

Short Note

Not peer-reviewed version

Upper and Lower Bounds of the State Variable of M/G/1 Psffa Model of the Non-Stationary $m/E_k/1$ Queueing System

[Dr Ismail A Mageed](#) *

Posted Date: 31 January 2024

doi: 10.20944/preprints202401.2243.v1

Keywords: PSFFA Model; Non-stationary M/E_k/1 queueing system; threshold; upper and lower bounds; set of phases



Preprints.org is a free multidiscipline platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Short Note

Upper and Lower Bounds of the State Variable of $M/G/1$ Psffa Model of the Non-Stationary $M/E_k/1$ Queueing System

Ismail A Mageed

Member IAENG, IEEE; drismailamageed@yahoo.com

Abstract: The current work reports the upper bound of the state variable of the of the $M/G/1$ Pointwise Steady Fluid Flow Approximation (PSFFA) model of non-stationary $M/E_k/1$ queueing system.

Keywords $M/G/1$ PSFFA model; non-stationary $M/E_k/1$ queueing system; threshold; upper and lower bounds; k set of phases

1. Introduction

In the given context, a mathematical model called PSFFA is used to solve a non-linear differential equation that represents the average number of entities in a queueing system. The PSFFA method utilizes steady state queueing relationships to determine the structure of the fluid flow differential equation. This approach is accurate and offers advantages such as versatility, simplicity in simulating queueing systems, and computational efficiency. Additionally, these techniques have the potential to serve as a fundamental mathematical framework for developing dynamic network control mechanisms[1].

Think about a queueing system for a single server that has a non-stationary arrival process. Let $\mu(t)$ reads as the time-dependent mean service rate and $\lambda(t)$ denotes the time-dependent mean arrival rate. As the system's ensemble average number at time t , we define $x(t)$ as the state variable. The rate of change of the state variable with respect to time is $x(t) = \frac{dx(t)}{dt}$. Let $f_{in}(t)$ and $f_{out}(t)$ stand for, respectively, the time-dependent the system's flow in and out. $f_{in}(t)$, $f_{out}(t)$ and $x(t)$ are equationally related as:

$$x(t) = -f_{out}(t) + f_{in}(t) \quad (1.1)$$

$f_{out}(t)$ relates to server utilization, $\rho(t)$ by

$$f_{out}(t) = \mu\rho(t) \quad (1.2)$$

For an infinite queue waiting space, we have

$$f_{in}(t) = \lambda(t) \quad (1.3)$$

Thus, (1.1) rewrites to:

$$x(t) = -\mu\rho(t) + \lambda(t), \quad 1 > \rho(t) = \frac{\lambda(t)}{\mu(t)} > 0 \quad (1.4)$$

In (1.4), approaching a steady state zone, implies $x(t) = 0$, that is

$$x = G_1(\rho) \quad (1.5)$$

Additonally, supposing the numerical invertibility of $G_1(\rho)$,

$$\rho = G_1^{-1}(x) \quad (1.6)$$

Consequently,

$$x(t) = -\mu(G_1^{-1}(x(t))) + \lambda(t) \quad (1.7)$$

For the time varying $M/E_k/1$ queueing system, the functional relationship G_1 (c.f., [1]) is determined by:

$$G_1(x) = \left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1} \right) \quad (1.8)$$

Therefore, the PSFFA model of the time varying $M/E_k/1$ system with k set of phases is determined by:

$$x(t) = -\mu \left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1} \right) + \lambda(t) \quad (1.9)$$

with $\lambda(t) = A + B\sin(\omega t + C)$

Both Figures 1 and 2(c.f., [2]) provide two real life applications of Time Varying queueing systems. In Figure 1, the Mean number of calls per minute at a central office switch – measured in 15 minutes intervals averaged over 10 work days are observed. Furthermore, Figure 2 describes the associated Mean call holding times.

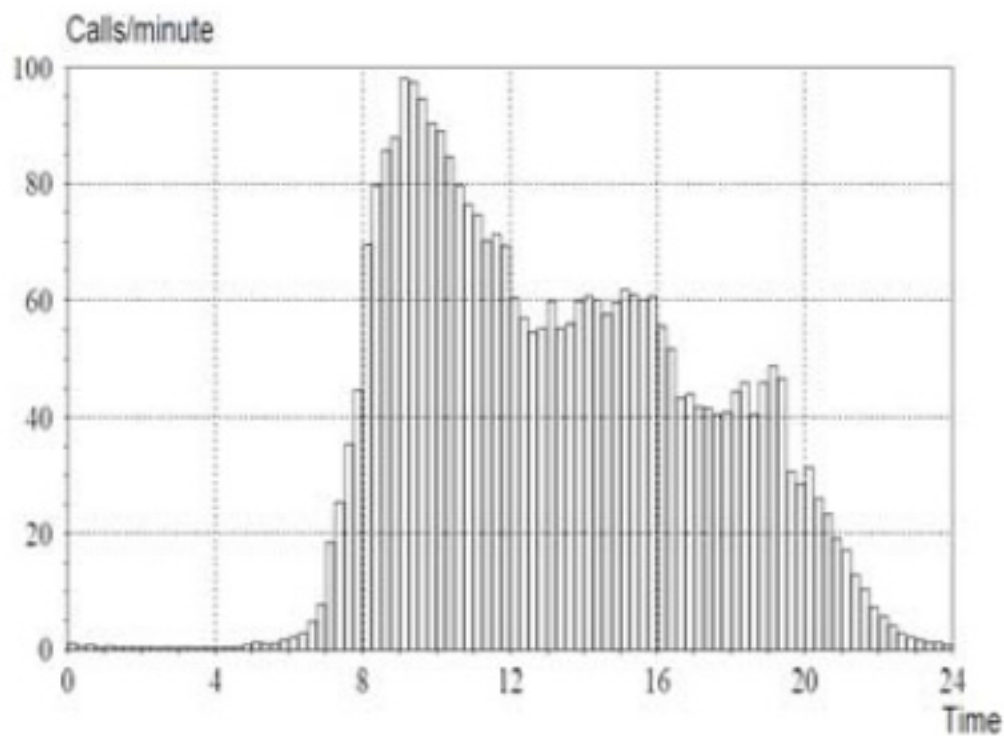


Figure 1.

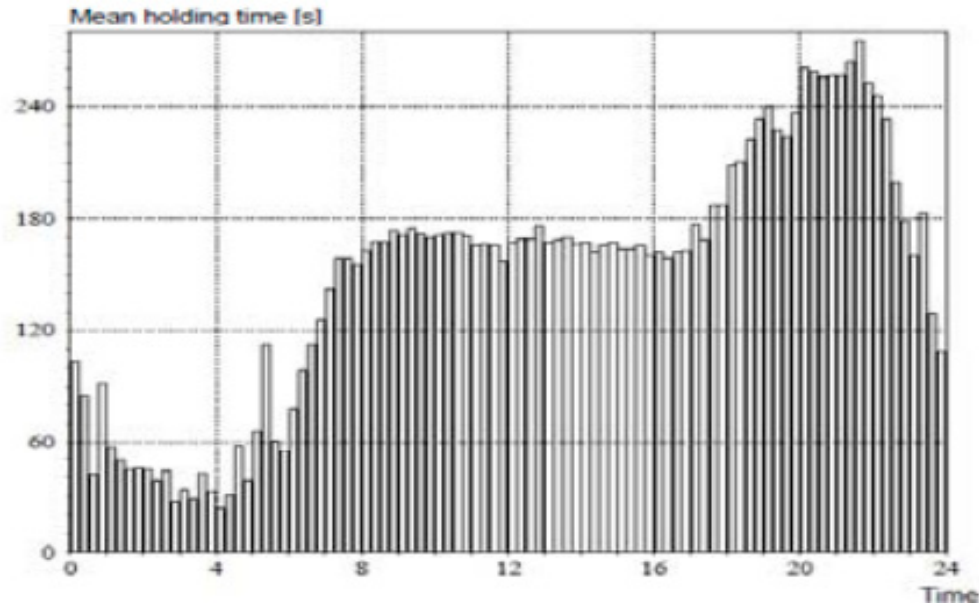


Figure 2.

Figure 3 draws a simulation comparison for the $M/E_2/1$ model corresponding to non-stationary traffic.

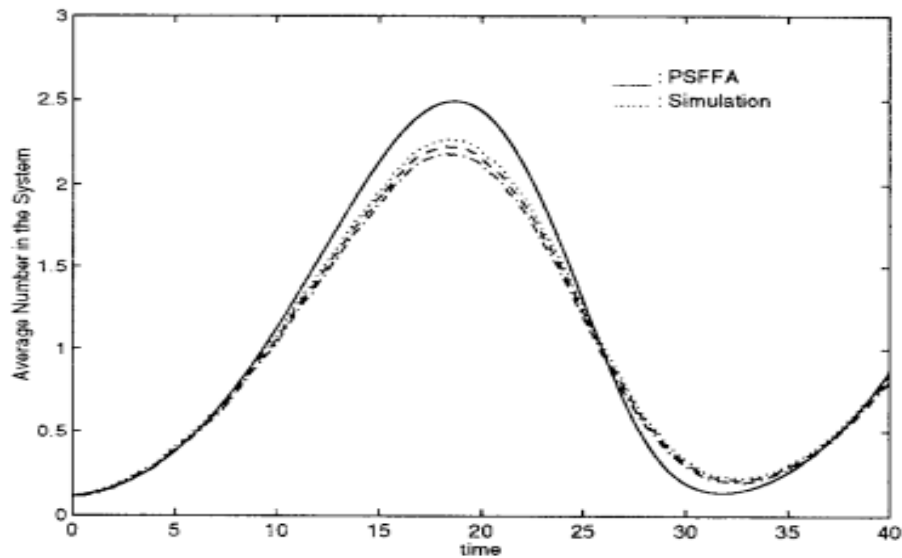


Figure 3. (c.f., [1]).

The following theorem is necessary to prove the new results in the current paper.

1. Preliminary Theorem (PT) [3]

Let f be a function that is defined and differentiable on an open interval (c, d) .

If $f'(x) > 0$ (< 0) for all $x \in (c, d)$, then f is increasing (decreasing) on (c, d) (1.10)

2. Upper and Lower Bounds of $x(t)$ (c.f., (1.9))

Theorem 1. $x(t)$ of the non-stationary $M/E_k/1$ queueing system satisfies the following inequality:

$$\left(1 + \frac{1}{k}\right) \frac{\rho(t)}{2} < x(t) < \frac{k\rho(t)}{1 - k\rho(t)}, \quad \rho(t) \in (0, 1) \quad (2.1)$$

Proof:

We have

$$x(t) = -\mu\left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) + \lambda(t) \quad (\text{c.f., (1.9)})$$

By PT, $x(t)$ is increasing $\Leftrightarrow x'(t) > 0$. Consequently

$$-\mu\left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) + \lambda(t) > 0 \quad (2.2)$$

Hence, it follows that:

$$\rho(t) = \frac{\lambda(t)}{\mu} > \left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) = \frac{2(k-1)x}{(k-1)(k(x+1)+\sqrt{(k^2+2kx+k^2x^2)})} = \frac{2x}{(k(x+1)+\sqrt{(k^2+2kx+k^2x^2)})} > \frac{x}{k(x+1)} \quad (2.3)$$

Therefore,

$$k\rho(t)[1+x(t)] > x(t) \quad (2.4)$$

$$k\rho(t) > (1-k\rho(t))x(t) \quad (2.5)$$

(2.5) implies:

$$x(t) < \frac{k\rho(t)}{1-k\rho(t)} \quad (\text{c.f., (2.1)})$$

By the Preliminary Theorem (PT), $x(t)$ is decreasing if and only if $x'(t) < 0$. Consequently

$$-\mu\left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) + \lambda(t) < 0 \quad (2.6)$$

Hence, it follows that:

$$\rho(t) = \frac{\lambda(t)}{\mu} < \left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2+2kx+k^2x^2)}}{k-1}\right) = \frac{2(k-1)x}{(k-1)(k(x+1)+\sqrt{(k^2+2kx+k^2x^2)})} < \frac{2kx}{k+1} \quad (2.7)$$

Therefore,

$$\left(1 + \frac{1}{k}\right) \frac{\rho(t)}{2} < x(t) \quad (\text{c.f., (2.1)})$$

Hence, the proof follows.

It is noted by (2.1), that at the instability phase ($\rho(t) \rightarrow 1$)

$$\lim_{\rho(t) \rightarrow 1} x(t) < \frac{k}{1-k} \quad (2.8)$$

(2.8) interprets as the limit as ($\rho(t) \rightarrow 1$) has an upper bound which is k dependent. As ($k \rightarrow 1$),

$$\lim_{\rho(t), k \rightarrow 1} x(t) < \infty \quad (2.9)$$

Moreover, assuming $k \rightarrow \infty$, (2.1) reduces to

$$\frac{\rho(t)}{2} < x(t) < \frac{\rho(t)}{1-\rho(t)}, \quad \rho(t) \in (0,1) \quad (2.10)$$

Clearly, (2.10) shows that the derived bounds inequality of the underlying $x(t)$ model of the non-stationary $M/D/1$ queueing system is a special case of that of the non-stationary $M/E_\infty/1$ queueing system.

3. Conclusions and Future Work

The upper bound of the state variable of the non-stationary $M/E_k/1$ queueing system's $M/G/1$ PSFFA model is reported in the current study for the first time ever. Additionally, it is found that the state variable of the $M/G/1$ PSFFA model of the non-stationary $M/E_\infty/1$ queueing system is a special case of the state variable of the derived bounds inequality of the state variable of the $M/G/1$ PSFFA model of the non-stationary $M/D/1$ queueing system. The boundaries of the state variable in the non-stationary $M/G/1$ PSFFA model will be determined in future study directions.

References

1. A Mageed, D.I. Effect of the root parameter on the stability of the Non-stationary D/M/1 queue's GI/M/1 model with PSFFA applications to the Internet of Things (IoT). Preprints 2024, 2024011835. <https://doi.org/10.20944/preprints202401.1835.v1>
2. A Mageed, D.I., Q Zhang, D. K. Solving the open problem for GI/M/1 pointwise stationary fluid flow approximation model (PSFFA) of the non-stationary D/M/1 queueing system. electronic Journal of Computer Science and Information Technology. 2023 Apr 23;9(1):1-6.
3. A Mageed, D.I., Q Zhang, D. K. The Rényiian-Tsallisian Formalisms of the Stable M/G/1 Queue with Heavy Tails Entropic Threshold Theorems for the Squared Coefficient of Variation. electronic Journal of Computer Science and Information Technology. 2023 Apr 23;9(1):7-14.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.