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M.S. Abu Zaytoon and M.H. Hamdan *

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Derivatives, Integrals and Polynomials Arising from the Inhomogeneous Airy's Equation

M.S. Abu Zaytoon [1] and M.H. Hamdan [2,*]

[1] Faculty of Mathematics and Data Science, Emirates Aviation University, Dubai UAE
[2] Department of Mathematics and Statistics, University of New Brunswick, Canada E2L4L5
* Correspondence: hamdan@unb.ca

Abstract: The various forms of Airy's differential equation are discussed in this work together with the special functions that arise in the processes of their solutions. Further properties of the arising integral functions are introduced and their connections to existing special functions are derived. A generalized form of the Scorer functions is obtained and expressed in terms of the generalized Nield-Kuznetsov functions. Complementary functions to the Nield-Kuznetsov functions are obtained together with higher derivatives of all generalized functions arising in this work. Airy's polynomials and generalized Airy's polynomials are derived and defined iteratively.

Keywords: inhomogeneous Airy's equation; generalized scorer; Airy's polynomials

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1. Introduction

Airy's differential equation and its associated Airy's functions date back to the nineteenth century, [1]. They are as relevant today as they were then due to their many applications in mathematical physics, circuit theory, systems theory, signal processing, electromagnetism and in fluid dynamic modeling (cf. [2,3,4] and the references therein). A large number of differential equations in quantum theory can be reduced to Airy's equation by an appropriate change of variables, thus adding to the importance and relevance of studies of Airy's functions and other arising and related special functions, [3,4].

Although Airy's equation has been largely studied in its homogeneous form, we consider it here in its more general, inhomogeneous form, written as, [5]:

$$u''_i - xu_i = f(x) \tag{1}$$

wherein the forcing function $f(x)$ is a continuous function of the non-negative, real variable x.

In equation (1), and throughout this work, prime notation denotes ordinary differentiation with respect to the independent variable.

When $f(x) \equiv 0$ general solution to the homogeneous Airy's equation (1) can be expressed in the form:

$$u_i = a_1 A_i(x) + b_1 B_i(x) \tag{2}$$

where a_1 and b_1 are arbitrary constants, and $A_i(x)$ and $B_i(x)$ are Airy's functions of the first- and second-kind, respectively, defined by:

$$A_i(x) = \frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{t^3}{3}\right)dt \tag{3}$$

$$B_i(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{t^3}{3}\right)dt + \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{t^3}{3}\right)dt \tag{4}$$

The Wronskian of $A_i(x)$ and $B_i(x)$ is given by, [2,4]:

$$W\big(A_i(x), B_i(x)\big) = \frac{1}{\pi} \tag{5}$$

When $f(x) = \mp\frac{1}{\pi}$, Scorer [6] obtained the following general solutions using variation of parameters. For $f(x) = -\frac{1}{\pi}$, general solution to equation (1) is given by

$$u_i = a_1 A_i(x) + b_1 B_i(x) + G_i(x) \tag{6}$$

and when $f(x) = \frac{1}{\pi}$, general solution to equation (1) is given by

$$u_i = a_1 A_i(x) + b_1 B_i(x) + H_i(x) \tag{7}$$

The functions $G_i(x)$ and $H_i(x)$ are known as Scorer's functions, [4], or the inhomogeneous Airy's functions, and arise in Raman scattering in chemical physics [7,8]. The Scorer functions are defined by:

$$G_i(x) = \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right) dt \tag{8}$$

$$H_i(x) = \frac{1}{\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right) dt \tag{9}$$

and are related to Airy's functions by

$$G_i(x) = A_i(x)\int_0^x B_i\,(t)dt + B_i(x)\int_x^\infty A_i\,(t)dt \tag{10}$$

$$H_i(x) = B_i(x)\int_{-\infty}^x A_i\,(t)dt - A_i(x)\int_{-\infty}^x B_i\,(t)dt \tag{11}$$

$$G_i(x) + H_i(x) = B_i(x) \tag{12}$$

In a generalization to the above, when $f(x) = \kappa$, where κ is any constant, general solution to equation (1) has been obtained in the following form, [9]:

$$u_i = a_1 A_i(x) + b_1 B_i(x) - \kappa\pi N_i(x) \tag{13}$$

where

$$N_i(x) = A_i(x)\int_0^x B_i\,(t)dt - B_i(x)\int_0^x A_i\,(t)dt \tag{14}$$

The function $N_i(x)$ was introduced by Nield and Kuznetsov, [10], in their analysis of flow in the transition layer where the governing Brinkman's equation was reduced to the inhomogeneous Airy's equation by a special choice of the permeability function. The function $N_i(x)$ is referred to as the *standard Nield-Kuznetsov function of the first-kind*, and its main properties were studied by Hamdan and Kamel [9].

In order to offer modelling flexibility in the study of flow through the transition layer, Abu Zaytoon et.al. [11] introduced a permeability model that reduced Brinkman's equation to the generalized inhomogeneous Airy's equation of index n, which takes the form

$$u''_n - x^n u_n = f(x) \tag{15}$$

The homogeneous part of (15), that is when $f(x) \equiv 0$, was studied by Swanson and Headley [12], who expressed its general solution as

$$u_n = a_n A_n(x) + b_n B_n(x) \tag{16}$$

wherein a_n and b_n are arbitrary constants, and the functions $A_n(x)$ and $B_n(x)$ are the *generalized Airy's functions of the first- and second-kind*, respectively, defined by:

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$$A_n(x) = p\,(x)^{1/2}\,[I_{-p}(\zeta) - I_p(\zeta)] \tag{17}$$

$$B_n(x) = (px)^{1/2}\,[I_p(\zeta) + I_{-p}(\zeta)] \tag{18}$$

The Wronskian of $A_n(x)$ and $B_n(x)$ is given by:

$$W\big(A_n(x), B_n(x)\big) = \frac{2}{\pi} p^{\frac{1}{2}} \sin(p\pi) \tag{19}$$

and the function I_p is the modified Bessel function defined as:

$$I_p(\zeta) = \sum_{m=1}^{\infty} \frac{1}{m!\,\Gamma(m+p+1)} (\frac{\zeta}{2})^{2m+p} \tag{20}$$

with $p = \frac{1}{n+2}$, $\zeta = 2p(x)^{\frac{1}{2p}}$, and $\Gamma(.)$ being the gamma function. It should be noted that when the index $n = 1$ in equations (15)-(19), we recover Airy's equation and its solutions (although we use subscript "i" instead of 1 for consistency with notation in the literature).

When $f(x) = \kappa$, Abu Zaytoon et.al. [11] obtained and expressed the general solution to the inhomogeneous generalized Airy's equation (15) as

$$u_n = a_n A_n(x) + b_n B_n(x) - \frac{\kappa\pi}{2\sqrt{p}\,\sin(p\pi)} N_n(x) \tag{21}$$

where the function $N_n(x)$ is the *generalized Nield-Kuznetsov function of the first-kind*, defined by:

$$N_n(x) = A_n(x) \int_0^x B_n\,(t)dt - B_n(x) \int_0^x A_n\,(t)dt \tag{22}$$

Analysis of the Nield-Kuznetsov functions were discussed and documented, [9,10,13], and include solution methodologies and methods of computations to the inhomogeneous Airy's equation, inhomogeneous generalized Airy's equation, and inhomogeneous Weber's equations, with initial and boundary conditions. Recent work in this field also includes the elegant work of Dunster, [14], on the Nield-Kuznetsov functions and the use of Laplace transform and uniform asymptotic expansions, and the analysis of Airy's polynomials that arise when higher derivatives are involved, and has been carried out in the elaborate work of Abramochkin and Razueva, [15]. The same Airy's polynomials, and other polynomials, have been shown to arise in the higher derivatives of the standard Nield-Kuznetsov function of the first-kind, and are important from both a theoretical and a practical point of view, as discussed by Hamdan et.al. [16].

The above discussion of the importance of the inhomogeneous Airy's and generalized Airy's equations motivates the current work in which we derive further properties of the Nield-Kuznetsov functions. In particular, we initiate discussion, derivation and analysis of the generalized Scorer functions and study their properties, and their relationships to the Nield-Kuznetsov functions and to the modified Bessel functions. We also define and analyze properties of the complementary Nield-Kuznetsov functions in the sense defined by Dunster [14]. Higher derivatives of the generalized Airy's, Nield-Kuznetsov, and Scorer functions, are then discussed and introduced together with their associated generalized polynomials. These generalized functions and polynomials might find their way in analysis of the Stark equation, [17], Schrodinger equation and Tricomi's inhomogeneous equation.

2. Airy's Inhomogeneous Equation

Solutions to Airy's equation (1) are streamlined into the following cases.

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Case 1: When $f(x) \equiv 0$, general solution to Airy's homogeneous equation is given by equation (2) as a linear combination of Airy's functions of the first- and second-kind, $A_i(x)$ and $B_i(x)$, respectively, that are defined by (3) and (4), and whose nonzero Wronskian is given by (5).

Airy's functions $A_i(x)$ and $B_i(x)$ take the following forms in terms of the modified Bessel functions, [2,4], obtained from (17) and (18) with $n = 1,\ p = \frac{1}{n+2} = \frac{1}{3}$, and $\zeta = 2p(x)^{\frac{1}{2p}} = \frac{2}{3}x^{\frac{3}{2}}$:

$$A_i(x) = \frac{\sqrt{x}}{3}[I_{-\frac{1}{3}}(\zeta) - I_{\frac{1}{3}}(\zeta)] \tag{23}$$

$$B_i(x) = \sqrt{\frac{x}{3}}\,[I_{\frac{1}{3}}(\zeta) + I_{-\frac{1}{3}}(\zeta)] \tag{24}$$

where the function $I_{\mp 1/3}$ is obtained from (20) as:

$$I_{\mp 1/3}(\zeta) = \sum_{m=1}^{\infty} \frac{1}{m!\,\Gamma(m \mp 1/3 + 1)} (\frac{\zeta}{2})^{2m \mp 1/3} \tag{25}$$

Using (23) and (24), derivatives and integrals of $A_i(x)$ and $B_i(x)$, take the following forms in terms of the modified Bessel function:

$$A'_i(x) = -\frac{x}{3}\left[I_{-\frac{2}{3}}(\zeta) - I_{\frac{2}{3}}(\zeta)\right] \tag{26}$$

$$B'_i(x) = \frac{x}{\sqrt{3}}\left[I_{-\frac{2}{3}}(\zeta) + I_{\frac{2}{3}}(\zeta)\right] \tag{27}$$

$$\int_0^x A_i(t)dt = \frac{1}{3}\int_0^{\zeta}\left[I_{-\frac{1}{3}}(t) - I_{\frac{1}{3}}(t)\right]dt \tag{28}$$

$$\int_0^x B_i(t)dt = \frac{1}{\sqrt{3}}\int_0^{\zeta}\left[I_{-\frac{1}{3}}(t) + I_{\frac{1}{3}}(t)\right]dt \tag{29}$$

Furthermore, the following integrals are important properties of Airy's functions, [4]:

$$\int_{-\infty}^0 B_i(t)dt = 0 \tag{30}$$

$$\int_{-\infty}^{\infty} A_i(t)\,dt = 1 \tag{31}$$

$$\int_0^{\infty} A_i(t)\,dt = \frac{1}{3} \tag{32}$$

Case 2: When $f(x) = \mp\frac{1}{\pi}$, general solutions to Airy's inhomogeneous equation (1) are given by equations (6) and (7). The particular solutions are expressed in terms of the Scorer functions: $G_i(x)$ when $f(x) = -\frac{1}{\pi}$, and $H_i(x)$ when $f(x) = \frac{1}{\pi}$. Scorer functions are defined by equations (8) and (9), and are related to Airy's functions by equations (10)-(12).

Further relationships between Scorer functions and integrals of Airy's functions can be obtained by defining the following Wronskians:

$$W_1 = W\big(A_i(x), G_i(x)\big) = A_i(x)G'_i(x) - G_i(x)A'_i(x) \tag{33}$$

$$W_2 = W\big(A_i(x), H_i(x)\big) = A_i(x)H'_i(x) - H_i(x)A'_i(x) \tag{34}$$

$$W_3 = W\big(B_i(x), G_i(x)\big) = B_i(x)G'_i(x) - G_i(x)B'_i(x) \tag{35}$$

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$$W_4 = W\left(B_i(x), H_i(x)\right) = B(x)H'_i(x) - H_i(x)B'_i(x) \tag{36}$$

The right-hand-sides of (33)-(36) have been expressed in terms of $\int_0^x A_i(t)dt$ and $\int_0^x B_i(t)dt$, [4]. We can then write:

$$\int_0^x A_i(t)dt = \frac{1}{3} + \pi\{G_i(x)A'_i(x) - A_i(x)G'_i(x)\} = \frac{1}{3} - \pi W_1 \tag{37}$$

$$\int_0^x A_i(t)dt = -\frac{2}{3} - \pi\{H_i(x)A'_i(x) - A_i(x)H'_i(x)\} = -\frac{2}{3} + \pi W_2 \tag{38}$$

$$\int_0^x B_i(t)dt = \pi\{G_i(x)B'_i(x) - B_i(x)G'_i(x)\} = -\pi W_3 \tag{39}$$

$$\int_0^x B_i(t)dt = -\pi\{H_i(x)B'_i(x) - B_i(x)H'_i(x)\} = \pi W_4 \tag{40}$$

Case 3: If $f(x) = \kappa$, where κ is any real constant, then a particular solution to (1) can be constructed using variation of parameters and takes the form

$$(u_i)_p = -\kappa\pi N_i(x) \tag{41}$$

and general solution to (1) is given by equation (13), wherein $N_i(x)$ is the standard Nield-Kuznetsov function of the first-kind, defined by equation (14). First derivative of $N_i(x)$ is obtained from (14) as:

$$N'_i(x) = A'_i(x)\int_0^x B_i(t)dt - B'_i(x)\int_0^x A_i(t)dt \tag{42}$$

Using (23)-(29), we can express $N_i(x)$ and its derivative, defined in equations (14) and (42), respectively, in terms of modified Bessel function as

$$N_i(x) = \frac{2}{3}\sqrt{\frac{x}{3}}\left\{I_{-\frac{1}{3}}(\zeta)\int_0^\zeta I_{\frac{1}{3}}(t)dt - I_{\frac{1}{3}}(\zeta)\int_0^\zeta I_{-\frac{1}{3}}(t)dt\right\} \tag{43}$$

$$N'_i(x) = \frac{2}{3}\sqrt{\frac{x}{3}}\left\{I_{\frac{2}{3}}(\zeta)\int_0^\zeta I_{\frac{1}{3}}(t)dt - I_{-\frac{2}{3}}(\zeta)\int_0^\zeta I_{-\frac{1}{3}}(t)dt\right\} \tag{44}$$

Connections between $N_i(x)$ and the Scorer functions $G_i(x)$ and $H_i(x)$ are established as follows.

Using (37)-(40) and (32) in (43), we obtain the following expressions for $Ni(x)$:

$$N_i(x) = \pi\{W_1B_i(x) - W_3A_i(x)\} - \frac{1}{3}B_i(x) \tag{45}$$

$$N_i(x) = \pi\{W_4A_i(x) - W_2B_i(x)\} + \frac{2}{3}B_i(x) \tag{46}$$

Furthermore, equation (10) can be written as

$$G_i(x) = A_i(x)\int_0^x B_i(t)dt + B_i(x)[\int_0^\infty A_i(t)dt - \int_0^x A_i dt] \tag{47}$$

Upon using (32) in (47), and invoking (14), we obtain

$$G_i(x) = N_i(x) + \frac{1}{3}B_i(x) \tag{48}$$

and upon using (14) in (48), we obtain

$$H_i(x) = \frac{2}{3}B_i(x) - N_i(x) \tag{49}$$

Equations (48) and (49) yield

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$$N_i(x) = \frac{2}{3}G_i(x) - \frac{1}{3}H_i(x) \tag{50}$$

and upon using (8) and (9) in (50), we obtain

$$N_i(x) = \frac{2}{3\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right)dt - \frac{1}{3\pi}\int_0^\infty \exp\left(xt - \frac{1}{3}t^3\right)dt \tag{51}$$

Scorer functions $G_i(x)$ and $H_i(x)$ take the following forms in terms of modified Bessel function, obtained using (23), (24), (43), (48) and (49):

$$G_i(x) = \frac{2}{3}\sqrt{\frac{x}{3}}\left\{I_{-\frac{1}{3}}(\zeta)\int_0^\zeta \left[I_{\frac{1}{3}}(t)\right]dt - I_{\frac{1}{3}}(\zeta)\int_0^\zeta \left[I_{-\frac{1}{3}}(t)\right]dt\right\} - \frac{1}{3}\sqrt{\frac{x}{3}}\left[I_{\frac{1}{3}}(\zeta) + I_{-\frac{1}{3}}(\zeta)\right] \tag{52}$$

$$H_i(x) = \frac{2}{3}\sqrt{\frac{x}{3}}\left[I_{\frac{1}{3}}(\zeta) + I_{-\frac{1}{3}}(\zeta)\right] - \frac{2}{3}\sqrt{\frac{x}{3}}\left\{I_{-\frac{1}{3}}(\zeta)\int_0^\zeta I_{\frac{1}{3}}(t)dt + I_{\frac{1}{3}}(\zeta)\int_0^\zeta I_{\frac{1}{3}}(t)dt\right\} \tag{53}$$

and equations (48)-(53) help furnish the following results.

Result 1: General solution to the inhomogeneous Airy's equation (1) when $f(x) = -\frac{1}{\pi}$ is given by equation (6), or equivalently by

$$u_i = a_1 A_i(x) + c_1 B_i(x) + N_i(x) \tag{54}$$

where $c_1 = b_1 + \frac{1}{3}$.

Result 2: General solution to the inhomogeneous Airy's equation (1) when $f(x) = \frac{1}{\pi}$ is given by equation (7), or equivalently by

$$u_i = a_1 A_i(x) + c_1 B_i(x) - N_i(x) \tag{55}$$

where $c_1 = b_1 + \frac{2}{3}$.

Result 3: The standard Nield-Kuznetsov function of the first-kind, $N_i(x)$, is defined in terms of Scorer functions by equation (50), and in terms of the improper integral definitions of Airy's functions by equation (51).

Result 4: Scorer functions $G_i(x)$ and $H_i(x)$ are defined in terms of modified Bessel functions by equations (52) and (53).

We conclude this section by tabulating values at zero of $A_i(x)$, $B_i(x)$, $G_i(x), H_i(x)$, and $N_i(x)$, and their derivatives:

Table 1. Values of the integral functions and their derivatives at *x=0*.

Function Value at $x = 0$	**First Derivative Value at $x = 0$**
$A_i(0) = \sqrt{3}G_i(0) = \frac{\sqrt{3}}{3^{\frac{7}{6}}\Gamma(\frac{2}{3})}$	$A'_i(0) = -\sqrt{3}G'_i(0) = \frac{-\sqrt{3}}{3^{\frac{5}{6}}\Gamma(\frac{1}{3})}$
$B_i(0) = 3G_i(0) = \frac{3}{3^{\frac{7}{6}}\Gamma(\frac{2}{3})}$	$B'_i(0) = 3G'_i(0) = \frac{3}{3^{\frac{5}{6}}\Gamma(\frac{1}{3})}$
$G_i(0) = \frac{A_i(0)}{\sqrt{3}} = \frac{1}{3^{\frac{7}{6}}\Gamma(\frac{2}{3})}$	$G'_i(0) = -\frac{1}{\sqrt{3}}A'_i(0) = \frac{1}{3^{\frac{5}{6}}\Gamma(\frac{1}{3})}$
$H_i(0) = \frac{2}{3}B_i(0) = \frac{2}{3^{\frac{7}{6}}\Gamma(\frac{2}{3})}$	$H'_i(0) = \frac{2}{3}B'_i(0) = \frac{2}{3^{\frac{5}{6}}\Gamma(\frac{1}{3})}$

$N_i(0) = 0$	$N'_i(0) = 0$

3. Complementary Function of $N_i(x)$:

Dunster [14] introduced the concept of a complementary function to the standard Nield-Kuznetsov parametric function by extending its definition to $[x, +\infty)$. Aspnes [18,19] introduced a function $A_{i1}(x)$, defined by, [4]:

$$A_{i1}(x) = \int_x^\infty A_i(x) = \pi\{A_i(x)G'_i(x) - A'_i(x)G_i(x)\} \tag{56}$$

Following the concept introduced by Dunster, [14], we define the complement of $N_i(x)$ as

$$\tilde{N}_i(x) = A_i(x)\int_x^\infty B_i\,(t)dt - B_i(x)\int_x^\infty A_i\,(t)dt \tag{57}$$

Now, using (37), we can express (56) as

$$\int_x^\infty A_i\,(t)dt = \pi W_1 \tag{58}$$

and use (58) to write (57) as

$$\tilde{N}_i(x) = A_i(x)\int_x^\infty B_i\,(t)dt - \pi W_1 B_i(x) \tag{59}$$

The sum of $N_i(x)$ and its complement $\tilde{N}_i(x)$, given by

$$N_i(x) + \tilde{N}_i(x) = A_i(x)\int_0^\infty B_i\,(t)dt - B_i(x)\int_0^\infty A_i\,(t)dt \tag{60}$$

can be written, with the help of (32), as

$$N_i(x) + \tilde{N}_i(x) = A_i(x)\int_0^\infty B_i\,(t)dt - \frac{1}{3}B_i(x) \tag{61}$$

Furthermore, upon implementing (48), (49) and (50), we obtain the following three expressions for $\tilde{N}_i(x)$, respectively:

$$\tilde{N}_i(x) = A_i(x)\int_0^\infty B_i\,(t)dt - G_i(x) \tag{62}$$

$$\tilde{N}_i(x) = A_i(x)\int_0^\infty B_i\,(t)dt - B_i(x) + H_i(x) \tag{63}$$

$$\tilde{N}_i(x) = A_i(x)\int_0^\infty B_i\,(t)dt - \frac{1}{3}\{B_i(x) + 2G_i(x) - H_i(x)\} \tag{64}$$

Upon using (3), (4), (51) and (61), we obtain the following two expressions for $\tilde{N}_i(x)$:

$$\tilde{N}_i(x) = A_i(x)\int_0^\infty B_i\,(t)dt - \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right)dt \tag{65}$$

$$\tilde{N}_i(x) = \left\{\frac{1}{\pi}\int_0^\infty \cos\left(xt + \frac{t^3}{3}\right)dt\right\}\int_0^\infty B_i\,(t)dt - \frac{1}{\pi}\int_0^\infty \sin\left(xt + \frac{1}{3}t^3\right)dt \tag{66}$$

The above derivations yield the following result.

Result 5: The complementary function $\tilde{N}_i(x)$ to the standard Nield-Kuznetsov function of the first-kind, $N_i(x)$, is defined by the equivalent expressions (57) and (59)-(66).

4. Generalized Airy's Inhomogeneous Equation

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4.1. Generalized Airy's and Nield-Kuznetsov Functions

Consider the generalized, inhomogeneous Airy's equation (15) in which $n > -2$ is an integer and $f(x)$ is a smooth function of its real, non-negative variable x. Solution to the homogeneous part of this equation is given by (16), with the generalized Airy's functions given by (17) and (18). The following two cases arise.

Case 1: If $f(x) = \kappa$, where κ is any real constant, then a particular solution to (15) can be constructed using variation of parameters and takes the form:

$$(u_n)_{particular} = -\frac{\kappa\pi}{2\sqrt{p}\,\sin(p\pi)} N_n(x) \tag{66}$$

where $N_n(x)$ is the generalized Nield-Kuznetsov function of the first-kind, defined by equation (22). General solution to (15) thus takes the form given by (21).

From equations (17), (18) and (22) the following derivatives of the functions $A_n(x)$, $B_n(x)$ and $N_n(x)$ are obtained, respectively:

$$A'_n(x) = -p\,(x)^{\frac{n+1}{2}}[(I_{1-p}(\zeta) - I_{p-1}(\zeta))] \tag{68}$$

$$B'_n(x) = p^{1/2}\,(x)^{\frac{n+1}{2}}[I_{p-1}(\zeta) + I_{1-p}(\zeta)] \tag{69}$$

$$N'_n(x) = A'_n(x)\int_0^x B_n\,(t)dt - B'_n(x)\int_0^x A_n\,(t)dt \tag{70}$$

Integrals of the generalized Airy's functions are obtained from (17) and (18) as

$$\int_0^x A_n(t)dt = p\int_0^\zeta [I_{-p}(t) - I_p(t)]dt \tag{71}$$

$$\int_0^x B_n(t)dt = \sqrt{p}\int_0^\zeta [I_p(t) + I_{-p}(t)]dt \tag{72}$$

Upon using (17), (18), (68), (69), (71) and (72) in (22) and (70), we obtain the following expressions for $N_n(x)$ and $N'_n(x)$:

$$N_n(x) = 2p\sqrt{p}\,(x)^{\frac{1}{2}}\left\{I_{-p}(\zeta)\int_0^\zeta [I_p(t)]dt - I_p(\zeta)\int_0^\zeta [I_{-p}(t)]dt\right\} \tag{73}$$

$$N'_n(x) = 2p\sqrt{p}\,(x)^{\frac{n+1}{2}}\left\{I_{p-1}(\zeta)\int_0^\zeta I_p(t)dt - I_{1-p}(\zeta)\int_0^\zeta I_{-p}(t)dt\right\} \tag{74}$$

Complementary function to the generalized Nield-Kuznetsov function of the first-kind is defined by:

$$\tilde{N}_n(x) = A_n(x)\int_x^\infty B_n\,(t)dt - B_n(x)\int_x^\infty A_n\,(t)dt \tag{75}$$

and the sum of functions $N_n(x) + \tilde{N}_n(x)$ is given by:

$$N_n(x) + \tilde{N}_n(x) = A_n(x)\int_0^\infty B_n\,(t)dt - B_n(x)\int_0^\infty A_n\,(t)dt \tag{76}$$

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4.2. Generalized Scorer Functions

For the special cases of $f(x) = \kappa = \mp\frac{1}{\pi}$, general solutions to the inhomogeneous Airy's equation (1) are expressed in terms of Airy's and Scorer's functions, as given by (6) and (7). Scorer [6] obtained these solutions using variation of parameters and defining the functions $G_i(x)$ and $H_i(x)$ using (10)-(12). Following Scorer's approach, we assume that the general solutions to (15) are given as follows. When $\kappa = -\frac{1}{\pi}$, general solution to (15) is given by

$$u_n = a_n A_n(x) + b_n B_n(x) + G_n(x) \tag{77}$$

and when $\kappa = \frac{1}{\pi}$, general solution to (15) is given by

$$u_n = a_n A_n(x) + b_n B_n(x) + H_n(x) \tag{78}$$

The functions $G_n(x)$ and $H_n(x)$ are termed the generalized Scorer functions, defined and related to the generalized Airy's functions by

$$G_n(x) = A_n(x)\int_0^x B_n(t)dt + B_n(x)\int_x^\infty A_n(t)dt \tag{79}$$

$$H_n(x) = B_n(x)\int_{-\infty}^x A_n(t)dt - A_n(x)\int_{-\infty}^x B_n(t)dt \tag{80}$$

We can relate $G_n(x)$ and $H_n(x)$ to $N_n(x)$, defined by (22) as follows. When $f(x) = \kappa = -\frac{1}{\pi}$, solution to (15) as given by (21) takes the form

$$u_n = a_n A_n(x) + c_n B_n(x) + \frac{1}{2\sqrt{p}\sin(p\pi)} N_n(x) \tag{81}$$

When $f(x) = \kappa = \frac{1}{\pi}$, solution to (15) as given by (21) takes the form

$$u_n = a_n A_n(x) + c_n B_n(x) - \frac{1}{2\sqrt{p}\sin(p\pi)} N_n(x) \tag{82}$$

In order for solutions (81) and (82) to reduce to the corresponding solutions when $n = 1$, namely solutions (54) and (55), we select $c_n = b_n + p$ in (81) and $c_n = b_n + 2p$ in (82). General solutions (81) and (82) can thus be written as:

$$u_n = a_n A_n(x) + b_n B_n(x) + \frac{1}{2\sqrt{p}\sin(p\pi)} N_n(x) + pB_n(x) \tag{83}$$

$$u_n = a_n A_n(x) + c_n B_n(x) - \frac{1}{2\sqrt{p}\sin(p\pi)} N_n(x) + 2pB_n(x) \tag{84}$$

The particular solutions in (83) and (84) can thus be compared to the particular solutions in (77) and (78) to yield:

$$G_n(x) = \frac{1}{2\sqrt{p}\sin(p\pi)} N_n(x) + pB_n(x) \tag{85}$$

$$H_n(x) = 2pB_n(x) - \frac{1}{2\sqrt{p}\sin(p\pi)} N_n(x) \tag{86}$$

with first derivatives given by

$$G'_n(x) = \frac{1}{2\sqrt{p}\sin(p\pi)} N'_n(x) + pB'_n(x) \tag{87}$$

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$$H'_n(x) = 2pB'_n(x) - \frac{1}{2\sqrt{p}\sin(p\pi)}N'_n(x) \tag{88}$$

Adding (85) and (86), we get

$$G_n(x) + H_n(x) = 3pB_n(x) \tag{89}$$

Solving (85) and (86) for $N_n(x)$, gives:

$$N_n(x) = \left\{\frac{2}{3}G_n(x) - \frac{1}{3}H_n(x)\right\}2\sqrt{p}\sin(p\pi) \tag{90}$$

When n=1, $p = \frac{1}{3}$ and (85), (86), (89) and (90) reduce to (12), (48), (49) and (50), respectively.

Upon using (18) and (73) in (85) and (86), the generalized Scorer functions are expressed in the following forms in terms of Bessel's modified functions:

$$G_n(x) = p\sqrt{px}\left[I_p(\zeta) + I_{-p}(\zeta)\right]$$
$$+\frac{p\sqrt{x}}{\sin(p\pi)}\left\{I_{-p}(\zeta)\int_0^{\zeta}[I_p(t)]dt - I_p(\zeta)\int_0^{\zeta}[I_{-p}(t)]dt\right\} \tag{91}$$

$$H_n(x) = 2p\sqrt{px}\left[I_p(\zeta) + I_{-p}(\zeta)\right]$$
$$-\frac{p\sqrt{x}}{\sin(p\pi)}\left\{I_{-p}(\zeta)\int_0^{\zeta}[I_p(t)]dt - I_p(\zeta)\int_0^{\zeta}[I_{-p}(t)]dt\right\} \tag{92}$$

The above yields the following results.

Result 6: Particular solutions to the generalized inhomogeneous Airy's equations: $u''_n - x^n u_n = -\frac{1}{\pi}$ and $u''_n - x^n u_n = \frac{1}{\pi}$, are given, respectively, by either:

$G_n(x)$ *or* $\frac{1}{2\sqrt{p}\sin(p\pi)}N_n(x)$

and

$H_n(x)$ *or* $-\frac{1}{2\sqrt{p}\sin(p\pi)}N_n(x)$.

Result 7: The generalized Scorer functions $G_n(x)$ and $H_n(x)$ are defined in terms of the generalized Nield-Kuznetsov function of the first-kind by equations (85) and (86).

We parallel properties (30), (31) and (32) by stating the following conjecture, where when $n = 1$, the integrals reduce to (30)-(32).

Conjecture 1: The following integrals follow from properties (30)-(32):

$$\int_{-\infty}^{0} B_n(x)dx = 0 \tag{93}$$

$$\int_{-\infty}^{\infty} A_n(x)\,dx = 3p \tag{94}$$

$$\int_{0}^{\infty} A_n(x)\,dx = p \tag{95}$$

We conclude this section by tabulating values at zero of $A_n(x)$, $B_n(x)$, $G_n(x), H_n(x)$ *and* $N_n(x)$, and their derivatives:

Table 2. Values of the generalized integral functions and their. **derivatives at *x* = *0*.**

$A_n(0) = \frac{(p)^{1-p}}{\Gamma(1-p)}$	$A'_n(0) = -\frac{(p)^p}{\Gamma(p)}$

$B_n(0) = \frac{(p)^{1/2-p}}{\Gamma(1-p)}$	$B'_n(0) = -\frac{(p)^{p-\frac{1}{2}}}{\Gamma(p)}$
$N_n(0) = 0$	$N'_n(0) = 0$
$G_n(0) = pB_n(0) = \frac{(p)^{3/2-p}}{\Gamma(1-p)}$	$G'_n(0) = -\frac{(p)^{p+\frac{1}{2}}}{\Gamma(p)}$
$H_n(0) = 2G_n(0) = 2\frac{(p)^{\frac{3}{2}-p}}{\Gamma(1-p)}$	$H'_n(0) = 2G'_n(0) = -2\frac{(p)^{p+\frac{1}{2}}}{\Gamma(p)}$

4.3. Computational Algorithm of Generalized Functions

Following Swanson and Headley, [12], the generalized Airy's functions are evaluated using the following relationships:

$$\rho_n = \frac{(p)^{1-p}}{\Gamma(1-p)} \quad \text{and} \quad \varphi_n = \frac{(p)^p}{\Gamma(p)} \tag{96}$$

$$g_{n1}(x) = 1 + \sum_{k=1}^{\infty} p^{2k} \prod_{j=1}^{k} \frac{x^{(n+2)k}}{j(j-p)} \tag{97}$$

$$g_{n2}(x) = x\left[1 + \sum_{k=1}^{\infty} p^{2k} \prod_{j=1}^{k} \frac{x^{(n+2)k}}{j(j+p)}\right] \tag{98}$$

$$A_n(x) = \rho_n g_{n1}(x) - \varphi_n g_{n2}(x) \tag{99}$$

$$B_n(x) = \frac{1}{\sqrt{p}}[\rho_n g_{n1}(x) + \varphi_n g_{n2}(x)] \tag{100}$$

Based on the above algorithm, the generalized Nield-Kuznetsov function of the first-kind and the generalized Scorer functions can be evaluated using the following expression:

$$N_n(x) = \frac{2}{\sqrt{p}}\rho_n\varphi_n\left\{g_{n1}(x)\int_0^x g_{n2}(t)dt - g_{n2}(y)\int_0^x g_{n1}(t)dt\right\} \tag{101}$$

$$G_n(x) = \frac{\rho_n\varphi_n}{p\sin(p\pi)}\left\{g_{n1}(x)\int_0^x g_{n2}(t)dt - g_{n2}(y)\int_0^x g_{n1}(t)dt\right\} + \sqrt{p}[\rho_n g_{n1}(x) + \varphi_n g_{n2}(x)] \tag{102}$$

$$H_n(x) = -\frac{\rho_n\varphi_n}{p\sin(p\pi)}\left\{g_{n1}(x)\int_0^x g_{n2}(t)dt - g_{n2}(y)\int_0^x g_{n1}(t)dt\right\} + 2\sqrt{p}[\rho_n g_{n1}(x) + \varphi_n g_{n2}(x)] \tag{103}$$

When $n = 1$, equations (96)-(103) reduce to the following:

$$\rho_1 = \frac{\left(\frac{1}{3}\right)^{\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)} \quad and \quad \varphi_1 = \frac{\left(\frac{1}{3}\right)^{\frac{1}{3}}}{\Gamma\left(\frac{1}{3}\right)} \tag{104}$$

$$g_{11}(x) = 1 + \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{2k} \prod_{j=1}^{k} \frac{x^{3k}}{j\left(j-\frac{1}{3}\right)} \tag{105}$$

$$g_{12}(x) = x\left[1 + \sum_{k=1}^{\infty}(\frac{1}{3})^{2k}\prod_{j=1}^{k}\frac{x^{3k}}{j\left(j+\frac{1}{3}\right)}\right] \tag{106}$$

$$A_i(x) = \rho_1 g_{11}(x) - \varphi_1 g_{12}(x) \tag{107}$$

$$B_1(x) = \sqrt{3}[\rho_1 g_{11}(x) + \varphi_1 g_{12}(x)] \tag{108}$$

$$N_i(x) = 2\sqrt{3}\rho_1\varphi_1\left\{g_{11}(x)\int_0^x g_{12}(t)dt - g_{12}(y)\int_0^x g_{11}(t)dt\right\} \tag{109}$$

$$G_i(x) = 2\sqrt{3}\rho_1\varphi_1\left\{g_{11}(x)\int_0^x g_{12}(t)dt - g_{12}(y)\int_0^x g_{11}(t)dt\right\} + \frac{1}{\sqrt{3}}[\rho_1 g_{11}(x) + \varphi_1 g_{12}(x)] \tag{110}$$

$$H_i(x) = -\sqrt{3}\rho_1\varphi_1\left\{g_{11}(x)\int_0^x g_{12}(t)dt - g_{12}(y)\int_0^x g_{11}(t)dt\right\} + \frac{2}{\sqrt{3}}[\rho_1 g_{11}(x) + \varphi_1 g_{12}(x)] \tag{111}$$

5. Higher Derivatives of $N_i(x)$, $A_n(x)$, $B_n(x)$ and $N_n(x)$

5.1. Higher Derivatives of $N_i(x)$

In a recent article Hamdan et.al. [16] discussed higher derivatives of the standard Nield-Kuznetsov function of the first kind, $N_i(x)$, and arrived at the following iterative definition of its k+1st derivative, where $k \geq 1$:

$$N_i^{(k+1)}(x) = P_{k+1}(x)N_i(x) + Q_{k+1}(x)N'_i(x) - \frac{R_{k+1}(x)}{\pi} \tag{112}$$

where $P_{k+1}(x)$, $Q_{k+1}(x)$, and $-R_{k+1}(x)$ are the polynomial coefficients of $N_i(x)$, $N'_i(x)$ and the Wronskian $W(A_i(x), B_i(x)) = \frac{1}{\pi}$, respectively, in the *k+1st* derivative of $N_i(x)$. These polynomial coefficients are obtained from the known polynomial coefficients in the *kth* derivative of $N_i(x)$ using the following relationships:

$$P_{k+1} = P'_k(x) + xQ_k(x) \tag{113}$$

$$Q_{k+1} = Q'_k(x) + P_k(x) \tag{114}$$

$$R_{k+1}(x) = R'_k(x) + Q_k(x) \tag{115}$$

5.2. Higher Derivatives of $A_n(x)$ and $B_n(x)$

In what follows, we derive expressions for higher derivatives of the generalized Airy's functions, $A_n(x)$ and $B_n(x)$, and the generalized Nield-Kuznetsov function of the first-kind, $N_n(x)$. The Wronskian of $A_n(x)$ and $B_n(x)$ is given by equation (19) as: $W(A_n(x), B_n(x)) = \frac{2}{\pi}p^{\frac{1}{2}}\sin(p\pi)$.

Consider the homogeneous generalized Airy's equation, written in the form

$$u''_n = x^n u_n \tag{116}$$

The first few derivatives of (116) are:

$$u'''_n = x^n u'_n + nx^{n-1}u_n \tag{117}$$

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$$u^{iv}{}_n = 2nx^{n-1}u'_n + [x^{2n} + n(n-1)x^{n-2}]u_n \tag{118}$$

$$u^{v}{}_n = [x^{2n} + 3n(n-1)x^{n-2}]u'_n + [4nx^{2n-1} + n(n-1)(n-2)x^{n-3}]u_n \tag{119}$$

Each of the above derivatives of u_n is expressed in terms of u_n and u'_n. Their coefficients are polynomials for any given index $n > -2$. The generalized Airy's functions, $A_n(x)$ and $B_n(x)$ satisfy the homogeneous generalized Airy's equation and the derivatives above. We can thus express the *kth* derivatives of $A_n(x)$ and $B_n(x)$ in the following forms:

$$A_n^{(k)}(x) = P_k(x)A'_n(x) + Q_k(x)A_n(x) \tag{120}$$

$$B_n^{(k)}(x) = P_k(x)B'_n(x) + Q_k(x)B_n(x) \tag{121}$$

where $P_k(x)$ is the polynomial coefficient of $A'_n(x)$ and $B'_n(x)$, and $Q_k(x)$ is the polynomial coefficient of $A_n(x)$ and $B_n(x)$ in the kth derivatives of $A_n(x)$ and $B_n(x)$. A few of these polynomials are shown in the Table below.

Table 3. The Polynomials $\boldsymbol{P_n(x)}$ and $\boldsymbol{Q_n(x)}$.

k	$P_n(x)$	$Q_n(x)$
0	0	1
1	1	0
2	0	x^n
3	x^n	nx^{n-1}
4	$2nx^{n-1}$	$x^{2n} + n(n-1)x^{n-2}$
5	$x^{2n} + 3n(n-1)x^{n-2}$	$4nx^{2n-1} + n(n-1)(n-2)x^{n-3}$

Now, from (120) and (121), we obtain the *k+1st* derivatives, and express them as:

$$A_n^{(k+1)}(x) = P_{k+1}(x)A'_n(x) + Q_{k+1}(x)A_n(x) \tag{122}$$

$$B_n^{(k+1)}(x) = P_{k+1}(x)B'_n(x) + Q_{k+1}(x)B_n(x) \tag{123}$$

where the polynomial coefficients in the *k+1st* derivatives are obtained with the knowledge of the polynomial coefficients in the kth derivatives using the following relationships:

$$P_{k+1}(x) = P'_k(x) + Q_k(x) \tag{124}$$

$$Q_{k+1}(x) = x^n P_k(x) + Q'_k(x) \tag{125}$$

The above derivations furnish the following results.

Result 7: Higher derivatives, of all orders, of $A_n(x)$ are expressible in terms of $A_n(x)$ and $A'_n(x)$, and higher derivatives, of all orders, of $B_n(x)$ are expressible in terms of $B_n(x)$ and $B'_n(x)$. Furthermore, the k+1st derivatives of the generalized Airy's functions are defined iteratively by equations (122) and (123).

Result 8: The generalized Airy's polynomials arising from higher derivatives of the generalized Airy's functions are defined iteratively by equations (124) and (125).

5.3. Higher Derivatives of $N_n(x), G_n(x)$ and $H_n(x)$

The inhomogeneous generalized Airy's equation (1), when $f(x) = \kappa$, can be written as:

$$u''_n = x^n u_n + \kappa \tag{126}$$

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The function $N_n(x)$ satisfies the particular solution (67) to the inhomogeneous generalized Airy's equation (126). We thus have

$$N''_n(x) = x^n N_n(x) - W(A_n(x), B_n(x)) \tag{127}$$

Repeated differentiation of (127) gives

$$N'''_n(x) = x^n N'_n(x) + n x^{n-1} N_n(x) \tag{128}$$

$$N^{iv}_n(x) = 2n x^{n-1} N'_n(x) + [x^{2n} + n(n-1)x^{n-2}] N_n(x) - x^n W(A_n(x), B_n(x)) \tag{129}$$

$$\begin{aligned} N^{v}_n(x) = {} & [x^{2n} + 3n(n-1)x^{n-2}] N'_n(x) \\ & + [4n x^{2n-1} + n(n-1)(n-2)x^{n-3}] N_n(x) \\ & - 3n x^{n-1} W(A_n(x), B_n(x)) \end{aligned} \tag{130}$$

$$N_n^{(k)}(x) = P_k(x) N'_n(x) + Q_k(x) N_n(x) - R_k(x) W(A_n(x), B_n(x)) \tag{131}$$

The *k+1st* derivative of $N_n(x)$ is obtained from (131) and takes the form

$$N_n^{(k+1)}(x) = P_{k+1}(x) N'_n(x) + Q_{k+1}(x) N_n(x) - R_{k+1}(x) W(A_n(x), B_n(x)) \tag{132}$$

where the polynomial coefficients $P_{k+1}(x)$, $Q_{k+1}(x)$ and $R_{k+1}(x)$, in the *k+1st* derivative are obtained from $P_k(x)$, $Q_k(x)$ and $R_k(x)$, in the *kth* derivative using the relationships:

$$P_{k+1}(x) = P'_k(x) + Q_k(x) \tag{133}$$

$$Q_{k+1}(x) = Q'_k(x) + x^n P_k(x) \tag{134}$$

$$R_{k+1}(x) = R'_k(x) + P_k(x) \tag{135}$$

The *k+1st* derivatives of generalized Scorer functions can be obtained using (85), (86), (123) and (132), as:

$$\begin{aligned} G_n^{(k+1)}(x) &= \frac{1}{2\sqrt{p}\sin(p\pi)} N_n^{(k+1)}(x) + p B_n^{(k+1)}(x) \\ &= \frac{1}{2\sqrt{p}\sin(p\pi)} [P_{k+1}(x) N'_n(x) + Q_{k+1}(x) N_n(x) - R_{k+1}(x) W(A_n(x), B_n(x))] \\ &\quad + p[P_{k+1}(x) B'_n(x) + Q_{k+1}(x) B_n(x)] \end{aligned} \tag{136}$$

$$\begin{aligned} H_n^{(k+1)}(x) &= 2p B_n^{(k+1)}(x) - \frac{1}{2\sqrt{p}\sin(p\pi)} N_n^{(k+1)}(x) \\ &= -\frac{1}{2\sqrt{p}\sin(p\pi)} [P_{k+1}(x) N'_n(x) + Q_{k+1}(x) N_n(x) - R_{k+1}(x) W(A_n(x), B_n(x))] \\ &\quad + 2p[P_{k+1}(x) B'_n(x) + Q_{k+1}(x) B_n(x)] \end{aligned} \tag{137}$$

The above derivations furnish the following results.

Result 9: Higher derivatives, of all orders, of $N_n(x)$ are expressible in terms of $N_n(x)$ and $N'_n(x)$. Furthermore, the k+1st derivatives of the generalized Nield-Kuznetsov function of the first-

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kind is defined iteratively by equation (132), and those of the generalized Scorer functions by equations (136) and (137).

Result 10: The generalized polynomials arising from higher derivatives of the generalized Nield-Kuznetsov function of the first-kind are defined iteratively by equations (133), (134) and (135).

Using (17), (18), (26) and (27) in (122) and (123), and using (73) and (73) in (132) we obtain the following expressions for the *k*+1*st* derivatives of $A_n(x)$, $B_n(x)$ $N_n(x)$ in terms of the modified Bessel functions:

$$A_n^{(k+1)}(x) = -\frac{x}{3}P_{k+1}(x)\left[I_{-\frac{2}{3}}(\zeta) - I_{\frac{2}{3}}(\zeta)\right] + p\,(x)^{1/2}Q_{k+1}(x)\,[I_{-p}(\zeta) - I_p(\zeta)] \quad (138)$$

$$B_n^{(k+1)}(x) = \frac{x}{\sqrt{3}}P_{k+1}(x)\,\left[I_{-\frac{2}{3}}(\zeta) + I_{\frac{2}{3}}(\zeta)\right] + (px)^{1/2}Q_{k+1}(x)\,[I_p(\zeta) + I_{-p}(\zeta)] \quad (139)$$

$$N_n^{(k+1)}(x) = 2p\sqrt{p}(x)^{\frac{n+1}{2}}P_{k+1}(x)\left\{I_{p-1}(\zeta)\int_0^{\zeta} I_p(t)dt - I_{1-p}(\zeta)\int_0^{\zeta} I_{-p}(t)dt\right\}$$
$$+ 2p\sqrt{p}\,(x)^{\frac{1}{2}}Q_{k+1}(x)\left\{I_{-p}(\zeta)\int_0^{\zeta}[I_p(t)]dt - I_p(\zeta)\int_0^{\zeta}[I_{-p}(t)]dt\right\} \quad (140)$$
$$- R_{k+1}(x)W(A_n(x), B_n(x))$$

Using the above derivatives, we tabulate in **Table 4** the values of the k+1st derivative at $x = 0$ for each of the integral functions.

Table 4. Values at $\boldsymbol{x = 0}$ of the k+1st derivatives of generalized integral functions.

$A_n^{(k+1)}(0)$	$-\frac{(p)^p}{\Gamma(p)}P_{k+1}(0) + \frac{(p)^{1-p}}{\Gamma(1-p)}Q_{k+1}(0)$
$B_n^{(k+1)}(0)$	$-\frac{(p)^{p-\frac{1}{2}}}{\Gamma(p)}P_{k+1}(0) + \frac{(p)^{1/2-p}}{\Gamma(1-p)}Q_{k+1}(0)$
$N_n^{(k+1)}(0)$	$-\frac{2\sqrt{p}}{\pi}\sin(p\pi)\,R_{k+1}(0)$
$G_n^{(k+1)}(0)$	$\frac{1}{2\sqrt{p}\,\sin(p\pi)}N_n^{(k+1)}(0) + pB_n^{(k+1)}(0)$
$H_n^{(k+1)}(0)$	$2pB_n^{(k+1)}(0) - \frac{1}{2\sqrt{p}\,\sin(p\pi)}N_n^{(k+1)}(0)$

6. Conclusions

The main theme of this work has been the study and analysis of Airy's and generalized Airy's differential equations, and the integral functions that define their particular solutions. We attempted to fill in gaps that exit in the knowledge-base of the Nield-Kuznetsov and the Scorer functions. This includes defining the generalized Scorer functions, studying some of their properties and their relations to special functions that exist in the literature. We also provided further analysis and properties of the Nield-Kuznetsov and the generalized Nield-Kuznetsov functions and their relationships to the generalized Airy's functions. All functions have been expressed in terms of modified Bessel functions. Furthermore, as higher derivatives of the Nield-Kuznetsov and Scorer functions give rise to important Airy's polynomials and generalized Airy's polynomials, we provided in this work iterative definitions of the higher derivatives together with an iterative method of generating these polynomials. It is expected that this work will make its way into the many applications of the Airy's family of differential equations and their emerging integral functions.

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