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Not peer-reviewed version

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[K. Mahesh Krishna](#) *

Posted Date: 16 March 2026

doi: 10.20944/preprints202601.0908.v3

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Short Note

Non-Archimedean Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality

K. Mahesh Krishna

School of Mathematics and Natural Sciences, Chanakya University Global Campus, NH-648, Haralur Village, Devanahalli Taluk, Bengaluru North District, Karnataka State, 562 110, India; kmaheshak@gmail.com

Abstract

Massera and Schaffer [Ann. Math. (2), 1958] derived a breakthrough upper bound for the Clarkson angle between two nonzero vectors in a normed linear space, which was later improved by Maligranda [Am. Math. Mon., 2006]. Pecaric and Rajic [Math. Inequal. Appl., 2007] extended Maligranda's inequality to finitely many nonzero vectors. We derive a non-Archimedean version of Massera-Schaffer-Maligranda-Pecaric-Rajic inequality.

Keywords: normed linear space; triangle inequality; non-Archimedean linear space; ultra-norm

MSC: 12J25; 46S10

1. Introduction

Let $\mathcal{X}$ be a normed linear space (NLS). Recall that for $x, y \in \mathcal{X} \setminus \{0\}$, the **Clarkson angle** [1] between x and y is defined as

$$\alpha[x, y] := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

Important Massera-Schaffer inequality [2] says that

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\max\{\|x\|, \|y\|\}}, \quad \forall x, y \in \mathcal{X} \setminus \{0\}. \quad (1)$$

Maligranda improved Inequality (1) in 2006 [3].

Theorem 1 ([3,4] Massera-Schaffer-Maligranda Inequality). Let $\mathcal{X}$ be a NLS. Then for all $x, y \in \mathcal{X} \setminus \{0\}$,

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{\|x - y\| + \|\|x\| - \|y\|\|}{\max\{\|x\|, \|y\|\}} \leq \frac{2\|x - y\|}{\max\{\|x\|, \|y\|\}}. \quad (2)$$

We note that, in 1964 Dunkl and Williams [5] independently showed that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{4\|x - y\|}{\|x\| + \|y\|}, \quad \forall x, y \in \mathcal{X} \setminus \{0\}. \quad (3)$$

Note that

$$\frac{2\|x - y\|}{\max\{\|x\|, \|y\|\}} \leq \frac{4\|x - y\|}{\|x\| + \|y\|}, \quad \forall x, y \in \mathcal{X} \setminus \{0\}.$$

Therefore Inequality (1) improves Inequality (3). Inequality (3) is famously known as Dunkl-Williams inequality in the literature. In 2007, Pecaric and Rajic extended Inequality (2) to finitely many nonzero elements [6].

Theorem 2 ([6] Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality). Let $\mathcal{X}$ be a NLS and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$,

$$\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \leq \min_{1 \leq k \leq n} \left\{ \frac{1}{\|x_k\|} \left(\left\| \sum_{j=1}^n x_j \right\| + \sum_{j=1}^n \left| \|x_j\| - \|x_k\| \right| \right) \right\}.$$

It is important to ask what are non-Archimedean versions of Theorems 1 and 2? We answer the question by deriving non-Archimedean Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality (Theorem 4).

2. Non-Archimedean Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality

Let $\mathbb{K}$ be a field. A map $|\cdot| : \mathbb{K} \rightarrow [0, \infty)$ is said to be a non-Archimedean valuation if following conditions holds.

- (i) If $\lambda \in \mathbb{K}$ is such that $|\lambda| = 0$, then $\lambda = 0$.
- (ii) $|\lambda\mu| = |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{K}$.
- (iii) (Ultra-triangle inequality) $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ for all $\lambda, \mu \in \mathbb{K}$.

In this case, $\mathbb{K}$ is called as non-Archimedean valued field [7]. Let $\mathcal{X}$ be a vector space over a non-Archimedean valued field $\mathbb{K}$ with valuation $|\cdot|$. A map $\|\cdot\| : \mathcal{X} \rightarrow [0, \infty)$ is said to be a non-Archimedean norm if following conditions holds.

- (i) If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.
- (ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{K}$, for all $x \in \mathcal{X}$.
- (iii) (Ultra-norm inequality) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{X}$.

In this case, $\mathcal{X}$ is called as non-Archimedean linear space (NALS) [8]. Let $\mathcal{X}$ be a NALS over $\mathbb{K}$. Let $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{K}$. We define the **non-Archimedean Clarkson angle** between x and y as

$$\alpha[x, y] := \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

Non-Archimedean version of Theorem 1 now reads as follows.

Theorem 3 (Non-Archimedean Massera-Schaffer Inequality). Let $\mathcal{X}$ be a NALS over $\mathbb{K}$. Then for all $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{K}$ it holds

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \min \left\{ \frac{1}{\|x\|} \max \left\{ \|x - y\|, \left| 1 - \frac{\|x\|}{\|y\|} \right| \|y\| \right\}, \frac{1}{\|y\|} \max \left\{ \|x - y\|, \left| 1 - \frac{\|y\|}{\|x\|} \right| \|x\| \right\} \right\}.$$

Proof. Let $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{K}$. Then

$$\begin{aligned} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &= \left\| \frac{x - y}{\|x\|} + \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) y \right\| \\ &\leq \max \left\{ \left\| \frac{x - y}{\|x\|} \right\|, \left\| \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) y \right\| \right\} \\ &\leq \max \left\{ \frac{\|x - y\|}{\|x\|}, \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|y\| \right\} \\ &= \max \left\{ \frac{\|x - y\|}{\|x\|}, \frac{1}{\|x\|} \left| 1 - \frac{\|x\|}{\|y\|} \right| \|y\| \right\} \end{aligned}$$

and

$$\begin{aligned} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &= \left\| \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) x + \frac{x-y}{\|y\|} \right\| \\ &\leq \max \left\{ \left\| \left(\frac{1}{\|x\|} - \frac{1}{\|y\|} \right) x \right\|, \left\| \frac{x-y}{\|y\|} \right\| \right\} \\ &\leq \max \left\{ \left| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right| \|x\|, \frac{\|x-y\|}{\|y\|} \right\} \\ &\leq \max \left\{ \frac{1}{\|y\|} \left| 1 - \frac{\|y\|}{\|x\|} \right| \|x\|, \frac{\|x-y\|}{\|y\|} \right\}. \end{aligned}$$

Therefore

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{1}{\|x\|} \max \left\{ \|x-y\|, \left| 1 - \frac{\|x\|}{\|y\|} \right| \|y\| \right\} \quad (4)$$

and

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{1}{\|y\|} \max \left\{ \|x-y\|, \left| 1 - \frac{\|y\|}{\|x\|} \right| \|x\| \right\}. \quad (5)$$

Inequalities (4) and (5) give

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \min \left\{ \frac{1}{\|x\|} \max \left\{ \|x-y\|, \left| 1 - \frac{\|x\|}{\|y\|} \right| \|y\| \right\}, \frac{1}{\|y\|} \max \left\{ \|x-y\|, \left| 1 - \frac{\|y\|}{\|x\|} \right| \|x\| \right\} \right\}.$$

□

Note the additional assumption $\|x\|, \|y\| \in \mathbb{K}$ in the previous theorem. This is necessary because, since the norm is a real number, we generally cannot guarantee that it belongs to the given non-Archimedean field. We now derive the non-Archimedean version of Theorem 2.

Theorem 4. (Non-Archimedean Massera-Schaffer-Maligranda-Pecaric-Rajic Inequality) Let $\mathcal{X}$ be a NALS over $\mathbb{K}$ and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{K}$ it holds

$$\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \leq \min_{1 \leq k \leq n} \left\{ \frac{1}{\|x_k\|} \max \left\{ \left\| \sum_{j=1}^n x_j \right\|, \max_{1 \leq j \leq n} \left| 1 - \frac{\|x_k\|}{\|x_j\|} \right| \|x_j\| \right\} \right\}.$$

Proof. Let $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{K}$. Let $1 \leq k \leq n$ be fixed. Then

$$\begin{aligned}
 \left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| &= \left\| \frac{x_k}{\|x_k\|} + \sum_{j=1, j \neq k}^n \frac{x_j}{\|x_j\|} \right\| \\
 &= \left\| \sum_{j=1}^n \frac{x_j}{\|x_k\|} - \sum_{j=1, j \neq k}^n \frac{x_j}{\|x_k\|} + \sum_{j=1, j \neq k}^n \frac{x_j}{\|x_j\|} \right\| \\
 &= \left\| \sum_{j=1}^n \frac{x_j}{\|x_k\|} - \sum_{j=1, j \neq k}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \\
 &= \left\| \sum_{j=1}^n \frac{x_j}{\|x_k\|} - \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \\
 &\leq \max \left\{ \left\| \sum_{j=1}^n \frac{x_j}{\|x_k\|} \right\|, \left\| \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right\} \\
 &= \max \left\{ \frac{1}{\|x_k\|} \left\| \sum_{j=1}^n x_j \right\|, \left\| \sum_{j=1}^n \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right\} \\
 &\leq \max \left\{ \frac{1}{\|x_k\|} \left\| \sum_{j=1}^n x_j \right\|, \max_{1 \leq j \leq n} \left\| \left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|} \right) x_j \right\| \right\} \\
 &= \max \left\{ \frac{1}{\|x_k\|} \left\| \sum_{j=1}^n x_j \right\|, \frac{1}{\|x_k\|} \max_{1 \leq j \leq n} \left| 1 - \frac{\|x_k\|}{\|x_j\|} \right| \|x_j\| \right\} \\
 &= \frac{1}{\|x_k\|} \max \left\{ \left\| \sum_{j=1}^n x_j \right\|, \max_{1 \leq j \leq n} \left| 1 - \frac{\|x_k\|}{\|x_j\|} \right| \|x_j\| \right\}.
 \end{aligned}$$

By varying k and taking minimum in the right side of previous inequality gives

$$\left\| \sum_{j=1}^n \frac{x_j}{\|x_j\|} \right\| \leq \min_{1 \leq k \leq n} \left\{ \frac{1}{\|x_k\|} \max \left\{ \left\| \sum_{j=1}^n x_j \right\|, \max_{1 \leq j \leq n} \left| 1 - \frac{\|x_k\|}{\|x_j\|} \right| \|x_j\| \right\} \right\}.$$

□

3. Conclusions

1. In 1958, Massera and Schaffer derived a surprising upper bound for the Clarkson angle between two nonzero elements in a normed linear space [2].
2. In 2006, Maligranda improved Massera-Schaffer inequality [3].
3. In 2007, Pecaric and Rajic extended Maligranda inequality for finitely many nonzero elements [6].
4. In this article, we introduced the notion of non-Archimedean Clarkson angle and derived non-Archimedean version of Massera-Schaffer-Maligranda-Pecaric-Rajic inequality.

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