, and an electric field $\boldsymbol{E}(t,\boldsymbol{x})$. The aim is in evaluating the electric field, $\boldsymbol{E}(\boldsymbol{x})$, and the charge density, $r_q(\boldsymbol{x})$, at equilibrium, (i.e. when the current density $\boldsymbol{J_q} = \rho_q \boldsymbol{v} = 0$, or the charge velocity $\boldsymbol{v} = 0$), if such an equilibrium exists. Generally, Maxwell equations produce electromagnetic waves that can be presented in the following general form

$$\boldsymbol{E}(t,\boldsymbol{x}) = \boldsymbol{E_o}(\boldsymbol{x}) + \boldsymbol{E_1}(t,\boldsymbol{x}) \quad (1)$$

$$r_q(t,\boldsymbol{x}) = r_{q,o}(\boldsymbol{x}) + r_{q,1}(t,\boldsymbol{x}) \quad (2)$$

where $\boldsymbol{E_o}(\boldsymbol{x})$ and $r_{q,o}(\boldsymbol{x})$ are stationary equilibrium solution components and $\boldsymbol{E_1}(t,\boldsymbol{x})$, $r_{q,1}(t,\boldsymbol{x})$ are oscillatory components that can be periodic, quasi-periodic, or even chaotic in the most general case. The stationary equilibrium values of $\boldsymbol{E_o}(\boldsymbol{x})$ and $r_{q,o}(\boldsymbol{x})$ can be zero. However, zero values of $\boldsymbol{E_o}(\boldsymbol{x})$ and $r_{q,o}(\boldsymbol{x})$ imply that the oscillations occur around $\boldsymbol{E_o}(\boldsymbol{x}) = 0$ and $r_{q,o}(\boldsymbol{x}) = 0$, a result that causes both the electric field $\boldsymbol{E}(t,\boldsymbol{x})$ as well as the charge density $r_q(t,\boldsymbol{x})$ to change signs as part of the solution. This means that if a charge were an electron, i.e. $q = e^- < 0$, it will change to positron, i.e. $q = e^+ > 0$ and back to electron in an oscillatory fashion, i.e. charge conjugation as an inherent part of the oscillation. Such a behavior was not observed and it is highly unlikely to occur in reality. The electron can annihilate when colliding with a positron producing (at low energy) two (or more) gamma particles, but does not change into a positron and back in a oscillatory fashion. Therefore the existence of a trivial stationary equilibrium is to be rejected and at least one non-trivial stationary equilibrium $\boldsymbol{E_o}(\boldsymbol{x})$ ¹ 0 and $r_{q,o}(\boldsymbol{x})$ ¹ 0 is anticipated. Also, the amplitude of the oscillations of $\boldsymbol{E_1}(t,\boldsymbol{x})$ and $r_{q,1}(t,\boldsymbol{x})$ need to be smaller than $|\boldsymbol{E_o}(\boldsymbol{x})|$ and $|r_q(\boldsymbol{x})|$ for all $\boldsymbol{x} \in \tilde{V}_o$.

The governing equations associated with the problem are Maxwell equations in free space as well as the inviscid Navier-Stokes equations presented in the form

Coulomb Law in field form

$$\nabla \circ \boldsymbol{E} = \frac{1}{e_o} r_q \quad (3)$$

Ampere Law

$$c_o^2 \nabla \times \boldsymbol{B} = \frac{1}{\varepsilon_o} \rho_q \boldsymbol{v} + \frac{\partial \boldsymbol{E}}{\partial t} \quad (4)$$

Faraday Law of Induction

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad (5)$$

Gauss Law for the Magnetic Field

$$\nabla \circ \boldsymbol{B} = 0 \quad (6)$$

Charge Continuity equation (charge conservation)

$$\frac{\partial \rho_q}{\partial t} + \nabla \bullet (\rho_q \boldsymbol{v}) = 0 \quad (7)$$

Mass Continuity Equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \boldsymbol{v}) = 0 \quad (8)$$

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Momentum Equation (linear momentum conservation) or Euler equation (inviscid)

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\bullet\nabla)\boldsymbol{v}\right]=-\nabla p-\rho_q\boldsymbol{E}-\rho_q(\boldsymbol{v}\times\boldsymbol{B})+\rho\boldsymbol{g} \tag{9}$$

where e_o is the permittivity of vacuum, c_o is the speed of light in free space, $\boldsymbol{B}$ is the induced magnetic field, p is pressure, and $\boldsymbol{g}$ the gravitational field.

The charge continuity equation (7) is equivalent to combining Coulomb and Ampere laws as applying the divergence operator on equation (4) and the time derivative operator on equation (3) and adding them produces equation (7). Also the mass continuity equation (8) can be presented in the form similar to equations (3) and (4) as presented by Vadasz (2024a)

$$\nabla \circ \boldsymbol{g} = \frac{1}{4pG} r \tag{10}$$

$$v_o^2\nabla\times\boldsymbol{\xi}=-4\pi G\rho\boldsymbol{v}+\frac{\partial \boldsymbol{g}}{\partial t} \tag{11}$$

where $v_o=\sqrt{\partial p/\partial\rho}$ is the constant speed of propagation of the pressure wave (speed of sound as special case but allowed to vary up to the value of the speed of light in vacuum), G is Newton's universal gravitational constant, and $\boldsymbol{\xi}=-\nabla\times\boldsymbol{v}$ is the counter-vorticity. The momentum equation (9) can be presented in the form similar to equations (5) and (6) as presented by Vadasz (2024a) subject to a Beltrami condition (Rousseaux *et al.* 2006, Marmanis 1998, Yoshida *et al.* 2003, Mahajan and Yoshida 1998, Gerner 2021, Amari *et al.* 2009, Bhattacharjee 2022, Lakhatakia 1994) for the Lamb vector (Lamb 1877)

$$\nabla\times\boldsymbol{g}=-\frac{\partial\boldsymbol{\xi}}{\partial t} \tag{12}$$

$$\nabla\bullet\boldsymbol{\xi}=0 \tag{13}$$

Equation (3) representing Coulomb law in field form produces, when integrated, the familiar algebraic form of Coulomb law for the electric field, i.e.

$$\boldsymbol{E}=\frac{1}{4pe_o}\frac{q}{r^2}\hat{\boldsymbol{e}}_r \qquad " \; r \, 3 \; r_o \tag{14}$$

where $\hat{\boldsymbol{e}}_r$ is a unit vector in the radial direction, and r_o is the radius of the spherical volume $\tilde{V}_o$ containing the charge creating the field. This is similar to the way the Newton's law of universal gravitation in field form, equation (10) leads upon integration the algebraic form of Newton's law of universal gravitation

$$\boldsymbol{g}=-G\frac{m}{r^2}\hat{\boldsymbol{e}}_r \qquad " \; r \, 3 \; r_o \tag{15}$$

where

$$q=\int_{\tilde{V}_o}\rho_q\,\mathrm{d}\tilde{V} \qquad \text{and} \qquad m=\int_{\tilde{V}_o}\rho\,\mathrm{d}\tilde{V} \tag{16}$$

and where $\tilde{V}_o$ is the volume occupied by the continuously distributed mass and electric charge. The sign change between equations (3) and (10) or between (14) and (15) is due to the fact that masses always attract due to gravitation, while charges of the same sign repel under Coulomb law, and attract only when they are of opposite signs. Therefore, it seems sensitive to have the signs of the electromagnetic repulsive terms (Lorentz terms) in the momentum equation (9) opposite to the sign of the gravity term. This sign allocation is distinct to the current mainstream convention in plasma dynamics or magneto-fluid-dynamics (MFD/MHD). Bittencourt (2004), and Vadasz (2024b) are examples of this incorrect convention although the derivations and conclusions in the latter are not affected by this sign change. The present paper demonstrates among others that the correct sign in

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front of the Lorentz terms in the momentum equation are indeed as presented in equation (9), i.e. opposite to the sign of the gravity term.

3. Equilibrium Solutions for a Free Continuously Distributed Electric Charge

The first observation is the fact that equations (3), (4), (8), and (9) satisfy a trivial solution for the stationary equilibrium, i.e. $\boldsymbol{E_o}(\boldsymbol{x})=0$ and $\rho_{q,o}(\boldsymbol{x})=0$, implying that any oscillations occur around zero, a result that causes both the electric field as well as the charge density to change signs as part of the solution. This solution was rejected already and therefore the focus will be on non-trivial stationary equilibria. The following statement will be proved and is essential to the further interpretation of the results.

STATEMENT

The term $\rho_q \boldsymbol{E}$ in the momentum equation (9) is always positive within the domain occupied by the distributed charge creating the field, i.e.

$$\rho_q \boldsymbol{E} > 0 \quad \forall \ \boldsymbol{x} \in \tilde{V}_o \qquad (17)$$

where $\tilde{V}_o$ is the volume containing the distributed charge creating the field. The term $\rho_{q2}\boldsymbol{E}$ is also positive beyond the domain occupied by a distributed charge ρ_{q1} creating the field as long as the sign of ρ_{q2} is the same as the charge creating the field, ρ_{q1}, i.e.

$$\begin{cases} \rho_{q2}\boldsymbol{E} > 0 \quad \forall \ \boldsymbol{x} \notin \tilde{V}_o \ \Leftrightarrow \ \operatorname{sgn}\left[\rho_{q2}\right] = \operatorname{sgn}\left[\rho_{q1}\right] \\ \rho_{q2}\boldsymbol{E} < 0 \quad \forall \ \boldsymbol{x} \notin \tilde{V}_o \ \Leftrightarrow \ \operatorname{sgn}\left[\rho_{q2}\right] \neq \operatorname{sgn}\left[\rho_{q1}\right] \end{cases} \qquad (18)$$

The proof of this statement is as follows.

PROOF

Since equation (3) produces equation (14) it implies that

$$\begin{cases} \boldsymbol{E} > 0 \ \Leftrightarrow \ q > 0 \\ \boldsymbol{E} < 0 \ \Leftrightarrow \ q < 0 \end{cases} \qquad (19)$$

But since $q = \int_{\tilde{V}_o} \rho_q \, \mathrm{d}\tilde{V}$ and within $\tilde{V}_o$ the sign of the charge does not change it follows that

$$\begin{cases} q > 0 \ \Leftrightarrow \ \rho_q > 0 \\ q < 0 \ \Leftrightarrow \ \rho_q < 0 \end{cases} \qquad (20)$$

Combining (19) with (20) leads to

$$\begin{cases} \boldsymbol{E} > 0 \ \Leftrightarrow \ \rho_q > 0 \\ \boldsymbol{E} < 0 \ \Leftrightarrow \ \rho_q < 0 \end{cases} \qquad (21)$$

These conditions apply for any domain within $\tilde{V}_o$, or even outside $\tilde{V}_o$ if the charge sign outside $\tilde{V}_o$ does not change either, although it might be different than the sign of ρ_q within $\tilde{V}_o$. Then based on (21) we conclude that

$$\rho_q \boldsymbol{E} > 0 \quad \forall \ \boldsymbol{x} \in \tilde{V}_o \qquad \text{Q.E.D.} \quad (17)$$

When ρ_{q1} is the charge density distribution creating the field within $\tilde{V}_o$ and ρ_{q2} is a test charge density distribution outside $\tilde{V}_o$, then (21) directly yields

$$\begin{cases} \rho_{q2}\boldsymbol{E} > 0 \quad \forall \ \boldsymbol{x} \notin \tilde{V}_o \ \Leftrightarrow \ \operatorname{sgn}\left[\rho_{q2}\right] = \operatorname{sgn}\left[\rho_{q1}\right] \\ \rho_{q2}\boldsymbol{E} < 0 \quad \forall \ \boldsymbol{x} \notin \tilde{V}_o \ \Leftrightarrow \ \operatorname{sgn}\left[\rho_{q2}\right] \neq \operatorname{sgn}\left[\rho_{q1}\right] \end{cases} \qquad \text{Q.E.D.} \ (18)$$

These conditions become useful in the following derivations.

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Switching now to evaluate the stationary equilibrium solutions and assuming the existence of at least one non-trivial (r_q ¹ 0, $\boldsymbol{E}$ ¹ 0) such stationary equilibrium solution ($\boldsymbol{v} = 0$), and neglecting gravity effects as they are substantially weaker than the electromagnetic ones, leads from equation (9) to

$$\nabla p + r_q \boldsymbol{E} = 0 \qquad (22)$$

Introducing a linear approximation between pressure and mass density in the form

$$p = p_o + v_o^2 (\rho - \rho_o) \qquad (23)$$

produces

$$\nabla p = v_o^2 \nabla \rho \qquad (24)$$

where $v_o = \sqrt{\partial p / \partial \rho}$ is the constant speed of propagation of the pressure wave (speed of sound as special case but allowed to vary up to the value of the speed of light in vacuum), and defining the mass to charge density ratio, assumed constant, in the form

$$b_q = \frac{r}{s_q r_q} = \frac{m}{s_q q} \qquad (25)$$

where

$$s_q = \mathrm{sgn}(r_q) = \begin{cases} +1 \;\text{"}\; r_q > 0 \\ -1 \;\text{"}\; r_q < 0 \end{cases} \qquad (26)$$

leading to

$$r = b_q s_q r_q \qquad (27)$$

Substituting (24) and (27) into (22) yields

$$\beta_q s_q v_o^2 \nabla \rho_q + \rho_q \boldsymbol{E} = 0 \qquad (28)$$

Substituting equation (3) into (28) produces an equation for the electric field in the form

$$\beta_q s_q v_o^2 \nabla (\nabla \bullet \boldsymbol{E}) + \boldsymbol{E} \nabla \bullet \boldsymbol{E} = 0 \qquad (29)$$

Assuming spherical symmetry with

$$\boldsymbol{E} = E_r \hat{\boldsymbol{e}}_r \qquad (30)$$

one obtains from (29)

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r^2}\frac{\mathrm{d}(r^2 E_r)}{\mathrm{d}r}\right] + \frac{E_r}{\beta_q s_q v_o^2 r^2}\frac{\mathrm{d}(r^2 E_r)}{\mathrm{d}r} = 0 \qquad (31)$$

and from (3)

$$\frac{e_o}{r^2}\frac{\mathrm{d}(r^2 E_r)}{\mathrm{d}r} = r_q \qquad (32)$$

Introducing the notation

$$h = r^2 E_r \qquad (33)$$

into (31) and (32) leads to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r^2}\frac{\mathrm{d}\eta}{\mathrm{d}r}\right] + \frac{\eta}{\beta_q s_q v_o^2 r^4}\frac{\mathrm{d}\eta}{\mathrm{d}r} = 0 \qquad (34)$$

which can be expanded in the form

$$\frac{1}{r^2}\frac{\mathrm{d}^2\eta}{\mathrm{d}r^2} - \left[2 - \frac{1}{\beta_q s_q v_o^2}\frac{\eta}{r}\right]\frac{1}{r^3}\frac{\mathrm{d}\eta}{\mathrm{d}r} = 0 \qquad (35)$$

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Equation (35) is a nonlinear ordinary differential equation that may have multiple solutions. One possible solution is by setting the term in the brackets equal to zero. This causes the first term in the equation to be zero too, i.e. $\mathrm{d}^2 h / \mathrm{d}r^2 = 0$. The result is then

$$2 - \frac{1}{\beta_q s_q v_o^2} \frac{\eta}{r} = 0 \tag{36}$$

which yields the solution

$$\eta = 2 s_q \beta_q v_o^2 r \tag{37}$$

Returning to the original variable E_r by using (33) into (37) produces the solution for the electric field at stationary equilibrium in the form

$$E_r = s_q 2 \beta_q v_o^2 \frac{1}{r} \tag{38}$$

From (38) it is evident by using (26) that

$$\begin{cases} E_r > 0 \ \text{"} \ r_q > 0 \\ E_r < 0 \ \text{"} \ r_q < 0 \end{cases} \tag{39}$$

Then obviously it follows that

$$r_q E_r > 0 \ \text{within} \ \tilde{V}_o \tag{40}$$

implying that the electrostatic force acts in the positive r-direction, i.e. a repulsion from inner layers out. The outer layers have no effect on the inner layers as this effect cancels due to symmetry. Substituting the solution (38) or (37) into (32) yields the charge density distribution solution in the form

$$\rho_q = s_q \frac{2 \varepsilon_o \beta_q v_o^2}{r^2} \tag{41}$$

producing a charge density distribution r_q carrying a sign consistent with s_q as expected. Should we had applied a positive sign in front of the term $r_q \boldsymbol{E}$ in equation (9) we would have obtained a r_q carrying a sign opposite to s_q introducing an impossible inconsistency.

4. Non-Trivial Equilibrium Solution for a Continuously Distributed Negative Charge Surrounding a Continuously Distributed Positive Charge with Spherical Symmetry

The problem of finding the non-trivial equilibrium solution for a continuously distributed negative charge surrounding a continuously distributed positive charge resembles the electron-proton situation in the hydrogen atom if we assume that the electron is continuously distributed around the nucleus like a charge density cloud behaving as an inviscid compressible fluid. Since in quantum electrodynamics the electron is treated as a "cloud of probabilities" it is sensible to check its behavior should it be deterministically set as a continuous distributed charge. For this problem, neglecting gravity on the account of being negligibly small compared to electromagnetic effects, equation (9) becomes

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \bullet \nabla) \boldsymbol{v} \right] = -\nabla p - \left[\rho_e \boldsymbol{E_e} + \rho_e (\boldsymbol{v} \times \boldsymbol{B}) \right] + \rho_e \boldsymbol{E_p} \tag{42}$$

where $\boldsymbol{E_p}$ is the proton electric field. The Coulomb law in field form, equation (3), applied to this situation is

$$\nabla \circ \boldsymbol{E_e} = \frac{1}{e_o} r_e \qquad \& \qquad \nabla \circ \boldsymbol{E_p} = \frac{1}{e_o} r_p \tag{43}$$

where r_p is the proton electric charge density distribution within the nucleus. For a stationary equilibrium $v = 0$ and $\boldsymbol{E_e} \,^{\circ}\, \boldsymbol{E_e}(\boldsymbol{x})$, $\boldsymbol{E_p} \,^{\circ}\, \boldsymbol{E_p}(\boldsymbol{x})$, i.e. they are not functions of time t. Then equations (42) and (43) become

$$\nabla p + r_e \boldsymbol{E_e} - r_e \boldsymbol{E_p} = 0 \qquad (44)$$

and by using (24), (25), (26) and (27) into (44) one obtains with $s_q = s_e = -1$

$$-\beta_e v_o^2 \nabla \rho_e + \rho_e \boldsymbol{E_e} - \rho_e \boldsymbol{E_p} = 0 \qquad (45)$$

where $b_e = m_e / |e|$, and m_e, e are the electron mass and charge, respectively. Substituting (43) into (45) leads to

$$-\beta_e v_o^2 \nabla\left(\nabla \bullet \boldsymbol{E_e}\right) + \left(\boldsymbol{E_e} - \boldsymbol{E_p}\right)\left(\nabla \bullet \boldsymbol{E_e}\right) = 0 \qquad (46)$$

Using now the spherical symmetry assumption on (46) it yields

$$\frac{d}{dr}\left[\frac{1}{r^2}\frac{d\left(r^2 E_r\right)}{dr}\right] - \frac{\left(E_{e,r} - E_{p,r}\right)}{\beta_q v_o^2 r^2}\frac{d\left(r^2 E_{e,r}\right)}{dr} = 0 \qquad (47)$$

and applying the same assumption on (43) it yields

$$\frac{1}{r^2}\frac{d\left(r^2 E_{e,r}\right)}{dr} = \frac{1}{e_o} r_e \quad \text{(a)} \quad \& \quad \frac{1}{r^2}\frac{d\left(r^2 E_{p,r}\right)}{dr} = \frac{1}{e_o} r_p \quad \text{(b)} \qquad (48)$$

Since the proton charge density distribution is

$$r_p = \begin{cases} r_p(r) & " \; r \in [0, r_N] \\ 0 & " \; r > r_N \end{cases} \qquad (49)$$

integrating (48b) produces the solution for $E_{p,r}$ in the form

$$E_{p,r} = \frac{|e|}{4pe_o}\frac{1}{r^2} > 0 \qquad (50)$$

where r_N is the nucleus radius. Then, equation (47), when using the definition of h from (33) i.e. $h = r^2 E_{e,r}$, becomes

$$\frac{d}{dr}\left[\frac{1}{r^2}\frac{d\eta}{dr}\right] - \frac{1}{\beta_q v_o^2}\left[\eta - \frac{|e|}{4\pi\varepsilon_o}\right]\frac{1}{r^4}\frac{d\eta}{dr} = 0 \qquad (51)$$

which can be expanded into

$$\frac{1}{r^2}\frac{d^2\eta}{dr^2} - \left[2 + \frac{1}{\beta_e v_o^2}\left(\eta - \frac{|e|}{4\pi\varepsilon_o}\right)\frac{1}{r}\right]\frac{1}{r^3}\frac{d\eta}{dr} = 0 \qquad (52)$$

A possible solution to equation (52) is obtained by setting the brackets equal to zero. The latter causes the first term in the equation, i.e. $d^2 h / dr^2 = 0$, to be zero too. The result is then

$$\eta = -2\beta_e v_o^2 r + \frac{|e|}{4\pi\varepsilon_o} \qquad (53)$$

and returning to the original variable $E_{e,r}$ by using $h = r^2 E_{e,r}$ yields

$$E_{e,r} = -\frac{2\beta_e v_o^2}{r} + \frac{|e|}{4\pi\varepsilon_o r^2} \qquad (54)$$

Substituting (53) or (54) into (48a) produces the equilibrium solution for the electron charge distribution r_e in the form

$$\rho_e = \frac{\varepsilon_o}{r^2}\frac{\mathrm{d}\eta}{\mathrm{d}r} = -\frac{2\beta_e\varepsilon_o v_o^2}{r^2} < 0 \qquad (55)$$

We can at this point integrate this solution of $\rho_e(r)$ over the domain $[r_N,\infty)$ and impose the requirement that the total charge is as we know the total one electron charge e, i.e.

$$\int_{r_N}^{\infty} \rho_e r^2\,\mathrm{d}r = \frac{e}{4\pi} \qquad (56)$$

Substituting the solution (55) into (56) shows that the integral diverges. It therefore leads to the conclusion that there must be a finite external radius r_∞, such that

$$\int_{r_N}^{r_\infty} \rho_e r^2\,\mathrm{d}r = \frac{e}{4\pi} \qquad (57)$$

Substituting (55) into (57) and integrating produces the result

$$(r_\infty - r_N) = -\frac{e}{8\pi\beta_e v_o^2\varepsilon_o} = \frac{e^2}{8\pi m_e v_o^2\varepsilon_o} \qquad (58)$$

defining r_∞ for a known value of v_o^2, or defining v_o^2 for a known value of r_∞

$$v_o^2 = \frac{e^2}{8\pi m_e\varepsilon_o(r_\infty - r_N)} \qquad (59)$$

If r_∞ is twice the Bohr radius, i.e. $r_\infty = 2r_B = 1.06\times 10^{-10}\,[\mathrm{m}]$, then

$$v_o^2 = \frac{e^2}{8\pi m_e(2r_B - r_N)\varepsilon_o} = 1.19\cdot 10^{12}\,[\mathrm{m}^2/\mathrm{s}^2] \qquad (60)$$

leading to $v_o = 1.093\cdot 10^6$ [m/s].

At this point it is of interest to find the condition for $\rho_e E_{e,r} < 0$, i.e. attracting towards the nucleus, or $\rho_e E_{e,r} > 0$ repulsed away from the nucleus. By using (54) one checks the condition for $E_{e,r} \le 0$ or $E_{e,r} \ge 0$ producing

$$r \ge \frac{e^2}{8\pi\varepsilon_o m_e v_o^2} \quad \text{for} \quad E_{e,r} \le 0 \qquad (61)$$

and

$$r \le \frac{e^2}{8\pi\varepsilon_o m_e v_o^2} \quad \text{for} \quad E_{e,r} \ge 0 \qquad (62)$$

Substituting equation (59) into (61) and (62) yields

$$r \ge r_\infty - r_N \quad \text{for} \quad E_{e,r} \le 0 \qquad (63)$$

and

$$r \le r_\infty - r_N \quad \text{for} \quad E_{e,r} \ge 0 \qquad (64)$$

Since $r_N \le r \le r_\infty$ conditions (63) and (64) lead to

$$r_\infty \le 2r_N \quad \text{for} \quad E_{e,r} \le 0 \quad \text{and} \quad \rho_e E_{e,r} \ge 0 \qquad (65)$$

i.e. repulsive, pulling away from the nucleus, and

$$r_\infty \ge 2r_N \quad \text{for} \quad E_{e,r} \ge 0 \quad \text{and} \quad \rho_e E_{e,r} \le 0 \qquad (66)$$

i.e. attractive, pushing towards the nucleus.

Since $r_\infty >> r_N$ the case of $r_\infty \ge 2r_N$ is selected and therefore one anticipates $\rho_e E_{e,r} \le 0$. However, this is only the force per unit volume due to the electron field. The total force per unit volume due to both the proton as well as the electron is from equation (42)

$$f_{Er} = -\rho_e E_{e,r} + \rho_e E_{p,r} \qquad (67)$$

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Substituting (50), (54), (55) and (59) into (67) produces

$$f_{Er} = -\frac{e^2}{4\pi^2 \left(r_\infty - r_N\right)^2} \frac{1}{r^3} < 0 \tag{68}$$

leading to the conclusion that the force per unit volume on the electron at static equilibrium is attractive towards the nucleus as expected.

5. The Wave Electromagnetic Equations and their Solution

A dynamic equilibrium is also possible with oscillations around the non-trivial stationary equilibrium that was derived in previous sections. Such a dynamic equilibrium applies when there is no motion of charges (no electric current) i.e. $\boldsymbol{v} = 0$ but with $\boldsymbol{E}(t,\boldsymbol{x})$ and $\boldsymbol{B}(t,\boldsymbol{x})$ that are related to equations (4), (5) and (6) which we did not deal with yet. For $\boldsymbol{v} = 0$ equations (4) and (5) are

$$\frac{1}{c_o^2}\frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{B} \tag{60}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \tag{61}$$

Applying the curl operator ($\nabla \times$) on equation (60) and the time derivative operator ($\partial/\partial t$) on equation (61), using the vector identity $\nabla \times \nabla \times \boldsymbol{B} = \nabla\left(\nabla \cdot \boldsymbol{B}\right) - \nabla^2 \boldsymbol{B}$ and equation (6), as well as adding the two equations leads to

$$\frac{1}{c_o^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \nabla^2 \boldsymbol{B} \tag{62}$$

Applying then the time derivative operator ($\partial/\partial t$) on equation (60) and the curl operator ($\nabla \times$) on equation (61), using the vector identity $\nabla \times \nabla \times \boldsymbol{E} = \nabla\left(\nabla \cdot \boldsymbol{E}\right) - \nabla^2 \boldsymbol{E}$ and equation (3), as well as subtracting the equations produces

$$\frac{1}{c_o^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \nabla^2 \boldsymbol{E} - \frac{1}{\varepsilon_o}\nabla \rho_q \tag{63}$$

In addition equation (9) becomes

$$v_o^2 \beta_q s_q \nabla \rho_q + \rho_q \boldsymbol{E} = 0 \tag{64}$$

Equation (62) is the classical wave equation, while equation (63) is also a wave equation that depends on ρ_q too, and is therefore coupled with equation (64). This coupling and the fact that equation (64) is a nonlinear equation due to the term $\rho_q \boldsymbol{E}$ transform the problem of the electromagnetic waves at equilibrium into a nonlinear problem.

6. Conclusions

The problem of stationary equilibrium of a continuously distributed charge was formulated in the form of finding both the electric field as well as the charge distribution, as distinct from typical corresponding problems dealing with finding either the electric field or the charge distribution given one of them as input data. Maxwell as well as the momentum equations were solved for the non-trivial stationary equilibrium. While oscillations do occur around the non-trivial equilibrium, they yield non-realistic results if they occur around the trivial equilibrium. The significance of allocating a negative sign in front of the repulsive Lorentz force terms in the momentum equation was demonstrated and proven necessary to avoid inconsistencies.

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