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*Article*

# Four-Dimensional Spaces of Complex Numbers and Unitary States of Two-Qubit Quantum Systems

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**Abstract**

Pure states of two-qubit quantum systems are described by a four-dimensional vector of complex numbers, and unitary operators transferring a two-qubit quantum system from one state to another have the form of a $4 \times 4$ matrix with complex elements. This fact brings to mind the idea of studying the spaces of four-dimensional numbers with complex components. Moreover, the results obtained by the authors for four-dimensional numbers with real components inspire some optimism. In this paper we construct four-dimensional spaces of complex numbers by analogy with four-dimensional spaces of real numbers. Each four-dimensional number is mapped to a matrix formed from its components and it is proved that the constructed mapping is a bijection and a homomorphism. In the space of four-dimensional numbers, of the eight basis elements, half are real and half are imaginary. The presence of such symmetry distinguish these spaces from the space of quaternions, in which one basis element is real and the rest are imaginary. The symmetry of the basis numbers makes these spaces a natural generalization of one-dimensional and two-dimensional (complex) algebra. The conditions under which the corresponding matrices are gates for two-qubit quantum systems are defined. The notion of unitary state of a two-qubit quantum system is introduced, to which various gates from commutative groups of gates correspond. It is shown that any gate of a unitary state transforms a unitary state into a unitary state and a non-unitary state into a non-unitary state. Almost all gates used in the construction of quantum circuits, in particular H, SWAP, CX, CY, CZ, have the same properties. The problem of searching for a gate that transfers a quantum system from one unitary state to another unitary state has been solved. Thus, with the help of four-dimensional spaces of complex numbers it was possible to construct whole classes of two-qubit gates, which opens new possibilities for the construction of quantum algorithms. The results obtained have important theoretical and practical implications for quantum computing.

**Keywords:** four-dimensional mathematics; four-dimensional numbers; four-dimensional complex numbers; two-qubit quantum systems; quantum states; two-qubit gates; quantum computing; abstract algebra

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## 1. Introduction

The pure states of two-qubit quantum systems are described by a four-dimensional vector of complex numbers, and the unitary operators that transfer a two-qubit quantum system from one state to another have the form of a $4 \times 4$ matrix with complex elements. This fact brings to mind the idea of studying the spaces of four-dimensional numbers with complex components. Moreover, the results, obtained by the authors for four-dimensional numbers with real components [1], inspire some optimism.

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In this paper quantum states of two-qubit quantum systems and corresponding gates, which transfer a two-qubit quantum system from one state to another, are considered. In this paper the theory of anisotropic spaces of four-dimensional numbers with complex components is developed to describe the states of two-qubit quantum systems, and the theory of matrix spaces for four-dimensional numbers is developed to describe two-qubit gates. 64 commutative groups of gates for two-qubit quantum systems are obtained and their explicit descriptions are given. The notion of unitary state of a two-qubit quantum system is introduced, to which different gates from commutative groups of gates correspond. It is shown that any gate of a unitary state transforms a unitary state into a unitary state and a non-unitary state into a non-unitary state. Almost all gates used in the construction of quantum circuits, in particular *H, SWAP, CX, CY, CZ,* have the same properties. We give an algorithm for transferring a two-qubit quantum system from one unitary state to another unitary state in one step.

A two-qubit quantum system consists of two qubits and has four basis states in the form of the following ket vectors:

$$|00\rangle = (1,0,0,0)^T,\ |01\rangle = (0,1,0,0)^T,\ |10\rangle = (0,0,1,0)^T,\ |11\rangle = (0,0,0,1)^T,$$

where the index $T$ stands for the transpose sign. Then an arbitrary state of the two-qubit quantum system can be written in the form [2,3]

$$|\varphi\rangle = x_1|00\rangle + x_2|01\rangle + x_3|10\rangle + x_4|11\rangle,$$

where $x_k = \operatorname{Re} x_k + i \operatorname{Im} x_k = x_k^{re} + i x_k^{im} \in \mathbb{C}, k = 1,2,3,4$, are complex numbers satisfying the condition $\sum_{k=1}^{4} |x_k|^2 = 1$. In other notations the arbitrary state of a two-qubit quantum system can be represented as

$$|\varphi\rangle = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

that is, any state of the two-qubit quantum system is uniquely determined by complex amplitudes $x_1, x_2, x_3, x_4$. Thus, an arbitrary state of a two-qubit quantum system is a four-dimensional number with complex components. In this paper we consider only pure states of two-qubit quantum systems.

A two-qubit quantum system is a minimal system in which such quantum mechanical effects as entanglement, separability, coherence, quantum teleportation, etc. begin to appear. Modern studies of two-qubit quantum systems are devoted precisely to the study of these quantum effects [4–9]. One of the important aspects is the visualization of two-qubit quantum systems on Bloch spheres [10,11]. Let us note the book by Nikitin et al [12], in which the axiomatics of quantum mechanics is stated. Works devoted to the development of new classes of gates for two-qubit quantum systems are practically absent. It is not surprising, as for this purpose it is necessary to develop a new mathematical apparatus, which was the apparatus of multidimensional mathematics, the foundations of which were laid by the famous Kazakh mathematician Abenov M.M. [13,14].

In the recent work of the authors [1] the anisotropic spaces of four-dimensional numbers with real components and corresponding to them spaces of matrices of size $4 \times 4$, the elements of which are formed from the coordinates of four-dimensional numbers are considered. All spaces of four-dimensional numbers with associative and commutative addition and multiplication are obtained. Such spaces turned out to be eight, and in six spaces the spectral norm is defined, i.e. these spaces are normalized spaces. A natural generalization of these results is the study of anisotropic four-dimensional spaces with complex components with associative and commutative multiplication and the corresponding matrix spaces. The interest in this generalization is also supported by the fact that the states of a two-qubit quantum system are described by a vector of four complex numbers, and the gates for two-qubit quantum systems are a $4 \times 4$ matrix with complex elements.

The paper has the following structure. Section 2 constructs spaces of four-dimensional numbers with complex components in general form. The operations of addition, subtraction, multiplication and division are introduced, the operations of addition and multiplication being associative and commutative In Section 3, the basis spaces of four-dimensional numbers with associative and commutative multiplication are distinguished. In Section 4, we construct matrix spaces

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corresponding to the spaces of four-dimensional numbers and construct bijective mappings between these spaces, which are homomorphisms by addition and multiplication. In Section 5 we define the conditions under which the constructed matrices are gates of two-qubit quantum systems. Section 6 introduces the concept of a unitary state of a two-qubit quantum system and studies its properties. Section 7 discusses further development prospects of the proposed theory. The last section provides a proof of Lemma 1.

## 2. Spaces of Four-Dimensional Numbers with Complex Components

We will write any complex number x in the form $x = x^{re} + x^{im}i$, where $x^{re}$ and $x^{im}$ are the real and imaginary parts of the complex number $x$, $i^2 = -1$. Non-standard notation for the real and imaginary parts of a complex number is adopted to avoid cumbersome expressions, to save space.

Let us denote by $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4)$ four-dimensional numbers with complex components. Then the sum X+Y of four-dimensional numbers X and Y is called a four-dimensional number

$X + Y = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4)$.

The introduced addition operation is, as is easy to see, an associative and commutative operation. The difference of two four-dimensional numbers X and Y is called a four-dimensional number

$X - Y = (x_1 - y_1, x_2 - y_2, x_3 - y_3, x_4 - y_4)$.

Now we will define the general form of multiplication, which will be an associative and commutative operation.

Associative and commutative multiplication of four-dimensional numbers with real components was first introduced in the work of M. Abenov [13] for one of the four-dimensional spaces. Further, the general form of associative and commutative multiplication of four-dimensional numbers with real components is described in [1,14]. Particular spaces of four-dimensional numbers with real components are studied in [15,16]. Associative and commutative multiplication of four-dimensional numbers with complex components is considered for the first time.

Our further goal is to define a multiplication operation of four-dimensional numbers with complex components which will be associative and commutative. To this end, for given four-dimensional numbers with complex components X and Y, we write the general definition of the multiplication Z = XY as

$$\begin{cases} z_1 = a_{11}x_1y_1 + a_{12}x_2y_2 + a_{13}x_3y_3 + a_{14}x_4y_4 \\ z_2 = a_{21}x_2y_1 + a_{22}x_1y_2 + a_{23}x_4y_3 + a_{24}x_3y_4 \\ z_3 = a_{31}x_3y_1 + a_{32}x_4y_2 + a_{33}x_1y_3 + a_{34}x_2y_4{}' \\ z_4 = a_{41}x_4y_1 + a_{42}x_3y_2 + a_{43}x_2y_3 + a_{44}x_1y_4 \end{cases} \quad (1)$$

Where $z_k = z_k^{re} + z_k^{im}i \in ?, k = 1,2,3,4$ and the complex elements $a_{kl} = a_{kl}^{re} + a_{kl}^{im}i$, $k, l = 1,2,3,4$ of the matrix $A = (a_{kl})_{k,l=1}^4$ must be defined so that the multiplication (1) becomes associative and commutative. The associativity and commutativity conditions impose certain restrictions on the elements of the matrix A.

**Lemma 1.** The multiplication operation defined in (1) is associative and commutative if and only if

$$a_{11} = a_{21} = a_{22} = a_{31} = a_{33} = a_{41} = a_{44} = \alpha, \quad (2)$$

$$a_{23} = a_{24} = \beta,\ a_{232} = a_{34} = \gamma,\ a_{42} = a_{43} = \delta, \quad (3)$$

$$a_{12} = \frac{\gamma\delta}{\alpha},\ a_{13} = \frac{\beta\delta}{\alpha},\ a_{14} = \frac{\beta\gamma}{\alpha}, \quad (4)$$

where $\alpha = \alpha_1 + \alpha_2 i$, $\beta = \beta_1 + \beta_2 i$, $\gamma = \gamma_1 + \gamma_2 i$, $\delta = \delta_1 + \delta_2 i$ are arbitrary non-zero complex numbers.

The proof of the lemma is given in **Appendix 1**.

According to this lemma, the general form of associative and commutative multiplication of four-dimensional numbers with complex components has the form (35).

Let us call the number $X^{re} = (x_1^{re} + x_1^{im}i, 0 + 0i, 0 + 0i, 0 + 0i)$ a quasi-real number. If $x_1^{im} = 0$, then the quasi-real number is called a real number. Multiply an arbitrary four-dimensional number $Y = (y_1^{re} + y_1^{im}i, y_2^{re} + y_2^{im}i, y_3^{re} + y_3^{im}i, y_4^{re} + y_4^{im}i)$ by a quasi-real number $X^{re}$:

$X^{re}Y = (\alpha_1 + \alpha_2 i)(x_1^{re} + x_1^{im}i)Y.$

Let us take a real number (-1,0,0,0,0) as $X^{re}$. Then $(-1,0,0,0)Y = -(\alpha_1 + \alpha_2 i)Y$. For consistency of addition and multiplication operations, we must require $X - Y = X + (-1,0,0,0)Y$, whence it follows that $\alpha_1 + \alpha_2 i = 1 + 0i$, or $\alpha_1 = 1, \alpha_2 = 0$. Thus, according to formula (35), the final form of associative and commutative multiplication of four-dimensional numbers with complex components is written as $Z = \left(z_1^{re} + z_1^{im}i, z_2^{re} + z_2^{im}i, z_3^{re} + z_3^{im}i, z_4^{re} + z_4^{im}i\right) = (z_1, z_2, z_3, z_4) = XY$, where

$$\begin{cases} z_1 = x_1y_1 + \gamma\delta x_2y_2 + \beta\delta x_3y_3 + \beta\gamma x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + \beta x_4y_3 + \beta x_3y_4 \\ z_3 = x_3y_1 + \gamma x_4y_2 + x_1y_3 + \gamma x_2y_4 \\ z_4 = x_4y_1 + \delta x_3y_2 + \delta x_2y_3 + x_1y_4 \end{cases}. \qquad (5)$$

As can be seen from the course of our reasoning other associative and commutative products of four-dimensional numbers with complex components do not exist. Thus, each point $(\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2) \in R^6$ of the six-dimensional space defines a particular multiplication that has the property of associativity and commutativity, as well as being consistent with the operation of addition. The obtained definition of multiplication (5) is exactly the same as the definition of multiplication of four-dimensional numbers with real components obtained in [1,5].

The following eight numbers are called basis numbers: $J_1^1 = (1,0,0,0)$, $J_1^2 = (i,0,0,0)$, $J_2^1 = (0,1,0,0)$, $J_2^2 = (0,i,0,0)$, $J_3^1 = (0,0,1,0)$, $J_3^2 = (0,0,i,0)$, $J_4^1 = (0,0,0,1)$, $J_4^2 = (0,0,0,i)$. Then any four-dimensional number with complex components can be represented as an expansion over the basis numbers

$$X = x_1^{re}J_1^1 + x_1^{im}J_1^2 + x_2^{re}J_2^1 + x_2^{im}J_2^2 + x_3^{re}J_3^1 + x_3^{im}J_3^2 + x_4^{re}J_4^1 + x_4^{im}J_4^2. \qquad (6)$$

Let us construct the table of multiplication of basis numbers (Table 1).

**Table 1.** Multiplications of basis numbers.

| | $J_1^1$ | $J_1^2$ | $J_2^1$ | $J_2^2$ | $J_3^1$ | $J_3^2$ | $J_4^1$ | $J_4^2$ |
|---|---|---|---|---|---|---|---|---|
| $J_1^1$ | $J_1^1$ | $J_1^2$ | $J_2^1$ | $J_2^2$ | $J_3^1$ | $J_3^2$ | $J_4^1$ | $J_4^2$ |
| $J_1^2$ | $J_1^2$ | $-J_1^1$ | $J_2^2$ | $-J_2^1$ | $J_3^2$ | $-J_3^1$ | $J_4^2$ | $-J_4^1$ |
| $J_2^1$ | $J_2^1$ | $J_2^2$ | $\gamma\delta J_1^1$ | $\gamma\delta J_1^2$ | $\delta J_4^1$ | $\delta J_4^2$ | $\gamma J_3^1$ | $\gamma J_3^2$ |
| $J_2^2$ | $J_2^2$ | $-J_2^1$ | $\gamma\delta J_1^2$ | $-\gamma\delta J_1^1$ | $\delta J_4^2$ | $-\delta J_4^1$ | $\gamma J_3^2$ | $-\gamma J_3^1$ |
| $J_3^1$ | $J_3^1$ | $J_3^2$ | $\delta J_4^1$ | $\delta J_4^2$ | $\beta\delta J_1^1$ | $\beta\delta J_1^2$ | $\beta J_2^1$ | $\beta J_2^2$ |
| $J_3^2$ | $J_3^2$ | $-J_3^1$ | $\delta J_4^2$ | $-\delta J_4^1$ | $\beta\delta J_1^2$ | $-\beta\delta J_1^1$ | $\beta J_2^2$ | $-\beta J_2^1$ |
| $J_4^1$ | $J_4^1$ | $J_4^2$ | $\gamma J_3^1$ | $\gamma J_3^2$ | $\beta J_2^1$ | $\beta J_2^2$ | $\beta\gamma J_1^1$ | $\beta\gamma J_1^2$ |
| $J_4^2$ | $J_4^2$ | $-J_4^1$ | $\gamma J_3^2$ | $-\gamma J_3^1$ | $\beta J_2^2$ | $-\beta J_2^1$ | $\beta\gamma J_1^2$ | $-\beta\gamma J_1^1$ |

Let $x = (x_1, x_2, x_3, x_4)$ be a four-dimensional number with complex components. Consider together with it the following numbers:

$$x^{(2)} = (x_1, x_2, -x_3, -x_4) = \sum_{k=1}^{4}(-1)^{\left[\frac{k}{3}\right]}\left(x_k^{re}J_k^1 + x_k^{im}J_k^2\right),$$

$$x^{(3)} = (x_1, -x_2, x_3, -x_4) = \sum_{k=1}^{4}(-1)^{k-1}\left(x_k^{re}J_k^1 + x_k^{im}J_k^2\right),$$

$$x^{(4)} = (x_1, -x_2, -x_3, x_4) = \sum_{k=1}^{4}(-1)^{\left[\frac{k}{2}\right]}\left(x_k^{re}J_k^1 + x_k^{im}J_k^2\right),$$

where square brackets mean the integer part of the number. Let's calculate the product of $x \cdot x^{(2)} \cdot x^{(3)} \cdot x^{(4)}$

$x \cdot x^{(2)} = (x_1^2 + \gamma\delta x_2^2 - \beta\delta x_3^2 - \beta\gamma x_4^2,\ 2x_1x_2 - 2\beta x_3x_4,\ 0,\ 0)$ ,

$x \cdot x^{(2)} \cdot x^{(3)} = (x_1^3 - \gamma\delta x_1x_2^2 - \beta\delta x_1x_3^2 - \beta\gamma x_1x_4^2 + 2\beta\gamma\delta x_2x_3x_4,\ x_1^2x_2 - \gamma\delta x_2^3 + \beta\delta x_2x_3^2 + \beta\gamma x_2x_4^2 - 2\beta x_1x_3x_4,\ x_1^2x_3 + \gamma\delta x_2^2x_3 - \beta\delta x_3^3 + \beta\gamma x_3x_4^2 - 2\gamma x_1x_2x_4,\ 2\delta x_1x_2x_3 - \beta\delta x_3^2x_4 - x_1^2x_4 - \gamma\delta x_2^2x_4 + \beta\gamma x_4^3)$,

$x \cdot x^{(2)} \cdot x^{(3)} \cdot x^{(4)} = (x_1^4 + \gamma^2\delta^2 x_2^4 + \beta^2\delta^2 x_3^4 + \beta^2\gamma^2 x_4^4 - 2\gamma\delta x_1^2 x_2^2 - 2\beta\delta x_1^2 x_3^2 - 2\beta\gamma x_1^2 x_4^2 - 2\beta\gamma\delta^2 x_2^2 x_3^2 - 2\beta\gamma^2\delta x_2^2 x_4^2 - 2\beta^2\gamma\delta x_3^2 x_4^2 + 8\beta\gamma\delta x_1 x_2 x_3 x_4,\ 0,\ 0,\ 0)$.

Thus, $x \cdot x^{(2)} \cdot x^{(3)} \cdot x^{(4)}$ is a quasi-real number.

**Definition**. The number $x^* = x^{(2)} \cdot x^{(3)} \cdot x^{(4)}$ is called the conjugate number to the number $x$.

Then $x \cdot x^* = A^{re}(X) J_1^1 + A^{im}(X) J_1^2$ ,
where

$$A(X) = x_1^4 + \gamma^2\delta^2 x_2^4 + \beta^2\delta^2 x_3^4 + \beta^2\gamma^2 x_4^4 - 2\gamma\delta x_1^2 x_2^2 - 2\beta\delta x_1^2 x_3^2 - 2\beta\gamma x_1^2 x_4^2 - 2\beta\gamma\delta^2 x_2^2 x_3^2 - 2\beta\gamma^2\delta x_2^2 x_4^2 - 2\beta^2\gamma\delta x_3^2 x_4^2 + 8\beta\gamma\delta x_1 x_2 x_3 x_4 \quad . \qquad (7)$$

The real part of the number A consists of 352 summands, and the imaginary part consists of 348 summands, so we do not give them here. Let us call the modulus of the number $A = A^{re} + A^{im} i$ the symplectic modulus of the number $X = (x_1, x_2, x_3, x_4)$.

By direct calculation we find the conjugate number $x^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ to the four-dimensional number $x = (x_1, x_2, x_3, x_4)$

$$x_1^* = x_1(x_1^2 - \gamma\delta x_2^2 - \beta\delta x_3^2 - \beta\gamma x_4^2) + 2\beta\gamma\delta x_2 x_3 x_4,$$
$$x_2^* = x_2(-x_1^2 + \gamma\delta x_2^2 - \beta\delta x_3^2 - \beta\gamma x_4^2) + 2\beta x_1 x_3 x_4,$$
$$x_3^* = x_3(-x_1^2 - \gamma\delta x_2^2 + \beta\delta x_3^2 - \beta\gamma x_4^2) + 2\gamma x_1 x_2 x_4,$$
$$x_4^* = x_4(-x_1^2 - \gamma\delta x_2^2 - \beta\delta x_3^2 + \beta\gamma x_4^2) + 2\delta x_1 x_2 x_3.$$

Accordingly, the conjugate numbers to the basis numbers are of the form:

$J_1^{1*} = J_1^1, J_1^{2*} = -J_1^2, J_2^{1*} = \gamma\delta J_2^1, J_2^{2*} = -\gamma\delta J_2^2, J_3^{1*} = \beta\delta J_3^1, J_3^{2*} = -\beta\delta J_3^2, J_4^{1*} = \beta\gamma J_4^1, J_4^{2*} = -\beta\gamma J_4^2.$

Let the symplectic modulus of the four-dimensional number $X = (x_1, x_2, x_3, x_4)$ be different from zero. Then there is a single number $x^{-1}$, called the inverse of $x$, such that $x \cdot x^{-1} = x^{-1} \cdot x = J_1^1$. Multiplying both parts of this equality by $x^*$ yields $x^{-1} \cdot x \cdot x^* = x^*$ or $(A, 0,0,0) x^{-1} = x^*$. Multiplying both parts of this equality by the number $\left(\frac{\bar{A}}{|A|^2}, 0,0,0\right)$, then, given that $\left(\frac{\bar{A}}{|A|^2}, 0,0,0\right) \cdot (A, 0,0,0) = J_1^1$ , we get $x^{-1} = x^* \cdot \left(\frac{\bar{A}}{|A|^2}, 0,0,0\right)$ , where $\bar{A}$ is a conjugate number to A. That is

$$X^{-1} = \left(\frac{\bar{A}}{|A|^2} x_1^*, \frac{\bar{A}}{|A|^2} x_2^*, \frac{\bar{A}}{|A|^2} x_3^*, \frac{\bar{A}}{|A|^2} x_4^*\right) \qquad (8)$$

Then we define the operation of division of four-dimensional numbers as $\frac{y}{x} = y \cdot x^{-1} = \frac{\bar{A}}{|A|^2} x^* y$, if the symplectic modulus of a four-dimensional number $x$ is different from zero.

Thus, we have defined the operations of addition, subtraction, multiplication and division of four-dimensional numbers with complex components. Moreover, the number of multiplications is infinitely many for each value of the triple $(\beta, \gamma, \delta) \in$ ▯$^3$. Next, let us construct basis spaces of four-dimensional numbers in which the multiplication operation is defined concretely and all possible multiplications can be obtained from these basis multiplications. In this way, we will construct some number of basis spaces of four-dimensional numbers with specific addition and multiplication operations that have the associativity and commutativity properties.

## 3. Basis Spaces of Four-Dimensional Numbers

We want to obtain some finite basis on which all possible products of the form (5) are decomposed. For this purpose, similarly to the case of four-dimensional numbers with real components, we simplify this definition by getting rid of arbitrary numbers $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$. Let us make the following substitution of variables:

$$\begin{cases} \tilde{x}_1^{re} + \tilde{x}_1^{im} i = x_1^{re} + x_1^{im} i \\ \tilde{x}_2^{re} + \tilde{x}_2^{im} i = \sqrt{(\gamma_1 + \gamma_2 i)(\delta_1 + \delta_2 i)}\left(x_2^{re} + x_2^{im} i\right) \\ \tilde{x}_3^{re} + \tilde{x}_3^{im} i = \sqrt{(\beta_1 + \beta_2 i)(\delta_1 + \delta_2 i)}\left(x_3^{re} + x_3^{im} i\right) \\ \tilde{x}_4^{re} + \tilde{x}_4^{im} i = \sqrt{(\beta_1 + \beta_2 i)(\gamma_1 + \gamma_2 i)}\left(x_4^{re} + x_4^{im} i\right) \end{cases} . \qquad (9)$$

The inverse transformation has the form

$$\begin{cases} x_1^{re}+x_1^{im}i=\tilde{x}_1^{re}+\tilde{x}_1^{im}i \\ x_2^{re}+x_2^{im}i=\frac{1}{\sqrt{(\gamma_1+\gamma_2 i)(\delta_1+\delta_2 i)}}\left(\tilde{x}_2^{re}+\tilde{x}_2^{im}i\right) \\ x_3^{re}+x_3^{im}i=\frac{1}{\sqrt{(\beta_1+\beta_2 i)(\delta_1+\delta_2 i)}}\left(\tilde{x}_3^{re}+\tilde{x}_3^{im}i\right) \\ x_4^{re}+x_4^{im}i=\frac{1}{\sqrt{(\beta_1+\beta_2 i)(\gamma_1+\gamma_2 i)}}\left(\tilde{x}_4^{re}+\tilde{x}_4^{im}i\right) \end{cases}.$$

Let us take the product of $z=x\cdot y$ and go to the transformed space with a wave. Then

$$\left(y_1^{re}+y_1^{im}i,y_2^{re}+y_2^{im}i,y_3^{re}+y_3^{im},y_4^{re}+y_4^{im}i\right)=$$

$$\left(\tilde{y}_1^{re}+\tilde{y}_1^{im}i,\frac{1}{\sqrt{(\gamma_1+\gamma_2 i)(\delta_1+\delta_2 i)}}\left(\tilde{y}_2^{re}+\tilde{y}_2^{im}i\right),\frac{1}{\sqrt{(\beta_1+\beta_2 i)(\delta_1+\delta_2 i)}}\left(\tilde{y}_3^{re}+\tilde{y}_3^{im}i\right),\frac{1}{\sqrt{(\beta_1+\beta_2 i)(\gamma_1+\gamma_2 i)}}\left(\tilde{y}_4^{re}+\tilde{y}_4^{im}i\right)\right),$$

$$\left(z_1^{re}+z_1^{im}i,z_2^{re}+z_2^{im}i,z_3^{re}+z_3^{im},z_4^{re}+z_4^{im}i\right)=$$

$$\left(\tilde{z}_1^{re}+\tilde{z}_1^{im}i,\frac{1}{\sqrt{(\gamma_1+\gamma_2 i)(\delta_1+\delta_2 i)}}\left(\tilde{z}_2^{re}+\tilde{z}_2^{im}i\right),\frac{1}{\sqrt{(\beta_1+\beta_2 i)(\delta_1+\delta_2 i)}}\left(\tilde{z}_3^{re}+\tilde{z}_3^{im}i\right),\frac{1}{\sqrt{(\beta_1+\beta_2 i)(\gamma_1+\gamma_2 i)}}\left(\tilde{z}_4^{re}+\tilde{z}_4^{im}i\right)\right).$$

Substituting this into formula (5) we have

$$\begin{cases} \tilde{z}_1^{re}+\tilde{z}_1^{im}i=\left(\tilde{x}_1^{re}+\tilde{x}_1^{im}i\right)\left(\tilde{y}_1^{re}+\tilde{y}_1^{im}i\right)+\left(\tilde{x}_2^{re}+\tilde{x}_2^{im}i\right)\left(\tilde{y}_2^{re}+\tilde{y}_2^{im}i\right)+ \\ \quad +\left(x_3^{re}+x_3^{im}i\right)\left(\tilde{y}_3^{re}+\tilde{y}_3^{im}i\right)+\left(x_4^{re}+x_4^{im}i\right)\left(\tilde{y}_4^{re}+\tilde{y}_4^{im}i\right) \\ \tilde{z}_2^{re}+\tilde{z}_2^{im}i=\left(\tilde{x}_2^{re}+\tilde{x}_2^{im}i\right)\left(\tilde{y}_1^{re}+\tilde{y}_1^{im}i\right)+\left(\tilde{x}_1^{re}+\tilde{x}_1^{im}i\right)\left(\tilde{y}_2^{re}+\tilde{y}_2^{im}i\right)+ \\ \quad +\left(x_4^{re}+x_4^{im}i\right)\left(\tilde{y}_3^{re}+\tilde{y}_3^{im}i\right)+\left(x_3^{re}+x_3^{im}i\right)\left(\tilde{y}_4^{re}+\tilde{y}_4^{im}i\right) \\ \tilde{z}_3^{re}+\tilde{z}_3^{im}i=\left(\tilde{x}_3^{re}+\tilde{x}_3^{im}i\right)\left(\tilde{y}_1^{re}+\tilde{y}_1^{im}i\right)+\left(\tilde{x}_4^{re}+\tilde{x}_4^{im}i\right)\left(\tilde{y}_2^{re}+\tilde{y}_2^{im}i\right)+ \\ \quad +\left(x_1^{re}+x_1^{im}i\right)\left(\tilde{y}_3^{re}+\tilde{y}_3^{im}i\right)+\left(x_2^{re}+x_2^{im}i\right)\left(\tilde{y}_4^{re}+\tilde{y}_4^{im}i\right) \\ \tilde{z}_4^{re}+\tilde{z}_4^{im}i=\left(\tilde{x}_4^{re}+\tilde{x}_4^{im}i\right)\left(\tilde{y}_1^{re}+\tilde{y}_1^{im}i\right)+\left(\tilde{x}_3^{re}+\tilde{x}_3^{im}i\right)\left(\tilde{y}_2^{re}+\tilde{y}_2^{im}i\right)+ \\ \quad +\left(x_2^{re}+x_2^{im}i\right)\left(\tilde{y}_3^{re}+\tilde{y}_3^{im}i\right)+\left(x_1^{re}+x_1^{im}i\right)\left(\tilde{y}_4^{re}+\tilde{y}_4^{im}i\right) \end{cases}.$$

or passing to the original notations without the wave

$$\begin{cases} z_1=x_1y_1+x_2y_2+x_3y_3+x_4y_4 \\ z_2=x_2y_1+x_1y_2+x_4y_3+x_3y_4 \\ z_3=x_3y_1+x_4y_2+x_1y_3+x_2y_4 \\ z_4=x_4y_1+x_3y_2+x_2y_3+x_1y_4 \end{cases}. \tag{10}$$

Equation (10) defines the canonical multiplication of four-dimensional numbers with complex components.

The given substitution of variables (9) is not the only possible substitution. Since the square root of a complex number has two roots we can define the following substitution of variables:

$$\begin{cases} \tilde{x}_1^{re}+\tilde{x}_1^{im}i=x_1^{re}+x_1^{im}i \\ \tilde{x}_2^{re}+\tilde{x}_2^{im}i=-\sqrt{(\gamma_1+\gamma_2 i)(\delta_1+\delta_2 i)}\left(x_2^{re}+x_2^{im}i\right) \\ \tilde{x}_3^{re}+\tilde{x}_3^{im}i=\sqrt{(\beta_1+\beta_2 i)(\delta_1+\delta_2 i)}\left(x_3^{re}+x_3^{im}i\right) \\ \tilde{x}_4^{re}+\tilde{x}_4^{im}i=\sqrt{(\beta_1+\beta_2 i)(\gamma_1+\gamma_2 i)}\left(x_4^{re}+x_4^{im}i\right) \end{cases}. \tag{11}$$

Carrying out similar transformations as in (10) we obtain another canonical form of multiplication of four-dimensional numbers with complex components

$$\begin{cases} z_1=x_1y_1+x_2y_2+x_3y_3+x_4y_4 \\ z_2=x_2y_1+x_1y_2-x_4y_3-x_3y_4 \\ z_3=x_3y_1-x_4y_2+x_1y_3-x_2y_4 \\ z_4=x_4y_1-x_3y_2-x_2y_3+x_1y_4 \end{cases}. \tag{12}$$

Now we make the following substitution of variables

$$\begin{cases} \tilde{x}_1^{re}+\tilde{x}_1^{im}i=x_1^{re}+x_1^{im}i \\ \tilde{x}_2^{re}+\tilde{x}_2^{im}i=\sqrt{(\gamma_1+\gamma_2 i)(\delta_1+\delta_2 i)}\left(x_2^{re}+x_2^{im}i\right) \\ \tilde{x}_3^{re}+\tilde{x}_3^{im}i=\sqrt{(\beta_1 i-\beta_2)(\delta_1+\delta_2 i)}\left(x_3^{re}+x_3^{im}i\right) \\ \tilde{x}_4^{re}+\tilde{x}_4^{im}i=\sqrt{(\beta_1 i-\beta_2)(\gamma_1+\gamma_2 i)}\left(x_4^{re}+x_4^{im}i\right) \end{cases}. \tag{13}$$

Then the inverse transformation has the form

$$\begin{cases} x_1^{re} + x_1^{im} i = \tilde{x}_1^{re} + \tilde{x}_1^{im} i \\ x_2^{re} + x_2^{im} i = \frac{1}{\sqrt{(\gamma_1+\gamma_2 i)(\delta_1+\delta_2 i)}} \left(\tilde{x}_2^{re} + \tilde{x}_2^{im} i\right) \\ x_3^{re} + x_3^{im} i = \frac{1}{\sqrt{(\beta_1 i-\beta_2)(\delta_1+\delta_2 i)}} \left(\tilde{x}_3^{re} + \tilde{x}_3^{im} i\right) \\ x_4^{re} + x_4^{im} i = \frac{1}{\sqrt{(\beta_1 i-\beta_2)(\gamma_1+\gamma_2 i)}} \left(\tilde{x}_4^{re} + \tilde{x}_4^{im} i\right) \end{cases}.$$

Let us rewrite the product (5) in variables with waves $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$. Substituting $(y_1, y_2, y_3, y_4) = \left(\tilde{y}_1, \frac{1}{\sqrt{(\gamma_1+\gamma_2 i)(\delta_1+\delta_2 i)}}\tilde{y}_2, \frac{1}{\sqrt{(\beta_1 i-\beta_2)(\delta_1+\delta_2 i)}}\tilde{y}_3, \frac{1}{\sqrt{(\beta_1 i-\beta_2)(\gamma_1+\gamma_2 i)}}\tilde{y}_4\right)$, $(z_1, z_2, z_3, z_4) = \left(\tilde{z}_1, \frac{1}{\sqrt{(\gamma_1+\gamma_2 i)(\delta_1+\delta_2 i)}}\tilde{z}_2, \frac{1}{\sqrt{(\beta_1 i-\beta_2)(\delta_1+\delta_2 i)}}\tilde{z}_3, \frac{1}{\sqrt{(\beta_1 i-\beta_2)(\gamma_1+\gamma_2 i)}}\tilde{z}_4\right)$ into (5) we obtain

$$\begin{cases} \tilde{z}_1 = \tilde{x}_1\tilde{y}_1 + \tilde{x}_2\tilde{y}_2 - i\tilde{x}_3\tilde{y}_3 - i\tilde{x}_4\tilde{y}_4 \\ \tilde{z}_2 = \tilde{x}_2\tilde{y}_1 + \tilde{x}_1\tilde{y}_2 - i\tilde{x}_4\tilde{y}_3 - i\tilde{x}_3\tilde{y}_4 \\ \tilde{z}_3 = \tilde{x}_3\tilde{y}_1 + \tilde{x}_4\tilde{y}_2 + \tilde{x}_1\tilde{y}_3 + \tilde{x}_2\tilde{y}_4 \\ \tilde{z}_4 = \tilde{x}_4\tilde{y}_1 + \tilde{x}_3\tilde{y}_2 + \tilde{x}_2\tilde{y}_3 + \tilde{x}_1\tilde{y}_4 \end{cases}. \quad (14)$$

The above transformations suggest that the basis multiplications are the multiplications in which the coefficients of $\beta, \gamma, \delta$ take the values $[1, -1, i, -i]$. There are only 64 such multiplications, which are summarized in Table 2. Each cell in Table 2 contains a triple $(a, b, c)$, where *a is the* value of $\beta$ , *b is the* value of $\gamma$, *c is the* value of $\delta$.

**Table 2.** All possible basis multiplications of four-dimensional numbers.

| | | □ = □ | □ = −□ | □ = □ | □ = −□ |
|---|---|---|---|---|---|
| □ = □ | $\gamma = 1$ | $(1,1,1)$ | $(1,1,-1)$ | $(1,1,i)$ | $(1,1,-i)$ |
| | $\gamma = -1$ | $(1-,1,1)$ | $(1,-1,-1)$ | $(1,-1,i)$ | $(1-,1,-i)$ |
| | $\gamma = i$ | $(1,i,1)$ | $(1,i,-1)$ | $(1,i,i)$ | $(1,i,-i)$ |
| | $\gamma = -i$ | $(1,-i,1)$ | $(1,-i,-1)$ | $(1,-i,i)$ | $(1,-i,-i)$ |
| □ = −□ | $\gamma = 1$ | $(-1,1,1)$ | $(-1,1,-1)$ | $(-1,1,i)$ | $(-1,1,-i)$ |
| | $\gamma = -1$ | $(-1,-1,1)$ | $(-1,-1,-1)$ | $(-1,-1,i)$ | $(-1,-1,-i)$ |
| | $\gamma = i$ | $(-1,i,1)$ | $(-1,i,-1)$ | $(-1,i,i)$ | $(-1,i,-i)$ |
| | $\gamma = -i$ | $(-1,-i,1)$ | $(-1,-i,-1)$ | $(-1,-i,i)$ | $(-1,-i,-i)$ |
| □ = □ | $\gamma = 1$ | $(i,1,1)$ | $(i,1,-1)$ | $(i,1,i)$ | $(i,1,-i)$ |
| | $\gamma = -1$ | $(i,-1,1)$ | $(i,-1,-1)$ | $(i,-1,i)$ | $(i,-1,-i)$ |
| | $\gamma = i$ | $(i,i,1)$ | $(i,i,-1)$ | $(i,i,i)$ | $(i,i,-i)$ |
| | $\gamma = -i$ | $(i,-i,1)$ | $(i,-i,-1)$ | $(i,-i,i)$ | $(i,-i,-i)$ |
| □ = −□ | $\gamma = 1$ | $(-i,1,1)$ | $(-i,1,-1)$ | $(-i,1,i)$ | $(-i,1,-i)$ |
| | $\gamma = -1$ | $(-i,-1,1)$ | $(-i,-1,-1)$ | $(-i,-1,i)$ | $(-i,-1,-i)$ |
| | $\gamma = i$ | $(-i,i,1)$ | $(-i,i,-1)$ | $(-i,i,i)$ | $(-i,i,-i)$ |
| | $\gamma = -i$ | $(-i,-i,1)$ | $(-i,-i,-1)$ | $(-i,-i,i)$ | $(-i,-i,-i)$ |

For each cell of this Table 2, substituting the corresponding values of $\beta, \gamma, \delta$ into formula (5) we obtain some basis multiplication of four-dimensional numbers with complex components. Let us consider as an example the cells of the first row of the table for which $\beta = \gamma = 1$. For cell $(1,1,1)$ the multiplication is defined by formula (10), for cell $(1,1,-1)$ by formula

$$\begin{cases} z_1 = x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 = x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 = x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{cases}, \quad (15)$$

for cell $(1,1,i)$ the multiplication is defined by formula

$$\begin{cases} z_1 = x_1y_1 + ix_2y_2 + ix_3y_3 + x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 = x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 = x_4y_1 + ix_3y_2 + ix_2y_3 + x_1y_4 \end{cases}, \quad (16)$$

and for the cell $(1,1,-i)$ the multiplication is defined by formula

$$\begin{cases} z_1 = x_1y_1 - ix_2y_2 - ix_3y_3 + x_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 = x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4 \\ z_4 = x_4y_1 - ix_3y_2 - ix_2y_3 + x_1y_4 \end{cases}. \quad (17)$$

The definition of multiplication (12) corresponds to the cell $(-1,-1,-1)$, and the definition (14) corresponds to the cell $(-i,1,1)$. The other definitions of base multiplications can be written out similarly. Obviously, any other definition of multiplication can be reduced through a substitution of variables analogous to (9), (11) or (13) to one of the basis multiplications. Thus, for four-dimensional numbers with complex components, 64 different multiplications can be defined which are associative and commutative, and consistent with the addition operation. As it is known [1] in the case of four-dimensional numbers with real components, there are 8 different multiplication operations, for six of which a pre-norm can be defined. These definitions correspond to the cells $(1,1,1), (1,1,-1), (1,-1,1), (1,-1,-1), (-1,1,1), (-1,1,-1), (-1,-1,1), (-1,-1,-1)$ of Table 2.

Each basis multiplication defines some linear vector space of four-dimensional numbers with complex components over the field of complex numbers. For further convenience, we introduce notation for these spaces. The most natural notation for them, by analogy with the case with real components, is $M_j$, where $j$ varies from 1 to 64. Thus we will number the spaces in Table 2 by rows, and within rows by columns. The corresponding notations are given in Table 3. It may be more convenient to use designations of the form $M(\beta,\gamma,\delta)$, where $\beta,\gamma,\delta$ takes the values $[1,-1,i,-i]$, especially for multidimensional numbers with the number of components more than four, for example, for eight-dimensional, sixteen-dimensional, etc. With these notations, the space $\mathrm{M}_2$ for the case of four-dimensional numbers with real components [1] corresponds to the space $M(-1,1,1)$, the space $\mathrm{M}_3$ corresponds to the space $M(1,-1,1)$, etc., the space $\mathrm{M}_7$ corresponds to the space $M(1,-1,-1)$. These notations of spaces of four-dimensional numbers with complex components are also given in Table 3. Each cell of Table 3 contains two designations of the corresponding basis space of four-dimensional numbers with complex components.

**Table 3.** Denotations of basis spaces of four-dimensional numbers.

| □ | □ | □ = □ | □ = −□ | □ = □ | □ = −□ |
|---|---|---|---|---|---|
| □ | $1$ | $M_1 = M(1,1,1)$ | $M_2 = M(1,1,-1)$ | $M_3 = M(1,1,i)$ | $M_4 = M(1,1,-i)$ |
| | $-1$ | $M_5 = M(1,-1,1)$ | $M_6 = M(1,-1,-1)$ | $M_7 = M(1,-1,i)$ | $M_8 = M(1,-1,-i)$ |
| | $i$ | $M_9 = M(1,i,1)$ | $M_{10} = M(1,i,-1)$ | $M_{11} = M(1,i,i)$ | $M_{12} = M(1,i,-i)$ |
| | $-i$ | $M_{13} = M(1,-i,1)$ | $M_{14} = M(1,-i,-1)$ | $M_{15} = M(1,-i,i)$ | $M_{16} = M(1,-i,-i)$ |
| −□ | $1$ | $M_{17} = M(-1,1,1)$ | $M_{18} = M(-1,1,-1)$ | $M_{19} = M(-1,1,i)$ | $M_{20} = M(-1,1,-i)$ |
| | $-1$ | $M_{21} = M(-1,-1,1)$ | $M_{22} = M(-1,-1,-1)$ | $M_{23} = M(-1,-1,i)$ | $M_{24} = M(-1,-1,-i)$ |
| | $i$ | $M_{25} = M(-1,i,1)$ | $M_{26} = M(-1,i,-1)$ | $M_{27} = M(-1,i,i)$ | $M_{28} = M(-1,i,-i)$ |
| | $-i$ | $M_{29} = M(-1,-i,1)$ | $M_{30} = M(-1,-i,-1)$ | $M_{31} = M(-1,-i,i)$ | $M_{32} = M(-1,-i,-i)$ |
| □ | $1$ | $M_{33} = M(i,1,1)$ | $M_{34} = M(i,1,-1)$ | $M_{35} = M(i,1,i)$ | $M_{36} = M(i,1,-i)$ |
| | $-1$ | $M_{37} = M(i,-1,1)$ | $M_{38} = M(i,-1,-1)$ | $M_{39} = M(i,-1,i)$ | $M_{40} = M(i,-1,-i)$ |
| | $i$ | $M_{41} = M(i,i,1)$ | $M_{42} = M(i,i,-1)$ | $M_{43} = M(i,i,i)$ | $M_{44} = M(i,i,-i)$ |
| | $-i$ | $M_{45} = M(i,-i,1)$ | $M_{46} = M(i,-i,-1)$ | $M_{47} = M(i,-i,i)$ | $M_{48} = M(i,-i,-i)$ |
| −□ | $1$ | $M_{49} = M(-i,1,1)$ | $M_{50} = M(-i,1,-1)$ | $M_{51} = M(-i,1,i)$ | $M_{52} = M(-i,1,-i)$ |
| | $-1$ | $M_{53} = M(-i,-1,1)$ | $M_{54} = M(-i,-1,-1)$ | $M_{55} = M(-i,-1,i)$ | $M_{56} = M(-i,-1,-i)$ |
| | $i$ | $M_{57} = M(-i,i,1)$ | $M_{58} = M(-i,i,-1)$ | $M_{59} = M(-i,i,i)$ | $M_{60} = M(-i,i,-i)$ |
| | $-i$ | $M_{61} = M(-i,-i,1)$ | $M_{62} = M(-i,-i,-1)$ | $M_{63} = M(-i,-i,i)$ | $M_{64} = M(-i,-i,-i)$ |

Thus, we obtained the basis spaces of four-dimensional numbers with complex components, in which the associative and commutative operations of addition and multiplication are defined. In each space, the multiplication table of basis numbers is easily written out by substituting into Table 1 the corresponding values of $(\beta,\gamma,\delta)$. For example, for the space $M(1,-i,-1)$ or $M_{14}$, the multiplication table of basis numbers is given in Table 4.

**Table 4.** Multiplications of basis numbers in the space $M_{14}$.

| | $\square_{\square}^{\square}$ | $\square_{\square}^{\square}$ | $\square_{\square}^{\square}$ | $\square_{\square}^{\square}$ | $\square_{\square}^{\square}$ | $\square_{\square}^{\square}$ | $\square_{\square}^{\square}$ | $\square_{\square}^{\square}$ |
|---|---|---|---|---|---|---|---|---|
| $\square_{\square}^{\square}$ | $J_1^1$ | $J_1^2$ | $J_2^1$ | $J_2^2$ | $J_3^1$ | $J_3^2$ | $J_4^1$ | $J_4^2$ |
| $\square_{\square}^{\square}$ | $J_1^2$ | $-J_1^1$ | $J_2^2$ | $-J_2^2$ | $J_3^2$ | $-J_3^1$ | $J_4^2$ | $-J_4^1$ |
| $\square_{\square}^{\square}$ | $J_2^1$ | $J_2^2$ | $J_1^2$ | $-J_1^1$ | $-J_4^1$ | $-J_4^2$ | $-J_3^2$ | $J_3^1$ |
| $\square_{\square}^{\square}$ | $J_2^2$ | $-J_2^1$ | $-J_1^1$ | $-J_1^2$ | $-J_4^2$ | $J_4^1$ | $J_3^1$ | $J_3^2$ |
| $\square_{\square}^{\square}$ | $J_3^1$ | $J_3^2$ | $-J_4^1$ | $-J_4^2$ | $-J_1^1$ | $-J_1^2$ | $J_2^1$ | $J_2^2$ |
| $\square_{\square}^{\square}$ | $J_3^2$ | $-J_3^1$ | $-J_4^2$ | $J_4^1$ | $-J_1^2$ | $J_1^1$ | $J_2^2$ | $-J_2^1$ |
| $\square_{\square}^{\square}$ | $J_4^1$ | $J_4^2$ | $-J_3^2$ | $J_3^1$ | $J_2^1$ | $J_2^2$ | $-J_1^2$ | $J_1^1$ |
| $\square_{\square}^{\square}$ | $J_4^2$ | $-J_4^1$ | $J_3^1$ | $J_3^2$ | $J_2^2$ | $-J_2^1$ | $J_1^1$ | $J_1^2$ |

The constructed basis spaces have a remarkable symmetry. Indeed, as can be seen from the diagonal elements of Table 4, in the space $M_{14}$, of the eight basis numbers, four are real and four numbers are imaginary. It is easy to realize that a similar property is possessed by the basis numbers of any space $M_j$, where $j \in [1,64]$. The presence of such symmetry distinguish these spaces from the space of quaternions, in which one basis element is real and the rest are imaginary. The symmetry of the basis numbers makes these spaces a natural generalization of one-dimensional and two-dimensional (complex) algebra.

## 4. Matrix Spaces for Four-Dimensional Numbers

Let us introduce the operator $F: X \to F(X)$, which maps a matrix $X = (x_1, x_2, x_3, x_4) \in \square^4$ to each four-dimensional number.

$$F(X) = \begin{pmatrix} x_1 & \gamma\delta x_2 & \beta\delta x_3 & \beta\gamma x_4 \\ x_2 & x_1 & \beta x_4 & \beta x_3 \\ x_3 & \gamma x_4 & x_1 & \gamma x_2 \\ x_4 & \delta x_3 & \delta x_2 & x_1 \end{pmatrix}, \tag{18}$$

Where $\beta = \beta_1 + \beta_2 i, \gamma = \gamma_1 + \gamma_2 i, \delta = \delta_1 + \delta_2 i$ are the coefficients in the definition of multiplication (5). All elements of the matrix $F(X)$ are formed from the coordinates of the number $X$.

The mapping $F: X \to F(X)$ is one-to-one and on. Indeed, two different numbers $x$ and $y$ correspond to different matrices and for any matrix of the above kind we can find the corresponding four-dimensional number with complex components.

**Theorem 1**. The set of all matrices of the form (18) is closed with respect to the operations of addition, subtraction and multiplication of matrices, as well as multiplication of a matrix by a scalar. The inverse matrix to a nondegenerate matrix has the same form (18).

Proof. It is proved by direct inspection.

The multiplication of two four-dimensional numbers $X = (x_1, x_2, x_3, x_4)$ and $Y = (y_1, y_2, y_3, y_4)$ with complex components can be represented as $X \cdot Y = F(X) \cdot Y$, where the sign of the multiplication in the left-hand side is understood in the sense of (5), and the sign of the multiplication in the right-hand side is understood as the multiplication of a matrix by a vector. Thus, we have defined an alternative definition of multiplication of four-dimensional numbers using matrix (18).

Considering as $(\beta, \gamma, \delta)$ their possible values for the basis spaces of four-dimensional numbers we obtain 64 different mappings $F: X \to F(X)$. Let us denote these mappings by $F_j, j = 1,2, \dots, 64$, or $F[\beta, \gamma, \delta]$, where $\beta, \gamma$ and $\delta$ take the values $[1, -1, i, -i]$. For example, for the space $M(1, -i, -1)$ or $M_{14}$, the mapping $F$ *is* as follows:

$$F_{14}(X) = \begin{pmatrix} x_1 & ix_2 & -x_3 & -ix_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -ix_4 & x_1 & -ix_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}, \tag{19}$$

Similarly for four-dimensional number spaces with real components, the mapping $F$ has unique properties, as given in the following theorem.

**Theorem 2.** Every mapping $F_j$ $(j = 1,2,\dots,64)$ for arbitrary four-dimensional numbers $X, Y \in ▯^4$ has the following properties:

1) $F_j(X \pm Y) = F_j(X) \pm F_j(Y)$;
2) $F_j(cX) = cF_j(X)$ for any $c \in ▯$;
3) $F_j(XY) = F_j(X)F_j(Y)$;
4) $F_j(X^{-1}) = F_j^{-1}(X)$;
5) $\det\left(F_j(X)\right) = A(X)$, where A(X) is defined in (7);
6) $\det\left(F_j(X) \pm F(Y)\right) = A(X \pm Y)$, where A(X$\pm$ Y) is defined in (7);
7) $\det\left(F_j(X)F_j(Y)\right) = A(XY)$, where A(XY) is defined in (7);
8) $\det\left(F_j^{-1}(X)\right) = A(X^{-1})$, if the symplectic module of *X is* different from zero.

Proof. Let us prove the theorem for the space of numbers $M_{14}$, for the other spaces it is proved analogously. Properties 1) and 2) are obvious. Let us prove property 3). In the basis space $M_{14}$ the multiplication operation has the form

$$\begin{cases} z_1 = x_1y_1 + ix_2y_2 - x_3y_3 - ix_4y_4 \\ z_2 = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ z_3 = x_3y_1 - ix_4y_2 + x_1y_3 - ix_2y_4 \\ z_4 = x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{cases}. \quad (20)$$

in which $(z_1, z_2, z_3, z_4) = XY$.

$$F_{14}(X)F_{14}(Y) = \begin{pmatrix} x_1 & ix_2 & -x_3 & -ix_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -ix_4 & x_1 & -ix_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 & iy_2 & -y_3 & -iy_4 \\ y_2 & y_1 & y_4 & y_3 \\ y_3 & -iy_4 & y_1 & -iy_2 \\ y_4 & -y_3 & -y_2 & y_1 \end{pmatrix} = B,$$

where $B$ is the resulting matrix. Let us calculate the elements of the matrix B.

$b_{11} = x_1y_1 + ix_2y_2 - x_3y_3 - ix_4y_4 = z_1$,

$b_{12} = ix_1y_2 + ix_2y_1 + ix_3y_4 + ix_4y_3 = iz_2$,

$b_{13} = -x_1y_3 + ix_2y_4 - x_3y_1 + ix_4y_2 = -z_3$,

$b_{14} = -ix_1y_4 + ix_2y_3 + ix_3y_2 - ix_4y_1 = -iz_4$,

$b_{21} = x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 = z_2$,

$b_{22} = ix_2y_2 + x_1y_1 - ix_4y_4 - x_3y_3 = z_1$,

$b_{23} = -x_2y_3 + x_1y_4 + x_4y_1 - x_3y_2 = z_4$,

$b_{24} = -ix_2y_4 + x_1y_3 - ix_4y_2 + x_3y_1 = z_3$,

$b_{31} = x_3y_1 - ix_4y_2 + x_1y_3 - ix_2y_4 = z_3$,

$b_{32} = ix_3y_2 - ix_4y_1 - ix_1y_4 + ix_2y_3 = -iz_4$,

$b_{33} = -x_3y_3 - ix_4y_4 + x_1y_1 + ix_2y_2 = z_1$,

$b_{34} = -ix_3y_4 - ix_4y_3 - ix_1y_2 - ix_2y_1 = -iz_2$,

$b_{41} = x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 = z_4$,

$b_{42} = ix_4y_2 - x_3y_1 + ix_2y_4 - x_1y_3 = -z_3$,

$b_{43} = -x_4y_3 - x_3y_4 - x_2y_1 - x_1y_2 = -z_2$,

$b_{44} = -ix_4y_4 - x_3y_3 + ix_2y_2 + x_1y_1 = z_1$.

Thus, the matrix B coincides with the matrix (19) in which $x$ *is* replaced by $z$. Let us prove property 4). By formula (8) $X^{-1} = \left(\frac{\bar{A}}{|A|^2}x_1^*, \frac{\bar{A}}{|A|^2}x_2^*, \frac{\bar{A}}{|A|^2}x_3^*, \frac{\bar{A}}{|A|^2}x_4^*\right)$ , and by formula (7)

$$A(X) = x_1^4 - x_2^4 + x_3^4 - x_4^4 - 2ix_1^2x_2^2 + 2x_1^2x_3^2 + 2ix_1^2x_4^2 + 2ix_2^2x_3^2 - 2x_2^2x_4^2 - 2ix_3^2x_4^2 + 8ix_1x_2x_3x_4, \tag{21}$$

hence

$$(X^{-1})_1 = \frac{1}{A(X)}x_1(x_1^2 - ix_2^2 + x_3^2 + ix_4^2) + \frac{1}{A(X)}2ix_2x_3x_4,$$

$$(X^{-1})_2 = \frac{1}{A(X)}x_2(-x_1^2 + ix_2^2 + x_3^2 + ix_4^2) + \frac{1}{A(X)}2x_1x_3x_4,$$

$$(X^{-1})_3 = \frac{1}{A(X)}x_3(-x_1^2 - ix_2^2 - x_3^2 + ix_4^2) - \frac{1}{A(X)}2ix_1x_2x_4,$$

$$(X^{-1})_4 = \frac{1}{A(X)}x_4(-x_1^2 - ix_2^2 + x_3^2 - ix_4^2) - \frac{1}{A(X)}2x_1x_2x_3.$$

Then we have the matrix

$$F_{14}(X^{-1}) = \begin{pmatrix} (X^{-1})_1 & i(X^{-1})_2 & -(X^{-1})_3 & -i(X^{-1})_4 \\ (X^{-1})_2 & (X^{-1})_1 & (X^{-1})_4 & (X^{-1})_3 \\ (X^{-1})_3 & -i(X^{-1})_4 & (X^{-1})_1 & -i(X^{-1})_2 \\ (X^{-1})_4 & -(X^{-1})_3 & -(X^{-1})_2 & (X^{-1})_1 \end{pmatrix}.$$

Multiplying the matrices $F_{14}(X^{-1})$ and $F_{14}(X)$ by each other we obtain $F_{14}(X^{-1})F_{14}(X) = E$, where $E$ *is a* unit matrix, which proves the required.

Let us prove property 5). Let us calculate the determinant of the matrix $F_{14}(X)$.

$$detF_{14}(X) = x_1\begin{vmatrix} x_1 & x_4 & x_3 \\ -ix_4 & x_1 & -ix_2 \\ -x_3 & -x_2 & x_1 \end{vmatrix} - ix_2\begin{vmatrix} x_2 & x_4 & x_3 \\ x_3 & x_1 & -ix_2 \\ x_4 & -x_2 & x_1 \end{vmatrix} - x_3\begin{vmatrix} x_2 & x_1 & x_3 \\ x_3 & -ix_4 & -ix_2 \\ x_4 & -x_3 & x_1 \end{vmatrix} +$$
$$ix_4\begin{vmatrix} x_2 & x_1 & x_4 \\ x_3 & -ix_4 & x_1 \\ x_4 & -x_3 & -x_2 \end{vmatrix} = x_1^2(x_1^2 - ix_2^2 + x_3^2 + ix_4^2) + 2ix_1x_2x_3x_4 - ix_2^2(x_1^2 - ix_2^2 - x_3^2 - ix_4^2) +$$
$$2ix_1x_2x_3x_4 - x_3^2(-x_1^2 - ix_2^2 - x_3^2 + ix_4^2) + 2ix_1x_2x_3x_4 + ix_4^2(x_1^2 + ix_2^2 - x_3^2 + ix_4^2) + 2ix_1x_2x_3x_4 =$$
$A(X)$, according to (21).

Property 6) follows automatically from properties 1) and 5). Property7 follows from properties 3) and 5), and property 8) follows from 4) and 5).

Thus, between the space of four-dimensional numbers with complex components and the space of $(4 \times 4)$-matrices of the form (18) there exist bijections $F_j$ $(j = 1,2,\ldots,64)$, which preserve arithmetic operations, that is, the existing bijections are homomorphisms by both addition and multiplication. That is, we have constructed 64 matrix spaces, each of which is isomorphic and homomorphic to the space of four-dimensional numbers with complex components. The elements of each matrix space form an abelian group by multiplication and by addition. In [1] it is shown that to each four-dimensional number with real components correspond 8 matrix spaces. To illustrate the results of Theorems 1 and 2, we give the following example.

**Example**. Take two four-dimensional numbers $X = (1, i, 0 - 1)$ and $Y = (-1,0,0,i)$ in the space $M_{14} = M(1, -i, -1)$. The product of these numbers, according to formula (5), is equal to $Z = XY = (-2, -i, i, 1 + i)$. The corresponding matrices $F_{14}(X)$, $F_{14}(Y)$ and $F_{14}(Z)$, according to formula (19), have the form respectively

$$\begin{pmatrix} 1 & -1 & 0 & i \\ i & 1 & -1 & 0 \\ 0 & i & 1 & 1 \\ -1 & 0 & -i & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & i & 0 \\ 0 & 1 & -1 & 0 \\ i & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 1 & -i & 1-i \\ -i & -2 & 1+i & i \\ i & 1-i & -2 & -1 \\ 1+i & -i & i & -2 \end{pmatrix}.$$

Hence it is easy to verify that $F_{14}(X)F_{14}(Y) = F_{14}(Z)$. The matrix inverse of $F_{14}(Y)$ has the form

$$F_{14}^{-1}(Y) = -\frac{1+i}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}.$$

Comparing this matrix with the matrix (19) we notice that in the space $M_{14}$ it is the matrix of the four-dimensional number $\tilde{Y} = (-\frac{1+i}{2}, 0, 0, \frac{1-i}{2})$. Multiplying the numbers $Y$ and $\tilde{Y}$ by each other we see that $\tilde{Y} = Y^{-1}$. Hence, $F_{14}^{-1}(Y) = F_{14}(Y^{-1})$ .

## 5. Commutative Gate Groups for Two-Qubit Quantum Systems

Quantum computation consists in successively applying unitary operators $U$ to the quantum state of a quantum system, which are called gates. The unitary operators or gates applied to a $n$-qubit quantum system are represented as a matrix of size $2^n \times 2^n$. For example, unitary operators for a two-qubit system take the form of a matrix of size $4 \times 4$:

$$U = \begin{pmatrix} x_{11}^{re} + x_{11}^{im} i & x_{12}^{re} + x_{12}^{im} i & x_{13}^{re} + x_{13}^{im} i & x_{14}^{re} + x_{14}^{im} i \\ x_{21}^{re} + x_{21}^{im} i & x_{22}^{re} + x_{22}^{im} i & x_{23}^{re} + x_{23}^{im} i & x_{24}^{re} + x_{24}^{im} i \\ x_{31}^{re} + x_{31}^{im} i & x_{32}^{re} + x_{32}^{im} i & x_{33}^{re} + x_{33}^{im} i & x_{34}^{re} + x_{34}^{im} i \\ x_{41}^{re} + x_{41}^{im} i & x_{42}^{re} + x_{42}^{im} i & x_{43}^{re} + x_{43}^{im} i & x_{44}^{re} + x_{44}^{im} i \end{pmatrix}, \quad (21)$$

Where $x_{ij}^{re}$ and $x_{ij}^{im}$ are real and imaginary parts of the elements of the matrix $U$. It is known that if the matrix (21) is a gate, then the Hermite conjugate matrix $U^*$ is also a gate, i.e. the matrix

$$U^* = \begin{pmatrix} x_{11}^{re} - x_{11}^{im} i & x_{21}^{re} - x_{21}^{im} i & x_{31}^{re} - x_{31}^{im} i & x_{41}^{re} - x_{41}^{im} i \\ x_{12}^{re} - x_{12}^{im} i & x_{22}^{re} - x_{22}^{im} i & x_{32}^{re} - x_{32}^{im} i & x_{42}^{re} - x_{42}^{im} i \\ x_{13}^{re} - x_{13}^{im} i & x_{23}^{re} - x_{23}^{im} i & x_{33}^{re} - x_{33}^{im} i & x_{43}^{re} - x_{43}^{im} i \\ x_{14}^{re} - x_{14}^{im} i & x_{24}^{re} - x_{24}^{im} i & x_{34}^{re} - x_{34}^{im} i & x_{44}^{re} - x_{44}^{im} i \end{pmatrix} \quad (22)$$

is a gate for a two-qubit quantum system. By definition $U \cdot U^* = E$, where $E$ is a unit matrix of size $4 \times 4$.

The main two-qubit gates used in the construction of quantum circuits are the Hadamard gate $H$, with unitary matrix

$$H = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

SWAP gate with matrix

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

CNOT or CX gate, with matrix

$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

as well as gates CY, CZ, with matrices

$$CY = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

In practice, quantum circuits for quantum computations are built from existing gates. A quantum circuit is a sequence of gates applied to one or more qubits in a quantum register. A quantum register is a collection of qubits that we use for computation [8]. At the hardware level, experimental physicists and engineers are working on optimizing the main gates. In addition, other physicists and computer scientists are trying to create the most efficient high-level gates [17]. Our goal is to create new groups of two-qubit gates. In the previous section, we constructed 64 abelian

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groups of matrices, with elements formed from the components of a four-dimensional number. Under some additional conditions, the constructed matrices turn into unitary operators.

Let the four-dimensional number $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ be the quantum state $|\varphi\rangle$ of a two-qubit quantum circuit, i.e.

$$|\varphi\rangle = x_1|00\rangle + x_2|01\rangle + x_3|10\rangle + x_4|11\rangle. \quad (23)$$

Then to each such state we can map 64 matrices $F_j(X), j = 1,2, \dots, 64$, of size $4 \times 4$ to each such state:

$$F_j(X) = \begin{pmatrix} x_1^{re} + x_1^{im} i & \gamma_j\delta_j(x_2^{re} + x_2^{im} i) & \beta_j\delta_j(x_3^{re} + x_3^{im} i) & \beta_j\gamma_j(x_4^{re} + x_4^{im} i) \\ x_2^{re} + x_2^{im} i & x_1^{re} + x_1^{im} i & \beta_j(x_4^{re} + x_4^{im} i) & \beta_j(x_3^{re} + x_3^{im} i) \\ x_3^{re} + x_3^{im} i & \gamma_j(x_4^{re} + x_4^{im} i) & x_1^{re} + x_1^{im} i & \gamma_j(x_2^{re} + x_2^{im} i) \\ x_4^{re} + x_4^{im} i & \delta_j(x_3^{re} + x_3^{im} i) & \delta_j(x_2^{re} + x_2^{im} i) & x_1^{re} + x_1^{im} i \end{pmatrix}, \quad (24)$$

where $\beta_j, \gamma_j, \delta_j$ takes one of the values of $\{+1, -1, +i, -i\}$ depending on the value of $j$ according to Table 3. Note that since $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ *is a* state of a two-qubit quantum system, it is true that

$$\sum_{j=1}^{4} |x_j|^2 = \sum_{j=1}^{4} \left( \left(x_j^{re}\right)^2 + \left(x_j^{im}\right)^2 \right) = 1. \quad (25)$$

**Theorem 3**. Let $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ be the state of the two-qubit quantum system and $F_j(X)$ be the matrix (24) corresponding to $X$, for some $j$. Let the elements of the matrix $F_j(X)$ satisfy the conditions:

$$\begin{cases} \rho_1(X) \equiv x_1\overline{x_2} + \gamma\delta\overline{x_1}x_2 + \delta x_3\overline{x_4} + \gamma\overline{x_3}x_4 = 0 \\ \rho_2(X) \equiv x_1\overline{x_3} + \delta x_2\overline{x_4} + \beta\delta\overline{x_1}x_3 + \beta\overline{x_2}x_4 = 0, \\ \rho_3(X) \equiv x_1\overline{x_4} + \gamma x_2\overline{x_3} + \beta\overline{x_2}x_3 + \beta\gamma\overline{x_1}x_4 = 0 \end{cases} \quad (26)$$

Where $\overline{x_j}$ means complex conjugate to $x_j$. Then the matrix $F_j(X)$ is a two-qubit gate.

Proof. Consider the Hermite conjugate matrix $F_j^*(X)$ to the matrix $F_j(X)$

$$F_j^*(X) = \begin{pmatrix} x_1^{re} - x_1^{im} i & x_2^{re} - x_2^{im} i & x_3^{re} - x_3^{im} i & x_4^{re} - x_4^{im} i \\ \overline{\gamma_j}\overline{\delta_j}(x_2^{re} - x_2^{im} i) & x_1^{re} - x_1^{im} i & \gamma_j(x_4^{re} - x_4^{im} i) & \delta_j(x_3^{re} - x_3^{im} i) \\ \beta_j\delta_j(x_3^{re} - x_3^{im} i) & \beta_j(x_4^{re} - x_4^{im} i) & x_1^{re} - x_1^{im} i & \delta_j(x_2^{re} - x_2^{im} i) \\ \beta_j\gamma_j(x_4^{re} - x_4^{im} i) & \beta_j(x_3^{re} - x_3^{im} i) & \gamma_j(x_2^{re} - x_2^{im} i) & x_1^{re} - x_1^{im} i \end{pmatrix}.$$

Multiply the matrices $F_j^*(X)$ and $F_j(X)$ by each other:

$$F_j(X)F_j^*(X) = \begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 \\ \bar{\gamma}\bar{\delta}\rho_1 & \rho_0 & \bar{\gamma}\rho_3 & \bar{\delta}\rho_2 \\ \bar{\beta}\bar{\delta}\rho_2 & \bar{\beta}\rho_3 & \rho_0 & \bar{\delta}\rho_2 \\ \bar{\beta}\bar{\gamma}\rho_3 & \bar{\beta}\rho_2 & \bar{\gamma}\rho_2 & \rho_0 \end{pmatrix},$$

Where $\rho_0 = \sum_{j=1}^{4} |x_j|^2$, here we took advantage of the fact that $\beta^2 = \gamma^2 = \delta^2 = 1$. Using relations (25) and (26) we have $F_j(X)F_j^*(X) = E$, where $E$ *is a* unit matrix, which proves the theorem.

Remark. The conditions (26) in the notation of matrix (24) can be written in the form of the following six equations:

$$x_1^{re}x_2^{re} + x_1^{im}x_2^{im} + (\gamma^{re}\delta^{re} - \gamma^{im}\delta^{im})(x_1^{re}x_2^{re} + x_1^{im}x_2^{im}) + (\gamma^{im}\delta^{re} + \gamma^{re}\delta^{im})(x_1^{im}x_2^{re} - x_1^{re}x_2^{im}) + \delta^{re}(x_3^{re}x_4^{re} + x_3^{im}x_4^{im}) - \delta^{im}(x_3^{im}x_4^{re} - x_3^{re}x_4^{im}) + \gamma^{re}(x_3^{re}x_4^{re} + x_3^{im}x_4^{im}) + \gamma^{im}(x_3^{im}x_4^{re} - x_3^{re}x_4^{im}) = 0.$$

$$x_1^{im}x_2^{re} - x_1^{re}x_2^{im} + (\gamma^{im}\delta^{re} + \gamma^{re}\delta^{im})(x_1^{re}x_2^{re} + x_1^{im}x_2^{im}) - (\gamma^{re}\delta^{re} - \gamma^{im}\delta^{im})(x_1^{im}x_2^{re} - x_1^{re}x_2^{im}) + \delta^{re}(x_3^{im}x_4^{re} - x_3^{re}x_4^{im}) + \delta^{im}(x_3^{re}x_4^{re} + x_3^{im}x_4^{im}) - \gamma^{re}(x_3^{im}x_4^{re} - x_3^{re}x_4^{im}) + \gamma^{im}(x_3^{re}x_4^{re} + x_3^{im}x_4^{im}) = 0.$$

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$x_1^{re}x_3^{re} + x_1^{im}x_3^{im} + \delta^{re}(x_2^{re}x_4^{re} + x_2^{im}x_4^{im}) - \delta^{im}(x_2^{im}x_4^{re} - x_2^{re}x_4^{im}) + (\beta^{re}\delta^{re} - \beta^{im}\delta^{im})(x_1^{re}x_3^{re} + x_1^{im}x_3^{im}) + (\beta^{im}\delta^{re} + \beta^{re}\delta^{im})(x_1^{im}x_3^{re} - x_1^{re}x_3^{im}) + \beta^{re}(x_2^{re}x_4^{re} + x_2^{im}x_4^{im}) + \beta^{im}(x_2^{im}x_4^{re} - x_2^{re}x_4^{im}) = 0.$

$x_1^{im}x_3^{re} - x_1^{re}x_3^{im} + \delta^{im}(x_2^{re}x_4^{re} + x_2^{im}x_4^{im}) + \delta^{re}(x_2^{im}x_4^{re} - x_2^{re}x_4^{im}) + (\beta^{im}\delta^{re} + \beta^{re}\delta^{im})(x_1^{re}x_3^{re} + x_1^{im}x_3^{im}) - (\beta^{re}\delta^{re} - \beta^{im}\delta^{im})(x_1^{im}x_3^{re} - x_1^{re}x_3^{im}) + \beta^{im}(x_2^{re}x_4^{re} + x_2^{im}x_4^{im}) - \beta^{re}(x_2^{im}x_4^{re} - x_2^{re}x_4^{im}) = 0.$

$x_1^{re}x_4^{re} + x_1^{im}x_4^{im} + \gamma^{re}(x_2^{re}x_3^{re} + x_2^{im}x_3^{im}) - \gamma^{im}(x_2^{im}x_3^{re} - x_2^{re}x_3^{im}) + \beta^{re}(x_2^{re}x_3^{re} + x_2^{im}x_3^{im}) + \beta^{im}(x_2^{im}x_3^{re} - x_2^{re}x_3^{im}) + (\beta^{re}\gamma^{re} - \beta^{im}\gamma^{im})(x_1^{re}x_4^{re} + x_1^{im}x_4^{im}) + (\beta^{im}\gamma^{re} + \beta^{re}\gamma^{im})(x_1^{im}x_4^{re} - x_1^{re}x_4^{im}) = 0.$

$x_1^{im}x_4^{re} - x_1^{re}x_4^{im} + \gamma^{re}(x_2^{im}x_3^{re} - x_2^{re}x_3^{im}) + \gamma^{im}(x_2^{re}x_3^{re} + x_2^{im}x_3^{im}) - \beta^{re}(x_2^{im}x_3^{re} - x_2^{re}x_3^{im}) + \beta^{im}(x_2^{re}x_3^{re} + x_2^{im}x_3^{im}) + (\beta^{im}\gamma^{re} + \beta^{re}\gamma^{im})(x_1^{re}x_4^{re} + x_1^{im}x_4^{im}) - (\beta^{re}\gamma^{re} - \beta^{im}\gamma^{im})(x_1^{im}x_4^{re} - x_1^{re}x_4^{im}) = 0.$

Thus, we have found 64 commutative groups of gates, when conditions (25), (26) are satisfied. For example, for the space of matrices $F_{14}(X)$, conditions (26) have the form

$$\begin{cases} \rho_1 \equiv x_1\overline{x_2} + i\overline{x_1}x_2 - x_3\overline{x_4} - i\overline{x_3}x_4 = 0 \\ \rho_2 \equiv x_1\overline{x_3} - x_2\overline{x_4} - \overline{x_1}x_3 + \overline{x_2}x_4 = 0 \\ \rho_3 \equiv x_1\overline{x_4} - ix_2\overline{x_3} + \overline{x_2}x_3 - i\overline{x_1}x_4 = 0 \end{cases}, \tag{27}$$

and in the matrix notation (24), the above six equations turn into the following three equations:

$$x_1^{im}x_2^{re} - x_1^{re}x_2^{im} - x_1^{re}x_2^{re} - x_1^{im}x_2^{im} + x_3^{re}x_4^{re} + x_3^{im}x_4^{im} + x_3^{im}x_4^{re} - x_3^{re}x_4^{im} = 0.$$

$$x_1^{im}x_3^{re} - x_1^{re}x_3^{im} + x_2^{re}x_4^{re} + x_2^{im}x_4^{im} + x_1^{re}x_3^{re} + x_1^{im}x_3^{im} - x_2^{im}x_4^{re} + x_2^{re}x_4^{im} = 0.$$

$$x_1^{im}x_4^{re} - x_1^{re}x_4^{im} - x_2^{im}x_3^{re} + x_2^{re}x_3^{im} = 0.$$

Similarly it is possible to write conditions (26) for all other groups, substituting instead of $\beta, \gamma, \delta$ the corresponding values.

If the quantum state consists of real components, the conditions (26) take the form

$$\begin{cases} \rho_1 \equiv (1 + \gamma\delta)x_1x_2 + (\delta + \gamma)x_3x_4 = 0 \\ \rho_2 \equiv (1 + \beta\delta)x_1x_3 + (\delta + \beta)x_2x_4 = 0, \\ \rho_3 \equiv (1 + \beta\gamma)x_1x_4 + (\gamma + \beta)x_2x_3 = 0 \end{cases} \tag{28}$$

As can be seen from these equalities, if $\beta = \gamma = 1, \delta = -1$, then the first two equalities are fulfilled automatically and the conditions (28) turn into one condition $x_1x_4 + x_2x_3 = 0$. Indeed, this case corresponds to the matrix

$$F_2(X) = F(1,1,-1) = \begin{pmatrix} x_1 & -x_2 & -x_3 & x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}.$$

Multiplying this matrix by its transpose we verify that it is a two-qubit gate if conditions (25) and $x_1x_4 + x_2x_3 = 0$ . are satisfied. In other cases, for example, when $\beta = \gamma = \delta = -1$, which corresponds to the matrix $F_{22}(X)$, condition (28) consists of three equalities. Similarly, we can investigate different variants of quantum gates for quantum states with purely imaginary components.

So, we have constructed an infinite number of different two-qubit gates that lie in 64 commutative groups of gates. Furthermore, multiplication of two gates from two different commutative groups is also a gate. It is easy to see that a gate obtained by multiplying two gates from two different groups is, in general, not an element of any commutative group $F_j(X)$. As for the gates *H, SWAP, CX, CY, CZ, which* are often used in practice, they are also not elements of $F_j(X)$. But note that

$$H \cdot SWAP = F_{22}(X_{\frac{1}{2}}) \text{ or } F_{22}\left(X_{\frac{1}{2}}\right) SWAP = H, \tag{29}$$

where $X_{\frac{1}{2}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^{\mathrm{T}}$.

The quantum algorithm assumes sequential application of different gates to qubits in the quantum circuit. It follows from the obtained results that if in a quantum algorithm there are gates belonging to one of 64 groups, this sequence can be replaced by one gate from the same group, since sequential application of gates from one group does not lead out of this group. Hence an optimal quantum algorithm should consist of gates which belong to different groups $F_j(X)$, where $j \in [1,64]$ .

## 6. Unitary States of two-Qubit Quantum Systems

The ultimate meaning of any quantum algorithm is to translate the state of a quantum system into the desired state for solving the problem. Therefore, finding a fan that translates a quantum system from one given state to another given state is an important practical problem. Let us consider this problem for a two-qubit quantum system. The proposed approach can be generalized to $n$-qubit system

**Definition**. A state of a two-qubit quantum system $X = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4$ is called a unitary state if there exists a triple $(\beta, \gamma, \delta) \in \mathbb{C}^3$, such that $\beta, \gamma, \delta \in \{+1, -1, +i, -i\}$ and the components $X$ satisfy (26).

In other words, a state of a two-qubit quantum system is unitary if it is defined by the space $M_j$, where $j \in [1,64]$, in which the matrix $F_j(X)$, corresponding to this state is a two-qubit gate.

**Remark**. The conditions $\beta, \gamma, \delta \in \{+1, -1, +i, -i\}$ in the definition of a unitary state are imposed only to consider states only from basis spaces. They are not mandatory.

Note that all basis states of a quantum system are unitary states in all 64 basis spaces.

**Definition**. Quantum states having one of the following $\mathrm{X}_1 = (x_1, 0,0,0)^T$, $\mathrm{X}_2 = (0, x_2, 0,0)^T$, $\mathrm{X}_3 = (0,0, x_3, 0)^T$, $\mathrm{X}_4 = (0,0,0, x_4)^T$, where $x_j \in \mathbb{C}, |x_j| = 1, j = 1,2,3,4$, are called quasi-basic states.

Obviously, all quasi-basic states satisfy (26) for any $j \in [1,64]$, that is, they are unitary states.

**Lemma 2**. A quasi-basic state $\mathrm{X}_1$ corresponds to a single gate in all spaces $M_j, j = 1,2, \dots ,64$. Each of the quasi-basic states $\mathrm{X}_2, \mathrm{X}_3, \mathrm{X}_4$ corresponds to 16 different gates in all spaces $M_j, j = 1,2, \dots ,64$.

Proof. The first statement concerning $X_1$ is obvious. Consider the quasi-basic state $X_2$. The corresponding matrix has the form

$$F_j(X_2) = \begin{pmatrix} 0 & \gamma\delta x_2 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma x_2 \\ 0 & 0 & \delta x_2 & 0 \end{pmatrix}.$$

Since $\gamma$ and $\delta$ take values from the set $\{1, -1, i, -i\}$ , there are only 16 different choices. Similarly, consider the states $X_3$ and $X_4$.

Each unitary state $X$ from $M_j$ defines a unitary operator (gate) from $F_j(X)$, which we will call a gate of the unitary state $X$. Let $Y$ be *an* arbitrary state of a two-qubit quantum system. We apply the gate $F_j(X)$ to the state $Y$: $F_j(X)Y = X \cdot Y$, that is, applying the gate $F_j(X)$ is equivalent to a four-dimensional multiplication of the unitary state $X$ by the state $Y$ in the space $M_j$. If $Y$ is also a unitary state from the same space, then we will get a unitary state from the same space. To what state will the state $Y$ go if it is not unitary? The following theorem gives the answer to this question.

**Theorem 4**. A unitary state gate from the space $M_j(\beta, \gamma, \delta)$ translates any unitary (non-unitary) state into a unitary (non-unitary) state.

Proof. For simplicity we prove the theorem for the space $M_{14}$, for other spaces it is proved analogously. Let $X = (x_1, x_2, x_3, x_4) \in$ ▯$^4$ be a unitary state from the space $M_{14}$ and $F_{14}(X)$, the corresponding gate of a unitary state $X$, which has the form (19), and the components of the vector $X$ satisfy (27). Let $Y$ be *an* arbitrary state of a two-qubit quantum system. Then the gate $F_{14}(X)$ moves the quantum system from the state $Y$ to the state of

$$F_{14}(X)Y = \begin{pmatrix} x_1 & ix_2 & -x_3 & -ix_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -ix_4 & x_1 & -ix_2 \\ x_4 & -x_3 & -x_2 & x_1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1y_1 + ix_2y_2 - x_3y_3 - ix_4y_4 \\ x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ x_3y_1 - ix_4y_2 + x_1y_3 - ix_2y_4 \\ x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4 \end{pmatrix}.$$

Let us compute $\rho_1(F_{14}(X)Y)$, where $\rho_1$ is defined in (27).

$\rho_1(F_{14}(X)Y) = (x_1y_1 + ix_2y_2 - x_3y_3 - ix_4y_4)(\overline{x_2y_1} + \overline{x_1y_2} + \overline{x_4y_3} + \overline{x_3y_4}) + (i\overline{x_1y_1} + \overline{x_2y_2} - i\overline{x_3y_3} - \overline{x_4y_4})(x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4) - (x_3y_1 - ix_4y_2 + x_1y_3 - ix_2y_4)(\overline{x_4y_1} - \overline{x_3y_2} - \overline{x_2y_3} + \overline{x_1y_4}) - (i\overline{x_3y_1} - \overline{x_4y_2} + i\overline{x_1y_3} - \overline{x_2y_4})(x_4y_1 - x_3y_2 - x_2y_3 + x_1y_4) = (x_1\overline{x_2} + i\overline{x_1}x_2 - x_3\overline{x_4} - i\overline{x_3}x_4)(|y_1|^2 + |y_2|^2 + |y_3|^2 + |y_4|^2) + (y_1\overline{y_2} + i\overline{y_1}y_2 - y_3\overline{y_4} - i\overline{y_3}y_4)(|x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2) + (x_1\overline{x_4} - ix_2\overline{x_3} + \overline{x_2}x_3 - i\overline{x_1}x_4)(y_1\overline{y_3} - y_2\overline{y_4} - \overline{y_1}y_3 + \overline{y_2}y_4) + (x_1\overline{x_3} - x_2\overline{x_4} - \overline{x_1}x_3 + \overline{x_2}x_4)(y_1\overline{y_4} - iy_2\overline{y_3} + \overline{y_2}y_3 - i\overline{y_1}y_4)$.

Passing to the notation $\rho_k$ from (27) we obtain

$$\rho_1(F_{14}(X)Y) = \rho_1(X)\rho_0(Y) + \rho_1(Y)\rho_0(X) + \rho_3(X)\rho_2(Y) + \rho_3(Y)\rho_2(X),$$

where $\rho_0(Y) = |y_1|^2 + |y_2|^2 + |y_3|^2 + |y_4|^2 = 1$, $\rho_0(X) = |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 = 1$. Since $X$ is a unitary state in the space $M_{14}$, by virtue of, (27) $\rho_1(X) = \rho_2(X) = \rho_3(X) = 0$. Hence, $\rho_1(F_{14}(X)Y) = \rho_1(Y)$. Performing similar calculations we obtain $\rho_2(F_{14}(X)Y) = \rho_2(Y)$ and $\rho_3(F_{14}(X)Y) = \rho_3(Y)$. It follows from these equalities that if $Y$ *is* a unitary state in $M_{14}$, then $F_{14}(X)Y$ is also a unitary state in $M_{14}$, and conversely, if $Y$ *is* a non-unitary state in $M_{14}$, then $F_{14}(X)Y$ is also a non-unitary state in $M_{14}$.

Thus, the unitary state gate translates unitary states into unitary states, and non-unitary states into non-unitary states. All the above gates *H, SWAP, CX, CY, CZ* have similar properties.

**Lemma 3**. The gates *H, SWAP, CX, CY, CZ* convert a unitary (non-unitary) state of a two-qubit quantum system into a unitary (non-unitary) state.

Proof. Let us prove the lemma first for the *SWAP* fan. Let $X = (x_1, x_2, x_3, x_4) \in$ ▯$^4$ be a unitary state of a two-qubit quantum system from the space $M(\beta, \gamma, \delta)$. Let us apply the *SWAP* gate to the state $X$: $Y = SWAP \cdot X = (x_1, x_3, x_2, x_4)$. Since $X$ is a unitary state, its components satisfy (26). Consequently, the components of the state $Y$ satisfy equations

$$\begin{cases} x_1\overline{x_3} + \gamma\delta\overline{x_1}x_3 + \delta x_2\overline{x_4} + \gamma\overline{x_2}x_4 = 0 \\ x_1\overline{x_2} + \delta x_3\overline{x_4} + \beta\delta\overline{x_1}x_2 + \beta\overline{x_3}x_4 = 0. \\ x_1\overline{x_4} + \gamma x_3\overline{x_2} + \beta\overline{x_3}x_2 + \beta\gamma\overline{x_1}x_4 = 0 \end{cases} \tag{30}$$

But these equations represent equations (26) for the space $M(\gamma, \beta, \delta)$. Thus the *SWAP* gate translates a state from the space $M(\beta, \gamma, \delta)$ to a state from the space $M(\gamma, \beta, \delta)$. Suppose now that $X$ is a non-unitary state of a two-qubit quantum system. Suppose that $Y = SWAP \cdot X$ is a unitary state. But then, as just proved, the state $X = SWAP^1 \cdot Y$ is a unitary state, which contradicts the original assumption.

For the gates *CX, CY,* CZ the proof is completely analogous. Let us prove for the Hadamard gate *H*. For this purpose, let us use the equality (29). Since the statements of the lemma are true for the *SWAP* and $F_{22}\left(X_{\frac{1}{2}}\right)$ gates, they are true for *H* as well.

Unitary states play an important role at construction of quantum algorithms, since for them we can explicitly specify a gate, which transfers a quantum system from one given state to another.

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**Theorem 5**. Let $X \in M_j$ be a unitary state of a two-qubit quantum system, where $1 \leq j \leq 64$. Then for any quasi-basic state $X_k, k = 1,2,3,4$, there exists a gate $F_j(Y)$, such that $F_j(Y)X = X_k$, for some state $Y$.

Proof. Take as $Y = X_k X^{-1}$. Since $X \in M_j$, then $X^{-1} \in M_j$. $X_k$ also belongs to $M_j$, hence $Y \in M_j$. Let us multiply both parts of the equation by $X$: $YX = X_k$. As we saw above $YX = F_j(Y)X$. Whence $F_j(Y)X = X_k$. The gate we are looking for is $F_j(X_k X^{-1})$.

This theorem states that from any unitary state one can go from any unitary state to any quasi-basic state, including any basis state, in one step.

**Corollary**. Let $X, Y$ be two unitary states lying in the same space $M_j$, where $1 \leq j \leq 64$. Then there exists a gate $F_j(Z)$, which transfers a two-qubit quantum system from state $X$ to state $Y$.

Proof. According to Theorem 4, there exist gates $F_j(\tilde{X})$ and $F_j(\tilde{Y})$, translating $X$ and $Y$ to $X_k$ for some k. Then the gate $F_j^{-1}(\tilde{Y})F_j(\tilde{X})$ translates state $X$ into state $Y$.

**Theorem 6**. Let $X, Y$ be two unitary states of a two-qubit quantum system. Then there exists a gate translating the two-qubit quantum system from state $X$ to state $Y$.
Proof. Let state $X$ lies in the space $M_{j1}$, and state $Y$ lies in the space $M_{j2}$. The base state $X_1 = (1,0,0,0)^T$, according to lemma 2, lies in both spaces. Hence, by Theorem 5, there exists a gate $F_{j1}(Z_1)$, which translates $X$ into $X_1$ and a gate $F_{j2}(Z_2)$, which translates $Y$ into $X_1$. Then the gate $F_{j2}^{-1}(Z_2)F_{j1}(Z_1)$ is the desired gate translating the quantum system from state $X$ to state $Y$.

Thus, we have divided all possible states of a two-qubit quantum system into two classes: unitary and non-unitary. Unitary states include all quasi-basic states (hence all basis states) and play an important role in the construction of quantum algorithms. Theorem 6 makes it possible to go from any unitary state to any other unitary state in one step, and the gate of this transition can be explicitly described and there are infinitely many such gates (since there are infinitely many quasi-basic states).

## 7. Discussions

In this paper, we construct four-dimensional spaces of complex numbers by analogy with four-dimensional spaces of real numbers [1]. Each four-dimensional number is mapped to a matrix formed from its components and it is proved that the constructed mapping is bijection and homomorphism. The conditions under which the corresponding matrices are gates for two-qubit quantum systems are defined. The notion of unitary state of a two-qubit quantum system is introduced. It is shown that any gate of a unitary state transforms a unitary state into a unitary state and a non-unitary state into a non-unitary state. Almost all gates used in the construction of uvant circuits, in particular *H, SWAP, CX, CY, CZ,* have the same properties. The question of existence of a gate of a two-qubit quantum system, which translates a unitary state into a non-unitary state or vice versa, is of interest.

The problem of finding a gate that transfers a quantum system from one unitary state to another unitary state is solved. Thus, with the help of four-dimensional spaces of complex numbers it was possible to construct whole classes of two-qubit gates, which opens new possibilities for the construction of quantum algorithms. The results obtained have important theoretical and practical implications for quantum computing.

In this paper, we have only concerned ourselves with the four-dimensional algebra of complex numbers. It is possible to develop a four-dimensional mathematical analysis of complex numbers by analogy with the four-dimensional mathematical analysis of real numbers [13–16]. The papers [18,19] give applications of the four-dimensional analysis of real numbers to the solution of problems of mathematical physics.

Another direction of development of the proposed theory consists in development of eight-dimensional algebra of real and complex numbers. As we have seen the four-dimensional algebra of

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complex numbers is an ideal model for two-qubit quantum systems. Correspondingly the eight-dimensional algebra of complex numbers will describe the mathematical model of three-qubit quantum systems, the sixteen-dimensional algebra of complex numbers - of four-qubit quantum systems, etc.

And of course, the potential of four-dimensional algebra of complex numbers for studying two-qubit quantum systems is not fully revealed. It is interesting to study the properties of gates formed by multiplication of several gates from different commutative groups of matrices constructed in this paper. As follows from the results obtained, optimal quantum algorithms consist of a sequence of gates from different commutative groups. As shown above, gates from different commutative groups do not commute with each other, that is, in fact, a new gate formation mechanism for two-qubit quantum systems is proposed. The new gate formation mechanism is of great importance for quantum computing.

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## 8. Appendix 1. Proof of Lemma 1.

For clarity, let us rewrite equalities (1) in a more expanded form

$$\begin{cases} z_1 = (a_{11}^{re} + a_{11}^{im} i)(x_1^{re} + x_1^{im} i)(y_1^{re} + y_1^{im} i) + (a_{12}^{re} + a_{12}^{im} i)(x_2^{re} + x_2^{im} i)(y_2^{re} + y_2^{im} i) \\ \quad +(a_{13}^{re} + a_{13}^{im} i)(x_3^{re} + x_3^{im} i)(y_3^{re} + y_3^{im} i) + (a_{14}^{re} + a_{14}^{im} i)(x_4^{re} + x_4^{im} i)(y_4^{re} + y_4^{im} i) \\ z_2 = (a_{21}^{re} + a_{21}^{im} i)(x_2^{re} + x_2^{im} i)(y_1^{re} + y_1^{im} i) + (a_{22}^{re} + a_{22}^{im} i)(x_1^{re} + x_1^{im} i)(y_2^{re} + y_2^{im} i) \\ \quad +(a_{23}^{re} + a_{23}^{im} i)(x_4^{re} + x_4^{im} i)(y_3^{re} + y_3^{im} i) + (a_{24}^{re} + a_{24}^{im} i)(x_3^{re} + x_3^{im} i)(y_4^{re} + y_4^{im} i) \\ z_3 = (a_{31}^{re} + a_{31}^{im} i)(x_3^{re} + x_3^{im} i)(y_1^{re} + y_1^{im} i) + (a_{32}^{re} + a_{32}^{im} i)(x_4^{re} + x_4^{im} i)(y_2^{re} + y_2^{im} i) \\ \quad +(a_{33}^{re} + a_{33}^{im} i)(x_1^{re} + x_1^{im} i)(y_3^{re} + y_3^{im} i) + (a_{34}^{re} + a_{34}^{im} i)(x_2^{re} + x_2^{im} i)(y_4^{re} + y_4^{im} i) \\ z_4 = (a_{41}^{re} + a_{41}^{im} i)(x_4^{re} + x_4^{im} i)(y_1^{re} + y_1^{im} i) + (a_{42}^{re} + a_{42}^{im} i)(x_3^{re} + x_3^{im} i)(y_2^{re} + y_2^{im} i) \\ \quad +(a_{43}^{re} + a_{43}^{im} i)(x_2^{re} + x_2^{im} i)(y_3^{re} + y_3^{im} i) + (a_{44}^{re} + a_{44}^{im} i)(x_1^{re} + x_1^{im} i)(y_4^{re} + y_4^{im} i) \end{cases} .(31)$$

Let us first satisfy the commutativity condition, i.e. write down the equation $XY$ = YX, give similar terms of the form $x_i^{re} y_j^{re}$, $x_i^{re} y_j^{im}$, $x_i^{im} y_j^{re}$, $x_i^{im} y_j^{im}$, and equate the coefficients at them to zero. Removing the same equations we obtain 12 equations with respect to the elements $a_{kl}^{re}, a_{kl}^{im} i, k, l = 1,2,3,4$ , solving which we have

$a_{21}^{re} = a_{22}^{re}$, $a_{21}^{im} = a_{22}^{im}$, $a_{23}^{re} = a_{24}^{re}$, $a_{23}^{im} = a_{24}^{im}$, $a_{31}^{re} = a_{33}^{re}$, $a_{31}^{im} = a_{33}^{im}$, $a_{32}^{re} = a_{34}^{re}$, $a_{32}^{im} = a_{34}^{im}$, $a_{41}^{re} = a_{44}^{re}$, $a_{41}^{im} = a_{44}^{im}$, $a_{42}^{re} = a_{43}^{re}$, $a_{42}^{im} = a_{43}^{im}$.

Substituting these values into (31) we obtain a general form of commutative multiplication of four-dimensional numbers with complex components:

$$\begin{cases} z_1 = (a_{11}^{re} + a_{11}^{im} i)(x_1^{re} + x_1^{im} i)(y_1^{re} + y_1^{im} i) + (a_{12}^{re} + a_{12}^{im} i)(x_2^{re} + x_2^{im} i)(y_2^{re} + y_2^{im} i) \\ \quad +(a_{13}^{re} + a_{13}^{im} i)(x_3^{re} + x_3^{im} i)(y_3^{re} + y_3^{im} i) + (a_{14}^{re} + a_{14}^{im} i)(x_4^{re} + x_4^{im} i)(y_4^{re} + y_4^{im} i) \\ z_2 = (a_{22}^{re} + a_{22}^{im} i)(x_2^{re} + x_2^{im} i)(y_1^{re} + y_1^{im} i) + (a_{22}^{re} + a_{22}^{im} i)(x_1^{re} + x_1^{im} i)(y_2^{re} + y_2^{im} i) \\ \quad +(a_{23}^{re} + a_{23}^{im} i)(x_4^{re} + x_4^{im} i)(y_3^{re} + y_3^{im} i) + (a_{23}^{re} + a_{23}^{im} i)(x_3^{re} + x_3^{im} i)(y_4^{re} + y_4^{im} i) \\ z_3 = (a_{33}^{re} + a_{33}^{im} i)(x_3^{re} + x_3^{im} i)(y_1^{re} + y_1^{im} i) + (a_{32}^{re} + a_{32}^{im} i)(x_4^{re} + x_4^{im} i)(y_2^{re} + y_2^{im} i) \\ \quad +(a_{33}^{re} + a_{33}^{im} i)(x_1^{re} + x_1^{im} i)(y_3^{re} + y_3^{im} i) + (a_{32}^{re} + a_{32}^{im} i)(x_2^{re} + x_2^{im} i)(y_4^{re} + y_4^{im} i) \\ z_4 = (a_{44}^{re} + a_{44}^{im} i)(x_4^{re} + x_4^{im} i)(y_1^{re} + y_1^{im} i) + (a_{42}^{re} + a_{42}^{im} i)(x_3^{re} + x_3^{im} i)(y_2^{re} + y_2^{im} i) \\ \quad +(a_{42}^{re} + a_{42}^{im} i)(x_2^{re} + x_2^{im} i)(y_3^{re} + y_3^{im} i) + (a_{44}^{re} + a_{44}^{im} i)(x_1^{re} + x_1^{im} i)(y_4^{re} + y_4^{im} i) \end{cases} .(32)$$

Thus, the number of independent coefficients $a_{kl}^{re}, a_{kl}^{im}, k, l = 1,2,3,4,$ is reduced from 32 to 20. Let us now write the equations *(XY)Z= X(YZ),* where $Z = (z_1, z_2, z_3, z_4)$, $z_k \in \mathbb{C}, k = 1,2,3,4,$ using (2), and equating the coefficients at the same products $x_i^l y_j^m z_k^n, l, m, n = re, im,$ we obtain 48 different equations with respect to $a_{kl}^{re}, a_{kl}^{im}$. Consider the equations obtained by equating to zero the coefficients at products $z_1^{re} x_2^{re} y_2^{re}$ and $z_1^{re} x_2^{im} y_2^{re}$

$a_{11}^{re} a_{12}^{re} - a_{11}^{im} a_{12}^{im} - a_{12}^{re} a_{22}^{re} + a_{12}^{im} a_{22}^{im} = 0,$

$-a_{11}^{re}a_{12}^{im} - a_{12}^{re}a_{11}^{im} + a_{22}^{re}a_{12}^{im} + a_{12}^{re}a_{22}^{im} = 0.$

Let us transform these equations as follows:

$a_{12}^{re}(a_{11}^{re} - a_{22}^{re}) - a_{12}^{im}(a_{11}^{im} - a_{22}^{im}) = 0,$

$-a_{12}^{im}(a_{11}^{re} - a_{22}^{re}) - a_{12}^{re}(a_{11}^{im} - a_{22}^{im}) = 0.$

Multiply the first equation by $a_{12}^{re}$, multiply the second equation by $a_{12}^{im}$ and subtract from each other:

$|a_{12}|^2(a_{11}^{re} - a_{22}^{re}) = 0.$

Now multiply the first equation by $a_{12}^{im}$, multiply the second equation by $a_{12}^{re}$ and add the resulting equations:

$|a_{12}|^2(a_{11}^{im} - a_{22}^{im}) = 0.$

Hence, $a_{11}^{re} = a_{22}^{re}$, $a_{11}^{im} = a_{22}^{im}$.

Let's equate to zero the coefficients at the products $z_1^{re}x_3^{re}y_3^{re}$ and $z_1^{re}x_3^{im}y_3^{re}$

$a_{11}^{re}a_{13}^{re} - a_{11}^{im}a_{13}^{im} - a_{13}^{re}a_{33}^{re} + a_{13}^{im}a_{33}^{im} = 0,$

$-a_{11}^{re}a_{13}^{im} - a_{13}^{re}a_{11}^{im} + a_{33}^{re}a_{13}^{im} + a_{13}^{re}a_{33}^{im} = 0$ .

Performing similar transformations we obtain $a_{11}^{re} = a_{33}^{re}$, $a_{11}^{im} = a_{33}^{im}$.

Further equating to zero the coefficients of $z_1^{re}x_4^{re}y_4^{re}$ and $z_1^{re}x_4^{im}y_4^{re}$ we obtain $a_{11}^{re} = a_{44}^{re}$, $a_{11}^{im} = a_{44}^{im}$.

Substituting the found values into (32), getting rid of $a_{22}^{re}, a_{33}^{re}, a_{44}^{re}, a_{22}^{im}, a_{33}^{im}, a_{44}^{im}$, we obtain a new definition of multiplication of four-dimensional numbers with complex components.

$$\begin{cases} z_1 = (a_{11}^{re} + a_{11}^{im}i)(x_1^{re} + x_1^{im}i)(y_1^{re} + y_1^{im}i) + (a_{12}^{re} + a_{12}^{im}i)(x_2^{re} + x_2^{im}i)(y_2^{re} + y_2^{im}i) \\ \quad +(a_{13}^{re} + a_{13}^{im}i)(x_3^{re} + x_3^{im}i)(y_3^{re} + y_3^{im}i) + (a_{14}^{re} + a_{14}^{im}i)(x_4^{re} + x_4^{im}i)(y_4^{re} + y_4^{im}i) \\ z_2 = (a_{11}^{re} + a_{11}^{im}i)[(x_2^{re} + x_2^{im}i)(y_1^{re} + y_1^{im}i) + (x_1^{re} + x_1^{im}i)(y_2^{re} + y_2^{im}i)] \\ \quad +(a_{23}^{re} + a_{23}^{im}i)[(x_4^{re} + x_4^{im}i)(y_3^{re} + y_3^{im}i) + (x_3^{re} + x_3^{im}i)(y_4^{re} + y_4^{im}i)] \\ z_3 = (a_{11}^{re} + a_{11}^{im}i)[(x_3^{re} + x_3^{im}i)(y_1^{re} + y_1^{im}i) + (x_1^{re} + x_1^{im}i)(y_3^{re} + y_3^{im}i)] \\ \quad +(a_{32}^{re} + a_{32}^{im}i)[(x_2^{re} + x_2^{im}i)(y_4^{re} + y_4^{im}i) + (x_4^{re} + x_4^{im}i)(y_2^{re} + y_2^{im}i)] \\ z_4 = (a_{11}^{re} + a_{11}^{im}i)[(x_4^{re} + x_4^{im}i)(y_1^{re} + y_1^{im}i) + (x_1^{re} + x_1^{im}i)(y_4^{re} + y_4^{im}i)] \\ \quad +(a_{42}^{re} + a_{42}^{im}i)[(x_2^{re} + x_2^{im}i)(y_3^{re} + y_3^{im}i) + (x_3^{re} + x_3^{im}i)(y_2^{re} + y_2^{im}i)] \end{cases} . \quad (33)$$

In definition (3), the number of independent coefficients is 14. Let us again write the equations *(XY)Z* = *X(YZ)* on the basis of definition (3) and by bringing similar terms we obtain 12 different equations. Let's equate to zero the coefficients at the products $z_4^{re}x_3^{im}y_3^{re}, z_4^{re}x_3^{re}y_3^{re}, z_3^{re}x_4^{re}y_4^{re}, z_3^{re}x_4^{im}y_4^{re}, z_4^{im}x_2^{im}y_2^{re}, z_4^{im}x_2^{im}y_2^{im}$

$a_{11}^{re}a_{13}^{re} - a_{11}^{im}a_{13}^{im} - a_{23}^{re}a_{42}^{re} + a_{23}^{im}a_{42}^{im} = 0,$

$-a_{11}^{re}a_{13}^{im} - a_{11}^{im}a_{13}^{re} + a_{23}^{im}a_{42}^{re} + a_{23}^{re}a_{42}^{im} = 0,$

$a_{11}^{re}a_{14}^{re} - a_{11}^{im}a_{14}^{im} - a_{23}^{re}a_{32}^{re} + a_{23}^{im}a_{32}^{im} = 0,$

$-a_{11}^{re}a_{14}^{im} - a_{11}^{im}a_{14}^{re} + a_{23}^{im}a_{32}^{re} + a_{23}^{re}a_{32}^{im} = 0,$

$a_{32}^{re}a_{42}^{re} - a_{32}^{im}a_{42}^{im} - a_{11}^{re}a_{12}^{re} + a_{11}^{im}a_{12}^{im} = 0,$

$-a_{32}^{re}a_{42}^{im} - a_{32}^{im}a_{42}^{re} + a_{11}^{im}a_{12}^{re} + a_{11}^{re}a_{12}^{im} = 0.$

Multiply the first equation by $a_{11}^{re}$, multiply the second equation by $a_{11}^{im}$, and from the first equation subtract the second equation:

$((a_{11}^{re})^2 + (a_{11}^{im})^2)a_{13}^{re} = a_{11}^{re}(a_{23}^{re}a_{42}^{re} - a_{23}^{im}a_{42}^{im}) + a_{11}^{im}(a_{23}^{re}a_{42}^{im} + a_{23}^{im}a_{42}^{re}),$

from where we have

$$a_{13}^{re} = \frac{a_{11}^{re}(a_{23}^{re}a_{42}^{re} - a_{23}^{im}a_{42}^{im}) + a_{11}^{im}(a_{23}^{re}a_{42}^{im} + a_{23}^{im}a_{42}^{re})}{(a_{11}^{re})^2 + (a_{11}^{im})^2} .$$

Next, multiply the first equation by $a_{11}^{im}$, multiply the second equation by $a_{11}^{re}$ and add the resulting equations:

$((a_{11}^{re})^2 + (a_{11}^{im})^2)a_{13}^{im} = a_{11}^{re}(a_{23}^{re}a_{42}^{im} + a_{23}^{im}a_{42}^{re}) + a_{11}^{im}(-a_{23}^{re}a_{42}^{re} + a_{23}^{im}a_{42}^{im}),$

whence

$$a_{13}^{im} = \frac{a_{11}^{re}\left(a_{23}^{re}a_{42}^{im}+a_{23}^{im}a_{42}^{re}\right)-a_{11}^{im}\left(a_{23}^{re}a_{42}^{re}-a_{23}^{im}a_{42}^{im}\right)}{(a_{11}^{re})^2+(a_{11}^{im})^2}.$$

Now notice that

$$a_{13}^{re}+a_{13}^{im}i = \frac{(a_{23}^{re}+a_{23}^{im}i)(a_{42}^{re}+a_{42}^{im}i)}{a_{11}^{re}+a_{11}^{im}i}.$$

Similarly, from the third and fourth equations we easily obtain

$$a_{14}^{re} = \frac{a_{11}^{re}\left(a_{23}^{re}a_{32}^{re}-a_{23}^{im}a_{32}^{im}\right)+a_{11}^{im}\left(a_{23}^{re}a_{32}^{im}+a_{23}^{im}a_{32}^{re}\right)}{(a_{11}^{re})^2+(a_{11}^{im})^2},$$

$$a_{14}^{im} = \frac{a_{11}^{re}\left(a_{23}^{re}a_{32}^{im}+a_{23}^{im}a_{32}^{re}\right)-a_{11}^{im}\left(a_{23}^{re}a_{32}^{re}-a_{23}^{im}a_{32}^{im}\right)}{(a_{11}^{re})^2+(a_{11}^{im})^2},$$

and respectively,

$$a_{14}^{re}+a_{14}^{im}i = \frac{(a_{23}^{re}+a_{23}^{im}i)(a_{32}^{re}+a_{32}^{im}i)}{a_{11}^{re}+a_{11}^{im}i},$$

And from the fifth and sixth equations we have

$$a_{12}^{re} = \frac{a_{11}^{re}\left(a_{32}^{re}a_{42}^{re}-a_{32}^{im}a_{42}^{im}\right)+a_{11}^{im}\left(a_{32}^{re}a_{42}^{im}+a_{32}^{im}a_{42}^{re}\right)}{(a_{11}^{re})^2+(a_{11}^{im})^2},$$

$$a_{12}^{im} = \frac{a_{11}^{re}\left(a_{32}^{re}a_{42}^{im}+a_{32}^{im}a_{42}^{re}\right)-a_{11}^{im}\left(a_{32}^{re}a_{42}^{re}-a_{32}^{im}a_{42}^{im}\right)}{(a_{11}^{re})^2+(a_{11}^{im})^2},$$

$$a_{12}^{re}+a_{12}^{im}i = \frac{(a_{32}^{re}+a_{32}^{im}i)(a_{42}^{re}+a_{42}^{im}i)}{a_{11}^{re}+a_{11}^{im}i}.$$

Substituting the found expressions into the other equations obtained from the equality $(XY)Z = X(YZ)$ we make sure that all 12 equations are satisfied. Thus, the general form of associative and commutative anisotropic multiplication of four-dimensional numbers with complex components is as follows

$$\begin{cases} z_1 = \left(a_{11}^{re}+a_{11}^{im}i\right)\left(x_1^{re}+x_1^{im}i\right)\left(y_1^{re}+y_1^{im}i\right)+ \\ +\frac{(a_{32}^{re}+a_{32}^{im}i)(a_{42}^{re}+a_{42}^{im}i)}{a_{11}^{re}+a_{11}^{im}i}\left(x_2^{re}+x_2^{im}i\right)\left(y_2^{re}+y_2^{im}i\right)+ \\ \frac{(a_{23}^{re}+a_{23}^{im}i)(a_{42}^{re}+a_{42}^{im}i)}{a_{11}^{re}+a_{11}^{im}i}\left(x_3^{re}+x_3^{im}i\right)\left(y_3^{re}+y_3^{im}i\right)+ \\ +\frac{(a_{23}^{re}+a_{23}^{im}i)(a_{32}^{re}+a_{32}^{im}i)}{a_{11}^{re}+a_{11}^{im}i}\left(x_4^{re}+x_4^{im}i\right)\left(y_4^{re}+y_4^{im}i\right) \\ z_2 = \left(a_{11}^{re}+a_{11}^{im}i\right)\left[\left(x_2^{re}+x_2^{im}i\right)\left(y_1^{re}+y_1^{im}i\right)+\left(x_1^{re}+x_1^{im}i\right)\left(y_2^{re}+y_2^{im}i\right)\right] \\ +\left(a_{23}^{re}+a_{23}^{im}i\right)\left[\left(x_4^{re}+x_4^{im}i\right)\left(y_3^{re}+y_3^{im}i\right)+\left(x_3^{re}+x_3^{im}i\right)\left(y_4^{re}+y_4^{im}i\right)\right] \\ z_3 = \left(a_{11}^{re}+a_{11}^{im}i\right)\left[\left(x_3^{re}+x_3^{im}i\right)\left(y_1^{re}+y_1^{im}i\right)+\left(x_1^{re}+x_1^{im}i\right)\left(y_3^{re}+y_3^{im}i\right)\right] \\ +\left(a_{32}^{re}+a_{32}^{im}i\right)\left[\left(x_2^{re}+x_2^{im}i\right)\left(y_4^{re}+y_4^{im}i\right)+\left(x_4^{re}+x_4^{im}i\right)\left(y_2^{re}+y_2^{im}i\right)\right] \\ z_4 = \left(a_{11}^{re}+a_{11}^{im}i\right)\left[\left(x_4^{re}+x_4^{im}i\right)\left(y_1^{re}+y_1^{im}i\right)+\left(x_1^{re}+x_1^{im}i\right)\left(y_4^{re}+y_4^{im}i\right)\right] \\ +\left(a_{42}^{re}+a_{42}^{im}i\right)\left[\left(x_2^{re}+x_2^{im}i\right)\left(y_3^{re}+y_3^{im}i\right)+\left(x_3^{re}+x_3^{im}i\right)\left(y_2^{re}+y_2^{im}i\right)\right] \end{cases} \quad (34)$$

The number of independent coefficients is eight. Let us introduce the following notations: $a_{11}^{re}+a_{11}^{im}i = \alpha_1+\alpha_2 i, a_{23}^{re}+a_{23}^{im}i = \beta_1+\beta_2 i, a_{32}^{re}+a_{32}^{im}i = \gamma_1+\gamma_2 i, a_{42}^{re}+a_{42}^{im}i = \delta_1+\delta_2 i.$

Then the general formula of associative and commutative multiplication is written in the following form

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$$\begin{cases} z_1 = (\alpha_1 + \alpha_2 i)(x_1^{re} + x_1^{im} i)(y_1^{re} + y_1^{im} i) + \\ + \frac{(\gamma_1+\gamma_2 i)(\delta_1+\delta_2 i)}{\alpha_1+\alpha_2 i}(x_2^{re} + x_2^{im} i)(y_2^{re} + y_2^{im} i) + \\ \frac{(\beta_1+\beta_2 i)(\delta_1+\delta_2 i)}{\alpha_1+\alpha_2 i}(x_3^{re} + x_3^{im} i)(y_3^{re} + y_3^{im} i) + \\ + \frac{(\beta_1+\beta_2 i)(\gamma_1+\gamma_2 i)}{\alpha_1+\alpha_2 i}(x_4^{re} + x_4^{im} i)(y_4^{re} + y_4^{im} i) \\ z_2 = (\alpha_1 + \alpha_2 i)(x_2^{re} + x_2^{im} i)(y_1^{re} + y_1^{im} i) + (\alpha_1 + \alpha_2 i)(x_1^{re} + x_1^{im} i)(y_2^{re} + y_2^{im} i), \\ +(\beta_1 + \beta_2 i)(x_4^{re} + x_4^{im} i)(y_3^{re} + y_3^{im} i) + (\beta_1 + \beta_2 i)(x_3^{re} + x_3^{im} i)(y_4^{re} + y_4^{im} i) \\ z_3 = (\alpha_1 + \alpha_2 i)(x_3^{re} + x_3^{im} i)(y_1^{re} + y_1^{im} i) + (\gamma_1 + \gamma_2 i)(x_4^{re} + x_4^{im} i)(y_2^{re} + y_2^{im} i) \\ +(\alpha_1 + \alpha_2 i)(x_1^{re} + x_1^{im} i)(y_3^{re} + y_3^{im} i) + (\gamma_1 + \gamma_2 i)(x_2^{re} + x_2^{im} i)(y_4^{re} + y_4^{im} i) \\ z_4 = (\alpha_1 + \alpha_2 i)(x_4^{re} + x_4^{im} i)(y_1^{re} + y_1^{im} i) + (\delta_1 + \delta_2 i)(x_3^{re} + x_3^{im} i)(y_2^{re} + y_2^{im} i) \\ +(\delta_1 + \delta_2 i)(x_2^{re} + x_2^{im} i)(y_3^{re} + y_3^{im} i) + (\alpha_1 + \alpha_2 i)(x_1^{re} + x_1^{im} i)(y_4^{re} + y_4^{im} i) \end{cases}$$

with eight independent coefficients $\alpha_j, \beta_j, \gamma_j, \delta_j \in R, j = 1,2$. It is easy to verify by direct inspection that the multiplication operation thus defined is associative and commutative for any values of the coefficients $\alpha_j, \beta_j, \gamma_j, \delta_j \in R, j = 1,2$.

The last equalities in a more compact form are as follows:

$$\begin{cases} z_1 = \alpha x_1 y_1 + \frac{\gamma\delta}{\alpha} x_2 y_2 + \frac{\beta\delta}{\alpha} x_3 y_3 + \frac{\beta\gamma}{\alpha} x_4 y_4 \\ z_2 = \alpha x_2 y_1 + \alpha x_1 y_2 + \beta x_4 y_3 + \beta x_3 y_4 \\ z_3 = \alpha x_3 y_1 + \gamma x_4 y_2 + \alpha x_1 y_3 + \gamma x_2 y_4 \\ z_4 = \alpha x_4 y_1 + \delta x_3 y_2 + \delta x_2 y_3 + \alpha x_1 y_4 \end{cases}. \tag{35}$$

Lemma 1 is proven.

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