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# A New Statistical Model Based on the Novel Generalized Odd Beta Prime Family of Continuous Probability Distributions with Applications to Cancer Disease Data Sets

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**Abstract:** Statistical modeling of lifetime data plays an essential role in a wide range of practical fields, such as health and engineering. There have been a lot of studies done to develop statistical models that can better describe health data than traditional models. For the first time, we pioneer a novel family of continuous probability distributions called the generalized odd beta prime generalized (GOBP-G) family of distributions. The cumulative distribution and probability density functions of the new family are presented. A new generalization of the Weibull distribution called "generalized odd beta prime-Weibull" (GOBPW) is proposed using the pioneered GOBP-G family. The mixture representations of the new distribution are defined and derived. Some formal statistical properties of the GOBPW distribution, such as the moments, moment generating function, incomplete moments, information generating function, entropies, stress-strength function, quantile function, and order statistics, are derived. The estimation of the parameters of the proposed distribution is evaluated using the maximum likelihood estimation approach. Different cancer disease data sets, such as the bladder, head and neck, acute bone, and blood cancers, are used to illustrate the applicability and usefulness of the new model and were compared using several statistical accuracy measures with that of well-established extended Weibull distributions, which are the beta modified Weibull distribution, Kumaraswamy modified Weibull distribution, gamma generalized modified Weibull distribution, gamma log-logistic Weibull distribution, and beta log-logistic Weibull distribution. The results show that the proposed model gives better results than the competitive models. This study could guide the relevant stakeholders in choosing a suitable statistical model for the health data instead of relying on traditional models to enhance decision-making.

**Keywords:** beta prime distribution; Weibull distribution; mixture representations; information generating function; entropies; order statistics; Monte Carlo simulation; blood cancer disease

**MSC:** 60E05;62E15

## 1. Introduction

Cancer is the second leading cause of death worldwide [1, 2], contributing to over 20% of all cause-specific deaths [3]. It is the leading cause of mortality in developed nations and the second leading cause of mortality in developing countries. The global burden of cancer continues to rise as the world population ages and grows, as do cancer-causing habits, particularly smoking, in developing countries. In 2008, there were 12.7

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million cancer cases and 7.6 million cancer deaths; 56% of cases and 64% of deaths occurred in developing countries [4]. Breast cancer accounts for 23% of all cancer cases and 14% of cancer deaths among women, while male lung cancer accounts for 17% of new cases and 23% of the total cancer deaths [5]. Cancers originate from accumulating epigenetic and genetic aberrations [6, 7]. Earlier studies developed a power law model based on multi-stage somatic mutation theory to describe age-dependent incidences for different cancer types [8, 9]. According to [10], most cancers have the same incidence pattern in classical epidemiological research, despite the complexity of carcinogenesis. The study conducted by [11] found a significant association between lifetime cancer risk and stem cell divisions. Some existing statistical models for evaluating a lifetime data set include time series [12], regression [13, 14], and more. More efficient method can provide a better estimate of the lifetime of an event.

Many researchers have extensively analyze lifetime data in a broad variety of practical domains, such as medical science, via statistical distributions. Medical experts typically use a statistical model to examine the distribution of their patients' lifetime data, including cancer patients. However, the chaotic nature of medical data makes it challenging to determine its underlying behavior. Consequently, it is necessary to study the nature of these data due to the fact that they directly affect people's lives and their health conditions. To determine reliable results in estimating medical data, it is often required to choose the appropriate statistical distribution of the data. To model biomedical data, it is possible to consider continuous probability distributions such as Weibull, gamma, log-normal, Rayleigh, logistic, exponential, and log-logistic distributions. The main rationale for this technique is the favorable properties of the distributions, including the existence of closed forms of probability density functions, which make it easy to estimate parameters for the data. However, these distributions are only represented by a few different distributional shapes and, consequently, are unable to describe the true behavior of the data. Moreover, it is evident that the biomedical data deviate from these traditional distributions because of the skewness and heavy tails or fat tails that are present in the data. This means that the aforementioned classical distributions are less suitable for modeling biomedical data, whose distribution is shown to be skewed. Therefore, there is a need to emphasize the importance of enhancing the classical distributions for modeling biomedical data.

There are popular distributions that are extensively utilized in modeling lifetime phenomena, including Weibull, exponential, gamma, and Rayleigh distributions [15]. In particular, the Weibull distribution is the most common and efficient model used for analyzing lifetime data. Another flexibility of the Weibull distribution is that it continues to be the most frequently used parameter distribution. However, this distribution is not adaptable enough to capture data types with considerable degrees of complexity. Almost all medical disorders, including neck, bladder, breast, and other cancers, have unimodal or modified unimodal hazard rates. After surgery, neck, bladder, and breast cancer risks are unimodal. For better information, we refer to [16] for bladder cancer, [17] for neck cancer, and [18] for breast cancer. Modeling such outcomes may not be suitable with exponential, Rayleigh, or Weibull distributions. As a result, researchers in the medical field are on the lookout for alternative distributions that can adequately model such lifetime data with a unimodal hazard function. The Weibull distribution is one of the common distributions for modeling mortality and failure [19]. However, the standard two-parameter Weibull distribution can only model monotonically increasing and decreasing hazard functions, making it less applicable for fitting when data indicate non-monotonic failure rates. Therefore, there is often a pressing need to improve the standard Weibull distribution for modeling biomedical data.

The drawbacks of the two-parameter Weibull distribution have motivated researchers to develop several generalizations and extended forms of it to obtain more flexible distributions in terms of modeling. Some of the recent development in modifications of the Weibull distribution mentioned in the literature include the exponentiated Weibull distribution by [20], Marshall–Olkin extended Weibull distribution by [21], the flexible

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Weibull extension by [22], the generalized modified Weibull distribution by [23], the Kumaraswamy Weibull distribution by [24], the beta modified Weibull distribution by [25], the beta generalized Weibull distribution [26], the beta inverse Weibull distribution by [27], the Kumaraswamy modified Weibull distribution by [28], the transmuted exponentiated generalized Weibull by [29], the Kumaraswamy transmuted exponentiated additive Weibull by [30], the Topp-Leone generated Weibull by [31]. Other studies that can be cited, including, among others, the Lindley Weibull distribution by [32], half-logistic generalized Weibull distribution by[33], the power generalized Weibull distribution by [34], the modified beta generalized Weibull distribution by [35], the generalized weighted Weibull distribution by [36], the beta exponentiated modified Weibull distribution by [37], the log-normal modified Weibull distribution by [38], the new Kumaraswamy Weibull distribution by [39], the generalized extended exponential Weibull distribution by [40], the Maxwell–Weibull distribution by [41], exponentiated additive Weibull distribution by [42], the flexible additive Weibull distribution by [43], the extended generalized inverted Kumaraswamy Weibull distribution by [44], the exponentiated generalized inverse flexible Weibull distribution by [45], the bivariate extended generalized inverted Kumaraswamy Weibull by [46], and the Khalil new generalized Weibull distribution by [47]. For a detailed review of extensions to the Weibull model, we refer to the works of [48] and [49].

Numerous studies have investigated the characteristics of different cancer data sets based on extended versions of Weibull distributions. For instance, head-and-neck cancer censored data was examined using the generalized power Weibull distribution proposed by [50]. In other studies, the beta-Weibull distribution introduced by [51] and the generalized modified Weibull distribution described by [52] were applied to fit a breast cancer data set. Similarly, a Weibull-based parametric model known as the log-beta Weibull distribution introduced by [53] was utilized to forecast the recurrence of prostate cancer for patients with clinically localized prostate cancer treated by open radical prostatectomy. The q-Weibull distribution by [54] was applied to data on cancer remission times, for which this distribution performed better than the standard Weibull distribution. The generalized Weibull distribution developed by [55] performed efficiently in modeling colorectal cancer. The performance of the beta-weighted Weibull distribution described by [56] was validated using bladder cancer data. The empirical proofs of the importance and flexibility of the transmuted exponentiated generalized Weibull distribution by [57], the Marshall-Olkin generalized-Weibull distribution by [58], the Marshall–Olkin power generalized Weibull distribution by [59], and the Gull alpha power Weibull distribution by [60] were assessed in modeling bladder cancer data sets. The modified Weibull extension distribution introduced by [61] was applied to model bile duct cancer data. The exponentiated log-inverse Weibull distribution established by [62] was analyzed using tongue cancer with an aneuploid DNA profile and bladder cancer. In a more recent study, the alpha power Kumaraswamy Weibull distribution proposed by [63] was presented to model different cancer data sets, including the blood and bone cancer data sets. The Weibull distribution has the following cumulative distribution function (cdf):

$$M\left(x;\alpha,\beta\right)=1-e^{-\alpha x^{\beta}},\quad x>0;\quad \alpha,\beta>0. \tag{1}$$

The corresponding probability density function (pdf) is defined as

$$m\left(x;\alpha,\beta\right)=\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}},\quad x>0;\quad \alpha,\beta>0, \tag{2}$$

where $\alpha$ and $\beta$ are scale and shape parameters, respectively.

The beta prime distribution, also known as the beta of the second kind, is a univariate continuous probability distribution used for modeling skewed data. This distribution can be used in a broad variety of scientific domains, including finance, hydrology, engineering, medical science, insurance, and machine learning. Despite its significance, it is not widely used by statisticians and has been little explored in the literature. For more details on beta distribution, see [64] and the references therein. In the current research, we attempt to offer a new family of continuous probability distributions yielded from the generalized beta prime distribution. As defined in [64], the beta prime has the following cdf:

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$$Q(x;\mathrm{a},\mathrm{b}) = I_{\frac{x}{(1+x)}} B(a,b), \quad x>0;\ a,b>0, \tag{3}$$

where $\mathrm{B}_x(a,b) = \int_0^x h^{a-1}(1-h)^{b-1}\,dh$ is the incomplete beta function.

The corresponding pdf is defined as

$$q(x;\mathrm{a},\mathrm{b}) = \frac{1}{B(a,b)}\frac{x^{a-1}}{(1+x)^{a+b}}, \quad x>, \tag{4}$$

where $a$ and $b$ shape parameters, respectively.

Over the past few years, several researchers have attempted to develop new families of continuous probability distributions that are more flexible than the conventional ones. It is a fact that statistical modeling can be more flexible with these new families, especially in practical fields such as health, engineering, the environment, and finance. Several methods are used in the statistical literature to generate a family of distributions from a well-known distribution by adding one or more parameters. When the parameters are added to the baseline distribution, the resulting distribution is the generalized distribution. The resulting distribution can be efficiently utilized in fitting lifetime data sets as it has the ability to accommodate monotonic and non-monotonic characteristics of the data [63, 65, 66]. Some families of continuous probability distributions are available in the literature, including the beta generalized-G family by [67], the beta-G by [65], a novel technique for generating families of continuous probability distributions by [68], the Lindley family of distributions by [69], the Zografos–Balakrishnan-G family of distributions by [70], the Topp–Leone family of distributions by [71], the Kumaraswamy transmuted-G family of distributions by [72], and the exponentiated Gompertz generated family of distributions by [73]. Other distribution families that can be cited are the generalized odd half-Cauchy family of distributions by [74], the odd-Burr generalized family of distributions by [75], the extended odd Fréchet-G family of distributions by [76], the generalized odd Weibull generated family of distributions by [77], the generalized odd gamma-G family of distributions by [78], the modified odd Weibull family of distributions by [79], the odd Dagum family of distributions by [80], the Zubair-G family of distributions by [81], the truncated Burr X-G family of distributions by [82], the alpha power Marshall-Olkin-G distributions by [83], the odd log-logistic Burr-X family of distributions by [84], the Teissier-G family of distributions [85], and the generalized alpha exponent power family of distributions by [86] among others.

Various authors argued that the new technique (T-X) of generating families of probability distributions developed by [68] allows the construction of more flexible families of probability distributions when modeling lifetime data and has proven itself in different settings. The odd Burr-III family of distributions was studied by [87], the odd Lomax generator of distributions proposed by [88], the odd log-logistic Poisson-G family of distributions introduced by [89], the odd log-logistic Topp–Leone-G family of distributions pioneered by [90], the odd Lomax generator of distributions established by [91], the odd Chen-G family of distributions constructed by [92], and the odd generalized N-H generated family of distributions introduced by [91], the odd extended exponential-G family of distributions defined by [93], the odd exponential-logarithmic family of distributions proposed by [94], the odd inverted Topp Leone–H family of distributions provided by [95], the odd log-logistic Weibull-G family of distributions developed [96], and the literature has provided some new groups, see [97].

Due to the importance of generating new families of probability distributions by adding new shape parameters, as recognized in the statistical literature. In this paper, we introduce the generalised odd beta prime generalised (GOBP-G for short) family, a novel family of continuous probability distributions based on the beta prime or beta second-kind distribution, using the T-X technique pioneered by[68]. In fact, based on the developed GOBP-G family, we proposed a new extended Weibull distribution, the so-called

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GOBPW distribution, that is more flexible than other existing versions. The primary justifications for introducing the GOBP-Weibull distribution are as follows:

i. To produce a more flexible extension of the Weibull distribution.
ii. To provide some statistical properties of the proposed model, such as the moments, moment generating function, incomplete moments, information generating function, entropies, stress-strength, quantile function, and order statistics.
iii. The density function of the new model can have decreasing, increasing, left-skewed, right-skewed, symmetric, or reversed-J shapes.
iv. It can be used as a useful model for modelling asymmetric data that cannot be well fitted by any of the popular statistical models, and it can be used to solve a wide range of problems in many applied sciences, including medicine, finance, and engineering.
v. To provide consistently better fits than some other well-reputed statistical models with good outcomes for some popular distributions.

The current paper is outlined as follow: In section 2, the new GOBP-G family of distributions is defined with the derivation of its validity test. Section 3 introduces the new special GOBPW distribution. Various mathematical and statistical properties of the new GOBP-Weibull distribution are obtained in Section 4. The maximum likelihood estimation for the model parameters is discussed in Section 5. Section 6 consists of simulation studies to assess the performance and consistency of the maximum likelihood estimators. Section 7 presents applications of the proposed GOBP-Weibull distribution, illustrated by means of four different cancer data sets. Finally, we offer some concluding remarks in Section 8.

## 2. Construction of the Generalized ODD Beta Prime Generalized Family

This section presents statistical descriptions of how the new GOBP-G family of distributions was developed via the $T-X$ method defined by [68].

Let us suppose that we have a random variable represented as $T$ with respective cdf and pdf as $Q\left(M\left(x;\phi\right)\right)$ and $r(t)$, such that $T\in\left[w,s\right]$ for $-\infty\le w<s<\infty$. Then the cdf of the distribution must satisfy the following conditions:

(i) $Q\left(M\left(x;\phi\right)\right)\in\left(w,s\right).$

(ii) $Q\left(M\left(x;\phi\right)\right)$ must be differentiable and monotonically decreasing function.

(iii) $Q\left(M\left(x;\phi\right)\right)\to w$ as $x\to-\infty,$ and

(iv) $Q\left(M\left(x;\phi\right)\right)\to s$ as $x\to\infty.$

We can obtain the cdf of the new $T-X$ family of distribution defined by [68] as

$$P\left(x\right)=\int_{w}^{Q\left(M\left(x;\phi\right)\right)}r(t)dt,\qquad x\in\square,\tag{5}$$

where $Q\left(M\left(x;\phi\right)\right)$ is the link function of distribution function $M\left(x;\phi\right)$ for any random variable $X$, $r(t)$ is the density function of random variable $T$, $\phi$ is the parameter of the distribution function $M\left(x;\phi\right)$. The link function $Q\left(M\left(x;\phi\right)\right)$ must satisfy the above conditions.

The corresponding pdf to (5) is given by

$$p\left(x\right)=\left\{\frac{\partial}{\partial x}Q\left(M\left(x;\phi\right)\right)\right\}r\left\{Q\left(M\left(x;\phi\right)\right)\right\},\quad x\in\square.$$

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Several authors use the T-X family method owing to its effectiveness; see [97] for more on this technique. The present study offered an additional family as follows.

Suppose a random variable X has a beta prime distribution with parameters $a,b>0$, $X\in[0,\infty]$, and X is any random variable with cdf $M(x;\phi)$, $\phi\in$ . The cdf of the GOBP-G family of distribution is defined by taking the pdf in (5) to be the pdf of (3) and the upper limit to be $Q[M(x;\phi)]=M(x;\phi)/[1-M(x;\phi)]$. Hence, the GOBP-G family has a cdf given by:

$$P_{GOBP-G}(x;a,b,\phi)=\frac{1}{B(a,b)}\int_0^{M^c(x;\phi)/[1-M^c(x;\phi)]}\frac{x^{a-1}}{(1+x)^{a+b}}dx,\quad x\in . \tag{6}$$

After some simplifications, the cdf of GOBP-G family is:

$$P_{GOBP-G}(x;a,b,\phi)=\frac{1}{B(a,b)}.B_{M^c(x;\phi)/[1-M^c(x;\phi)]}(a,b),\ a,b>0,\ x,\phi\in , \tag{7}$$

with pdf

$$\begin{aligned}p_{GOBP-G}(x;a,b,\phi)&=\frac{1}{B(a,b)}\frac{\left[M^c(x;\phi)/1-M^c(x;\phi)\right]^{a-1}}{\left[1+\left[M^c(x;\phi)/1-M^c(x;\phi)\right]\right]^{a+b}}\cdot\frac{cm(x;\phi)M^{c-1}(x;\phi)}{\left[1-M^c(x;\phi)\right]^2}\\&=\frac{cm(x;\phi)M^{ca-1}(x;\phi)}{B(a,b)\left[1-M^c(x;\phi)\right]^{a+1}\left[1+\left[M^c(x;\phi)/1-M^c(x;\phi)\right]\right]^{a+b}},\end{aligned} \tag{8}$$

where, $M(x;\phi)$ is cdf of any baseline distribution with parameter $\phi$, $m(x;\phi)$ is the corresponding pdf of any baseline distribution, and c additional shape parameter.

To evaluate the validity of GOBP-G family of distribution, we apply the integral

$$\int_{-\infty}^{\infty}f(x)dx=1. \tag{9}$$

If satisfies the condition given in (9), we can say that the GOBP-G is a valid family of statistical distributions.

The proof is as follows:

Substituting (8) in (9), we get

$$\frac{c}{B(a,b)}\int_0^{\infty}\frac{m(x;\phi)M^{ca-1}(x;\phi)}{\left[1-M^c(x;\phi)\right]^{a+1}\left[1+\left[M^c(x;\phi)/1-M^c(x;\phi)\right]\right]^{a+b}}dx. \tag{10}$$

Let $y=M^c(x;\phi)/1-M^c(x;\phi)$, $dy=cm(x;\phi)M^{c-1}(x;\phi)dx/\left[1-M^c(x;\phi)\right]^2$,

so that $dx=\left[1-M^c(x;\phi)\right]^2dy/cm(x;\phi)M^{c-1}(x;\phi)$. (11)

Putting (11) in (10), we have

$$\frac{1}{B(a,b)}\int_0^{\infty}\frac{M^{c(a-1)}(y;\phi)}{\left[1-M^c(y;\phi)\right]^{a-1}(1+y)^{a+b}}dy=1. \tag{12}$$

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Hence, GOBP-G family of distributions is indeed a family of statistical distributions.

### 3. Construction of the Generalized Odd Beta Prime-Weibull Distribution

In this section, a new lifetime Weibull model is introduced from the proposed family defined in (8), which is considered a new sub-model of the GOBP-G family, the so-called generalized odd beta prime-Weibull (GOBPW) distribution.

Consider the cdf and pdf of the Weibull with $\alpha,\beta>0$ as given in (1) and (2) a $M(x;\alpha,\beta)=1-e^{-\alpha x^{\beta}}$ and $m(x;\alpha,\beta)=\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}$; for $x>0$, where the cdf of X has the GOBPW distribution, say $X \square GOBPW(a,b,c,\alpha,\beta)$ is given by

$$P_{GOBPW}(x;a,b,c,\alpha,\beta)=\frac{1}{B(a,b)}.B_{\left(1-e^{-\alpha x^{\beta}}\right)^{c}\Big/\left[1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right]}(a,b);\quad a,b,c,\alpha,\beta>0,\ x>0, \tag{13}$$

with corresponding pdf

$$p_{GOBPW}(x;a,b,c,\alpha,\beta)=\frac{c\left[\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\right].\left(1-e^{-\alpha x^{\beta}}\right)^{ca-1}}{B(a,b)\left[1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right]^{a+1}\left[1+\left[\left(1-e^{-\alpha x^{\beta}}\right)^{c}\Big/1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right]\right]^{a+b}}, \tag{14}$$

The survival function for GOBPW distribution is

$$\begin{aligned}\varsigma_{GOBPW}(x;a,b,c,\alpha,\beta)&=1-P_{GOBPW}(x;a,b,c,\alpha,\beta)\\&=1-\frac{1}{B(a,b)}.B_{\left(1-e^{-\alpha x^{\beta}}\right)^{c}\Big/\left[1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right]}(a,b).\end{aligned} \tag{15}$$

The hazard function for GOBPW distribution is expressed as

$$\eta_{GOBPW}(x;a,b,c,\alpha,\beta)=\frac{\dfrac{c\left[\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\right].\left(1-e^{-\alpha x^{\beta}}\right)^{ca-1}}{B(a,b)\left[1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right]^{a+1}\left[1+\left[\left(1-e^{-\alpha x^{\beta}}\right)^{c}\Big/1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right]\right]^{a+b}}}{1-\dfrac{1}{B(a,b)}.B_{\left(1-e^{-\alpha x^{\beta}}\right)^{c}\Big/1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}}(a,b)}. \tag{16}$$

Graphs of the pdf and hazard function of the GOBPW distribution at several different distribution parameters are given in Figures 1 and 2, respectively.

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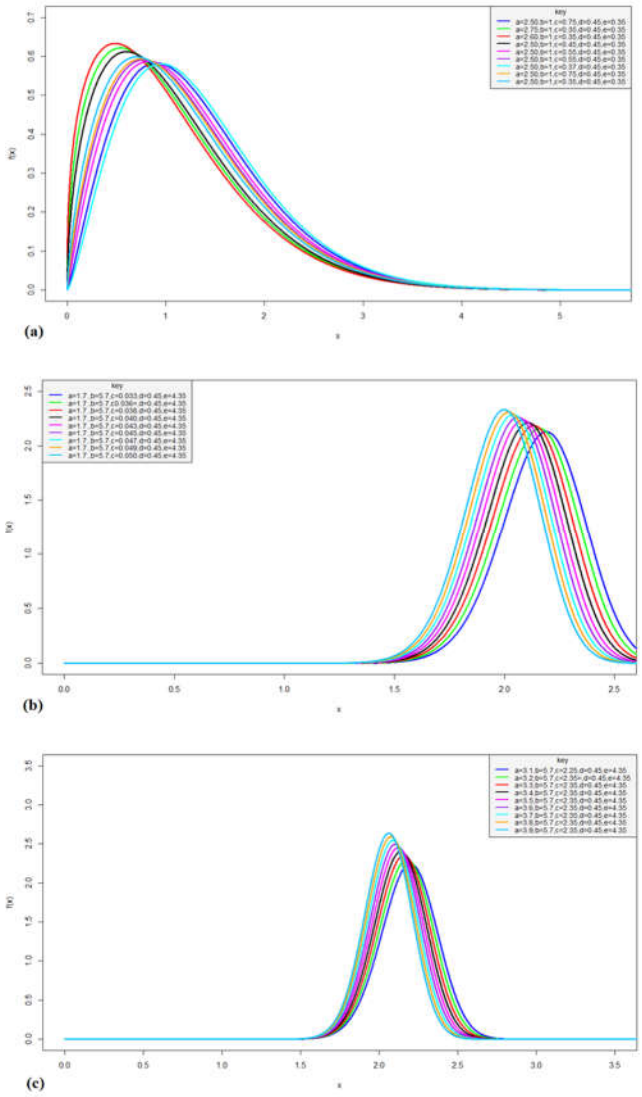


**Figure 1.** Plots of the GOBPW distribution density function.

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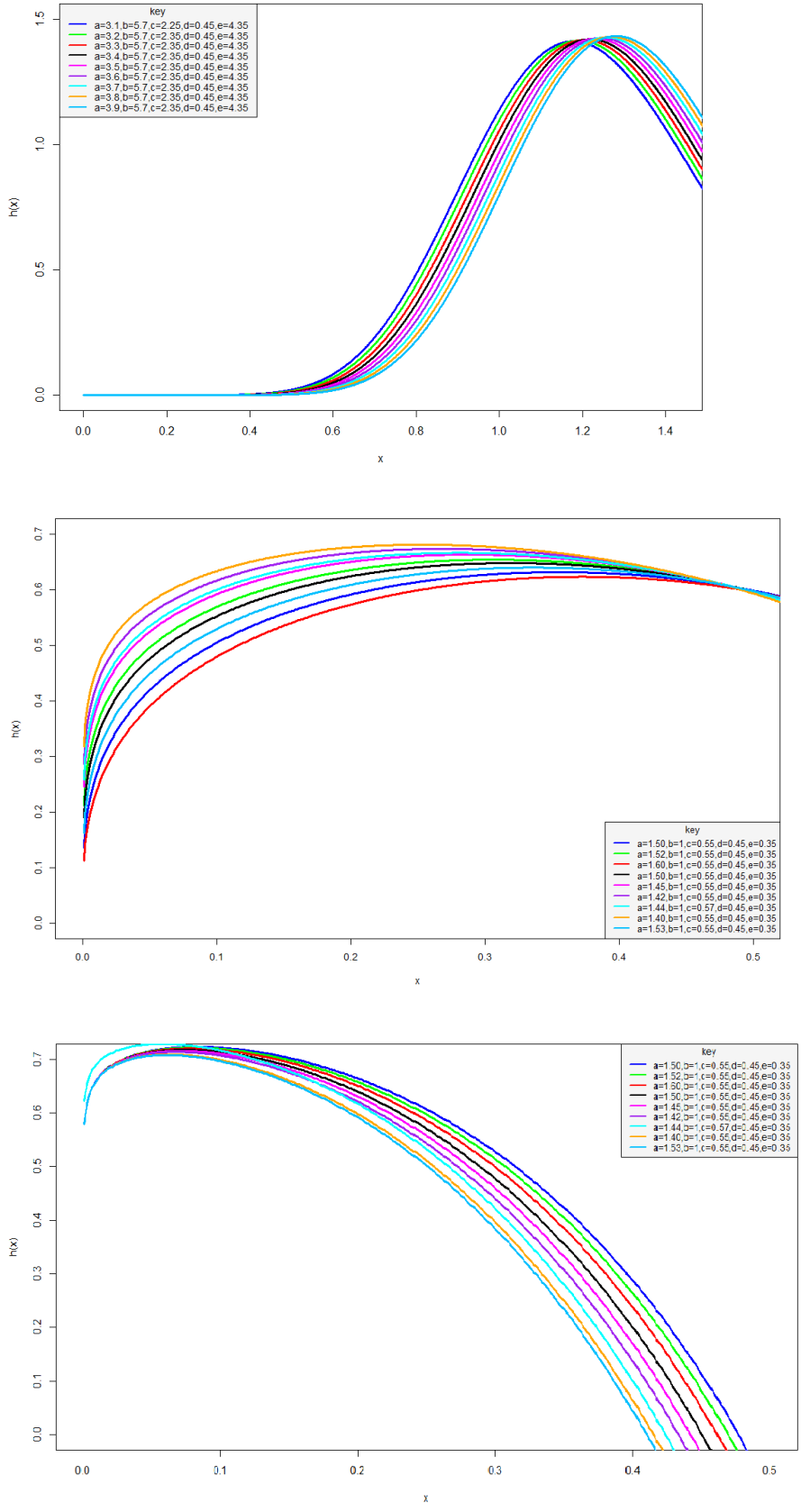


**Figure 2.** Plots of the GOBPW distribution hazard rate function.

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### 3.1. Mixture Representations of the GOBPW Distribution

Here, we derive the mixture representations of the proposed GOBPW distribution. Using the following binomial expansion given by [98] as

$$(1+m)^{-n} = \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} m^i. \tag{17}$$

Applying (17) in (14), lead to

$$\begin{aligned} p_{GOBPW}(x;a,b,c,\alpha,\beta) &= \frac{\alpha\beta c x^{\beta-1} e^{-\alpha x^\beta} \left(1-e^{-\alpha x^\beta}\right)^{ca-1}}{B(a,b)\left[1-\left(1-e^{-\alpha x^\beta}\right)^c\right]^{a+1}} \sum_{i=0}^{\infty} (-1)^i \binom{a+b+i-1}{i} \cdot \left(\frac{\left(1-e^{-\alpha x^\beta}\right)^c}{\left[1-\left(1-e^{-\alpha x^\beta}\right)^c\right]}\right)^i \\ &= \frac{\alpha\beta c x^{\beta-1} e^{-\alpha x^\beta}}{B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{a+b+i-1}{i} \frac{\left(1-e^{-\alpha x^\beta}\right)^{c(a+i)-1}}{\left[1-\left(1-e^{-\alpha x^\beta}\right)^c\right]^{a+i+1}}. \end{aligned} \tag{18}$$

For $|\psi|<1$, the power series holds

$$(1-\psi)^{-n} = \sum_{j=0}^{\infty} \frac{\Gamma(n+j)}{j!\Gamma(n)} \psi^j. \tag{19}$$

Then, based on (19), the denominator of (18) becomes

$$p_{GOBPW}(x;a,b,c,\alpha,\beta) = \frac{\alpha\beta c x^{\beta-1} e^{-\alpha x^\beta}}{B(a,b)} \sum_{i,j=0}^{\infty} (-1)^i \binom{a+b+i-1}{i} \cdot \frac{\Gamma(a+i+j+1)}{j!\Gamma(a+i+1)} \left(1-e^{-\alpha x^\beta}\right)^{c(a+1)+cj-1}. \tag{20}$$

After simplifications, we get

$$\begin{aligned} p_{GOBPW}(x;a,b,c,\alpha,\beta) &= \frac{\alpha\beta c x^{\beta-1} e^{-\alpha x^\beta}}{B(a,b)} \sum_{i,j=0}^{\infty} (-1)^i \binom{a+b+i-1}{i} \cdot \frac{\Gamma(a+i+j+1)}{j!\Gamma(a+i+1)} \\ &\quad \times \sum_{k=0}^{\infty} (-1)^k \binom{c(a+i+j)-1}{k} e^{-\alpha k x^\beta} \\ &= x^{\beta-1} \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} \cdot e^{-(1+k)\alpha x^\beta}, \end{aligned} \tag{21}$$

where $\omega_{i,j,k} = \frac{\alpha\beta c}{B(a,b)} (-1)^{i+k} \binom{a+b+i-1}{i} \binom{c(a+i+j)-1}{k} \cdot \frac{\Gamma(a+i+j+1)}{j!\Gamma(a+i+1)}$.

Which is the pdf of GOBPW distribution in terms of mixture representations.

## 4. Statistical Properties of GOBPW Distribution

Some important statistical properties of GOBPW distribution are supplied in this section, such as the moments, moment generating function, incomplete moments, information generating function, entropies, stress-strength function, quantile function, and order statistics.

### 4.1. Moments

The moments of any model plays an important role in any statistical analysis. They allow to determine essential features of the distribution, for example, tendency, dispersion, coefficients of variation, skewness, and kurtosis. Here, we derive the moments of the

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GOBPW distribution. Based on (21), the $r^{th}$ moment of the GOBPW distribution is given by

$$E\left(X^r\right)=\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\int_{-\infty}^{\infty}x^{\beta+r-1}e^{-(1+k)\alpha x^{\beta}}dx. \tag{22}$$

Let $y=(1+k)\alpha x^{\beta}$ we get $dx=dy/\alpha\beta(1+k)x^{\beta-1}$. (23)

Inserting (23) in (22), then the $r^{th}$ moment of the GOBPW distribution is

$$E\left(X^r\right)=\frac{1}{\alpha\beta(1+k)}\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\int_{-\infty}^{\infty}x^{\beta+r-1-\beta+1}e^{-y}dy$$

$$=\frac{1}{\alpha\beta(1+k)}\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}\int_{-\infty}^{\infty}\left[\frac{y}{\alpha(1+k)}\right]^{\frac{r}{\beta}}e^{-y}dy.$$

After some simplifications, the $r^{th}$ moment of the GOBPW distribution is

$$E\left(X^r\right)=\frac{1}{\beta\left\{\alpha(1+k)\right\}^{1+\frac{r}{\beta}}}\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}.\Gamma\left(\frac{r}{\beta}+1\right). \tag{24}$$

*4.2. Moment Generating Function*

The moment generating function plays an important role in probability theory and statistics. It allows to uniquely determines its probability distribution. That is, it is an alternative specification of its distribution. Here, we obtained the moment generating function of the GOBPW distribution based on (21) as follows:

$$M_X\left(t\right)=\int_{-\infty}^{\infty}e^{tx}.x^{\beta}\sum_{i,j,k=0}^{\infty}\omega_{i,j,k}.e^{-(1+k)\alpha x^{\beta}}dx. \tag{25}$$

Utilizing the exponential series expansion as

$$e^{t}=\sum_{l=0}^{\infty}\frac{t^l}{l!}. \tag{26}$$

The moment generating function of the GOBPW distribution is obtained by using (26) in (25) as:

$$M_X\left(t\right)=\sum_{i,j,k,l=0}^{\infty}\omega_{i,j,k}\frac{t^l}{l!}\int_{0}^{\infty}x^{\beta+l-1}.e^{-(1+k)\alpha x^{\beta}}dx. \tag{27}$$

Using (23), the moment generating function of the GOBPW distribution holds

$$M_X\left(t\right)=\frac{\sum_{i,j,k,l=0}^{\infty}\omega_{i,j,k}t^l}{l!\alpha\beta(1+k)}\int_{0}^{\infty}\left[\frac{y}{\alpha(1+k)}\right]^{\frac{l}{\beta}}e^{-y}dy$$

$$=\frac{\sum_{i,j,k,l=0}^{\infty}\omega_{i,j,k}t^l}{l!\beta\left\{\alpha(1+k)\right\}^{1+\frac{l}{\beta}}}.\Gamma\left(\frac{l}{\beta}+1\right). \tag{28}$$

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*4.3. Incomplete Moments*

The incomplete moments have applications in modeling lifetime data. They can be used to calculate important quantities, including the mean residual function, mean waiting time, mean deviations, Bonferroni and Lorenz curves. In this sub-section, we present the expression for the incomplete moments of the GOBPW distribution. Based on (21), the $r^{th}$ incomplete moment of the GOBPW distribution is

$$I_r(t) = \int_{-\infty}^{t} x^r . x^{\beta} \sum_{i,j,k=0}^{\infty} \omega_{i,j,k} . e^{-(1+k)\alpha x^{\beta}} dx. \tag{29}$$

Using (23), the incomplete moment of the GOBPW distribution yields

$$\begin{aligned} I_r(t) &= \frac{\sum_{i,j,k=0}^{\infty} \omega_{i,j,k}}{\alpha\beta(1+k)} \int_0^{\alpha(1+k)t^{\beta}} \left[\frac{y}{\alpha(1+k)}\right]^{\frac{r}{\beta}} e^{-y} dy \\ &= \frac{\sum_{i,j,k=0}^{\infty} \omega_{i,j,k}}{\beta\{\alpha(1+k)\}^{1+\frac{r}{\beta}}} . \gamma\left(\frac{r}{\beta}+1 \ , \alpha(1+k)t^{\beta}\right). \end{aligned} \tag{30}$$

*4.4. Information Generating Function*

In information theory and statistics, the information generating function has been utilized to generate some important information quantities, such as Kullback-Leibler divergence and Shannon entropy. It has been widely applied in physics and chemistry to analyse the atomic structure of a given phenomenon or system. Here, we define the information generating function of the GOBPW distribution as follow:

$$I_{\theta}(x) = \int_{-\infty}^{\infty} f^{\theta}(x) dx, \quad \theta > 0, \tag{31}$$

where $f(x)$ is the pdf of the GOBPW distribution defined in (14). The integrand in (31) can be expressed as:

$$f^{\theta}(x) = \frac{\{c\alpha\beta / B(a,b)\}^{\theta} x^{\theta(\beta-1)} e^{-\alpha\theta x^{\beta}} \left(1-e^{-\alpha x^{\beta}}\right)^{\theta(ca-1)}}{\left[1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right]^{\theta(a+1)} \left\{1+\left[\left(1-e^{-\alpha x^{\beta}}\right)^{c} / 1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right]\right\}^{\theta(a+b)}}. \tag{32}$$

Using the binomial expansion (17), then the integrand in (32) gives

$$\begin{aligned} f^{\theta}(x) &= \frac{k x^{\theta(\beta-1)} e^{-\alpha\theta x^{\beta}} \left(1-e^{-\alpha x^{\beta}}\right)^{\theta(ca-1)}}{\left[1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right]^{\theta(a+1)}} . \sum_{l=0}^{\infty} (-1)^{l} \binom{\theta(a+b)+l-1}{l} \left\{\frac{\left(1-e^{-\alpha x^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}}\right\}^{l} \\ &= k x^{\theta(\beta-1)} e^{-\alpha\theta x^{\beta}} \sum_{l=0}^{\infty} (-1)^{l} \binom{\theta(a+b)+l-1}{l} \frac{\left\{\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right\}^{\theta(a+b)+l}}{\left\{1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right\}^{\theta(a+b)+l}}, \end{aligned} \tag{33}$$

where $k = \left\{\frac{c\alpha\beta}{B(a,b)}\right\}^{\theta}$.

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Applying the power series (19) to last term of (33), it leads to

$$f^{\theta}(x)=kx^{\theta(\beta-1)}e^{-\alpha\theta x^{\beta}}\sum_{l=0}^{\infty}(-1)^{l}\binom{\theta(a+b)+l-1}{l}.\frac{\sum_{m=0}^{\infty}\Gamma\left[\theta(a+1)+l+m\right]}{m!\Gamma\left[\theta(a+1)+l\right]}\left\{\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right\}^{\theta(a+b)+l+m}. \tag{34}$$

After some simplifications, we get

$$\begin{aligned} f^{\theta}(x) &= kx^{\theta(\beta-1)}e^{-\alpha\theta x^{\beta}}\sum_{l=0}^{\infty}(-1)^{l}\binom{\theta(a+b)+l-1}{l}\frac{\Gamma\left[\theta(a+1)+l+m\right]}{m!\Gamma\left[\theta(a+1)+l\right]} \\ &\times\sum_{p=0}^{\infty}(-1)^{p}\binom{c\{\theta(a+1)+l+m\}}{p}e^{-\alpha px^{\beta}} \\ &=\sum_{l,m,p=0}^{\infty}\Lambda_{l,m,p}x^{\theta(\beta-1)}.e^{-\alpha(\theta+p)x^{\beta}}, \end{aligned} \tag{35}$$

where

$$\Lambda_{l,m,p}=\left\{\frac{c\alpha\beta}{B(a,b)}\right\}^{\theta}(-1)^{(l+p)}\binom{\theta(a+b)+l-1}{l}\binom{c\{\theta(a+1)+l+m\}}{p}\frac{\Gamma\left[\theta(a+1)+l+m\right]}{m!\Gamma\left[\theta(a+1)+l\right]}.$$

Inserting (35) in (31), the information generating function of the GOBPW distribution yields

$$I_{\theta}(x)=\int_{0}^{\infty}\sum_{l,m,p=0}^{\infty}\Lambda_{l,m,p}\,x^{\theta(\beta-1)}\,e^{-\alpha(\theta+p)x^{\beta}}dx. \tag{36}$$

Letting $w=\alpha(\theta+p)x^{\beta}$ we get $dx=dw/\alpha\beta(\theta+p)x^{\beta-1}$. (37)

Substituting (37) in (36), we have

$$\begin{aligned} I_{\theta}(x) &= \sum_{l,m,p=0}^{\infty}\Lambda_{l,m,p}\int_{0}^{\infty}x^{\theta(\beta-1)-\beta+1}e^{-w}\frac{dw}{\alpha\beta(\theta+p)} \\ &=\frac{\sum_{l,m,p=0}^{\infty}\Lambda_{l,m,p}}{\beta\{\theta+p\}^{\frac{\theta(\beta-1)+1}{\beta}}}\int_{0}^{\infty}w^{\frac{\theta(\beta-1)+1}{\beta}-1}e^{-w}dw \\ &=\frac{\sum_{l,m,p=0}^{\infty}\Lambda_{l,m,p}}{\beta\{\theta+p\}^{\frac{\theta(\beta-1)+1}{\beta}}}\Gamma\left(\frac{\theta(\beta-1)+1}{\beta}\right). \end{aligned} \tag{38}$$

*4.5. Entropies*

4.5.1. Rényi Entropy

The Rényi entropy of order $\delta$ for the GOBPW distribution is obtained as:

$$R_{\delta}(x)=\frac{1}{1-\delta}\log\left[\int_{-\infty}^{\infty}f^{\delta}(x)dx\right],\quad \delta>0,\ \delta\neq 1,\ x\in\mathbb{R}. \tag{39}$$

The integrand $\int_{-\infty}^{\infty}f^{\delta}(x)dx$ is as defined in (38), so that

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$$\int_{-\infty}^{\infty} f^{\delta}(x)dx = \frac{\sum_{l,m,p=0}^{\infty} \Lambda_{l,m,p}}{\beta\{\delta+p\}^{\frac{\delta(\beta-1)+1}{\beta}}} \Gamma\left(\frac{\delta(\beta-1)+1}{\beta}\right). \quad (40)$$

Substituting (40) in (39), we obtain the Rényi entropy of the GOBPW distribution as

$$R_{\delta}(x) = \frac{1}{1-\delta}\log\left[\frac{\sum_{l,m,p=0}^{\infty} \Lambda_{l,m,p}}{\beta\{\delta+p\}^{\frac{\delta(\beta-1)+1}{\beta}}} \Gamma\left(\frac{\delta(\beta-1)+1}{\beta}\right)\right]. \quad (41)$$

4.5.2. $q$ Entropy

The $q$ entropy of the GOBPW distribution is obtained as follows:

$$Q_{\delta}(x) = \frac{1}{1-\delta}\log\left[1-\int_{-\infty}^{\infty} f^{\delta}(x)dx\right], \quad \delta>0,\ \delta\neq 1,\ x\in\square. \quad (42)$$

The $q$ entropy of the GOBPW distribution is derived by substituting the integral in (42) with (38) yeilds

$$Q_{\delta}(x) = \frac{1}{1-\delta}\log\left[1-\frac{\sum_{l,m,p=0}^{\infty} \Lambda_{l,m,p}}{\beta\{\delta+p\}^{\frac{\delta(\beta-1)+1}{\beta}}} \Gamma\left(\frac{\delta(\beta-1)+1}{\beta}\right)\right]. \quad (43)$$

*4.6. Stress-Strength System Model*

The idea of stress-strength is one of the primary determinants of the failure of engineering systems. In the stress-strength system modelling, the reliability (R) quantifies how resistant a system is to failure under conditions of random stress ($X_2$) in comparison to the strength ($X_1$). Failures occur when the applied stress exceeds the system's strength. In this case, we can write $R = P(X_2 < X_1)$. In this subsection, we derived the stress-strength for the GOBPW distribution.

Suppose $X_1$ and $X_2$ are two independent random variables both follow the GOBPW distribution with parameters , then the stress-strength is expressed as:

$$\begin{aligned} R = P(X_2 < X_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} p(x_1;a_1,b,c,\alpha,\beta)p(x_2;a_2,b,c,\alpha,\beta)dx_1dx_2 \\ &= \int_{-\infty}^{\infty} p(x_1;a_1,b,c,\alpha,\beta)P(x_1;a_2,b,c,\alpha,\beta)dx_1, \end{aligned} \quad (44)$$

where $p(x_1;a_1,b,c,\alpha,\beta)$ and $P(x_1;a_2,b,c,\alpha,\beta)$ are pdf and cdf of the GOBPW distribution, respectively.

Applying the series expansion for the cdf of beta model studied by [99] as

$$I_{\vartheta}(a,b) = \frac{\vartheta^a}{B(a,b)}\sum_{m=0}^{\infty}\frac{(1-b)_m}{(a+m)_m!}\vartheta^m. \quad (45)$$

Applying (45) in (44), we have

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$$p(x_1;a_1,b,c,\alpha,\beta)P(x_1;a_2,b,c,\alpha,\beta)=p(x_1;a_1,b,c,\alpha,\beta)$$
$$\times\frac{\left\{\left(1-e^{-\alpha x_1^{\beta}}\right)^c\Big/1-\left(1-e^{-\alpha x_1^{\beta}}\right)^c\right\}^{a_2}}{B(a_2,b)}\sum_{m=0}^{\infty}\frac{(1-b)_m}{(a_2+m)_m!}\left\{\frac{\left(1-e^{-\alpha x_1^{\beta}}\right)^c}{1-\left(1-e^{-\alpha x_1^{\beta}}\right)^c}\right\}^m$$
$$=\frac{c\alpha\beta x_1^{\beta-1}e^{-\alpha x_1^{\beta}}}{B(a_1,b).B(a_2,b)}\sum_{m=0}^{\infty}\frac{(1-b)_m}{(a_2+m)_m!}\frac{\left(1-e^{-\alpha x_1^{\beta}}\right)^{ca_1-1+ca_2+cm}}{\left\{1-\left(1-e^{-\alpha x_1^{\beta}}\right)^c\right\}^{a_1+a_2+1+m}} \tag{46}$$
$$\times\frac{1}{\left\{1+\left[\left(1-e^{-\alpha x_1^{\beta}}\right)^c\Big/1-\left(1-e^{-\alpha x_1^{\beta}}\right)^c\right]\right\}^{a_1+b}}.$$

Applying the binomial expansion (17) in (46), we can write

$$p(x_1;a_1,b,c,\alpha,\beta)P(x_1;a_2,b,c,\alpha,\beta)=\frac{c\alpha\beta x_1^{\beta-1}e^{-\alpha x_1^{\beta}}}{B(a_1,b).B(a_2,b)}\sum_{m,p=0}^{\infty}(-1)^p\binom{a_1+b+p-1}{p}$$
$$\times\frac{(1-b)_m}{(a_2+m)_m!}\frac{\left(1-e^{-\alpha x_1^{\beta}}\right)^{c(a_1+a_2+m+p)-1}}{\left\{1-\left(1-e^{-\alpha x_1^{\beta}}\right)^c\right\}^{a_1+a_2+m+p+1}}. \tag{47}$$

Applying the power series (19) in (47), it becomes

$$p(x_1;a_1,b,c,\alpha,\beta)P(x_1;a_2,b,c,\alpha,\beta)=\sum_{m,p,s,q=0}^{\infty}\Omega_{m,p,s,q}\,x_1^{\beta-1}e^{-\alpha x_1^{\beta}(1+q)}, \tag{48}$$

where,

$$\Omega_{m,p,s,q}=\frac{c\alpha(-1)^{p+q}}{B(a_1,b).B(a_2,b)}\binom{a_1+b+p-1}{p}\frac{(1-b)_m}{(a_2+m)_m!}\binom{c(a_1+a_2+m+p+s)-1}{q}$$
$$\times\frac{\Gamma(a_1+a_2+m+p+s+1)}{s!\Gamma(a_1+a_2+m+p+1)}.$$

Substituting (48) in (44), it gives

$$R=\sum_{m,p,s,q=0}^{\infty}\Omega_{m,p,s,q}\int_0^{\infty}x_1^{\beta-1}e^{-\alpha x_1^{\beta}(1+q)}dx_1. \tag{49}$$

$$\text{Let}\quad \varsigma=\alpha x_1^{\beta}(1+q)\quad\text{we get}\quad dx_1=d\varsigma\big/\alpha\beta x_1^{\beta-1}(1+q). \tag{50}$$

Inserting (50) in in (49), we obtained the stress-strength for the GOBPW distribution as

$$R=\sum_{m,p,s,q=0}^{\infty}\frac{\Omega_{m,p,s,q}}{\alpha\beta(1+q)}. \tag{51}$$

*4.7. Quantile Function*

In probability and statistics, the quantile function is associated with the probability distribution of a random variable. Its purpose is to specify the value of the random variable such that the probability of the variable being less than or equal to that value equals

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the given probability. In this subsection, we obtained the quantile function of the GOBPW distribution.

Let the random variable X follow the GOBP-G family with cdf (7). Then the quantile function of the GOBPW distribution is obtained by inverting (7) as:

$$\frac{M^c(x;\phi)}{1-M^c(x;\phi)} = I^{-1}(u;a,b), \quad 0<u<1. \tag{52}$$

So that

$$\begin{aligned} M^c(x;\phi) &= \left[1-M^c(x;\phi)\right] I^{-1}(u;a,b) \\ &= \frac{I^{-1}(u;a,b)}{1+I^{-1}(u;a,b)} \\ &= \frac{\Phi}{1+\Phi}, \end{aligned} \tag{53}$$

where $\Phi = I^{-1}(u;a,b)$.

Therefore, the quantile of a baseline cdf is derived as:

$$M(x;\phi) = \left\{\frac{\Phi}{1+\Phi}\right\}^{\frac{1}{c}}. \tag{54}$$

Replacing $M(x;\phi)$ in (54) with the cdf of Weibull distribution in (1), we have

$$1-e^{-\alpha x^{\beta}} = \left\{\frac{\Phi}{1+\Phi}\right\}^{\frac{1}{c}}. \tag{55}$$

After some simplifications, the quantile function of the GOBPW distribution yields

$$x = \left\{-\frac{1}{\alpha}\log\left\{1-\hat{\lambda}\right\}\right\}^{\frac{1}{\beta}}, \tag{56}$$

where $\hat{\lambda} = \left\{\frac{\Phi}{1+\Phi}\right\}^{\frac{1}{c}}$.

*4.8. Order Statistics*

Order statistics are a very useful concept in many areas of statistical theory and practice. They have a broad variety of applications including modeling insurance policies, auctions, optimizing production processes, car races, estimating parameters of distributions, and others. Here, we provide the expression of order statistics for GOBPW distribution.

Suppose $X_1,...,X_n$ be a random sample from the GOBPW distribution $X_{1,n} < X_{2,n} < ... < X_{n,n}$ is a set of random variables of $n$ ordered, then the distribution of $\eta^{th}$ order statistics is

$$\begin{aligned} F_{\eta,n}(x) &= \frac{n!f(x)}{(\eta-1)!(n-\eta)!} F_{(x)}^{\eta-1}\left[1-F(x)\right]^{n-1} \\ &= \frac{n!f(x)}{(\eta-1)!(n-\eta)!}\sum_{v=0}^{\infty}(-1)^v\binom{n-\eta}{v}F_{(x)}^{\eta+v-1}, \end{aligned} \tag{57}$$

where $f(x)$ and $F(x)$ are pdf and cdf of GOBPW distribution.

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Now, (57) can simplified as

$$F_{\eta,n}(x)=\frac{n!f(x)}{(\eta-1)!(n-\eta)!}\sum_{v=0}^{\infty}(-1)^{v}\binom{n-\eta}{v}\times\sum_{e=0}^{\infty}\sum_{e=f}^{\infty}(-1)^{e+f}\binom{\eta+v-1}{e}\binom{e}{f}F_{(x)}^{f}. \quad (58)$$

Applying (45) in (58), we have

$$F_{\eta,n}(x)=f(x)\sum_{v,e=0}^{\infty}\sum_{e=f}^{\infty}\Lambda_{v,e,f}\left[\frac{\left\{\left(1-e^{-\alpha x^{\beta}}\right)^{c}\Big/1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}\right\}^{a}}{B(a,b)}\sum_{g=0}^{\infty}\frac{(1-b)_{g}}{(a+g)_{g_i}!}\times\left\{\frac{\left(1-e^{-\alpha x^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x^{\beta}}\right)^{c}}\right\}^{g}\right]^{f}. \quad (59)$$

After some simplifications, we have the order statistics of GOBPW distribution as

$$\begin{aligned}F_{\eta,n}(x)=&\frac{c\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}y^{a-\frac{1}{c}}}{B(a,b)(1+y)^{a+b-\frac{1}{c}-1}}\sum_{v,e=0}^{\infty}\sum_{e=f}^{\infty}\Lambda_{v,e,f}\times\\&\sum_{g_1=0}^{\infty}\cdots\sum_{f=0}^{\infty}\left[\frac{y^{af}}{B(a,b)^{f}}\times\frac{(1-b)_{g_1}\ldots(1-b)_{g_f}\times y^{g_1+\ldots+g_f-1}}{(a+g_1)\ldots(a+g_f)g_1!\ldots g_f!}\right],\end{aligned} \quad (60)$$

where $y=\dfrac{\left(1-e^{-\alpha x_1^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_1^{\beta}}\right)^{c}}$.

## 5. Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) is a method used in statistics to estimate the parameters of an assumed probability distribution based on some experimental data. This is done by maximizing a likelihood function that makes the experimental data likely under the assumed statistical model. Maximum likelihood estimate is the point in the parameter value that maximizes the likelihood. The purpose of maximum likelihood estimation is to obtain the model parameter values that maximise the likelihood function over the parameter space. There are several methods for estimating parameters that have been proposed in the literature, but the maximum likelihood method is the most commonly used. In this section, we examine the maximum likelihood estimates (MLEs) of the parameters of the GOBPW distribution.

Suppose $x_1,x_2,\ldots x_n$ be an experimental sample of size $n$ drawn from the GOBPW distribution with parameter vector $\xi=(a,b,c,\alpha,\beta)^{T}$. Then, the total likelihood $(L)$ and log-likelihood functions for $\xi$ are, respectively, expressed as:

$$L(x;\xi)=\left\{\frac{c\alpha\beta}{B(a,b)}\right\}^{n}\prod_{i=1}^{n}\frac{x_i^{\beta-1}e^{-\alpha x_i^{\beta}}\left\{1-e^{-\alpha x_i^{\beta}}\right\}^{ca-1}}{\left\{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right\}^{a+1}\left\{1+\left[\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\Big/1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right]\right\}^{a+b}}, \quad (61)$$

and

$$\begin{aligned}\ell=\log L(x;\xi)=&\,n\log\left(\frac{c\alpha\beta}{B(a,b)}\right)+\left(\beta-1\right)\sum_{i=1}^{n}\log(x_i)-\alpha\sum_{i=1}^{n}x_i^{\beta}\\&+(ca-1)\sum_{i=1}^{n}\log\left(1-e^{-\alpha x_i^{\beta}}\right)-(a+1)\sum_{i=1}^{n}\log\left\{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right\}\\&-(a+b)\sum_{i=1}^{n}\log\left\{1+\left[\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\Big/1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right]\right\},\end{aligned} \quad (62)$$

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where $B(a,b)=\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ (for reference see [100]).

The MLE of $\xi$ say $\hat{\xi}$, is obtained by differentiating (53) partially with respect to $\xi$. Then, we have

$$\frac{\partial \ell(x;\xi)}{\partial a}=n\Psi(a,b)-n\Psi(a)+c\sum_{i=1}^{n}\log\left(1-e^{-\alpha x_i^{\beta}}\right)-\sum_{i=1}^{n}\log\left\{1+\left[\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\Big/1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right]\right\}. \quad (63)$$

$$\frac{\partial \ell(x;\xi)}{\partial b}=n\Psi(a,b)-n\Psi(b)-\sum_{i=1}^{n}\log\left\{1+\left[\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\Big/1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right]\right\}. \quad (64)$$

$$\begin{aligned}\frac{\partial \ell(x;\xi)}{\partial c}=&\frac{n}{c}+a\sum_{i=1}^{n}\log\left(1-e^{-\alpha x_i^{\beta}}\right)-(a+1)\sum_{i=1}^{n}\left(1\Big/1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right)\frac{\partial}{\partial c}\left(1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right)\\&-(a+b)\sum_{i=1}^{n}\left(1\Big/\left\{1+\left(\frac{\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}\right)\right\}\right)\frac{\partial}{\partial c}\left\{1+\left(\frac{\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}\right)\right\},\end{aligned} \quad (65)$$

where $\Psi(.)$ is the digamma function,

$$\frac{\partial}{\partial c}\left(1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right)=-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\log\left(1-e^{-\alpha x_i^{\beta}}\right), \text{ and}$$

$$\frac{\partial}{\partial c}\left\{1+\left(\frac{\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}\right)\right\}=\frac{\left(1-e^{-\alpha x_i^{\beta}}\right)\log\left(1-e^{-\alpha x_i^{\beta}}\right)}{\left[1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right]^{2}}.$$

So that

$$\begin{aligned}\frac{\partial \ell(x;\xi)}{\partial \alpha}=&\frac{n}{\alpha}-\sum_{i=1}^{n}x_i^{\beta}+(ca-1)\sum_{i=1}^{n}\left(\frac{1}{1-e^{-\alpha x_i^{\beta}}}\right)\frac{\partial}{\partial \alpha}\left(1-e^{-\alpha x_i^{\beta}}\right)\\&-(a+1)\sum_{i=1}^{n}\left(\frac{1}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}\right)\frac{\partial}{\partial \alpha}\left(1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right)\\&-(a+b)\sum_{i=1}^{n}\left(1\Big/\left\{1+\left(\frac{\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}\right)\right\}\right)\frac{\partial}{\partial \alpha}\left\{\left(\frac{\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}\right)\right\},\end{aligned} \quad (66)$$

where $\dfrac{\partial}{\partial \alpha}\left(1-e^{-\alpha x_i^{\beta}}\right)=x_i^{\beta}e^{-\alpha x_i^{\beta}}$, $\dfrac{\partial}{\partial \alpha}\left(1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right)=-c\left(1-e^{-\alpha x_i^{\beta}}\right)^{c-1}x_i^{\beta}e^{-\alpha x_i^{\beta}}$, and

$$\frac{\partial}{\partial \alpha}\left\{\left(\frac{\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}\right)\right\}=\frac{x_i^{\beta}e^{-\alpha x_i^{\beta}}}{\left[1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right]^{2}}.$$

So that

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$$\frac{\partial \ell(x;\xi)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \log(x_i) - \alpha \sum_{i=1}^{n} x_i^{\beta} \log(x_i) + (ca-1) \times \sum_{i=1}^{n} \left( \frac{1}{1-e^{-\alpha x_i^{\beta}}} \right) \frac{\partial}{\partial \beta} \left(1-e^{-\alpha x_i^{\beta}}\right) - (a+1) \times \sum_{i=1}^{n} \left( \frac{1}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}} \right) \frac{\partial}{\partial \beta} \left(1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right) - (a+b) \times \sum_{i=1}^{n} \left( 1 \Big/ \left\{ 1 + \left( \frac{\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}} \right) \right\} \right) \frac{\partial}{\partial \beta} \left\{ 1 + \left( \frac{\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}} \right) \right\}, \tag{67}$$

where $\frac{\partial}{\partial \beta}\left(1-e^{-\alpha x_i^{\beta}}\right) = -\alpha x_i^{\beta} \log(x_i) e^{-\alpha x_i^{\beta}}$,

$$\frac{\partial}{\partial \beta}\left(1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right) = c\left(1-e^{-\alpha x_i^{\beta}}\right)^{c-1} \alpha x_i^{\beta} \log(x_i) e^{-\alpha x_i^{\beta}}, \text{ and}$$

$$\frac{\partial}{\partial \alpha}\left\{ \left( \frac{\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}}{1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}} \right) \right\} = \frac{-c\left(1-e^{-\alpha x_i^{\beta}}\right)^{c-1} \alpha x_i^{\beta} \log(x_i) e^{-\alpha x_i^{\beta}}}{\left[1-\left(1-e^{-\alpha x_i^{\beta}}\right)^{c}\right]^{2}}.$$

The estimators for $\xi$ can be obtained by equating (63), (64), (65), (66) and (67) to zero. Since $\hat{\xi}$ has not a closed form, the R statistical software [101] can be used to find the numerical solutions.

## 6. Simulation Study

In this section, some Monte Carlo simulation results are discussed for the various sample sizes to examine the accuracy of the MLE $\hat{\xi}$. The samples are simulated from the GOBPW model. The algorithm for numerical procedures can be performed as described below.

**Step 1:** Generate $N = 1000$ random samples of samples sizes n=100, 200, 300, 500, 700, 850, and 1000 from the GOBPW distribution based on the quantile function obtained from (56) as

$$x_i = \left\{ -\frac{1}{\alpha} \log\{1-\hat{\lambda}\} \right\}^{\frac{1}{\beta}}, \quad i = 1,2,...,n.$$

**Step 2:** Chose different values of the model parameters, with order $(a,b,c,\alpha,\beta)$:

Set 1: (0.5, 0.5,1.75,0.5,1.5), Set 2: (0.5, 0.5,2.5,0.5,0.5), Set 3: (0.5, 0.5,0.72,0.5,1.5).

**Step 3:** Measures such as the mean of the MLEs and mean squared errors (MSEs) for $\hat{\xi}$ are obtained for each n to evaluate the performance of estimates.

**Step 4:** The numerical results are obtained using R-studio package version 4.2.1. The outcomes of the results are given in Table 1.

From Table 1, we can draw the following conclusions:

- The mean estimates tend to be closer to the true values of parameters when the sample sizes increases.
- For all chosen true values of parameter sets, we notice that the MSEs decrease as the sample sizes increases.
- The results show that the MLE method performs quite well in estimating the parameters of the proposed GOBPW model.

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**Table 1.** Monte Carlo simulation results (mean MLE and MSE) for GOBPW distribution for several values.

| | | **Set1: 0.5, 0.5,1.75,0.5,1.5** | | **Set2: 0.5, 0.5,2.5,0.5,0.5** | | **Set3: 0.5, 0.5,0.72,0.5,1.5** | |
|---|---|---|---|---|---|---|---|
| | | Mean | MSE | Mean | MSE | Mean | MSE |
| n=100 | $a$ | 2.1918 | 0.1969 | 2.5799 | 0.0063 | 1.8305 | 1.2505 |
| | $b$ | 0.4395 | 0.0037 | 0.4507 | 0.0024 | 0.6092 | 0.0158 |
| | $c$ | 0.1286 | 2.1650 | 0.8126 | 0.2376 | 0.1891 | 2.1937 |
| | $\alpha$ | 0.2507 | 0.0623 | 0.3465 | 0.0235 | 0.6778 | 0.0420 |
| | $\beta$ | 0.4076 | 0.0087 | 0.6440 | 0.0207 | 0.3845 | 0.0164 |
| n=200 | $a$ | 1.7258 | 0.1525 | 2.1726 | 0.0052 | 1.3989 | 1.2007 |
| | $b$ | 0.4435 | 0.0032 | 0.4491 | 0.0022 | 0.5977 | 0.0142 |
| | $c$ | 0.1326 | 2.1534 | 0.9406 | 0.2111 | 0.2039 | 2.1893 |
| | $\alpha$ | 0.2734 | 0.0515 | 0.3571 | 0.0204 | 0.6448 | 0.0393 |
| | $\beta$ | 0.4157 | 0.0072 | 0.6305 | 0.0170 | 0.5996 | 0.0162 |
| n=300 | $a$ | 1.4242 | 0.1244 | 1.8580 | 0.0048 | 1.2559 | 1.1067 |
| | $b$ | 0.4486 | 0.0026 | 0.4569 | 0.0018 | 0.5816 | 0.0126 |
| | $c$ | 0.5935 | 2.1463 | 1.0806 | 0.1769 | 0.4216 | 2.1857 |
| | $\alpha$ | 0.2882 | 0.0454 | 0.3510 | 0.0221 | 0.6068 | 0.0275 |
| | $\beta$ | 0.4191 | 0.0066 | 0.6259 | 0.0131 | 0.7202 | 0.0122 |
| n=500 | $a$ | 1.2612 | 0.1026 | 1.2894 | 0.0046 | 1.0072 | 0.9894 |
| | $b$ | 0.4556 | 0.0020 | 0.4655 | 0.0011 | 0.5576 | 0.0060 |
| | $c$ | 0.7354 | 2.1452 | 1.5048 | 0.1203 | 0.4230 | 2.1817 |
| | $\alpha$ | 0.3027 | 0.0390 | 0.3855 | 0.0210 | 0.5724 | 0.0123 |
| | $\beta$ | 0.8220 | 0.0061 | 0.6017 | 0.0126 | 0.9439 | 0.0061 |
| n=700 | $a$ | 1.0723 | 0.0735 | 1.0594 | 0.0044 | 0.8694 | 0.9558 |
| | $b$ | 0.4612 | 0.0015 | 0.4637 | 0.0010 | 0.5498 | 0.0026 |
| | $c$ | 1.3733 | 2.1398 | 1.8250 | 0.1055 | 0.5235 | 2.1814 |
| | $\alpha$ | 0.3119 | 0.0354 | 0.3869 | 0.0199 | 0.5685 | 0.0078 |
| | $\beta$ | 0.9423 | 0.0058 | 0.5902 | 0.0123 | 1.0426 | 0.0043 |
| n=850 | $a$ | 0.9747 | 0.0600 | 0.9632 | 0.0041 | 0.6901 | 0.9460 |
| | $b$ | 0.4627 | 0.0014 | 0.4668 | 0.0007 | 0.5463 | 0.0025 |
| | $c$ | 1.3846 | 2.1365 | 1.8389 | 0.1047 | 0.6293 | 2.1811 |
| | $\alpha$ | 0.3741 | 0.0346 | 0.3883 | 0.0113 | 0.5671 | 0.0058 |
| | $\beta$ | 1.0424 | 0.0058 | 0.5742 | 0.0117 | 1.1437 | 0.0034 |
| n=1000 | $a$ | 0.6277 | 0.0378 | 0.7597 | 0.0039 | 0.5922 | 0.9431 |
| | $b$ | 0.4842 | 0.0013 | 0.4722 | 0.0006 | 0.5459 | 0.0023 |
| | $c$ | 1.3788 | 2.1362 | 2.0057 | 0.1034 | 0.6732 | 2.1809 |
| | $\alpha$ | 0.3962 | 0.0339 | 0.3899 | 0.0324 | 0.5710 | 0.0051 |
| | $\beta$ | 1.3242 | 0.0057 | 0.5737 | 0.0113 | 1.4419 | 0.0030 |

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## 7. Applications to Cancer Disease Data sets

This section demonstrates the applicability of the proposed GOBPW distribution using four practical data sets involving cancer-related diseases, including bladder, acute bone, neck and head, as well as blood cancers. The performance of the GOBPW distribution is evaluated and compared with the other competing models also based the Weibull distribution with five parameters, including the beta modified Weibull (BMW) distribution by [102], Kumaraswamy modified Weibull (KumMW) distribution by [103], gamma generalized modified Weibull (GGMW) distribution by [104], gamma log-logistic Weibull (GLLoGW) distribution by [105], and beta log-logistic Weibull (BLLoGW) distribution by [106].

Basically, we use MLE to estimate the parameters of each model and then compare them using the maximum value of log-likelihood analyzed at MLEs $(-\hat{\ell})$ together with their standard errors (SEs). Also, some conventional goodness-of-fit measures are considered, including the Akaike Information Criterion (AIC), Bayesian information criterion (BIC), Cramer–von Mises (CM), and Anderson– Darling (AD) statistics. The value of the Kolmogorov-Smirnov (KS) statistic as well as its p-value are also presented to compare the GOBPW model with other competing models. The golden rule is as follows: the model that has the smallest values of these statistics must be chosen as the better one for fitting data [98, 107]. Consequently, model performance is ranked using these measured statistics. In the context of data analysis, it is well-established that the maximum likelihood approach and the aforementioned criteria have been demonstrated to be effective. All computations in this study were performed using R studio software (version 4.2.1). As mentioned, the GOBPW distribution is applied as model to investigate four different practical cancer data sets given below.

### *7.1. Bladder Cancer Data Set (CD1)*

The first data set is recently reported by [108]. It concerns the remission times (in months) of a random sample of 132 bladder cancer patients. The data are as follows: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.87, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.33, 4.34, 4.40, 4.50, 4.51, 4.65, 4.70, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.60, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 10.86, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.69, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 19.36, 20.28, 21.73, 22.69, 23.60.

### *7.2. Acute Bone Cancer Data Set (CD2)*

The second data set is obtained from [109]. The data represents the survival times (in days) of 73 patients who diagnosed with acute bone cancer, as follows: 0.09, 0.76, 1.81, 1.10, 3.72, 0.72, 2.49, 1.00, 0.53, 0.66, 31.61, 0.60, 0.20, 1.61, 1.88, 0.70, 1.36, 0.43, 3.16, 1.57, 4.93, 11.07, 1.63, 1.39, 4.54, 3.12, 86.01, 1.92, 0.92, 4.04, 1.16, 2.26, 0.20, 0.94, 1.82, 3.99, 1.46, 2.75, 1.38, 2.76, 1.86, 2.68, 1.76, 0.67, 1.29, 1.56, 2.83, 0.71, 1.48, 2.41, 0.66, 0.65, 2.36, 1.29, 13.75, 0.67, 3.70, 0.76, 3.63, 0.68, 2.65, 0.95, 2.30, 2.57, 0.61, 3.93, 1.56, 1.29, 9.94, 1.67, 1.42, 4.18, 1.37.

### *7.3. Head and Neck Cancer Data Set (CD3)*

The third data set is studied and analyzed by [110]. In involves the survival time for 44 patients diagnosed by Head and Neck cancer disease. The data set are: 12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194,195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776.

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*7.4. Blood Cancer Data Set (CD4)*

The fourth data set contains the life time (in years) of a 40 blood cancer (leukemia) patients from one of Ministry of health hospitals in Saudi Arabia reported by [111]. This actual data are as follows: 0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.370, 2.532, 2.693, 2.805, 2.910, 2.912, 3.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.381.

Table 2 shows the statistical descriptions of these four practical data sets. From Table 2, the statistical descriptions indicate that the data sets have distinct skewness and kurtosis features. In particular, right skewed with high kurtosis coefficients for CD2 and CD3 are observed, while left skewed for CD4 is observed. Hence, these data sets are suitable for skewed statistical models.

Figure 3 presents the boxplots of the four different cancer data sets. From Figure 3, it could be observed that the data sets, especially CD2 and CD3 indicate statistical behavior of extreme values. Therefore, these data sets are appropriate for extreme value distributions.

**Table 2.** Descriptive statistics for CD1 – CD4.

| Data | n | Min | Q1 | Q3 | Median | Mean | Max | Variance | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|---|---|---|
| CD1 | 132 | 0.080 | 3.348 | 9.537 | 5.665 | 7.149 | 23.600 | 27.673 | 1.059 | 0.613 |
| CD2 | 73 | 0.090 | 0.920 | 2.750 | 1.570 | 3.755 | 86.010 | 112.331 | 6.660 | 47.369 |
| CD3 | 44 | 12.20 | 67.21 | 219.00 | 128.50 | 223.48 | 1776.00 | 93286.41 | 3.269 | 12.816 |
| CD4 | 40 | 0.315 | 2.199 | 4.264 | 3.348 | 3.141 | 5.381 | 1.847 | -0.401 | -0.838 |

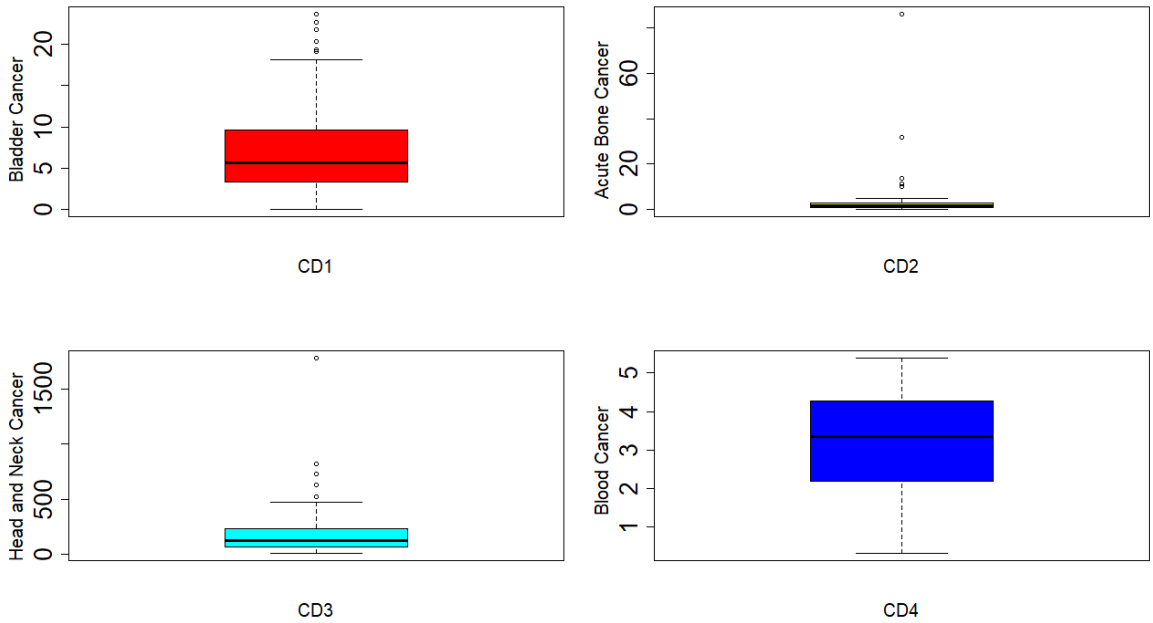


**Figure 3.** Boxplots for CD1 – CD4.

Furthermore, we compare the fits of the proposed GOBPW model with the other aforementioned competing models. Tables 3 – 6 provide the results of the MLEs and their corresponding SEs for CD1 – CD4, respectively. The values of the considered goodness-of-fit measures for the candidate models are listed in Tables 7 – 10 for CD1 – CD4, respectively. From these results, it is obvious that the proposed GOBPW model not only provided better fits to the cancer data sets but also superior to the other fitted models because

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it possesses the smallest AIC, BIC, CM, KS, and AD values. Once more, the p-value of the K-S statistic for all of the fitted models is more than the nominal 0.05 level of significance, suggesting that all the considered models fitted the cancer data sets quite well and that the proposed GOBPW model provided the best fit. Figures 4 – 7 depict the fits of the estimated pdfs over the histograms and estimated cdfs over the empirical cdfs for CD1 – CD4, respectively. It is very clear that the GOBPW model provides adequate fits than all other competing models.

**Table 3.** MLEs with corresponding standard errors (in parentheses) of competitive models for CD1.

| **Model** | $\hat{a}$ | $\hat{b}$ | $\hat{c}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
|---|---|---|---|---|---|
| GOBPW | 1.3690 | 7.8108 | 5.9224 | 3.0174 | 1.8748 |
| | (0.0933) | (0.5226) | (1.5777) | (2.7770) | (0.2537) |
| BMW | 6.5146 | 2.8781 | 4.3765 | 3.7531 | 0.9356 |
| | (0.4348) | (0.2107) | (0.6342) | (0.3421) | (0.2132) |
| KumMW | 7.1487 | 5.2405 | 2.1047 | 5.3845 | 2.4115 |
| | (0.4561) | (0.3225) | (0.3885) | (1.2753) | (1.1925) |
| GGMW | 1.9701 | 5.5735 | 2.1057 | 3.1993 | 1.4243 |
| | (0.1439) | (0.4247) | (0.2363) | (0.7051) | (0.5422) |
| GLLoGW | 1.6337 | 0.2285 | 8.3884 | 2.6473 | 1.6003 |
| | (0.1842) | (0.0301) | (3.6511) | (0.4025) | (2.2953) |
| BLLoGW | 5.4160 | 2.7557 | 5.3826 | 1.2122 | 1.3564 |
| | (0.3999) | (0.3086) | (0.6803) | (0.1093) | (0.0432) |

**Table 4.** MLEs with corresponding standard errors (in parentheses) of competitive models for CD2.

| **Model** | $\hat{a}$ | $\hat{b}$ | $\hat{c}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
|---|---|---|---|---|---|
| GOBPW | 1.8690 | 1.6163 | 1.9833 | 1.7413 | 0.0731 |
| | (0.1854) | (0.1725) | (0.2023) | (0.4131) | (0.0041) |
| BMW | 3.7552 | 10.5257 | 11.6736 | 2.6749 | 3.3548 |
| | (1.2319) | (0.8711) | (3.5769) | (1.5367) | (0.3547) |
| KumMW | 1.1174 | 1.1280 | 2.7285 | 1.0704 | 2.7069 |
| | (0.1644) | (0.2077) | (1.1981) | (0.2156) | (0.8717) |
| GGMW | 0.5196 | 1.0382 | 0.5653 | 0.0973 | 3.3457 |
| | (0.1215) | (0.0859) | (0.3201) | (0.0367) | (1.9585) |
| GLLoGW | 1.4342 | 0.7183 | 0.3788 | 0.3822 | 3.2247 |
| | (0.1269) | (0.1149) | (0.0583) | (0.0253) | (0.6007) |
| BLLoGW | 2.1117 | 2.2208 | 2.4679 | 0.4567 | 7.5321 |
| | (0.3951) | (0.2443) | (0.5543) | (0.8354) | (1.4567) |

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**Table 5.** MLEs with corresponding standard errors (in parentheses) of competitive models for CD3.

| Model | $\hat{a}$ | $\hat{b}$ | $\hat{c}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
|---|---|---|---|---|---|
| GOBPW | 1.6909 | 25.0378 | 3.4434 | 5.2567 | 1.7018 |
| | (0.2128) | (9.3944) | (0.4236) | (2.6743) | (1.1823) |
| BMW | 37.5243 | 23.4770 | 11.6349 | 7.8732 | 5.5325 |
| | (9.3245) | (6.9634) | (3.6453) | (2.6523) | (2.5342) |
| KumMW | 1.0234 | 0.4578 | 2.9743 | 1.3546 | 4.1036 |
| | (0.1827) | (0.0991) | (0.4583) | (0.6454) | (1.0343) |
| GGMW | 16.3572 | 6.7247 | 18.2546 | 8.3452 | 3.2436 |
| | (4.7443) | (2.8669) | (7.4532) | (3.2543) | (1.0124) |
| GLLoGW | 4.8469 | 1.0408 | 5.2208 | 0.2563 | 3.4563 |
| | (0.1569) | (0.1109) | (2.9744) | (0.1324) | (0.5341) |
| BLLoGW | 51.1783 | 29.7071 | 17.6324 | 5.7352 | 4.8322 |
| | (17.4352) | (8.8432) | (6.5362) | (2.1453) | (1.8743) |

**Table 6.** MLEs with corresponding standard errors (in parentheses) of competitive models for CD4.

| Model | $\hat{a}$ | $\hat{b}$ | $\hat{c}$ | $\hat{\alpha}$ | $\hat{\beta}$ |
|---|---|---|---|---|---|
| GOBPW | 3.2194 | 0.7882 | 5.6453 | 2.8463 | 1.4546 |
| | (0.2199) | (0.1024) | (2.4753) | (0.1734) | (0.1135) |
| BMW | 3.0698 | 3.0117 | 4.8721 | 2.6451 | 0.8721 |
| | (0.4192) | (0.2648) | (0.5391) | (0.1734) | (0.0645) |
| KumMW | 3.4646 | 1.1032 | 4.0832 | 2.1835 | 1.7429 |
| | (0.7404) | (0.2530) | (1.3541) | (0.6319) | (0.1439) |
| GGMW | 2.5003 | 3.5183 | 5.6572 | 3.2539 | 1.3926 |
| | (0.3378) | (0.2316) | (0.5493) | (0.2845) | (0.0745) |
| GLLoGW | 3.1407 | 1.3413 | 7.9453 | 4.9564 | 2.7034 |
| | (0.2121) | (0.1505) | (3.7462) | (1.7354) | (0.2035) |
| BLLoGW | 0.9932 | 0.6463 | 2.6293 | 1.9453 | 0.8453 |
| | (0.1021) | (0.0726) | (0.7453) | (0.4352) | (0.0936) |

**Table 7.** Goodness-of-fit results of GOBPW and other fitted models for CD1.

| Model | $-\hat{\ell}$ | AIC | BIC | CM | KS | AD | p-value (KS) |
|---|---|---|---|---|---|---|---|
| GOBPW | 382.53 | 769.06 | 774.83 | 0.0337 | 0.0443 | 0.2087 | 0.7146 |
| BMW | 404.12 | 812.23 | 817.99 | 0.2640 | 0.0965 | 2.4586 | 0.2721 |
| KumMW | 405.94 | 815.89 | 821.66 | 0.5655 | 0.1278 | 3.4594 | 0.1397 |
| GGMW | 391.06 | 786.12 | 791.89 | 0.0764 | 0.0677 | 0.8933 | 0.3401 |
| GLLoGW | 383.31 | 770.63 | 776.39 | 0.0368 | 0.0509 | 0.2455 | 0.6732 |
| BLLoGW | 418.34 | 840.68 | 846.44 | 0.5981 | 0.1517 | 4.3278 | 0.1259 |

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**Table 8.** Goodness-of-fit results of GOBPW and other fitted models for CD2.

| Model | $-\hat{\ell}$ | AIC | BIC | CM | KS | AD | p-value (KS) |
|---|---|---|---|---|---|---|---|
| GOBPW | 139.98 | 283.96 | 288.54 | 0.0524 | 0.0673 | 0.4687 | 0.5127 |
| BMW | 275.41 | 554.82 | 559.40 | 3.7427 | 0.3881 | 8.3428 | 0.0176 |
| KumMW | 147.90 | 299.80 | 304.38 | 0.2919 | 0.1411 | 1.8408 | 0.2386 |
| GGMW | 144.26 | 292.52 | 297.10 | 0.1649 | 0.0943 | 1.1650 | 0.4821 |
| GLLoGW | 153.29 | 310.59 | 315.17 | 0.5302 | 0.1589 | 3.5821 | 0.1359 |
| BLLoGW | 221.36 | 446.73 | 451.32 | 1.6176 | 0.2869 | 3.7573 | 0.0731 |

**Table 9.** Goodness-of-fit results of GOBPW and other fitted models for CD3.

| Model | $-\hat{\ell}$ | AIC | BIC | CM | KS | AD | p-value (KS) |
|---|---|---|---|---|---|---|---|
| GOBPW | 277.79 | 559.59 | 563.16 | 0.0137 | 0.0496 | 0.1164 | 0.8438 |
| BMW | 313.68 | 631.36 | 634.93 | 0.9574 | 0.2691 | 5.0483 | 0.024 |
| KumMW | 282.01 | 568.05 | 571.57 | 0.1844 | 0.1472 | 0.9861 | 0.6381 |
| GGMW | 289.39 | 582.78 | 586.35 | 0.3938 | 0.1882 | 2.6942 | 0.3081 |
| GLLoGW | 277.46 | 558.92 | 562.49 | 0.0218 | 0.0684 | 0.1403 | 0.7231 |
| BLLoGW | 302.91 | 609.82 | 613.39 | 0.4464 | 0.2180 | 3.0811 | 0.1721 |

**Table 10.** Goodness-of-fit results of GOBPW and other fitted models for CD4.

| Model | $-\hat{\ell}$ | AIC | BIC | CM | KS | AD | p-value (KS) |
|---|---|---|---|---|---|---|---|
| GOBPW | 68.51 | 141.03 | 144.41 | 0.0480 | 0.0830 | 0.4224 | 0.9158 |
| BMW | 73.39 | 151.78 | 154.16 | 0.1766 | 0.1441 | 1.5449 | 0.5967 |
| KumMW | 76.54 | 156.09 | 160.47 | 0.2518 | 0.1582 | 1.5570 | 0.3480 |
| GGMW | 70.15 | 143.21 | 146.59 | 0.1166 | 0.1184 | 0.8564 | 0.7316 |
| GLLoGW | 69.59 | 143.19 | 146.47 | 0.0637 | 0.0902 | 0.4472 | 0.8651 |
| BLLoGW | 79.02 | 162.05 | 165.43 | 0.3983 | 0.1771 | 2.3360 | 0.1478 |

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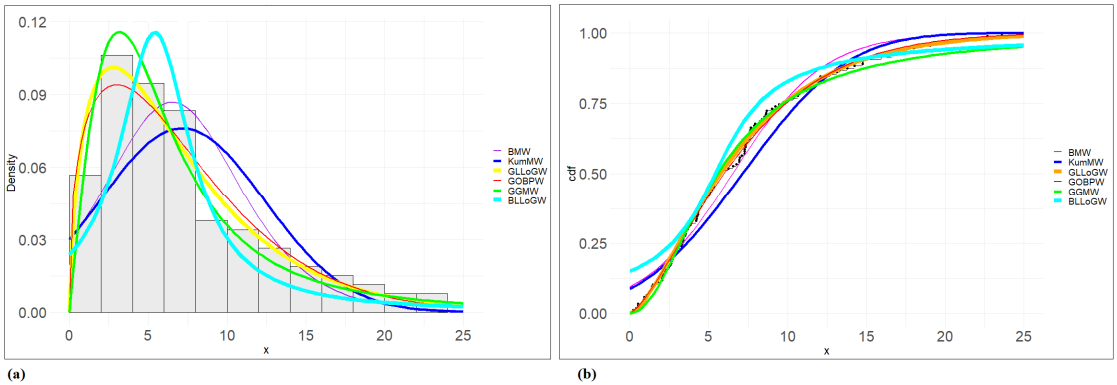


**Figure 4.** Plots of the (a) estimated pdfs over the histogram and (b) estimated cdf over the empirical cdf for CD1.

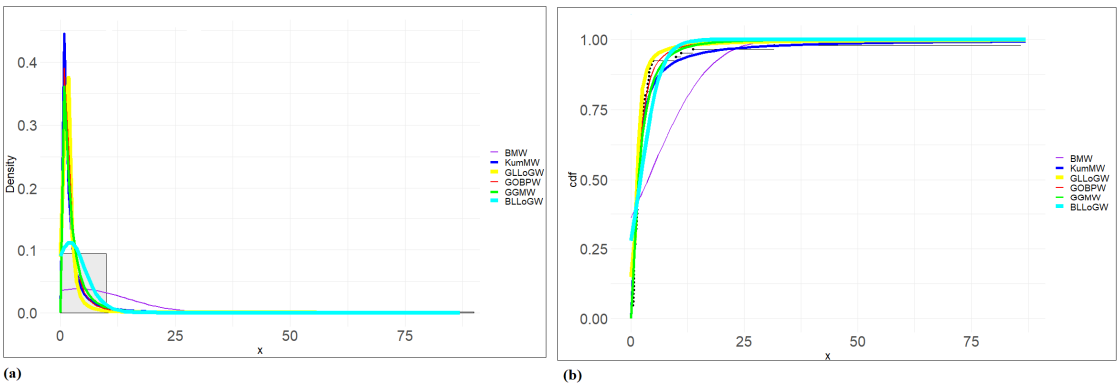


**Figure 5.** Plots of the (a) estimated pdfs over the histogram and (b) estimated cdf over the empirical cdf for CD2.

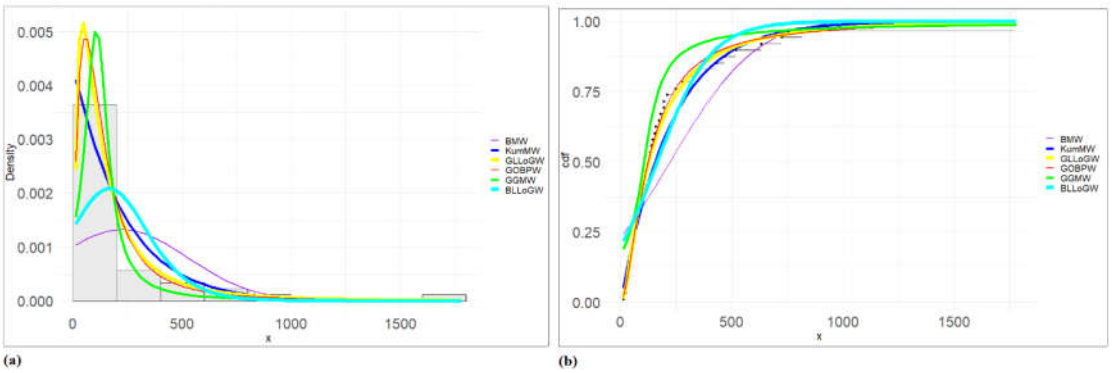


**Figure 6.** Plots of the (a) estimated pdfs over the histogram and (b) estimated cdf over the empirical cdf for CD3.

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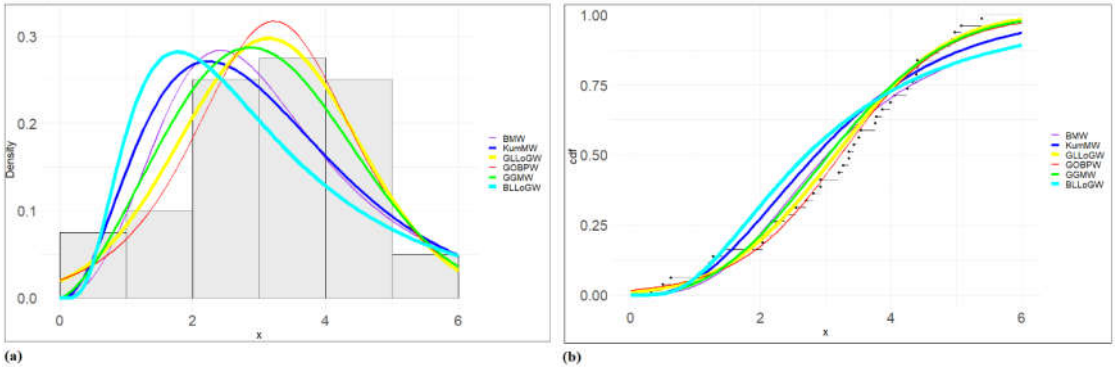


**Figure 7.** Plots of the (a) estimated pdfs over the histogram and (b) estimated cdf over the empirical cdf for CD4.

## 8. Conclusions

In this work, we pioneer a new family of continuous probability distributions obtained from the beta prime random variable called the generalized odd beta prime-G (GOBP-G) family. We then employ the pioneered family to introduce a new generalized Weibull univariate continuous probability distribution called the generalized odd beta prime-Weibull (GOBPW) distribution. The plots of the pdf and hazard rate functions of the GOBPW distribution showed that the distribution is capable of modeling skewed, heavy-tailed, and unimodal lifetime data sets. The probability density function of the proposed GOBPW distribution is derived in terms of mixture representations. We defined and derived a number of statistical properties of the new distribution, such as the moments, moment generating function, incomplete moments, information generating function, entropies, stress-strength function, quantile function, and order statistics. The parameters of the proposed distribution were estimated using the maximum likelihood estimation method. Applications of statistical distributions are vital to the field of medical research and can provide a significant contribution to enhancing public health, particularly in the case of cancer patients. Hence, the efficacy of the proposed GOBPW distribution is demonstrated by its applications to four different cancer data sets, including bladder, acute bone, neck and head, and blood cancers. The results indicate that the proposed GOBPW model not only provides better fits to the cancer data sets but also indicates superior performance compared to other fitted models based on selection criteria such as the log-likelihood, AIC, BIC, CM, KS, and AD statistics. We hoped that the proposed GOBPW distribution would be an alternative to other traditional distributions for modeling positively skewed lifetime data, especially for cancer research. The statistical properties of the family and their applications to lifetime data can be studied in the future.

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