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C^4 Space-Time...
A Window to New Physics?

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Abstract

We explore the possibility to form a physical theory in C^4. We argue that the expansion of our usual 4-d real space-time to a 4-d complex space-time, can serve us to describe geometrically electromagnetism and nuclear fields and unify it with gravity, in a different way that Kaluza-Klein theories do. Specifically, the electromagnetic field A_μ, is included in the free geodesic equation of C^4. By embedding our usual 4-d real space-time in the symplectic 8-d real space-time (symplectic R^8 is algebraically isomorphic to C^4), we derive the usual geodesic equation of a charged particle in gravitational field, plus new information which is interpreted. Afterwards, we formulate and explore the extended special relativity and extended general relativity an C^4 orR^8. After embedding our usual 4-d space-time in R^8, two new phenomena rise naturally, that are interpreted as "dark matter" and "dark energy". A new cosmological model is presented, while the geometrical terms associated with "dark matter" and "dark energy" are investigated. Similarities, patterns and differences between "dark matter", "dark energy", ordinary matter and radiation are presented, where "dark energy" is a dynamic entity and "dark matter" reveal itself as a "mediator" betwen ordinary matter and "dark energy". Moreover, "dark matter" is deeply connected with "dark energy". Furthermore, the extended Hamilton-Jacobi equation of the extended space-time, is transformed naturally as an extended Klein-Gordon equation, in order to get in contact with quantum theories. By solving the Klein-Gordon equation analytically, we derive an eigenvalue for Higg's boson mass value at 125,173945 Gev/c^2. The extended Klein-Gordon equation, also connects Higg's boson (or vacuum) with Cosmology, due to the existence of our second "time" T (cosmological time), which serve us to connect quantum theories with Cosmology. Afterwards, in the general case, we explore the symmetries of the curved Hamilton-Jacobi equation locally, in order to investigate the consequences of a C^4 space-time in Standard Model. An extension to Standard Model is revealed, especially in the sector of strong nuclear field. The Stiefel manifold $SU(4)/SU(2)$ seems capable not only to describe the strong nuclear field but give us,as well, enough room to explore in the future, the possibility to explain quark confinment. Our extension, flavors firstly the unification of nuclear fields and afterwards the unification of nuclear fields with electromagnetic field. The desired grand unification, is achieved locally, through the symmetry group $GL(4,C) \simeq SO(4,4) \cap U(4)$ and we present a potential mechanism to reduce the existing particle numbers to just six. Afterwards,23 present the extended Dirac equation in C^4 space-time (Majorana-Weyl representation) plus a preliminary attempt to introduce a pure geometric structure for fermions. Finally, we consider a new geometric structure through n-linear forms in order to give geometric explanation for quantisation

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2 Introduction

The most difficult problem in the present history of physics, is the hunt of a unified theory. A unified theory, which could incorporate general relativity and quantum theory and could explain the nature of dark energy and dark matter, as well. This task is on progress and several theories and suggestions exist in the literature of physics. But yet, a final satisfactory proposal is still missing. Of course, there are promising candidates, such as superstrings, loop quantum gravity and classic quantum gravity theories, which are still under development. At this point, we would like to suggest an alternative, which is pure geometric. We argue that an expansion of our usual 4-d real space-time to a 4-d complex space-time (or to the algebraically isomorphical symplectic 8-d real space-time), could be promising. In fact the extension to a complex space -time is not something new. A. Einstein has used several complex structures in order to unify gravity with electromagnetism [11], W. Pauli generalised the Kaluza-Klein theory to a six-dimensional space (3-d complex space) [12] and H. P. Soh , advised by A. Eddington, published a theory attempting to unifying gravitation and electromagnetism within a complex 4-dimensional Riemannian geometry [13]. Moreover, S. Hawking discussing mathematical models which involve imaginary time for the description of the Universe in [14], makes a comment [6] suggesting that the distinction between real and imaginary quantities is just a mind trap. In the past, several attempts were made in the direction of a classical unified field theory (UFT), (all our information about UFT can be found in a marvelous and extensive article by Hubert F. M. Goenner [18], [19]), where the basic idea was the generalization or the extension of the Riemannian geometry, in such a way that electromagnetism could be included in a geometrical way. The main efforts attempted were, Weyl's infinitesimal geometry in terms of gauge field, Kaluza- Klein's 5-d theories, Eddington's pire affine geometry, Schrodinger's affine geometry, Born's reciprocity theory, Sciama's attempt to define classical spin and finally Einstein's several attempts using one after the other possible geometries as mixed (metric-affine geometry), asymmetric or complex geometries. All these attempts were failed for many reasons, for instance Weyl's UFT failed to produce rational physical findings, but fortunately, his efforts and ideas guided scientists, through gauge invariance, to formulate quantum field theory. Schrodinger's affine geometry, which was a blend of a of Weyl's theory and Eddington's pure affine theory, was not gauge invariant plus his false intention to include mesons apart from the unification of gravitation and electromagnetism. On the other hand Eddington's affine geometry was mathematically difficult and his suggestions did not have the elegance and the presentation required. And then it was A. Einstein, who played with almost every possible geometric structure, during his hunt for UFT. In the period 1923-1933, he started investigating Eddington's affine geometry, Cartan's tere-parallel geometry, Kaluza's 5-d Riemannian geometry and finally a mixed one where he mixed affine geometry with a metric with a skew-symmetric part. In this mixed geometry, he argued, that the symmetric part of the metric tensor, is associated to inertia and gravitational field, while

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[6]"One might think this means that imaginary numbers are just a mathematical game having nothing to do with the real world. From the viewpoint of positivist philosophy, however, one cannot determine what is real. All one can do is find which mathematical models describe the universe we live in. It turns out that a mathematical model involving imaginary time predicts not only effects we have already observed but also effects we have not been able to measure yet nevertheless believe in for other reasons. So what is real and what is imaginary? Is the distinction just in our minds? "

the assymetric part of the metric would be linked to electromagnetism, where the field equations should be derived as limiting case. Afterwards, in 1948, he started to relate mathematical objects to physical observables,such that, the anti-symmetric density play the role of an electromagnetic potential. , there were many problems in the interpretation, from the torsion tensor which appeared in the definition of the metric, to the two versions of UFT, a weak and a strong one, or to the spherically symmetric solution derived from A.Papapetrou, which did coincide asymptotically with the solution of Einstein-Maxwell equation. Moreover, a clear connection between geometric objects and observables could not be found and finally, there were many problems to pass from continuous to discrete mass. But afterwards, in 1945 tried something new and wrote about it

> it "What i now do will seem a bit crazy.... consider a space the 4 coordinates x_1, x_2, x_3, x_4 which are complex such that in fact it is an 8-d space...In place of the Riemannian metric another one of the form g_{ik} obtains"

But eventually, in order to maintain the 4-d space, he merely abandon this idea and used only the field variables to be complex. In a same manner A. Eddington and H. P. Soh , used also a 4-d Riemannian geometry with real coordinates, but with a complex metric, where the real part of the metric corresponds with mass and gravitation, while the imaginary part corresponds with charge and electromagnetism. And afterwards, it was D.S.Sciama, who tried to give a new approach to UFT. Sharing the same opinion with A. Einstein, about a quantum theory derived directly from geometry, he returned to metric affine geometry. In order to geometrize the spin tensor, he abandoned the idea that the skew symmetric part of the connection is associated to electromagnetism but rather, with a classical spin angular momentum of matter. And finally, there where the Kaluza-Klein theory, where a fifth dimension was added, so that, this extra dimension would house the incorporation of the electromagnetism field into geometry. Eventually,Kaluza-Klein theory, also suffered with the problem of a static spherically symmetric solution. But all these attempts had bigger problems, that the above mentioned ones. The first of them, it was the lack of knowledge in that period i.e all these great scientists, did not know the existence of nuclear fields, they did not have the experimental data of today, nor the existence of what we call dark matter and dark energy fields. This lack of knowledge, did not give them the chance to properly connect and relate the mathematical objects of their theories to physical observables. Moreover, there was and there is, a certain belief that our usual 4-d space-time, should be derived as a limiting case or in reductive way or with compactification of the extra dimensions. In our attempt, fortunately, we take into account all our present knowledge in order to properly connect mathematical objects with observables, plus the fact, that we are willing to connect, even the extra dimensions with observables. Moreover, we asked ourselves, if Cosmos is not in reality 4-d and has a bigger dimension, how would a 4-d observer, would observe this higher dimensional space? Through this question, we do not any longer want to take limits or other similar techniques, but rather to embed our usual 4-d space-time to this higher dimensional space. We argue, that in our approach, we would have the possibility to get in touch with several existing problems of theoretical physics and as well, to give us the opportunity to seek for new physical phenomena. Two new terms will arise after the embedding procedure, apart from gravity, where these new terms are directly connected to geometry and could be linked with the problem of dark fields. In addition, through the embedding procedure, scales will arise, where a uniform scale is recognized as a bound in energy scale,which also looks like a geometric description of Higg's mechanism and the anti-symmetric part of the metric tensor give us enough room, in order to include nuclear forces. Finally the generalized spacial relativity that is formulated as a consequence of C^4 or R^8 and the two time consideration, suggests that there also exist a second invariant "velocity", apart the usual speed of light, that totally changes our beliefs about the propagation of information and everything to it and leads to new physical phenomena. Our main approach, is to repeat all the steps that were made in the past, but now not for the 4-d real space-time, but for the 4-d complex space-time. Specifically, we want to establish a theory of mechanics, a theory of "special relativity" and a "general relativity", directly in C^4. The extra dimensions of this formulation, can be served as additional degrees of freedom, which could can help us to describe geometrically the property of mass and "sources" in general. We want to present a "static" problem in C^4, which becomes "dynamic" after embedding our usual 4-d space-time in the 4-d complex space-time. Sources in general, will arise, as the lost information of this embedding. The advantage of such a consideration, is the ability to present a close theory, as it happens with mechanics and general relativity. Furthermore, we want to explore, the possibility to re-establish quantum theory, as a classic mechanics theory in C^4, giving us this way, the ability to alter the axiomatic demands of quantum theories, to axiomatic definitions of usual mechanics theory.

3 Method behind the choice of a C^4 space-time

There are two successful theories that are capable to describe the basic and elementary "forces" in Nature, General Relativity (GR) for gravity and Standard Model (SM) for electromagnetism, weak and strong nuclear interactions. We would like to find a method originated from these two theories, in order not to see or find a way to unify them, but rather to seek for a new frame, from which those two theories could arise. A nice way to start discussing about gravity and GR is the principle of general covariance, a principle that in our opinion expresses deep philosophical issues concerning the description, the existence and the understanding of the Universe or Cosmos. This principle as it is expressed in [19], is the idea that "every physical quantity must be describle by a geometric object and that the laws of physics must all be expressible as geometric relationships between these geometric objects". This principle, was originally expressed by Felix Klein (Erlanger program) and it was A. Einstein who successfully used in GR. Geometric objects are in general tensors such as vectors (1-tensors), metric tensor (2-tensor), Riemann-Christoffel 3-tensor R^i_{klm} etc, which exist independently of coordinate systems or reference frames but in general expressible by them. This way GR is usually called as a geometric theory and it has its foundation on three axioms

1. There is a metric tensor
2. The metric tensor fulfills the Einstein field equation
$$G_{ij} = 8\pi T_{ij}$$
3. All special relativistic laws pf physics are valid in local Lorentz frames of metric

Then curvature in geometry manifests itself as gravitation as the energy momentum tensor T_{ij}, is the "average' of curvature expressed by Einstein's tensor G_{ij}. Based on the above mentioned, we would like to impose a question as a new way of investigation. Can we expand the relationship between the energy-momentum tensor T_{ij} and geometry described by G_{ij}, to a new principle that even T_{ij} is not connected by relationship to geometry but T_{ij} can be described by geometry itself? Or in an other way, if T_{ij} generates an average curvature described by Einstein's tensor, can we find a higher dimensional space, let us call it X, implying this way a new extension of our usual 4-d real space-time to space-time X, where now T_{ij} can be described or connected with a new "average" curvature defined in the new expanding part of X and then the generalized "Einstein" tensor (this generalised "Einstein" tensor will follow the dimensionality of this space X, for instance if X is n dimensional then, this "generalised Einstein" tensor will be a matrix $n \times n$), let us call it for now $G^{'}_{ij}$, of the extended space-time X, fulfills the field equation

$$G^{'}_{ij} = 0$$

which suggests that the "average" curvature of this space-time X is 0 and the energy-momentum tensor is incorporated purely geometrical in $G^{'}_{ij}$? If the answer is yes, then $G^{'}_{ij} = 0$ is nothing else, than an equilibrium equation (not of "Poisson type" but rather a "Laplace type" equation) which means that Universe or Cosmos is an expanding dynamic system in equilibrium state , governed by the geometry of the space-time or manifold X, through a Ricci flow. Meanwhile, those extra dimensions can be seen also as additional degrees of freedom, from which the entities of our usual energy-momentum tensor expressed as matter, charges, currents or even undescribed by GR physical quantities such as pure quantum characteristics (spin, isospin, colours, etc), or sources in general could be defined or described pure geometrically. Of course extra dimensions and extension to a higher dimensional space-time is not something new, it was originally proposed by T. Kaluza and O. Klein and afterwards from string theories in general, where a 10+1 dimensional space is proposed as the necessary "arena" for the description of M-theory. At this part, we would like to examine, if there are any clues, from our well known and accepted theories, which could inform us about the type and dimensionality of space-time X. We think that there is no better candidate than the Standard Model (SM). Let us focus only in the electromagnetic part of SM, for simplicity, where the gauge symmetry is the abelian group U(1)and the covariant derivative associated with it, is

$$D_\mu = \partial_\mu - iA_\mu$$

There is a lot for someone to discuss about gauge theories, involving symmetry groups, Lie groups, Lie manifolds, tangent bundles, submersions etc, but we would like to focus on a different and more simple path. We want to examine this covariant derivative in a strict, in the beginning, mathematical or geometrical way and afterwards, we will try to evaluate the physical meaning and interpretation. In this covariant derivative, obviously ∂_μ is a 4-d real vector or vector field (the basic tangent vector of 4-d real space-time) and A_μ is a 4-d real vector or vector field as well. Now, if we consider that D_μ is a vector or even better a tangent vector, in the sense of a geometrical description, in which space does D_μ belongs to? The answer is very simple but awkward

$$D_\mu \text{ belongs in a } C^4 \text{ space}$$

due to the fact that the complex number "i" lies in the covariant derivative between the two 4-d real vectors! But what does it means, is it just a mathematical tric or can we give physical meaning? We argue that not only C^4 has physical meaning and interpretation but rather this is the key or clue we were looking for space X. We suggest a new extension of our usual 4-d real space-time to a 4-d complex space-time, or a we will see further to its geometrically equivalent 8-d real symplectic space. We shall see in sections (18), (19), that the choice of a C^4 space-time by the beginning, will explain not only how, but why as well, as concerned the choice of a complex field φ in quantum field theories and additionally, the causality of the existence of the symmetries described by the unitary groups $U(1)$, $SU(2)$, $SU(3)$ in gauge theories. This way we would not need to start by a God given field and God given symmetries (as R. Penrose comments in "The Road To Reality"), as they would arise naturally as properties of the choice of a C^4 space-time. For instance, we will see that the Hermitian metric, associated with C^4, is invariant under transformations described by the group $GL(4, C)$ which is isomorphic as

$$GL(4, C) \simeq SO(4, 4) \cap U(4)$$

Furthermore, we will see in section (18), that $U(4)$ breaks simultaneously after embedding our usual space-time in C^4 into desired unitary groups, giving us this way, the chance to explain the phenomenon of spontaneous symmetry breaking, as the causality of this embedding. Specifically, in section (18) we show that the symmetry group for nuclear and electromagnetic field should expanded to an extension of SM as $\dfrac{SU(4)}{SU(2)} \times SU(2) \times U(1)$, where the quotient is not anymore a group but rather a coset (orbit space), that is called in the literature of mathematics as Stieffel manifold, that contains an su(3) algebra, as a subalgebra. Furthermore, this coset is isomorphical to the product of spheres $S^7 \times S^5$ which is clearly "bigger" than the group SU(3)and as a result, it gives us room to seek for unexplained phenomena linked to strong nuclear field. In addition, all together the product naturally leads to an extension of SM. It is our desire throughout all our consideration to answer not only the "how" in physics , but the "why" as well. Additionally, if the choice of a C^4 space-time is valid, it could also explain the great success of quantum theories in general, as they would appear to have already used the fact of a complex space-time. Complex geometrical structures are not something new in physics, as we have already seen in the introduction, and already such structures are used in the sense of Kahler and Calabi-Yao manifolds. Moreover, symplectic and complex geometries are suggested as new tools in the connection of Yang-Mills theories and geometry in [17]. But our suggestion is not only some additional dimensions, serving as additional degrees of freedom, but we want to propose to give direct physical interpretation to these ones. There are two ways

1. We must find some physical quantities that will be related to this extra dimensions. Or, are there any physical necessities that could be introduced by the beginning and C^4 could be the right framework?
2. We can start with just a usual vector of C^4
$$z_i = x_i + iy_i$$
where x_i, $i = 0, 1, 2, 3$ are the usual coordinates of 4-d real space-time and leave y_i without any physical interpretation, in the beginning, and let the mathematical processing to lead us to a desired and suitable interpretation.

We have chosen the second way, due to the fact, that we can build and establish a more concrete framework and examine step by step the arisen structures and this way we keep in touch with the well known physical theories. Finally, from GR the key was geometry, from gauge theories and SM the key was C^4 space and the combination lead us to this consideration in the search of new physics (if someone believes in such a hunt)

"We suggest to investigate geometrically C^4 space-time. In C^4 space-time there must be a unified field (Gravity, electromagnetism, etc) which is a property of this 4-d complex space-time itself, as gravitational field is a property of our usual 4-d real space-time"

As a consequence, in the next paragraph, we will start with a pure geometrical picture, by investigating the elementary length in a curved C^4 space-time and afterwards, we will give the physical interpretation of these extra dimensions as a natural consequence of geometry processing. The key in order to take back our usual well known theories which are expressed in the "language" of a 4-d real space-time, will be the embedding of our usual 4-d space-time, in the 4-d complex space-time. Moreover, in this paper, we investigate the flat cases of C^4 and R^8, which leads to an extended special relativity and a second invariant constant is introduces, while the symmetry group $SO(8)$ is connected with the signatures $(4, 4)$, $(8, 0)$, $(0, 8)$ through Cartan's principle of triality. The field equations of the unified field in curved C^4 space-time is investigated in section (11). In sections (12),(14),(15) by releasing the end point of the action's integral, we pass to

Hamilton-Jacobi equations and we argue that the covariant derivative of SM is nothing else than a part of the Hamilton-Jacobi derivative as it comes straightforward, from the problem of least action, derived directly from the geometry of the curved C^4 space-time and the usual symmetries and groups of SM are related with the symmetry of this action, which is invariant as we shall see, under transformations of the group $GL(4, C)$ and $U(4)$. Afterwards, in section (12), complex time will help us to overcome the problems of the ADM formalism and express a suitable Hamilton- Jacobi equation for the curved C^4, defining this way a super-energy tensor connected to the complex time.

4 Geometry in C^4

There are several geometrical structures that we can equip a C^4 space such complex, almost complex, Hermitian, holomorphic,Kahler, Kalabi-Yao, etc. From these structures, we have chosen the Hermitian one because it is the most natural extension of the Riemann's spaces in a complex space. Specifically, we can define an elementary length of the type

$$ds^2 = G_{ij} dz^i d\bar{z}^j + hc \tag{1}$$

where G_{ij} is a Hermitian metric tensor (in analogy to a symmetric metric tensor in Riemann's spaces). It is obvious, that we treat to C^4 space as

$$C^4 \simeq X \times iY \simeq R^4 \times iR^4 \tag{2}$$

where $x^i \in X$ and $y^i \in Y$. Many authors write the Hermitian metric tensor $G_{i\bar{j}}$ instead of G_{ij} but we will keep the notation without the bra, in order to make the notation more simple. We can proceed by introducing the elements of the C^4 space as

$$z_i = x_i + iy_i \tag{3}$$

where $x_i \in R^4(X)$, $y_i \in R^4(Y)$. The x_i, y_i must be of the same type which means that x_0 and y_0 are both time-like while x_1, x_2, x_3 and y_1, y_2, y_3 are space-like. The corresponding Cauchy derivative will be

$$\partial_{z_i} = \frac{1}{2}(\partial_{x_i} - i\partial_{y_i}) \tag{4}$$

In addition, the metric tensor of C^4 will be a Hermitian 4 x 4 metric G_{ij}

$$G_{ij} = g_{ij} + iI_{ij} \tag{5}$$

with g_{ij} its symmetric and I_{ij} its anti-symmetric part. Obviously, g_{ij} plays the role of the metric tensor in X and Y consisting only of terms without any mixing of variables in X and Y, while I_{ij} contains only such mixing terms. If we introduce Eq. (3) in Eq. (1) we will move from the C^4 space to an R^8 space equipped with a symplectic geometry where the elementary length will then be

$$ds^2 = g_{ij}dx^i dx^j + g_{ij}dy^i dy^j + I_{ij}(dx^i dy^j - dy^j dx^i) \tag{6}$$

where g_{ij} is our common symmetric metric tensor and I_{ij} is a symplectic antisymmetric tensor. In the case that I_{ij} vanishes, we fall naturally in the case of a Riemann's space of type R^{2n} where $n = 4$. The Hermitian metric tensor has become in the case of real representation

$$G_{ij} = \begin{pmatrix} g_{ij} & I_{ij} \\ -I_{ij} & g_{ij} \end{pmatrix}$$

The symplectic term in Eq. (6) can be written also as

$$ds^2 = g_{ij}dx^i dx^j + g_{ij}dy^i dy^j + 2I_{ij}dx^i dy^j \tag{7}$$

because $I_{ij}dy^i dx^j = I_{ji}dx^i dy^j = -I_{ij}dy^i dx^j$. Our next step is to generalise the usual Christoffel symbols $\Gamma_{k,ij}$ to Christoffel symbols $\widehat{\Gamma}_{k,ij}$ with respect to the Hermitian metric tensor G_{ij}. So, we have to compute the partial derivatives $\frac{\partial G_{jk}}{\partial z^i}, \frac{\partial G_{ki}}{\partial z^j}, \frac{\partial G_{ij}}{\partial z^k}$ with respect to the Cauchy's derivative as

$$\frac{\partial G_{jk}}{\partial z^i} = \frac{1}{2}\left(\frac{\partial G_{jk}}{\partial x^i} - i\frac{\partial G_{jk}}{\partial y^i}\right) = \frac{1}{2}\Big(\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial I_{jk}}{\partial y^i}\right) + i\left(\frac{\partial I_{jk}}{\partial x^i} - \frac{\partial g_{jk}}{\partial y^i}\right)\Big) \tag{8}$$

thus, the Christoffel symbols $\widehat{\Gamma}_{k,ij}$ are

$$\widehat{\Gamma}_{k,ij} = \Gamma^{(x)}_{k,ij} + \Delta^{(x)}_{k,ij} - i\big(\Gamma^{(y)}_{k,ij} + \Delta^{(y)}_{k,ij}\big) \tag{9}$$

or in real representation R^8

$$\widehat{\Gamma}_{k,ij} = \big(\Gamma^{(x)}_{k,ij} + \Delta^{(x)}_{k,ij}, -\Gamma^{(y)}_{k,ij} + \Delta^{(y)}_{k,ij}\big) \tag{10}$$

where $\Gamma_{k,ij}$ are the usual Christoffel symbols with respect to the symmetric tensor g_{ij}, $\Delta_{k,ij}$ are the "Christoffel symbols" with respect to the antisymmetric tensor I_{ij} and by (x), (y) we denote the kind of the coordinates to which we find the partial derivative. As concerned the $\Delta_{k,ij}$ symbols it is easy to see that

$$\Delta^{(x)}_{k,ij} = -\Delta^{(x)}_{k,ji} \tag{11}$$

$$\Delta^{(y)}_{k,ij} = -\Delta^{(y)}_{k,ji} \tag{12}$$

which means, that they are antisymmetric with respect to the pair of indices ij . Now we can proceed to find the geodesics through the variation of an action of the form

$$\delta S = \delta \int ds \tag{13}$$

for ds as defined by Eq. (7) which can be written also as

$$\delta S = \delta \int \big(g_{ij}u^i u^j + g_{ij}v^i v^j + 2I_{ij}u^i v^j\big)ds \tag{14}$$

where $u^i = \frac{dx^i}{ds}$ and $v^i = \frac{dy^i}{ds}$. After some calculus we derive the pair of geodesic equations

$$\big(g_{kj}\frac{du^j}{ds} + \Gamma^{(x)}_{k,ij}u^i u^j\big) + \big(I_{ki}\frac{dv^i}{ds} + 2\Delta^{(x)}_{k,ij}v^i u^j\big) + \frac{\partial I_{jk}}{\partial x^i}v^i u^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}v^i v^j = 0 \tag{15}$$

$$\big(g_{kj}\frac{dv^j}{ds} + \Gamma^{(y)}_{k,ij}v^i v^j\big) + \big(I_{ki}\frac{du^i}{ds} + 2\Delta^{(y)}_{k,ij}u^i v^j\big) + \frac{\partial I_{jk}}{\partial y^i}u^i v^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}u^i u^j = 0 \tag{16}$$

the first parenthesis in both equations reminds us our usual geodesic equation of the space-time R^4, while we have other terms that we want to link them to electromagnetism so that the equations (15), (16) could gives us the geodesic equation of a charged particle in gravitational field and hopefully new elements! It is obvious now, that we want to link the symplectic term I_{ij} (antisymmetric tensor) with a generalized field K_μ which will represent a generalized "electromagnetism" which could contain not only the electromagnetic field A_μ but the weak nuclear field W_μ and the strong nuclear field G_μ as well, giving us the opportunity to describe those fields purely geometrically in a larger extended space-time. We must remember that even the electromagnetic field A_μ is not a pure geometric object of our usual space-time, but rather added (ad-hoc) to the geometric action (derived by the elementary length of R^4) by a term

$$-\int \frac{q}{c}A_i dx^i \tag{17}$$

The term $I_{ij}v^i v^j$ in Eq. (14) can be also seen as

$$I_{ij}u^i v^j ds = I_{ij}\frac{dx^i}{ds}\frac{dy^i}{ds}ds = -I_{ji}\frac{dy^j}{ds}\frac{dx^i}{ds}ds = -\big(I_{ji}\frac{dy^j}{ds}\big)dx^i \tag{18}$$

It is obvious that we could immediately recognize as

$$A_i = I_{ji}\frac{dy^j}{ds} \tag{19}$$

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but, these could be premature and as we have mentioned above we want to identify a "generalized unified electromagnetism" K_i firstly, but Eq. (19) can give us some clue. We introduce the anti-symmetric tensor K_{ij} defined as

$$K^{(x)}_{jk} = \frac{\partial K_k}{\partial x^j} - \frac{\partial K_j}{\partial x^k} \tag{20}$$

where $K_j = I_{ji}\dot{y}^i = -I_{ij}\dot{y}^i$ then Eq. (20) becomes

$$K^{(x)}_{jk} = \frac{\partial K_k}{\partial x^j} - \frac{\partial K_j}{\partial x^k} = \big(\frac{\partial I_{ki}}{\partial x^j} - \frac{\partial I_{ij}}{\partial x^k}\big)v^i \tag{21}$$

or with respect to Δ symbols

$$K^{(x)}_{jk} = \big(2\Delta^{(x)}_{k,ij} + \frac{\partial I_{jk}}{\partial x^i}\big)v^i \tag{22}$$

this way, the first pair of the geodesic equations can be written

$$\big(g_{kj}\frac{du^j}{ds} + \Gamma^{(x)}_{k,ij}u^iu^j + K^{(x)}_{jk}u^j\big) + I_{ki}\frac{dv^i}{ds} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}v^iv^j = 0 \tag{23}$$

The term in the parenthesis starts to look like the desired one, but we must remember that we have the second pair also which becomes

$$\big(g_{kj}\frac{dv^j}{ds} + \Gamma^{(y)}_{k,ij}v^iv^j + K^{(y)}_{jk}v^j\big) + I_{ki}\frac{du^i}{ds} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}u^iu^j = 0 \tag{24}$$

The tensor $K^{(y)}_{jk}$, $K^{(x)}_{jk}$ are nothing else than the "Christoffel symbols" $\Delta^{(y)}_{k,ij}$, $\Delta^{(x)}_{k,ij}$ multiplied by a velocity! This way, the analogue of the symmetric metric tensor "field" g_{ij} is the anti-symmetric tensor I_{ij} "field" and not the K_i (or A_i which is a sub case) as we have suspected as far now in the usual context of physics. Moreover, the 2-form K_{ij} (or F_{ij} for the sub case) is not equivalent with the curvature 2-form Riemann-Christoffel tensor R_{ij}. On the contrary the equivalence of K_{ij} is between the Christoffel symbols Γ. From our point of view, this is the reason that we have failed to unify successfully gravity and electromagnetism. Even in the case of the Kaluza -Klein theories, the g_{ij} was put in equal foot with the "field" A_i. As we have seen in our consideration g_{ij} and K_i are different with respect a velocity. And that was the reason that Kaluza-Klein theories where merely successful. This situation was merely saved, due to the fact that the variation of the action was taken with respect to the "field" A_i itself and not with respect a field analogue to the metric tensor, as we have done so far in our consideration. It is important to note though, that we could form "fields" with respect to the metric tensor g_{ij} in the same way as we have done for the "fields" K_i, combining the g_{ij} with a velocity, or even form a 2 tensor with respect to g_{ij} in the same way that we have done for K_{ij}, combining the Γ with a velocity. But all these, will be investigated later.

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5 Embeding R^4 in R^8

The main problem of the pair of geodesic equations (23), (24) is that they express some physics in the symplectic space R^8 which is very different from our usual space R^4. Specifically, these equations should be valuable only to R^8 observers! Unfortunately, we are 4-d dimensional observers and our physical theories are expressed in the mathematical language of a 4-d real space. In order to identify the observables of the 8-d space we can embed our usual 4-d space-time in the 8-d extended space-time. This way, it seems that 4-d observers live in one of the projection spaces of C^4 and by embedding the one projection R^4 in C^4 or R^8 symplectic space, we will recover the lost information. But, before the embedding we must clarify some important issues about the flat cases and the signature problem. The flat Hermitian metric tensor can take the following signatures (1,1,1,1), (-1,-1,-1,-1,), (1,1,-1,-1), (1,1,1,-1) and (-1,1,1,1) where the 2 first two are Hermitian, while the other two are pseudo-Hermitian, which gives in the real representation the signatures (8,0), (0,8), (4,4), (6,2), (2,6) accordingly and similarly the first two are Euclidean, while all the others are pseudo-Euclidean. The signatures (8,0), (0,8) share a duality property and (6,2), (2,6) as well. But there is a unique property that comes as first time in 8-d real spaces, the Cartan's triality property, which states that the three signatures (8,0), (4,4), (0,8) are all correlated (for more information about triality see Appendix 1). By Cartan's principle of triality we will try not only to choose the right signature but also to explain the choice of the 8-d space (according to Duff's viewpoint in [3] a fundamental theory of everything should explain not only the dimensionality but the signature of the space-time as well). In fact, we will be able to provide an independent signature framework in the same spirit general relativity provides a coordinate independent description. For that reason, we have the right to pick one of those three signatures and we have chosen the (4,4) one, due to the fact that it can be splitted to (1+3,3+1) signature, giving us the opportunity to present our usual Minkowski's space as we shall see below. For clarity, we must emphasize that Hermitian geometry will only provide us with the signatures (8,0) and (0,8), the (4,4) one which comes from a pseudo-hermitian geometry, can be used only as a consequence of Cartan's property of triality and if used, we must automatically change the sign of the second g_ij in equation (6) or (7) in the general case of the Hermitian geometry, from (+) to (-) by hand. Specifically, the signature (4,4) stands for in the flat case

$$ds^2 = dx_0^2 + \mathbf{dx^2} - dy_0^2 - \mathbf{dy^2} \tag{25}$$

where bold means 3-d. We can split the signature if we change place between x_0 and y_0 as

$$ds^2 = -dy_0^2 + \mathbf{dx^2} + dx_0^2 - \mathbf{dy^2} \tag{26}$$

The term $-dy_0^2 + \mathbf{dx^2}$ defines our usual Minkoskwi tensor n_{ij} with signature $(-1, 1, 1, 1)$. Moreover we would like to add some comments about the embedding procedure. In order to proceed with embedding, we must pass from the initial coordinate x_i and y_i that describe R^8, to a re-expression containing only x_i that describe the embedded space R^4. This way the y_i coordinates must be re-expressed with respect to the coordinates of the embedded space. The lost information referred to coordinates y_i, will be recovered, as we can see from equations of (33), (34) with additional terms in the final expression of the metric tensor. Now, we can proceed to the embedding which is a standard mathematic topic, similar to the parametrisation, equations (27)-(34) are part of this mathematical topic as it exists in the literature of diferrential geometry.

We start once again by equation (6) derived earlier in this section

$$ds^2 = g_{ij}dx^i dx^j + g_{ij}dy^i dy^j + I_{ij}(dx^i dy^j - dy^j dx^i) \tag{27}$$

If R^4 is embedded in R^8 and N_{ij} is the metric tensor of R^4 , then in R^4 we have

$$ds^2 = N_{ij}dx^i dx^j \tag{28}$$

We will write the metric tensor in R^8 using Greek indices α, β

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{\alpha\beta}dy^\alpha dy^\beta + 2I_{\alpha\beta}dx^\alpha dy^\beta \tag{29}$$

The elementary length ds of R^4 is the same in R^8 and as a result

$$N_{ij}dx^i dx^j = g_{\alpha\beta}dx^\alpha dx^\beta + g_{\alpha\beta}dy^\alpha dy^\beta + 2I_{\alpha\beta}dx^\alpha dy^\beta \tag{30}$$

If $y^\alpha = y^\alpha(x^0, x^1, x^2, x^3)$ and $dy^\alpha = \dfrac{\partial y^\alpha}{\partial \varrho} dx^\varrho$ we have

$$N_{ij}dx^i dx^j = g_{\alpha\beta}dx^\alpha dx^\beta + g_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^\varrho}dx^\varrho \frac{\partial y^\beta}{\partial x^\mu}dx^\varrho + 2I_{\alpha\beta}dx^\alpha \frac{\partial y^\beta}{\partial x^\mu}dx^\mu \tag{31}$$

Because, now we refer to the variables x^i, we can replace the Greek indices by Latin i,j wherever needed and therefore

$$N_{ij}dx^i dx^j = g_{ij}dx^i dx^j + g_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}dx^i dx^j + 2I_{i\beta}\frac{\partial y^\beta}{\partial x^j}dx^i dx^j \tag{32}$$

which actually means that

$$N_{ij} = g_{ij} + g_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + 2I_{i\beta}\frac{\partial y^\beta}{\partial x^j} \tag{33}$$

or even

$$N_{ij} = g_{ij} + g_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + I_{i\alpha}\frac{\partial y^\alpha}{\partial x^j} + I_{j\alpha}\frac{\partial y^\alpha}{\partial x^i} \tag{34}$$

The pair of the geodesic equation (23),(24) becomes as one as

$$N_{ij}\frac{d^2x^j}{ds^2}+\widehat{\Gamma}_{i,jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0 \tag{35}$$

where

$$\widehat{\Gamma}_{i,jk}=\frac{1}{2}\left(\frac{\partial N_{ki}}{\partial x^j}+\frac{\partial N_{ij}}{\partial x^k}-\frac{\partial N_{jk}}{\partial x^i}\right) \tag{36}$$

It is important to simplify a little bit the above mentioned equation by introducing a special case of the embedding functions

1. $y^{\alpha'}=\lambda\delta^{\alpha'}_{\varrho}x^{\varrho}$ for $\alpha'=1,2,3$ and $y^0=y^0(x^0)$. As we can see the space-like functions are linear while the time-like function is free and can be (as we can see in section (16)) of the form $y_0=Ae^{Bx_0}$. After some calculus, the metric tensor N_{ij} can be written as

$$N_{ij}=(1+\lambda^2)g_{ij}+\lambda D_{ij}\frac{\partial y^0}{\partial x^0}+2E_{ij}\left(\frac{\partial y^0}{\partial x^0}\right)^2+M_{ij}\frac{\partial y^0}{\partial x^0} \tag{37}$$

and if we want to split the signature in (1+3, 3+1) we just have to interchange x_0 with y_0. This way g_{ij} is our usual metric tensor and locally it is the Minkowsky's metric tensor. Moreover the tensors D_{ij}, E_{ij}, M_{ij} are

$$D_{ij}=\begin{pmatrix}2g_{00} & g_{01} & g_{02} & g_{03}\\ g_{10} & 0 & 0 & 0\\ g_{20} & 0 & 0 & 0\\ g_{03} & 0 & 0 & 0\end{pmatrix}$$

$$M_{ij}=\begin{pmatrix}0 & I_{01} & I_{02} & I_{03}\\ I_{10} & 0 & 0 & 0\\ I_{20} & 0 & 0 & 0\\ I_{03} & 0 & 0 & 0\end{pmatrix}$$

$$E_{ij}=g_{00}\delta^0_i\delta^0_j \tag{38}$$

E_{ij} can be nicely combined with M_{ij}, in order to form the scalar quantity of electromagnetism! If we proceed in the calculation of $\widehat{\Gamma}_{i,jk}$ with respect to the tensors D_{ij}, E_{ij}, M_{ij} we can see that it breaks into pieces as

- our usual Christoffel symbols formed by the first term of Eq. (37) which means that they are formed by g_{ij}
- some peculiar " Christoffel symbols " formed by the second term of Eq. (37) D_{ij} which are g_{ij} related and have the form

$$\Gamma^{(D)}_{i,jk} = \Big(\frac{\partial g_{k0}}{\partial x^j} + \frac{\partial g_{j0}}{\partial x^k}\Big)\delta^0_i + \Big(\frac{\partial g_{i0}}{\partial x^k} - \frac{\partial g_{k0}}{\partial x^i}\Big)\delta^0_j + \Big(\frac{\partial g_{i0}}{\partial x^j} - \frac{\partial g_{j0}}{\partial x^i}\Big)\delta^0_k \quad (39)$$

the first parenthesis is symmetric while the other two are antisymmetric, which is in contrast to the behaviour of our usual Christoffel symbols.

- the "Christoffel symbols" with respect to the antisymmetric tensor I_{ij} that we have called them as $\Delta_{i,jk}$

$$\Gamma^{(M)}_{i,jk} = \Delta_{i,jk} = \Big(\frac{\partial I_{k0}}{\partial x^j} + \frac{\partial I_{j0}}{\partial x^k}\Big)\delta^0_i + \Big(\frac{\partial I_{i0}}{\partial x^k} - \frac{\partial I_{k0}}{\partial x^i}\Big)\delta^0_j + \Big(\frac{\partial I_{i0}}{\partial x^j} - \frac{\partial I_{j0}}{\partial x^i}\Big)\delta^0_k \quad (40)$$

it is peculiar but the $\Gamma^{(D)}_{i,jk}$, $\Gamma^{(M)}_{i,jk} = \Delta_{i,jk}$ have exactly the same form, except the fact that the first one is with respect to the symmetric g_{ij} while the second one with respect to the antisymmetric I_{ij}.

All these terms will appear in the geodesic equation. Afterwards, we can express some cases concerning Eq. (37) . Firstly, it is interesting to note that in the case that $i, j \neq 0$ we have

$$N_{ij} = (1 + \lambda^2) g_{ij} \quad (41)$$

and for $i, j = 0$ we have

$$N_{00} = (1+\lambda^2)g_{00} + \lambda g_{00}\frac{\partial y^0}{\partial x^0} + 2g_{00}\Big(\frac{\partial y^0}{\partial x^0}\Big)^2 = \Big((1+\lambda^2) + \lambda\frac{\partial y^0}{\partial x^0} + 2\Big(\frac{\partial y^0}{\partial x^0}\Big)^2\Big)g_{00} \quad (42)$$

Equation (42) expresses energies, which means that the parenthesis in front g_{00} is a coupling constant. This term has a minimum in the scale $\frac{\lambda}{2} = -\frac{\partial y^0}{\partial x^0}$ suggesting that at this point the scale λ is unified with $\frac{\partial y^0}{\partial x^0}$ and that we cannot intrude this scale, all the permitted scales are only above this scale! It somewhat peculiar but it looks like we have a geometrical description of Higg's mechanism (without the interaction term that comes from φ^4 and can be recovered from the other papers) and that we have the possibility to enter in the area of high energy physics. We must proceed with the interpretation of Eq. (38) term by term in order to clarify what this energy scales mean.

- the first term of Eq. (37) is $(1-\lambda^2)g_{ij}$ where g_{ij} is our usual metric tensor of the 4-d space-time that expresses gravity and is connected with ordinary masses. Moreover, we will see that λ stands for Planck scale as it will be derived from general relativity. In this case, λ is fixed as it happens in General Relativity, but in the next case, the scale will be time depended.
- the last term represents the "unified generalised electromagnetism" as we have mentioned. But for $y^{\alpha'} = \lambda\delta^{\alpha'}_{\varrho}x^{\varrho}$ for $\alpha' = 1, 2, 3$ that we are studying, this should be our well known electromagnetism, due to the linearity of the embedding functions! Specifically, in this case the electromagnetic field tensor F_{ij} should stand for

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$$F_{ij} = \frac{\partial y^0}{\partial x^0}(\frac{\partial I_{k0}}{\partial x^j} - \frac{\partial I_{j0}}{\partial x^k})\frac{dy^0}{ds} \tag{43}$$

where $\frac{\partial y^0}{\partial x^0} = \frac{q}{c}$.

- the second term has a scale as the product of the scale of the first term and the last one. Moreover, the $\Gamma^{(D)}_{i,jk}$ have the same behaviour with the $\Delta_{i,jk}$ but with respect to the symmetric tensor g_{ij}. It looks like this term both "gravitates" and "electromagnitates" in behavioral way! It is a hybrid between those two fundamental elementary fields. We propose to interpretate or connect this field to what we use to call as dark field (or for the linear case and only "dark electromagnetism")!
- finally the third term that has only one element $E_{ij} = g_{00}\delta^0_i\delta^0_j$ (scalar), share the scale of electromagnetism squared. We shall see later that it is invariant to any transformation that generalises $y^{\alpha'} = \lambda\delta^{\alpha'}_{\varrho}x^{\varrho}$ for $\alpha' = 1,2,3$, which can be interpreted as dark energy field.

2. If we write y^α around a point $(x^0_0, x^1_0, x^2_0, x^3_0)$, where $\vec{x}_0 = (x^1_0, x^2_0, x^3_0)$ is a steady point or pole, we can have for the embedding functions

$$y^{\alpha'} = y^{\alpha'}(x_0, \vec{x_0}) + \frac{\partial y^{\alpha'}(x_0, \vec{x_0})}{\partial x^\gamma}(x^\gamma - x^\gamma_0) + ... \tag{44}$$

for $\alpha' = 1,2,3$ and $\gamma = 1,2,3$. If we keep only the two first terms of the expansion and if we set

$$\varepsilon^\kappa_\lambda = \begin{cases} 0, & \kappa = \lambda \\ 1, & \kappa \neq \lambda \end{cases} \tag{45}$$

the final embedding functions are

$$y^{\alpha'} = y^{\alpha'}(x_0, \vec{x_0}) + c^\alpha_\gamma(x^\gamma - x^\gamma_0)\varepsilon^\alpha_0 \tag{46}$$

for $\alpha = 1,2,3,4$ and $\gamma = 1,2,3$. We have the following cases as concerning the indices i,j

- for $i, j = 1,2,3$

$$N_{ij} = g_{ij} - +_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} + I_{i\alpha}\frac{\partial y^\alpha}{\partial x^j} + I_{j\alpha}\frac{\partial y^\alpha}{\partial x^i} \tag{47}$$

or

$$N_{ij} = g_{ij} + g_{\alpha\beta}c^\alpha_i c^\beta_j + I_{i\alpha}c^\alpha_i + I_{j\alpha}c^\beta_j \tag{48}$$

We have to mention that in contrast to $y^{\alpha'} = \lambda\delta^{\alpha'}_{\varrho}x^{\varrho}$ for $\alpha' = 1,2,3$ embedding transformations that we have studied earlier, we have terms generated by I_{ij}.

- $i = 0$ and $j = 1, 2, 3$ we have

$$N_{0j} = g_{0j} + g_{\alpha\beta}\Big(\frac{\partial y_0^{\alpha}}{\partial x^0} + \frac{\partial c_{\gamma}^{\alpha}}{\partial x^0}(x^{\gamma} - x_0^{\gamma})\Big)c_j^{\beta} + g_{0\beta}\frac{\partial y^0}{\partial x^0}c_j^{\beta} + I_{0\alpha}c_j^{\alpha} + ...$$

$$I_{j\alpha}\Big(\frac{\partial y_0^{\alpha}}{\partial x^0} + \frac{\partial c_{\gamma}^{\alpha}}{\partial x^0}(x^{\gamma} - x_0^{\gamma})\Big) + I_{j0}\frac{\partial y^0}{\partial x^0} \quad (49)$$

and if $x^{\gamma} \longrightarrow x_0^{\gamma}$ the above equation takes the simpler form

$$N_{0j} = g_{0j} + g_{\alpha\beta}\frac{\partial y_0^{\alpha}}{\partial x^0}c_j^{\beta} + g_{0\beta}\frac{\partial y^0}{\partial x^0}c_j^{\beta} + I_{0\alpha}c_j^{\alpha} + I_{j\alpha}\frac{\partial y_0^{\alpha}}{\partial x^0} + I_{j0}\frac{\partial y^0}{\partial x^0} \quad (50)$$

where in this equation we have time dependence for all the terms in contrast to the previous case $y^{\alpha'} = \lambda\delta_{\varrho}^{\alpha'}x^{\varrho}$ for $\alpha' = 1, 2, 3$ embedding transformations that we have studied earlier. This way even the scale for g_{ij} is time depended.

- finally the case $i, j = 0$ leads to

$$N_{00} = g_{00} + g_{\alpha\beta}\frac{\partial y^{\alpha}}{\partial x^0}\frac{\partial y^{\beta}}{\partial x^0} + 2g_{\alpha 0}\frac{\partial y^{\alpha}}{\partial x^0}\frac{\partial y^0}{\partial x^0} - 2g_{00}\Big(\frac{\partial y^0}{\partial x^0}\Big)^2 + 2I_{0\alpha} \quad (51)$$

if $x^{\gamma} \longrightarrow x_0^{\gamma}$ this equation take the form

$$N_{00} = g_{00}+g_{\alpha\beta}\frac{\partial y_0^{\alpha}}{\partial x^0}\frac{\partial y_0^{\beta}}{\partial x^0}+2g_{\alpha 0}\frac{\partial y_0^{\alpha}}{\partial x^0}\frac{\partial y_0^0}{\partial x^0}+2g_{00}\Big(\frac{\partial y_0^0}{\partial x^0}\Big)^2+2I_{0\alpha}\frac{\partial y_0^{\alpha}}{\partial x^0} \quad (52)$$

we can see that again the term $E_{00} = 2g_{00}\big(\frac{\partial y^0}{\partial x^0}\big)^2$ unchanged from the previous case $y^{\alpha'} = \lambda\delta_{\varrho}^{\alpha'}x^{\varrho}$ for $\alpha' = 1, 2, 3$. The last term splits into three scales for the $\alpha = 1, 2, 3$ where this term, as we have mentioned, expresses the "unified generalised electromagnetism". This split is exactly why we have called it this way. It would be formidable if we could interprate (in a first approach) this term as electromagnetism, weak nuclear field and strong unified nuclear in a unified pure geometrical way. Moreover, the two first terms that are gravity and ordinary mass related, splits into three scale where each one them splits into three sub-scales. The third term involves three energy scale splitting as the last term does, too. These energy scales will help us to enter in the area of particle physics. For $a = 0$, we have a uniform scale involving all the terms and is the same with the previous case. The case for $a = 0$, can serce us, as a base scale, which can be seen, as the vacuum state. Moreover, the spliting of the scales for $i = 1, 2, 3$, c an serve us to form different subscales, that could be connected with the mass hierarchy problem abd as well, the existing number of families in Nature. Moreover, we must say that before the embedding, C^4 space had an original symmetry (as we shall see in section (18)) which after the embedding has broken into several symmetries. This is exactly what we call in standard model and Higg's mechanism, spontaneous symmetry breaking. Of course, it is not spontaneous at all! There is a cause, the difference between how a 8-d observer and a 4-d one, observes Cosmos. The symmetry that is connected to our usual g_{ij} tensor is what we used to call external symmetries, while all the others , involving the g_{ij} connected with y_i and the I_{ij} involving both x_i andy_i,are what we use to call "internal". These symmetries, will be fyrher distinguished to global and local. But all these things will be extensively studied in the third paper of this series. Another comment for this paragraph is that the final case should be better be studied, involving not two but three parts, taking in account these way a term that is totally nonlinear and these non-linearity is that accompanies non-abelian theories.

6 Interpretation of the coordinates

The introduction of a C^4 as an extended space-time, automatically leads to the question, what is the physical interpretation of the coordinates of this space. We must admit that we have used more dimensions than four, but we do not wish to treat them as strings theories do. We want to connect the extra dimensions with already existing physical variables. Let us consider an element of C^4 space as

$$z^i = (z^0, z^1, z^2, z^3) = x^i + iy^i = (x^0, x^1, x^2, x^3) + i(y^0, y^1, y^2, y^3) \qquad (53)$$

As we have mentioned, x_i, y_i must be of the same type which means that x_0 and y_0 are both time-like while x_1, x_2, x_3 and y_1, y_2, y_3 are space-like. If x_1, x_2, x_3 are our usual length, width and height, time can be x_0 or even y_0. In the case that time is y_0 we could define an imaginary time! But before messing with times, it is wiser to see what happens with y_1, y_2, y_3. Let us consider an elementary particle, in order to describe it, we must introduce a lot of information concerning its basic characteristics such as mass value, charge , spin weak isospin, colour, flavour and what ever else is still hidden. All these characteristics are not well defined, but rather ad-hoc properties that came by logic, observation and inspiration. Now, if we go back to the geodesic equation of the first embedding functions, there is a term as

$$(1-\lambda^2)\big(g_{kj}\frac{du^j}{ds} + \Gamma_{k,ij}u^iu^j\big) \qquad (54)$$

and another term as

$$F_{ij}u^j\frac{dy^0}{ds} \qquad (55)$$

We can observe that $(1-\lambda^2)$ stands exactly at the point that a mass term should be and that $\frac{dy^0}{ds}$ where charge q should be (see also Eq. (173)). These terms appeared as an echo of the information that we lost through the embedding, or just the pay back of y^i. This way, we can say that we have a sort of geometrisation for mass (from the g_{ij} part) and geometrisation of "charges" (from the I_{ij} part). This geometrisation will reflect to the equivalence principle. Specifically, before embedding, we have a C^4 or a symplectic R^8 space-time. Let us consider the case that I_{ij} vanishes. Then, there is an equivalence between velocities and accelerations of the two projection spaces $X \simeq R^4$ and

$Y \simeq R^4$. But, space Y will reflect after the embedding to the definition of inertial mass, which finally in sections (14), (15) will give us the equivalence principle, as a consequence. Let us now generalise the picture, we will use the the 3-d space that is defined by y^i in order to define geometrically the characteristics that elementary particles have. We like to call y^i as mass-like vectors (in section (19), we can see the connection of y_i with mass eigenstates and that is the reason we called them mass-like) and the space that they are define as mass space. So, if y^i are mass-like, we need a physical quantity that is mass linked. In general relativity exists such a quantity the Schwarzschild radius r_g.

$$r_g = 2\frac{G}{c^2}m \longrightarrow r_g\frac{c^2}{G} = 2m \tag{56}$$

where m is the mass of a body. Every physical entity has a Schwarzschild radius . For instance for the Sun $r_g = 2,95 \times 10^3$, for Earth $r_g = 8,87 \times 10^{-3}$ and for an electron $r_g = 1,353 \times 10^{-57}$. The study of a massive object through Schwarzschild radius or its mass is equivalent. Thus, it is worth to try relate the geometrical space Y with the mass property. To this end let us write $y_i = r_i$

$$\parallel r_i \parallel = \sqrt{r_1^2 + r_2^2 + r_3^2} = \frac{1}{4}r_g^2 = \frac{G^2}{c^4}m^2 \tag{57}$$

leading to a mass-related vector

$$(r_1, r_2, r_3) = \frac{G}{c^2}(m_1, m_2, m_3) \tag{58}$$

where

$$\parallel m \parallel = m = \frac{1}{4}\frac{G}{c^2}r_g \tag{59}$$

Re-expressing r_i in spherical coordinates we get :

$$(r_1, r_2, r_3) \longrightarrow (r_g, \Theta, \Phi) = (\frac{G}{c^2}m, \Theta, \Phi) \tag{60}$$

where the angles Θ , Φ are related to mass states and therefore could be linked in the future to PMNS, CKM matrices in the context of a field theoretical description, combined with the scales of the previous paragraph. A vector in R^8 can be written as

$$\vec{k} = (x_1, x_2, x_3, ct, \frac{G}{c^2}m_1, \frac{G}{c^2}m_2, \frac{G}{c^2}m_3, T) \tag{61}$$

and setting G=c=1

$$\vec{k} = (x_1, x_2, x_3, t, m_1, m_2, m_3, T) \quad (62)$$

or even in C^4

$$\vec{k} = (x_1, x_2, x_3, t) + i(m_1, m_2, m_3, T) \quad (63)$$

At this part, in order to keep contact with the standard notation we perform a weak rotation in (t,T) subspace writing the metric as

$$dk^2 = dx_1^2 + dx_2^2 + dx_3^2 + dT^2 - dm_1^2 - dm_2^2 - dm_3^2 - dt^2 \quad (64)$$

giving a signature of (4,4). Writing Eq. (64) without the $d\vec{m}$ term we have:

$$dk^2 = d\vec{x}^2 + dT^2 - c^2 dt^2 \quad (65)$$

which looks like the De-Sitter metric and models the De-Sitter's Universe in vacuum without mass. This way, the peculiar situation where we have two qualitatively different observers, one travelling in space and another travelling under the cosmic expansion, attains a simple interpretation. Let us add here that two-time approaches became recently very popular in the context of string or M-theory [2] [4] [5] [6] [7]. But we have to note that two times physics also means as we have seen a complex time, which is after all the basis of our consideration. This approach gives us many advantages, but it totally alters the way that we must look, understand and approach physically and philosophically Cosmos. Already, S. Hawking had refereed to this subject many times. If a complex time exists, Cosmos is much more different than we have thought. Our usual image, as 4-d observers (this is where we have written our usual theories) is that Cosmos looks like a giant "ring bell". But if time is complex, Cosmos will be actually a "sphere" inside the C^4 space. If such a hypothesis holds, we were driven to another paradox, comparable with the one of Ptolemy. It is very different what things seem to be, to what things actually are. Many times in our history senses have tricked us. Moreover, a singularity problem in the C^4 space, will have totally different meaning and require different approach, compared to a singularity problem in our usual 4-d space-time.

7 Special relativity in R^8

Let us now start working on the flat metric with signature (4,4)

$$ds^2 = d\vec{r}^2 + dT^2 - f^2 d\vec{m}^2 - c^2 dt^2 \quad (66)$$

where $f = \frac{G}{c^2}$. Our next step is to formulate the associated "special relativity" in R^8, compatible with all the above mentioned considerations. The first step is to write an action S.

$$S = \int_t L dt \tag{67}$$

and try to obtain a link to Einstein's special relativity action. To this end we apply the transformation $T \longleftrightarrow it$. Then

$$ds^2 = \left(\frac{1}{c^2}\left(\frac{d\vec{r}}{dt}\right)^2 + \frac{1}{c^2}\left(\frac{dT}{dt}\right)^2 - \frac{1}{c^2}\left(\frac{d(f\vec{m})}{dt}\right)^2 - 1\right)c^2dt^2 \tag{68}$$

Introducing the notation

$$\vec{u} = \frac{d\vec{r}}{dt}, v = \frac{dT}{dt}, \vec{w} = \frac{d(f\vec{m})}{dt} \tag{69}$$

for the derivatives, the metric becomes

$$ds^2 = \left(\frac{u^2}{c^2} + \frac{v^2}{c^2} - \frac{w^2}{c^2} - 1\right)c^2dt^2 \tag{70}$$

then the Lagrangian of a free point-particle is written

$$S = -\int_t Dc\sqrt{1 - \frac{u^2}{c^2} + \frac{w^2}{c^2} - \frac{v^2}{c^2}}dt \tag{71}$$

where the constant D has dimensions of momentum. The canonical momenta are

$$p_u = \frac{\partial L}{\partial u} = -\frac{D^2u}{L} \tag{72}$$

$$p_w = \frac{\partial L}{\partial w} = \frac{D^2w}{L} \tag{73}$$

$$p_v = \frac{\partial L}{\partial v} = -\frac{D^2v}{L} \tag{74}$$

while the Hamiltonian H is

$$H = p_u u + p_v v + p + w - L = -\frac{D^2c^2}{L} \tag{75}$$

leading to

$$H = \frac{Dc}{\sqrt{1 - \frac{u^2}{c^2} + \frac{w^2}{c^2} - \frac{v^2}{c^2}}} \tag{76}$$

We can make the following observations concerning this Hamiltonian

1. If $\frac{w^2}{c^2} - \frac{v^2}{c^2} = 0 \longrightarrow f dm = dt \longrightarrow dm = \frac{1}{f} dT \longrightarrow m = \frac{1}{f} T + b$
   where m is the magnitude $m = |\vec{m}|$ and b is a constant. We can also write:

$$\int_{m_o}^{m} dm = \frac{1}{f} \int_{T_o}^{T} dT \longrightarrow m - m_o = \frac{1}{f}(t - t_o) \tag{77}$$

2. If $\vec{u} = \vec{w} = 0$ then $\frac{d\vec{m}}{dt} = 0 \longrightarrow m = m_o$

3. If $\frac{w^2}{c^2} - \frac{v^2}{c^2} = 0$ or $\vec{u} = \vec{w} = 0$ holds, the Hamiltonian coincides with the usual Hamiltonian of Einstein's special relativity for $D = m_o c$. The only free parameters are m_o and c

4. We have to give an interpretation to the velocity $\vec{w} = f\frac{\vec{m}}{dt}$. Let us write again the metric

$$dk^2 = d\vec{r}^2 + dT^2 - f^2 d\vec{m}^2 - c^2 dt^2 \tag{78}$$

Rotating in the (t,T) plane we get:

$$dk^2 = d\vec{r}^2 - c^2 dt^2 - f^2 d\vec{m}^2 + dT^2 \tag{79}$$

Since the light speed is constant, $d\vec{r}^2 - c^2 dt^2$ is an invariant quantity. For $dT^2 - f^2 d\vec{m}^2$ a similar invariant quantity should occur

$$dT^2 - f^2 d\vec{m}^2 = \left(1 - f^2 \frac{dm^2}{dT^2}\right) dT^2 = f^2 \left(\frac{1}{f^2} dT^2 - d\vec{m}^2\right) \tag{80}$$

The equation $\left(\frac{T}{f}\right)^2 - (m_1^2 + m_2^2 + m_3^2) = 0$ defines a cone (not a light-cone) in space $M^{3,4}$ (we refer to space Y as mass space M). Setting $m = |\vec{m}| = \sqrt{m_1^2 + m_2^2 + m_3^2}$ then
$\frac{m}{T} = \frac{1}{f}$. From the relation $\frac{1}{f^2} dT^2 - d\vec{m}^2 = 0$ we have $\frac{dm}{dt} = \frac{1}{f}$ where the quantity $\frac{m}{T}$ is a linear density. If T is the "Cosmos"(Universe) radius, then we get that this linear density (Cosmos' linear density) is an invariant. The above consideration holds on the cone. Consequently

$$d\vec{m} = \frac{1}{f}dT \longrightarrow m = \frac{1}{f}T + m_o \longrightarrow T = f(m - m_o) \tag{81}$$

then

$$d\vec{m} = \frac{1}{f}dT \longrightarrow \frac{\vec{m}}{dt} = \frac{1}{f}\frac{dT}{dt} \longrightarrow f\frac{\vec{m}}{dt} = \frac{dT}{dt} \longrightarrow \vec{u} = \vec{w} \tag{82}$$

which also holds on the cone. Then

$$H = \frac{Dc}{\sqrt{1 - \frac{u^2}{c^2} + \frac{w^2}{c^2} - \frac{v^2}{c^2}}} = \frac{Dc}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{83}$$

on the cone of space $M^{3,4}$. As a result, the equation $H = \frac{Dc}{\sqrt{1 - \frac{u^2}{c^2}}}$ is valid only on the cone of space $M^{3,4}$ or Einstein's special relativity is valid only on the cone of $M^{3,4}$. This way, we obtain a generalisation of Einstein's special relativity. This generalised picture gives us of course Einstein's special relativity plus information about matter and Cosmos' radius.

Axiom(Invariance principle): The linear density of Cosmos $\frac{d\vec{M}}{dT}$ (M is the mass of Cosmos) is constant and independent from observers in M^4. The quantity $\left(\frac{T}{f}\right)^2 - (m_1^2 + m_2^2 + m_3^2)$ is an invariance of space $M^{3,4}$ or in differential form the metric $ds_M^2 = dT^2 - f^2 d\vec{m}^2$ is invariant. Moreover $ds_R^2 = d\vec{r}^2 - c^2 dt^2$ is invariant in space $R^{3,4}$. Since the variables are not mixed (flat space) the total length $ds^2 = ds_R^2 + ds_M^2$ is invariant, as well. Then, ds^2 must be invariant for all observers in R^8.

Theorem: For any quadratic form in R^n there is a group of linear transformations of space R^n that leave the associated quadratic form invariant. In the case of R^8 this group is $SO(4,4)$ or $SO(3+1,1+3)$. The linear transformations of this group are the transformations that the observers of R^8 must use in order to communicate with each other so the quadratic form will remain unchanged. This way, the "pseudo-distance" between two different points of R^8 must be the same for all observers of R^8.

Now we must "evaluate" the constant f. We have already mentioned that f is $\frac{G}{c^2}$ and we have to figure out the consistency of this choice. Let us consider two different states of Cosmos. The first state is when Cosmos was in Planck state while the second is "now". In the first one, Cosmos is considered as the theoretical Planck particle with mass m_P and length-radius l_P. Then $\frac{M}{T} = \frac{m_P}{l_P} = \frac{G}{c^2}$. In the second one Cosmos is considered to have a mass $10^{52}kgr$ and radius 10^{26} then $\frac{T}{M} \simeq \frac{10^{26}}{10^{52}} \simeq 10^{-26} \simeq \frac{G}{c^2}$. As a conclusion, these two different and far apart states lead to $f = \frac{G}{c^2}$. Of course all the above statements are valid and applicable to a Cosmos that is flat and looks as a De-Sitter Cosmos. Note that this way the coordinates of M^3 are expressed as $\frac{G}{c^2}m$, which is the Schwarzschild's radius and must be interpreted with care! We must also say that f is a global invariance and all the above results holds in R^8, while M and T are quantities concerning Cosmos. Thus, T=constant defines hyper surfaces of R^8. Additionally m_0 ($m_0 \in R$) describes a mass moving in the usual space-time originating from the sub-space M^3. Different subspaces of R^7 express different m_1, m_2, ... that move inside different subspaces of the usual space-time, forming different "cosmic lines" for different masses m_i, which are connected through usual Lorentz transformations. As a conclusion, we have a local invariance, which is realized through the invariance of c and m_o. This picture extends Einstein's special relativity.

5. We considered what happens in the signature (3+1, 1+3) where we saw the existence of two cones. Trying a similar analysis for the signature (4,4) the (-) sign between the spaces M^4, R^4 will lead to three different "leave-spaces" which are separated since $S0(4,4)$ is not simply connected. We do not have cones of the type we are familiar with. For instance, if we are in $R^{1,3}$ we descend one dimension and we can find the cone as a hyper surface in $R^{1,3}$ ($c^2t^2 = x^2 + y^2 + z^2$). In our case, we have two spaces and we have to descend not one dimension but a whole dimensional space ($x_1^2 + x_2^2 + x_3^2 + T^2 = m_1^2 + m_2^2 + m_3^2 + c^2t^2$). We have to descend from R^8 to R^4 or M^4. This way, we have a "cone" like structure that cannot be handled as usual. We cannot formulate a "velocity" in order to proceed as we know. However, there is an alternative way through Casimir's and Pauli-Lubanki's invariants from which we can extract the existing invariance principle. If p_μ, $\mu = 1, 2, .., 8$ is the pure momentum vector then the expression $p_\mu p^\mu$ is an invariant

$$(p_R, p_M)(p^R, p^M) = p_R p^R - p_M p^M = -D^2 \tag{84}$$

where D has units of momentum $[kgr\frac{m}{sec}] = [m\frac{kgr}{sec}]$. A mass m that moves in the space R^4 is described by vectors of the type $(\vec{r}, t)$ and velocities that have the general form $\vec{u} = \frac{1}{c^2}\frac{d\vec{r}}{dt}$ where c is an invariant . A "length" l that moves in the space M^4 is described by vectors of the type $(\vec{m}, t)$ and velocities that have the general form $\vec{w} = \frac{G}{c^3}\frac{d\vec{m}}{dt}$ where $\frac{c^3}{G}$ has dimensions $[\frac{kgr}{sec}]$ being an invariant, too. Of course this two evolutions must be equivalent for consistency reasons. Let us discuss what does a local observer in R^4 and M^4 experiences. Let us represent local observers of usual space as (SO) and local observers of "mass" space as (MO). An (SO) observes a Cosmos with diameter $\simeq 10^{52}$ m and he needs $\simeq 10^{18}$ sec to fully trespass it with velocity c. On the other hand, (MO) observes a Cosmos with diameter $\simeq 10^{53}$ kgr and he needs 10^{18} sec to fully trespass it with velocity c^3/G. So the trespass time is the same for the two observers. This situation is more correct in Planck's picture. What does a velocity of $[\frac{kgr}{sec}]$ means? Unfortunately we are used to think velocity in $[\frac{m}{sec}]$ and a $[\frac{kgr}{sec}]$ "velocity" seems irrational. In order to understand the differences between the two velocities let us consider the following case. Let us imagine two (SO) observers in the space of Milky way and Andromeda ($2.5 \cdot 10^6$ light years distance) respectively. In order to communicate they must sent a signal. If this signal travels with velocity c it will need $2.5 \cdot 10^6$ years to trespass this distance. On the other hand, this space is almost empty (one hydrogen atom per cubic meter or mass of 1 kgr distributed in this area). Two (MO) observers can communicate in 10^{-34} sec by sending signals with $\frac{c^3}{G}$ velocity. An (MO) signal can travel between galaxies extremely "fast", almost instantaneously. Although all observers (MO, SO) need the same time to trespass all Cosmos, the time needed to trespass local structures in Cosmos may vary tremendously between the two different kinds of observers, due to the difference between how masses and the distances between them are distributed in Cosmos. We have huge concentrations of mass in small areas and small concentrations in huge areas. Thus, specific information travelling with velocity c^3/G could lead to correlations during the Planck period which may explain the horizon and isotropy problems.

6. The elementary length leads us two three possible cases, the first one is $ds^2 > 0$, the second one is $ds^2 < 0$ and the third one $ds^2 = 0$. The question is what these three cases will represent if we apply not for the flat metric tensor but for a spherical symmetrical metric tensor, in the same spirit as we apply in the usual context of general relativity with the Schwarzschild metric which of course leads us to black holes. What must happen in order to pass from the first case $ds^2 > 0$ to $ds^2 = 0$ and afterwards to $ds^2 < 0$? What energy barrier we must overseen and is it possible? Can this energy scale that is required in order to make the passages, linked to Chandrasekhar limit? This are some questions that is worth to investigate in the future, giving us the chance to enter into a black hole. The most certain fact is that through our consideration, black holes do not have an information paradox any more, because of the existence of C^4 space. The information that we think is lost, is there inside the Y space and then the geometry of C^4 must be taken literally, in order to enter and investigate the interior of a black hole. The embedding, provide us only with the information taken from our projection space and tell us what we can observe from here. The horizon of the black hole, seems to be this "geometric" barrier.

Now we can continue to calculate the squared Hamiltonian as :

$$H^2 = \frac{D^2c^2}{1 - \frac{u^2}{c^2} + \frac{w^2}{c^2} - \frac{v^2}{c^2}} = D^2c^2\Big(1 + \frac{u^2}{c^2 - u^2 + w^2 - v^2}\Big) \tag{85}$$

or after some calculus

$$H^2 = D^2c^2\Big(1 + \frac{u^2}{c^2 - u^2 + w^2 - v^2} - \frac{w^2}{c^2 - u^2 + w^2 - v^2} + \frac{v^2}{c^2 - u^2 + w^2 - v^2}\Big) \tag{86}$$

while conjugate momenta are

$$p_u^2 = \frac{D^2u^2}{c^2 - u^2 + w^2 - v^2} \tag{87}$$

$$p_w^2 = \frac{D^2w^2}{c^2 - u^2 + w^2 - v^2} \tag{88}$$

$$p_v^2 = \frac{D^2 v^2}{c^2 - u^2 + w^2 - v^2} \tag{89}$$

As a result the squared Hamiltonian can be written

$$H^2 = D^2c^2 + p_u^2c^2 - p_w^2c^2 + p_v^2c^2 \tag{90}$$

or if the energy is conserved

$$E^2 = D^2c^2 + p_u^2c^2 - p_w^2c^2 + p_v^2c^2 \tag{91}$$

the first and the second terms on the right for $D = m_o c$ are the familiar terms of the Einstein's equation of energy. Moreover, we can define the 8-d vector of energy-momentum as

$$\left(p_{iu}, p_v, p_{iw}, \frac{H}{c}\right) \tag{92}$$

The energy equation can be written also as

$$p_u^2 + p_v^2 - p_w^2 - \frac{H^2}{c^2} = -D^2 \tag{93}$$

where the left side of the equation coincides with the pseudo-measure of the 8-d vector of energy-momentum.

Definition: If $(A_1, B_1), (A_2, B_2)$ two 8-d vectors we define as the pseudo-internal product

$$(A_1, B_1) \bullet (A_2, B_2) = A_1A_2 - B_1B_2 \tag{94}$$

where A_1, A_2, B_1, B_2 are 4-D vectors and A_1A_2, B_1B_2 Euclidean internal products. Then the pseudo-measure of an 8-d vector is

$$(A, B)^2 = A^2 - B^2 \tag{95}$$

where A^2, B^2 Euclidean measures.

As a conclusion, the square of the 8-d vector of the 8-d momentum is constant. If we use the action S we can write

$$p_{iu} = \frac{\partial S}{\partial x^i}, \quad p_{iw} = \frac{\partial S}{\partial y^i}, \quad p_{iv} = \frac{\partial S}{\partial T}, \quad H = -\frac{\partial S}{\partial t} \tag{96}$$

leading to the the Hamilton-Jacobi equation

$$\left(\frac{\partial S}{\partial x^i}\right)^2 + \left(\frac{\partial S}{\partial T}\right)^2 - \frac{1}{C^2}\left(\frac{\partial S}{\partial m^i}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 + D^2 = 0 \tag{97}$$

if we set $y^i = Cm^i$.

8 Angular -momentum

If $a = (a_i)$, $b = (b_i)$ are two n-dimensional vectors then the exterior product $a \times b = \tau_{ij}$ is a second rank antisymmetric tensor with dimension 6. We can write this tensor as

$$\tau_{ij} = a_i b_j - a_j b_i \tag{98}$$

$$\tau_{ij} = 0 \tag{99}$$

$$\tau_{ij} = -\tau_{ji} \tag{100}$$

In the space $K = R^8 \equiv C^4$ or $K = R^4 + iM^4$ the vectors have the form

$$k = (\vec{r}, T, \vec{m}, t) \equiv \vec{r} + i\vec{m} + T + it = (\vec{r} + T) + i(\vec{m} + t) \tag{101}$$

If we keep only the "length-mass" part then we can define the total angular-momentum in K as

$$L = \vec{k} \times \vec{p}_k \tag{102}$$

where $\vec{p}_k = \left(p_{iu}, p_v, p_{iw}, \frac{H}{c}\right)$

This tensor $L = (L_{ij})$ has $\frac{n(n+1)}{2} = \frac{6 \times 5}{2} = 15$ components and can be written as a matrix

$$L_{ij} = \begin{pmatrix} 0 & l_{12} & l_{13} & l_{14} & l_{15} & l_{16} \\ -l_{12} & 0 & l_{23} & l_{24} & l_{25} & l_{26} \\ -l_{13} & -l_{23} & 0 & l_{34} & l_{35} & l_{36} \\ -l_{14} & -l_{24} & -l_{34} & 0 & l_{45} & l_{46} \\ -l_{15} & -l_{25} & -l_{35} & -l_{45} & 0 & l_{56} \\ -l_{16} & -l_{26} & -l_{36} & -l_{46} & -l_{56} & 0 \end{pmatrix}$$

or

$$L_{ij} = \begin{pmatrix} L_R & L_{RM} \\ -L_{RM}^T & L_M \end{pmatrix}$$

where L_R is our usual angular-momentum tensor in R^3, the L_M is the angular-momentum in M^3 and the L_{RM} is the mixture between them. The L_M can be interpreted as classical spin while the mixed L_{RM} as the interaction between angular-momentum and classical spin the same way that in quantum physics we have the spin-orbit coupling.

9 Poincare group

Before constructing the Poincare group in R^8 let us recall its structure as it appears in Minkowskian R^4 space-time. It consists of translations (P), rotations (J) and boosts (K). Specifically we have

1. translations (displacements) in time and space (P) which form the Abelian Lie group of translations in spacetime
2. rotations (J) in space which form the non Abelian Lie group of three dimensional rotations
3. boosts (K) which are transformations that connect two uniformly moving bodies

The symmetries J, K consist the homogeneous Lorentz group, while the semi-direct product of P and the Lorentz group, form the inhomogeneous Lorentz group or just the Poincare group. The Poincare group is a ten dimensional non-compact Lie group and actually is isometric to the group of Minkowski spacetime. We can write

$$\text{Poincare group} \cong ISO(3) \cong R^{1,3} \times SO(1,3) \tag{103}$$

where $SO(1,3)$ is the homogeneous Lorentz group and $ISO(1,3)$ the inhomogeneous one. Moreover if we set $J_i = -\varepsilon_{imn}\dfrac{M^{mn}}{2}$ and $K_i = M_{io}$

1. $[P_\mu, P_\nu] = 0$
2. $\frac{1}{i}[M_{\mu\nu}, P_\mu] = n_{\mu\rho}P_\nu - m_{\nu\rho}P_\mu$
3. $\frac{1}{i}[M_{\mu\nu}, P_{\rho\sigma}] = n_{\mu\rho}M_{\nu\sigma} - n_{\mu\sigma}M_{\nu\rho} - n_{\nu\rho}M_{\mu\sigma} + n_{\nu\sigma}M_{\mu\rho}$

where P is the generator of translations, M the generator of Lorentz transformations. The third relation is the homogeneous Lorentz group. Let us now form the Poincare group in the 8 dimensional space with signature (4.4). First of all we need to set our notation. We have two different indices with small letters $i, j = 0, 1, 2, 3$ and capital letters $I, J = R, M$ indicating the space in which we refer (using R for the usual length space and M for the mass space). From the Lagrangian we can observe that we have Galilean transformations for R^4, Galilean transformations for M^4 and Lorentzian transformations between R^4, M^4. In the case $(3+1, 1+3) \cong (4, 4)$ from the Lagrangian we have Lorentzian transformations in R^4, Lorentzian transformations in M^4 and Galilean ones between R^4, M^4. We find

1. $[P_I\mu, P_J\nu] = 0$
2. $\frac{1}{i}[M_{I\mu\nu}, P_{J\mu}] = \delta_{IJ}(n_{I\mu\rho}P_{I\nu} - m_{J\nu\rho}P_{J\mu})$
3. $\frac{1}{i}[M_{IJ\mu\nu}, P_{RS\rho\sigma}] = n_{MR\mu\rho}M_{NS\nu\sigma} - n_{MS\mu\sigma}M_{NR\nu\rho} - n_{NR\nu\rho}M_{MS\mu\sigma} + n_{NS\nu\sigma}M_{MR\mu\rho}$

where $M_{II} = M_I$, $M_{JJ} = M_J$, $P + II = P_I$, $P_{JJ} = P_J$ and δ_{IJ} is one for $I = J$ and zero for $I \neq J$. The flat metrics are for the cases:

1. $\text{sgn}(n_{I\mu\nu}) = (1, 1, 1, -1)$ and $\text{sgn}(n_{J\mu\nu}) = (-1, -1, -1, 1)$
2. $\text{sgn}(n_{I\mu\nu}) = (1, 1, 1, 1)$ and $\text{sgn}(n_{J\mu\nu}) = (-1, -1, -1, -1)$

The complete structure of the Poincare group can be found in Appendix A. Furthermore, in our usual space-time the Killing's vectors of Minkowski space-time have general solution $\xi_\mu = c_\mu + b_{\mu\gamma}x^\gamma$ where c_μ, $b_{\mu\gamma}$ are constants. The Minkowski's metric tensor has 10 unique components due to his symmetrical form. As a conclusion, it has ten linearly independent Killing vectors fields which corresponds to the 10 generators of the Poincare algebra. In the same spirit, in our case, the 8 dimensional real space, the flat metric N_{ij} is symmetric and has 36 unique components. Respectively, the 8 dimensional real space has 35 linearly independent Killing vectors which will correspond to the generators of the Poincare group, as it listed above. The Poincare group of the 8 dimensional space equipped with the metric tensor N_{ij} with signature $(4,4)$ is represented by 36 generators. Especially, we have 6 generators from the R^3 part, 6 generators from the M^3 part and $2\times2\times4 = 16$ generators from the $R^3 \times M^3$ (mixed components) and 8 generators determined by the dimension. The Poincare group can be written as

$$\text{Poincare group} \cong ISO(4,4) \cong R^{4,4} \times SO(4,4) \tag{104}$$

The group $SO(4,4)$ has $\dfrac{7\times 8}{2} = 28$ generators plus 8 generators from the $R^{4,4}$ (displacements). There is a connection of the algebra of those 36 generators of the Poincare group, to the algebra of the groups $U(6)$ (has 36 generators) or $Sp(4)$ ($n(2n+1)$ generators, for $n = 4$ we have 36 generators). Both $U(6)$ and $Sp(4)$ are compact Lie group and it would be interesting to match the $ISO(4,4)$ algebra to an algebra of a compact simply connected group.

10 Tensor calculus in C^4

If G_{ij} the Hermitian metric tensor and v^i a contra-variant complex vector, the covariant complex vector is defined as $v_j = G_{ij}v^i$ and the following relations holds

$$v^j = G^{ij}v_i = G^{ij}G_{pi}v^p = G_{pi}G^{ij}v^p = \delta^j_p v^p = v^j \tag{105}$$

$$v_j = G_{ij}v^i = G_{ij}G^{pi}v_p = G^{pi}G_{ij}v_p = \delta^p_j v_p = v_j \tag{106}$$

and the measure is

$$\mid\mid v \mid\mid^2 = G_{ij}v^i\overline{v}^j = v_j\overline{v}^j \tag{107}$$

but

$$v^j = G^{pj}v_p \longrightarrow \overline{v}^j = \overline{G^{pj}}\,\overline{v}_p = G^{jp}\overline{v}_p \tag{108}$$

so Eq. (108) can be written also as

$$\mid\mid v \mid\mid^2 = v_j\overline{v}^j = v_jG^{jp}\overline{v}_j = G^{pj}v_j\overline{v}_p \tag{109}$$

Now, let us consider in general a complex tensor A^{ij}. If we wish to lower the indice j we must use the relation $A^i_j = G_{pj}A^{ip}$ and in the same spirit $A^{ij} = G^{pj}A^i_p$ and as a consequence the following relation holds

$$A^{ij} = G^{pj}(G_{kp}A^{ik}) = G_{kp}G^{pj}A^{ik} = \delta^j_k A^{ik} = A^{ij} \tag{110}$$

We can further proceed with a complex tensor with three indices B_{ijk} where the mixed tensor can be written as $B^i_{jk} = G^{pi}B_{pjk}$ and $B_{ijk} = G_{pi}B^p_{jk}$ and moreover

$$B_{ijk} = G_{pi}B^p_{jk} = G_{pi}(G^{lp}B_{ljk}) = G^{lp}G_{pi}B_{ljk} = \delta^l_i B_{ljk} = B_{ijk} \tag{111}$$

and as a conclusion the above relations holds. We can also define the pseudo-product between the contra- variant complex vectors as

$$< u \mid v >= G_{ij}u^i\overline{v}^j \tag{112}$$

If we consider $u = \lambda_1 u_1 + \lambda_1 u_2$, then we have

$$< \lambda_1 u_1 + \lambda_1 u_2 \mid v >= G_{ij}(\lambda_1 u^i_1 + \lambda_1 u^2_2)\overline{v}^j = \lambda_1 < u_1 \mid v > +\lambda_2 < u_2 \mid v > \tag{113}$$

which tells us that it is linear with respect to the first variable. If we consider now as $v = \mu_1 u_1 + \mu_1 u_2$

$$< u \mid \mu_1 u_1 + \mu_1 u_2 >= \overline{\mu_1} < u \mid v_1 > +\overline{\mu_2} < u \mid v_2 > \tag{114}$$

which tells us that it is not linear with respect to the second variable.
Moreover, the determinant of a complex matrix exists and it is a real number.
Our next step is to define the "Christoffel symbols" in this complex geometry. We have already define in the first paper

$$\widehat{\Gamma}_{k,ij} = \Lambda_{k,ij} = \left(\Gamma^{(x)}_{k,ij} + \Delta^{(x)}_{k,ij}, -\Gamma^{(y)}_{k,ij} + \Delta^{(y)}_{k,ij}\right) \tag{115}$$

with respect to the Cauchy derivative. But we can also define the $\Lambda_{\overline{k},\overline{i},\overline{j}}$ symbols as

$$\Lambda_{\overline{k},\overline{i},\overline{j}} = \frac{1}{2}\left(\frac{\partial G_{jk}}{\partial \overline{z}^i} + \frac{\partial G_{jk}}{\partial \overline{z}^j} - \frac{\partial G_{jk}}{\partial \overline{z}^k}\right) \tag{116}$$

but we can also prove that

$$\overline{\Lambda_{\overline{k},\overline{i},\overline{j}}} = \Lambda_{k,ji} \tag{117}$$

The "Christoffel symbols" of the second kind will be respectfully

$$\Lambda_{ij}^{k} = G^{lk}\Lambda_{l,ij} \tag{118}$$

Now, we can define the associate "Riemmann- Christoffel" curvature tensor in the complex geometry as

$$Z_{ijkl} = \begin{vmatrix} \dfrac{\partial}{\partial z^k} & \dfrac{\partial}{\partial z^l} \\ \\ \Lambda_{i,jk} & \Lambda_{i,jl} \end{vmatrix} + \begin{vmatrix} \Lambda_{jk}^{b} & \Lambda_{jl}^{b} \\ \\ \Lambda_{i,jk} & \Lambda_{i,jl} \end{vmatrix}$$

After some calculation Z_{ijkl} can be written also as

$$Z_{ijkl} = \big(\overset{xx}{R}_{ijkl} - \overset{yy}{R}_{ijkl} + \overset{xy}{M}_{ijkl} + \overset{yx}{M}_{ijkl}\big) + i\big(\overset{xx}{M}_{ijkl} - \overset{yy}{M}_{ijkl} - \overset{xy}{R}_{ijkl} - \overset{yx}{R}_{ijkl}\big) \tag{119}$$

where with $R_{....}$ we symbolise the "Riemmann-Christoffel" tensor with respect to the symmetric tensor g_{ij} and with $M_{....}$ the "Riemmann-Christoffel" tensor with respect to the antisymmetric tensor I_{ij} and the symbols over shows us the kind of the partials i.e xx stands for $\dfrac{\partial}{\partial x^i}\dfrac{\partial}{\partial x^i}$. In the same spirit the mixed one is

$$Z_{jkl}^{i} = G^{pi}Z_{pjkl} = \big(\overset{xx}{R}{}_{jkl}^{i} - \overset{yy}{R}{}_{jkl}^{i} + \overset{xy}{M}{}_{jkl}^{i} + \overset{xy}{M}{}_{jkl}^{i}\big) + i\big(\overset{xx}{M}{}_{jkl}^{i} - \overset{yy}{M}{}_{jkl}^{i} + \overset{xy}{R}{}_{jkl}^{i} + \overset{yx}{R}{}_{jkl}^{i}\big) \tag{120}$$

We must keep in mind that all the tensors of type R and M in the covariant form are all real, while in the mixed form they are all complex due to the fact that we raise by the complex- Hermitian metric tensor G^{pj}. We can continue with the "Ricci tensor" $Z_{jk} = Z_{jkl}^{l} = G^{pl}Z_{pjkl}$

$$Z_{jk} = \big(\overset{xx}{R}_{jk} - \overset{yy}{R}_{jk} + \overset{xy}{M}_{jk} + \overset{yx}{M}_{jk}\big) + i\big(\overset{xx}{M}_{jk} - \overset{yy}{M}_{jk} + \overset{xY}{R}_{jk} + \overset{Yx}{R}_{jk}\big) \tag{121}$$

and finally the "Ricci scalar quantity" $Z = G^{jk}Z_{jk}$

$$Z = \big(\overset{xx}{R} - \overset{yy}{R} + \overset{xy}{M} + \overset{yx}{M}\big) + i\big(\overset{xx}{M} - \overset{yy}{M} + \overset{xy}{R} + \overset{yx}{R}\big) \tag{122}$$

all the scalar quantities R and M are complex. Furthermore, in the usual geometry of R^4 there exists the relation

$$\frac{\partial ln\sqrt{|g\,|}}{\partial x^l} = \Gamma_{il}^{i} \tag{123}$$

which takes the form for the Hermitian metric tensor G_{ij}

$$\frac{\partial ln\sqrt{|G\,|}}{\partial x^l} = G^{ji}\Lambda_{i,jl} + \Lambda_{lj}^{i} \tag{124}$$

11 Field equations in C^4

We can now proceed with the field equations in the same spirit of general relativity by an action of the form

$$S_\pi = \int Z\sqrt{G}d\Omega \tag{125}$$

where $d\Omega$ volume of C^4. The variation of the action will lead us to the equations

$$Z_{\mu\nu} - \frac{1}{2}ZG_{\mu\nu} = 0 \tag{126}$$

$$\nabla_l G^{\mu\nu} = 0 \tag{127}$$

Eq. (127) expresses the metric compatibility with respect to the Hermitian metric tensor. Moreover, Eq. (126) can be written as a pair with respect to R^8 as

$$\begin{aligned} Re\big(Z_{\mu\nu}-\frac{1}{2}ZG_{\mu\nu}\big) = 0 \\ Im\big(Z_{\mu\nu}-\frac{1}{2}ZG_{\mu\nu}\big) = 0 \end{aligned} \tag{128}$$

These equations must represent the unified geometric theory for all fields if our consideration is valid. It necessary to say that we will not need to add an energy-momentum tensor $T_{\mu\nu}$! We will create the energy-momentum tensor $T_{\mu\nu}$ by embedding R^4 in R^8 or C^4. This way we will find how the energy-momentum tensor $T_{\mu\nu}$ is formed, plus from what quantities it consists of. In order to understand, we must go back to the embedded metric tensor N_{ij} for the simplest case $y^{\alpha^{'}} = \lambda\delta^{\alpha^{'}}_{\varrho} x^{\varrho}$ for $\alpha^{'} = 1, 2, 3$ and $y^0 = y^0(x^0)$ as it was investigated in section (5)

$$N_{ij} = (1+\lambda^2)g_{ij} + \lambda D_{ij}\frac{\partial y^0}{\partial x^0} + 2E_{ij}\big(\frac{\partial y^0}{\partial x^0}\big)^2 + M_{ij}\frac{\partial y^0}{\partial x^0} \tag{129}$$

This tells us that the pair of Eq. (128) will become only one equation where all quantities will be refereed to N_{ij} instead G_{ij}. Thus, the "Ricci tensor" with respect to N_{ij} will break to pieces as

$$\overset{\text{N}}{R}_{\mu\nu} = (1+\lambda^2)^2\overset{\text{g}}{R}_{\mu\nu} + \lambda^2\kappa^2\overset{\text{D}}{R}_{\mu\nu} + 2\kappa^4\overset{\text{E}}{R}_{\mu\nu} + \kappa^2\overset{\text{M}}{R}_{\mu\nu} + ... \tag{130}$$

where we have set $\kappa = \dfrac{\partial y^0}{\partial x^0}$. It is important to observe and note the factor k^4 in front of the term $\overset{\text{E}}{R}_{\mu\nu}$ which is actually a scalar quantity (only g_{44} exists) and connected to dark energy term. The field equations will be then

$$\overset{\mathrm{N}}{R}_{\mu\nu} - \frac{1}{2}\overset{\mathrm{N}}{R}N_{\mu\nu} = 0 \tag{131}$$

Finally, the desired form of our usual general relativity will be

$$\overset{\mathrm{g}}{R}_{\mu\nu} - \frac{1}{2}\overset{\mathrm{g}}{R}g_{\mu\nu} = T_{\mu\nu} \tag{132}$$

where $T_{\mu\nu}$ consists of all the other parts that are left from Eq. (131) except the terms of the first part $\overset{\mathrm{g}}{R}_{\mu\nu} - \frac{1}{2}\overset{\mathrm{g}}{R}g_{\mu\nu}$ of Eq. (132). As a result we suggest that all the terms consist of the $T_{\mu\nu}$ have the form

2 indices "curvature tensor" − scalar tensor × "metric tensor"

This will be more simple if we remember what happens in the usual context of electromagnetism. Particularly, the $T_{\mu\nu}$ of electromagnetism is

$$T_{ik} = \frac{1}{4\pi}\big(- F_{il}F^{l}_{k} + \frac{1}{4}F_{lm}F^{lm}g_{ik}\big) \tag{133}$$

which follows the above mentioned scheme and must be compared to the term formulated by M. The same behaviour happens with the case of perfect fluid. The big difference is that in our consideration the variation is always with respect to a metric tensor, in contrast with the usual variation of electromagnetism which is with respect to the "field" A_μ. But the context of this different pictures must be finally equivalent, as we have seen in previous section. Morover, our usual $T_{\mu\nu}$ energy momontum tensor will be constructed not only by g_{ij} metric tensor but from the embedding functions as well. Different embedding functions will create a family of functions that has to be compatible with the pair Eq. (127)(127). If we write the "Ricci tensor"with respect to I_{ij}, we have

$$\overset{\mathrm{I}}{R}_{ij} = \begin{vmatrix} \dfrac{\partial}{\partial z^l} & \dfrac{\partial}{\partial z^j} \\ \Delta^m_{ij} & \Delta^l_{lm} \end{vmatrix} + \begin{vmatrix} \Delta^m_{ij} & \Delta^m_{il} \\ \Delta^l_{jm} & \Delta^l_{lm} \end{vmatrix}$$

the second determinant contains terms of the form $\Delta^m_{ij}\Delta^l_{lm} - \Delta^m_{il}\Delta^l_{lm}$. But, we have to remember that Δ symbols where connected with the K_{ij} which is finally in the sub case (and after embedded) our usual F_{ij}. This way, the first determinant represents (after embedded) our usual currents! If we repeat this step for the g_{ij} part, we can form a 2 tensor similar to K_{ij}, using now the Γ symbols (let us symbolise it B_{ij}), breaking this way $\overset{\mathrm{g}}{R}_{ij}$ into products of B_{ij} as we do for electromagnetism. Specifically, after we embedded, we can have a gravitation field tensor Gr_{ij}, a dark field tensor Dm_{ij} and a dark energy field tensor (actually a scalar) DE_{ij}. This way we can form Lagrangians containing terms of the form

$$L = \int \Omega_{ij}\Omega^{ij} \tag{134}$$

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where Ω_{ij} is the unified field tensor with respect to the Hermitian metric tensor G_{ij} which will break after embedded to terms as

$$Gr_{ij}Gr^{ij}, Dm_{ij}Dm^{ij}, DE_{ij}DE^{ij}, F_{ij}F^{ij}, W_{ij}W^{ij}, G_{ij}G^{ij} \tag{135}$$

where the variation must be with respect to the broken "fields" which follows the relation

$$\Omega_i = G_{ij}V^j \tag{136}$$

plus the terms associated with currents and are produced by the first determinant.

But, it is well known that in order to define the equation of general relativity we need the equation of states also. The equation of states must be derived from the volumes of R^8 or C^4 as they transformed to volumes of the embedded R^4. A theory of "primitive" thermodynamics must be formulated in R^8 or C^4, in order to fully understand the definitions of entropy and pressure as we meet them in our usual context of thermodynamics. As a final comment, we think that all the above mentioned considerations should be investigated as an expanded manifold of C^4 with respect to the dynamic parameter $\mathcal{T} = T + it$ in the literature of Poincare's conjecture by implying the Ricci flow with respect to the Hermitian tensor G_{ij} and keep up with the proof of Poincare's conjecture in Hermitian manifolds. In the next section we will continue with the fields considering actions with free end point, in order to define the associated Hamilton-Jacobi equation with respect to the unified field Ω_μ as

$$\Omega_\mu = G_{\mu\nu}V^\nu = g_{\mu\nu}V^\nu + iI_{\mu\nu}V^\nu = \mathcal{B}_\mu + i\mathcal{K}_\mu \tag{137}$$

where $\mathcal{B}_\mu$ is the unified field with respect to the symmetric tensor $g_{\mu\nu}$ and $\mathcal{K}_\mu$ the unified field with respect to the antisymmetric tensor $I_{\mu\nu}$. Eventually, we will show that our usual quantum theories are nothing else than ordinary (classic) theory, after embed our usual space-time directly in C^4. All axioms and demands of quantum theories will be just properties of this procedure!

12 Dynamic path in C^4-A new ADM treatment

In the previous , we have presented a new formulation for a unified physical theory in C^4 space. This formulation can alter many beliefs and strategies existing in the literature of physics. One of the most surprising element, is that we have distinguished two different approaches or pictures; the one that works with the metric tensor G_{ij} as a field and the one that works with $\Omega_i = G_{ij}V^j$ as a field. In the literature of classical quantum gravity the main attempt was to identify the metric tensor as a field and as a consequence we where looking for graviton from the field equations provided by general relativity. In our consideration we have identified the graviton from the geodesic equation and from the field equations provided by an action

$$S_\pi = \int Z\sqrt{g}d\Omega \tag{138}$$

we are seeking for field strengths for the various fields that have appeared, including graviton (if it exists). Specifically, we argue that the problems existing in the formulation of a classical quantum gravity as the time disappearance in the ADM treatment (in the dispersion relation and the time problem in the commutative relations needed, could be solved within our formulation. Of course, the word quantum is used as a relic of the typical language in physics, due to the fact that we argue that quantum theories are just classical theories derived from C^4 after the restriction to the usual 4-d space-time (section 14-15). The main goal of this section is to formulate a normal Hamiltonian derived from the action S_π after the release of the end-point of the integral, creating a dynamic path length. If S_γ is the action that will provide us (section 14-15) the geodesic equation, where S_γ is the action that comes from

$$ds^2 = G_{ij} dz^i d\overline{z}^j \tag{139}$$

or

$$ds^2 = g_{ij} dx^i dx^j + g_{ij} dy^i dy^j + I_{ij}(dx^i dy^j - dy^j dx^i) \tag{140}$$

and $\mathcal{S}_\gamma$ is the new action that comes from the second exremisation

$$\mathcal{S}_\gamma = \int G^{ij} \frac{\partial S}{\partial z_j} \frac{\partial S^*}{\partial \overline{z}_i} \tag{141}$$

we can make the following formulation; S_γ gives us the geodesic equation or the equation of motion, which means that the "lines" are drawn as geometry dictates. The geometry does not tell us what "it" moves on this lines, but for sure the background was produced and it is produced in a curved space. We have to remember that in S_γ the initial and end points of the integral are fixed. We have will release the end point of the integral and we will passe to a complex functional action (for the case described in section 14) or a real functional (for the case described by 15). But, throughout the procedure of releasing the end point, the geodesic equation still holds, which means that "lines" are drawn or the background is already made to receive the $\mathcal{S}_\gamma$ which is actually "moving" in this fixed background or even better, it describes the properties of what it moves. The Poisson brackets or even the associated to them Lie brackets are

$$\{z_i, z_j\} = 0 \qquad \{P_i, P_j\} = 0 \qquad \{z_i, P_j\} = \delta_{ij}$$

and tell us what happens with the geometry itself, while the Poisson brackets

$$\{\mathcal{S}_i, \mathcal{S}_j\} = 0 \qquad \{P_i, P_j\} = 0 \qquad \{\mathcal{S}_i, P_j\} = \delta_{ij}$$

or the Lie brackets

$$[\widehat{\mathcal{S}}_i, \widehat{\mathcal{S}}_j] = 0 \qquad [\widehat{\mathcal{P}}_i, \widehat{\mathcal{P}}_j] = 0 \qquad [\widehat{\mathcal{S}}_i, \widehat{\mathcal{P}}_j] = \delta_{ij}$$

tell us what happens with the fields $\mathcal{S}_\gamma = \hbar\varphi$, that moves on the "lines" where we have set $\mathcal{S}^i = \mathcal{S}(z^i) = \hbar\varphi(z^i)$. In the case that we restrict only to the usual 4-d space-time coordinates and eigenvalue the physical quantities connected with the other coordinates (as we will see in section 14 and 15) an "i" will appear in the commutative relations due to the fact that we will have only partial derivatives with respect to x_i, which when they act on the complex $\mathcal{S}_\gamma$ will give us an "i". In the same spirit, we can proceed with the action

$$S_\pi = \int Z\sqrt{G}d\Omega \tag{142}$$

which will tell us about curvatures and field strengths. The fixed points of the integral are actually, hypersurfaces surrounding 4-d complex volumes. We can also define an equivalent integral in the symplectic R^8 surrounding 8-volumes or even for the case of the embedding of usual 4-d space-time in R^8 or C^4. If we release the end point, we will be able to form a functional action $\mathcal{S}_\pi$. So if we keep fixed the initial hypersurface which is defined by G_{ij} and vary slightly the end point to a $G^{'}_{ij}$, we are making a new value for the extremisation of the integral and by repeating this procedure again and again we create a "dynamic path length" $\mathcal{S}_\pi(G_{ij})$ that connects the given geometries defined by the different Hermitian metric tensors and $\mathcal{S}_\pi(G_{ij})$ is only depended by those Hermitian metric tensors and only. The sliding on the "dynamic path" tell us how we pass from G_{ij} to another $G^{'}_{ij}$ and to another $G^{''}_{ij}$, ..., which eventually inform us how geometry changes "point" by "point" or how geometry is created and destroyed "point" by "point" or even how geometry flows throughout a "Ricci flow" mechanism in C^4 or R^8 or in the embedded R^4. The relations for $\mathcal{S}_\pi$ will be then

$$\frac{\delta\mathcal{S}_\pi}{\delta Z_{ij}} = G^{ij} \tag{143}$$

$$\frac{\delta\mathcal{S}_\pi}{\delta G^{ij}} = Z_{ij} = \Pi_{ij} \tag{144}$$

where the conjugated quantities are G_{ij} and the generalised super momentum Π_{ij} whereZ_{ij} is the Ricci curvature tensor in C^4. Eq. (145) tell us about the rate of change of the action with respect to the "field coordinates" G_{ij}. But, the existence of a complex time or a 2-d time, is saving us by the problems that exist in the ADM formulation. We will not lose time through the "sandwich" procedure. Time now, is a complex parameter described by an one dimensional complex space or manifold or by a 2-d real space or manifold We can define a super momentum π_{ij} and a super energy E_{ij} as

$$\frac{\delta\mathcal{S}_\pi}{\delta G^{ij}} = \pi_{ij}, \tag{145}$$

for $i, j \neq 0$, $i \neq 0$, $j \neq 0$ and

$$\frac{\delta \mathcal{S}_\pi}{\delta G^{ij}} = E_{ij}, \tag{146}$$

for $i, j = 0$, $i = 0$, $j = 0$. This way, we have

$$\delta \mathcal{S}_\pi = \pi_{ij} \delta \overset{6\text{d}}{G_{ij}} - E_{ij} \delta \overset{2\text{d}}{G_{ij}} \tag{147}$$

The dispersion relation reads as

$$E_{ij} = \mathcal{H} \tag{148}$$

or

$$-\frac{\delta \mathcal{S}_\pi}{\delta \overset{2\text{d}}{G_{ij}}} = \mathcal{H}\big(\pi_{ij}, \overset{6\text{d}}{G_{ij}}\big) \tag{149}$$

and the super Hamilton -Jacobi equation is

$$G_{ijlm} \frac{\delta \mathcal{S}_\pi}{\delta G_{ij}} \frac{\delta \mathcal{S}_\pi}{\delta G_{lm}} = \Omega^2 \tag{150}$$

where G_{ijlm} stands for

$$G_{ijlm} = \frac{1}{2} G_{ij} Glm - G_{il} G_{jm} \tag{151}$$

and Ω is a function of the the constant D as $\Omega = f(D^4)$. The super Hamilton -Jacobi equation tell us how "wave crests" "propagate" in the space. If we subject Eq. (151) to a new extremisation with respect to Ω we will form a new action Σ_π as

$$\Sigma_\pi = \int G_{ijlm} \frac{\delta \mathcal{S}_\pi}{\delta G_{ij}} \frac{\delta \mathcal{S}_\pi}{\delta G_{lm}} \tag{152}$$

The Poisson brackets for the geometry will be

$$\{G_{ij}, G_{lm}\} = 0 \qquad \{\Pi_{ij}, \Pi_{lm}\} = 0 \qquad \{G_{ij}, \Pi^{lm}\} = \delta^l_{(i} \delta^m_{j)}$$

while the Poisson brackets for the field $\mathcal{S}_\pi$ are

$$\{\mathcal{S}_\pi^{ij}, \mathcal{S}_\pi^{lm}\} = 0 \qquad \{\Pi_{ij}, \Pi_{lm}\} = 0 \qquad \{\mathcal{S}_\pi^{ij}, \Pi_{lm}\} = \delta^i_{(l}\delta^j_{m)}$$

and the Lie brackets

$$[\widehat{\mathcal{S}_\pi^{ij}}, \widehat{\mathcal{S}_\pi^{lm}}] = 0 \qquad [\widehat{\Pi_{ij}}, \widehat{\Pi_{lm}}] = 0 \qquad [\widehat{\mathcal{S}_\pi^{ij}}, \widehat{\Pi_{lm}}] = \delta^i_{(l}\delta^j_{m)}$$

where we have set $\mathcal{S}_\pi^{ij} = \mathcal{S}_\pi(G^{ij})$. By extracting $\mathcal{S}_\pi$, we have the operators

$$\widehat{\Pi}_{ij} = \frac{\delta}{\delta G^{ij}} \qquad \widehat{\pi}_{ij} = \frac{\delta}{\overset{6d}{\delta G_{ij}}} \qquad \widehat{E}_{ij} = \frac{\delta}{\overset{2d}{\delta G_{ij}}}$$

Now, $\mathcal{S}_\pi$ can be seen as a physical quantity or describes a physical quantity, which expresses the deviation between two "lines" of C^4 or the "curvature" which is created by those two "lines" . These "lines" are connected with bosons as we will see in section 14 and 15. The relation

$$\widehat{\Pi}_{ij}\mathcal{S}_\pi = \widehat{Z}_{ij}\mathcal{S}_\pi = Z\mathcal{S}_\pi \tag{153}$$

is the analogue eigenvalue equation.

The Lie algebras with respect to the fields $\mathcal{S}_\pi$ or $\mathcal{S}_\gamma$ are formed in the space of the solutions of the super H-J and usual H-J respectively. If we want to describe the properties that a boson has, we must look in the space of the solutions of the usual H-J, while the "area that the boson field generates" is described by the properties of the solutions of the super H-J.

13 Cosmology

In this section, we will investigate what are the consequences of our theory in the area of Cosmology. In specific, we want to compare our findings with the standard model of Cosmology (FRW). Recent developments in Cosmology as they come from the data of Hubble telescope, open a window for modification to (FRW) model. In particular, the recent data show a disagreement in the value of Hubble constant, which has puzzled many scientists. A lot have already been proposed in order to explain this papers [20], [21], [22], [23], [24], [25]

13.1 Introduction

The key element as it is presented in section (11) is our extended general relativity and its dynamic character. Especially, the existence of the second "time" T which is connected with Cosmos' radius (this connection is presented in this section and in section (16))give us the chance to present a dynamic cosmological "constant" $\Lambda(t)$ and thus a dynamic presentation of dark energy. If Λ is a dynamical entity, its energy density will change in time and soace, in contrast with a static one which will be homogeneous over time and soace. This way, dark energy will be dynamic and could give us explanations as concerned, for example, the different values of H_0 for late and early Cosmos. Moreover, in this section we will see that the relation of "time" T with Cosmos' radius $R(t)$, comes as result of the embedding procedure, while in section (16), we will see, in analytical way, that the mean value $< T >$ is deeply connected with $R(t)$, where $< T >$ is calculated from our extended "Klein-Gordon" equation in C^4 space-time. Espedially, in section (16),we will have the chance to connect the quantum vaccuum with the cosmological vaccuum, which will serve us to present a possible answer to the cosmological constant problem. At this part we will only solve the equation of velocities Eq. (68) in order to extract a cosmological equation similar to the first cosmological equation (Friedmann), which contains, as we expected, a part for radiation, a part of dark energy and a part of dark matter, directly from geometry. Of course, this first cosmological equation will be just a first approximation and a lot must be solved and investigated in the future. A second equation must be derived from the equations of our extended general relativity in C^4 or R^8 space-time and an equation which will involve the volumes between R^8 space-time, the embedded space-time and the volumes of the dual spaces R^4 and M^4, in order to define from the beginning entities such as pressure, temperature and entropy primitively and directly from geometry, as a result (once again) of the embedding procedure. This way, the equation of state will be also derived naturally from geometry. Moreover, we would like to clarify that in this paper, we use the expressions dark energy and dark matter freely, as popular expressions, in order to name the two unexplained phenomena in Cosmos. Someone else could use names such as dark gravity or modified gravity etc. Actually in our hypothesis, three different "quantities" or phenomena arise, that we call them gravity, dark matter and dark energy, with respect to the symmetric metric tensor g_{ij}, as it comes from C^4 space-time eqquiped with a Hermitian metric tensor as we have seen in previous sections. Gravity, dark matter and dark energy, are just three pieces of one unified field, that breaks after embedding our usual 4-d space-time, in R^8 space-time. As we have seen the elementary length, after embedding our usual 4-d space-time in R^8 can be also written as (we split times)

$$ds^2 = N_{ij}dx^i dx^j = N_{00}dx^0dx^0 + N_{0l}dx^0dx^l + N_{k0}dx^k dx^0 + N_{kl}dx^k dx^l \tag{154}$$

or due to the symmetry of N_{ij}

$$ds^2 = N_{00}dx^0dx^0 + 2N_{k0}dx^k dx^0 + N_{kl}dx^k dx^l \tag{155}$$

where k, l=1, 2, 3. From Eq. (42) and the matrices D_{ij}, M_{ij}, E_{ij} (section (5) we have

$$N_{00} = \big((1+\lambda^2) + \lambda\frac{\partial y^0}{\partial x^0} + 2(\frac{\partial y^0}{\partial x^0})^2\big)g_{00} \tag{156}$$

$$N_{k0} = \big(1 + \lambda\frac{\partial y^0}{\partial x^0} + \lambda^2\big)g_{k0} + I_{k0}\frac{\partial y^0}{\partial x^0} \tag{157}$$

$$N_{kl} = (1 + \lambda^2)g_{kl} \tag{158}$$

If we replace Eq. (156), Eq. (157), Eq. (158) in Eq. (155)

$$\begin{aligned} ds^2 = \big((1+\lambda^2) + \lambda\frac{\partial y^0}{\partial x^0} + 2(\frac{\partial y^0}{\partial x^0})^2\big)g_{00}(dx^0)^2 + \\ 2\big(1 + \lambda\frac{\partial y^0}{\partial x^0} + \lambda^2\big)g_{k0} + I_{k0}\frac{\partial y^0}{\partial x^0}\big)dx^k dx^0 + (1+\lambda^2)g_{kl}dx^k dx^l \end{aligned} \tag{159}$$

Change of variables: We consider $y^a = \frac{G}{c^2}m^a$, $y^0 = T$ and $x^0 = ct$ for a=1, 2, 3. But, from section (5) the embedding functions (we consider the simple case) are $y^a = \lambda\delta^a_r x^r$ for r=1, 2, 3. As a consequence we can derive the equations

$$u^a = \frac{dm^a}{dt} = \lambda\frac{G}{c^2}\frac{dx^a}{dt} = \lambda\frac{G}{c^2}v^a \tag{160}$$

$$\frac{\partial m^a}{\partial x^k} = \lambda\frac{G}{c^2}\delta^a_r\frac{\partial x^r}{\partial x^k} = \lambda\frac{G}{c^2}\delta^a_r\delta^r_k = \lambda\frac{G}{c^2} \tag{161}$$

From the last equation, we can see that the linear densities are constant and equal, towards all the direcrions of x^k. IF we divide Eq. (159) by dt^2 we have the equation of velocities as

$$V = \frac{ds^2}{dt^2} = \varphi(l)g_{00}c^2 + 2\big(f(\lambda)g_{k0} + I_{k0}\frac{1}{c}\frac{\partial T}{\partial t}\big)v^k c + (1+\lambda^2)g_{kl}v^k v^l \tag{162}$$

where we have set as $f(\lambda)$ and $g(\lambda)$ the quantities

$$\varphi(\lambda) = \big((1+\lambda^2) + \frac{\lambda}{c}\frac{T}{\partial t} + 2(\frac{\partial T}{\partial t})^2\big) \tag{163}$$

$$f(\lambda) = \big((1 + \lambda^2 + \frac{\lambda}{c}\frac{T}{\partial t}\big) \tag{164}$$

these two functions $\varphi(\lambda)$, $f(\lambda)$ have a minimum at the same point

$$\lambda = -\frac{1}{2c}\frac{T}{\partial t} \tag{165}$$

As a consequence, the velocity presents a minimum as

$$V^2 = \big(1 + \frac{5}{4}\frac{1}{c^2}(\frac{\partial T}{\partial t})^2\big)g_{00}c^2 + 2\big(1 - \frac{1}{2}\frac{1}{c^2}(\frac{\partial T}{\partial t})^2\big)g_{k0} - I_{ko}\frac{1}{c}\frac{T}{\partial t})v^k c + \big(1 + \frac{5}{4}\frac{1}{4c^2}(\frac{\partial T}{\partial t})^2\big)g_{kl}v^k v^l \tag{166}$$

At this minimum the relation between velocities becomes

$$u^a = \frac{1}{2}\frac{c}{G}\frac{\partial T}{\partial t}v^a \tag{167}$$

$$\frac{\partial m^a}{\partial x^k} = \frac{1}{2}\frac{c}{G}\frac{\partial T}{\partial t} \tag{168}$$

Eq. (165) becomes very interesting, as it tells us that Cosmos, experienced a minimum in the past, but it had an initial velocity. The consequences of this equation will be investigated further in this section. Our next step is to seperate the terms of Eq. (162) according to ordinary matter, dark matter, dark energy, velocities, radiation and curvature as

$$\begin{aligned} V^2 = (g_{00}c^2 + g_{kl}v^k v^l + 2g_{ko}v^k c) \\ +\lambda^2(g_{00}c^2 + g_{kl}v^k v^l + 2g_{ko}v^k c) \\ +\frac{\lambda}{c}\frac{\partial T}{\partial t}(g_{00}c^2 + g_{ko}v^k c) \\ +(\frac{\partial T}{\partial t})^2 g_{00} \\ +2I_{k0}\frac{\partial T}{\partial t}v^k \end{aligned} \tag{169}$$

The first line of Eq. (169) expresses geometry, the second line expresses ordinary matter, the third one dark matter, the fourth one dark energy and the last one radiation. In order to simplify Eq. (169), we will consider a flat g_{ij} (while I_{ij} remains this way). The only problem that must be clarified is the signature once again. Our original "flat" consideration, is as we have seen for $G = c = 1$

$$ds^2 = dx_0^2 + \mathbf{dx^2} - dy_0^2 - \mathbf{dm^2}$$

where $x^0 = T$ and $y^0 = t$. We have used T as y^0 due to the fact of the transformation from C^4 to R^8. This elementary length, is actually invariant to transformationso of the form $x \leftrightarrow m$ and $t \leftrightarrow T$, up to a sign (just looking the upside down). This is what our theory really tells. But, in order to get in contact with our familiar Minkowskian space-time and signature we interchange $T \leftrightarrow t$ as

$$\begin{aligned} ds^2 = (\mathbf{dx^2} - dt^2) + (dT^2 - \mathbf{dm^2}) \equiv \\ (dt^2 - \mathbf{dx^2}) + (\mathbf{dm^2} - dT^2) \end{aligned} \tag{170}$$

this waym we have the signatures $(1,3)+(3,1)$ or $(3,1)+(1,3)$. Now $x^0 = t$ and $y^0 = T$. The problem that we have to be cautious with, is that after enbeddieng, we do not "see" the signature of the "dual" space M^4. The choice of signature for R^4 affects the signature of M^4 and backwards. We must remember that in section (6), we presented an extended special relativity, with a "big cone" for R^8, which breaks into two maller cones, each one assigned to the dual 4-d spaces R^4 and M^4. As we have seen we had the elementary lenght to be

$$ds^2_{R^8} = ds^2_{R^4} + ds^2_{M^4}$$

If $ds^2_{R^8} > 0$, we are inside the big "cone", while for $ds^2_{R^8} = 0$, we are on the "two time like" or "comlex time-like" part of the "cone". Th big "cone" breaks into two smaller cones according to $ds^2_{R^4} ><= 0$ and $ds^2_{M^4} ><= 0$. The consequences of the signature $(4,4)$ will be that $\lambda^2 < 0$ (in a similar way that in Higg's mechanism $\mu^2 < 0$) for signature $(1,3)$ for the usual Minkoskwi space-time, while for the signature $(3,1)$ we will have $\lambda^2 < 0$ and the usual velocity $v = \frac{dx^a}{dt}$ will come with opposite sign in each case. The extendedd special relativity tells us for an 8-d "observer" that x,m and T,t are now relative. As a consequence of the above mentioned comments and for $(1,3)$ signature Eq. (151) becomes

$$V^2 = (1+\lambda^2)c^2 - v^2 + \lambda^2 v_g^2 + \mid \lambda \mid \frac{\partial T}{\partial t} c + 2(\frac{\partial T}{\partial t})^2 + 2I_{k0}\frac{\partial T}{\partial t} v_g^k \qquad (171)$$

where $V = \frac{ds}{dt}$, $v = \frac{dr}{dt}$, $v_g = \frac{dr_g}{dt}$ and we have already used that $\lambda^2 < 0$. The difference between v and v_g is that v comes from the g_{ij} part of R^4, while v_g comes from the parts involving M^4. The velocity v_g involves a 3-d Schwarzschild radius as we have already seen in section (6) connected with the mass-scale vector. Eq. (156) can be read as

$$V^2 + v^2 = \text{backround} + \text{ord.matter} + \text{dark matter} + \text{dark energy} + \text{radiation} \qquad (172)$$

Of course, this equation is the simplest approximation that someone could use as it valid for "flat" g_{ij} and we examine it in order to collect simple data. For instance, in the right part of this equation, the key is the presence of velocities v_g and $k = \frac{dT}{dt}$. For $k = v$ all terms excepr curvature behave the same. It is natural to consider that when these velocities are of the same order, all the "ingredients" of Cosmos where just a "soup", while after $k > v$ the photons decoupled ordinary matter. The presence of k, affects differently the terms d.matter, d.energy and radiation, which is easy to see in the general case. We must remember that in section (6) we saw that for just electromagnetism

$$I_{ij}u^j \frac{dT}{ds} = \frac{q}{c}E_i \qquad (173)$$

which means that the last part of Eq. (156), in the case of just electromagnetism, is connected with q or the "charge" of elecrtomagnetism or the coupling constant of electromagnetism plus the "number" of photons. In the same way, we must treat all the other terms, the d.energy term will involve the "number" of vaccua (as we will see in section

(15) related with Higg's boson) and the d.matter term should involve the coupling constant or the d.matter "charge" and the number of dark particles. It is very interesting the presence of λ in both the ordinary matter and dark matter terms. The equation Eq. (156) is the simplest analogue of the first Friedmann equation, as it comes from our theory. Velocity V is reffering to the velocity as it looks from R^8 space-time, while v is the projection to our 4-d usual space-time.

13.2 Relation to current Cosmology

The current cosmological model (FRW) depends on general relativity, as it formulated in our usual 4-d space-time, fper a specific choice of metric. There is no use to refer to the success of general relativity, but it is certain, that it does not tell us about dark matter and dark energy, as even for dark energy, the cosmological constant is still added ad-hoc. Today, in order to do Cosmology, we do not only need to define dark matter and dark energy but at the same time to fill our framework with extra inputs such as proper distance-time, comoving distance-time, comoving frames, peculiar velocities etc, in a pure Copernican-Newtonian way,as these inputs are not derived directly from general relativity or geometry itself. We need to imagine observers that travel along with expansion, local observers and we need some kind of geoemtry to connect them,as comoving and proper distances are not the same concept of distance as the concept of distance in special relativity. In our oppinion, it seems that we need a new type of special relativity or an extended special relativity, that could incorporate all these extra inputs and different type of observers and express them as naturally risen entities. Moreover, this extended special relativity and the extended general relativity should share a common ground. In this spirit, we consider that our presented extended special relativity (section 7) could serve us this way. In our oppinion, the current picture in Cosmology, remind us, two akward situations of the past, Ptolemy's picture about Cosmos and Galilean relativity, where the main problem could be seen as: *there is a bif difference between how things seem to us how things are actually are.* In Galilean relativity pseudo forces and velocities existed, while in Ptolemy model objects of Cosmos where seem to follow peculiar orbits. We believe that through our model, we can propose a way to bridge the differences and manage to reach "*how things are actually are.* In our consideration, the key entities that must be interpeted are

The second time T: The original time in C^4 is $\mathfrak{T} = T + it$. We can see that our usual time t is a periodical entity and we need periodical instruments to measure it, as clocks, orbits or even Cepheid stars. On the other hand T seems to be a straightful entity and we need some kind of "sand timer", in order to measure it. But, at the same time, we have already defined that T is some kind of "cosmic time" as it is related with Cosmos' radius. But we can imagine a "sand time" suitable for our purpose. Imagine a population of many local observers, very close to each other, that formulates a big chain and that our "sand timer" is the sum of the lenghts as it is measured by each one of them. This "sand timer", also defines cosmologicaly, the proper distance between two objects!

The velocity v: In the beginning, we start with a C^4 space-time and a velocity of the type $\frac{dS}{d\mathfrak{T}}$, which is the velocity that a 4-d complex observer would measure. Unfortunately, we are not 4-d complex observers, thus we embedded our usual 4-d space-time in R^8 in order to see, how usual 4-d observers, observe what happens in C^4 space-time. As a result, the velocity V. tell us what is the velocity od 4-d complex observers, in terms of our usual 4-d real observers. Alternatively, S is the lenght of a 4-d complex observer, as it looks to us the 4-d real observers, with respect to our "local" time t! The advantage is, that in the case that our Cosmos in not actually described by a 4-d real model, we could see how our Cosmos realyy looks. In the case, that our Cosmos is really described

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by a 4-d real model, the hypothesis of a 4-d complex one will fall. In the first case, we actuallyu consider that 4-d complex observers, can see properly Cosmos, while our usual 4-d real observers, experience Ptolemy-Galilean effects. Our extended special relativity will bridge the differences

The velocity $\frac{\partial T}{\partial t}$:As $\mathfrak{T} = T + it$ the following relation holds

$$\frac{\partial \mathfrak{T}}{\partial t} = \frac{\partial T}{\partial t} - i \rightarrow \frac{\partial T}{\partial t} = \frac{\partial \mathfrak{T}}{\partial t} + i \tag{174}$$

This velocity yell us that the rate of T and $\mathfrak{T}$ with respect to t, are different to an imaginary constant. Now, we must understand, what really T is. Let us consider the simplest case, where we have only dark energy, thus Eq. (171) will be

$$V^2 = v^2 + \left(\frac{\partial T}{\partial t}\right)^2 \tag{175}$$

or

$$\left(\frac{dS}{dt}\right)^2 = \left(\frac{dR}{dt}\right)^2 + \left(\frac{\partial T}{\partial t}\right)^2 \tag{176}$$

If $\frac{\partial T}{\partial t} = 0$, then the lenght S of the 4-d complex or 8-d real observers, will be equal to the lenght R of our usual 4-d real obserbers, and as expected nothing new is defined. But, if $\frac{\partial T}{\partial t} \neq 0$, then this term would seem to a 4-d real observer as a "Galilean" term, or a "peculiar" velocity. But, through our extended special relativity this term is well defined and expresses a second type of energy or a new momentum (as we have already seen in section 7). As a result, usual 4- real observers, can not define properly dark energy with respect to their 4-d space-time, but they need auxiliary parameters or ad-hoc terms to the standard geometry of the 4-d space-time. In the simplest scenario, a 4-d complex or an 8-d real observer, can see Cosmos as a "closed object", whose velocity V, breaks into two velocities, a natural to us 4-d real observers v and a peculiar to velocity $\frac{\partial T}{\partial t}$ as

$$V^2_{object} = v^2_{rec} + k^2_{pec} \tag{177}$$

Furthermore, if V is the true velocity, we can formulate a new Hubble law as

$$\left(\frac{dS}{dt}\right)^2 \frac{1}{S^2} = \frac{1}{S^2}\left(\frac{dR}{dt}\right)^2 + \frac{1}{S^2}\left(\frac{\partial T}{\partial t}\right)^2 \tag{178}$$

If R=R(t) our usual radius as it exists in the context of Cosmology and R is different to S to a factor b we can also have

$$\left(\frac{dS}{dt}\right)^2 \frac{1}{S^2} = \frac{b^2}{R^2}\left(\frac{dR}{dt}\right)^2 + \frac{1}{S^2}\left(\frac{\partial T}{\partial t}\right)^2 \tag{179}$$

$$\rightarrow \mathcal{H}^2 = b^2 H^2 + \frac{v_{pec}^2}{S^2} \tag{180}$$

where H our usual Hubble parameter. Three different "proper" times are exist in these equations, a first one with respect to the 4-d complex or 8-d real observer through the eyes of the usual 4-d observer $\frac{dS}{c}$, a second one with respect to the usual 4-d observer $\frac{dR}{c}$ and a third one with respect to the second time T. In addition from Eq. (177) we can also have

$$S^2 = R^2 + T^2 + f \tag{181}$$

where f is a constant of integration. But, as we shall see in section 16, $S =< T >$ and if f=0

$$< T >^2 = R^2 + T^2 \rightarrow R^2 =< T >^2 - T^2 \tag{182}$$

which can be connected with a variance of T.The new Hubble law, with respect to the "external" 4-d complex or 8-d real observer, is perfect with respect to him and must be the actual Hubble law. If $\mathcal{H}$ is constant then

$$S = S_0 e^{\mathcal{H}t} \tag{183}$$

Moreover, as we shall see in section 16, $S(t) \neq 0$, which automatically means, that not only we can divide freely with S but, also that there exists no singularity. The term $k = \frac{\partial T}{\partial t}$ can not be considered as constant, thus dark energy is not a constant as Λ dark energy model proposes. The only case that dark energy is constant at any time t and exists only in this simplest scenario (only dark energy exists or dark energy extremely dominates), is the scenario that k can be considered as a true peculiar velocity, which means that a "light" exists so that $k = \pm c$. In general, as T will be always related to R(t) (the relation between T and R(t) varies according to the components of Cosmos i.e dark matter, ordinary matter, radiation and interactions) dark energy will be an increasing entity or maybe is some epoques nearly constant! In the case that we want to establish a Λ representation for dark energy, we could consider the term

$$\frac{1}{S^2}\Big(\frac{\partial T}{\partial t}\Big)^2 \equiv \Lambda(t)c^2 \equiv H_T \tag{184}$$

where H_T is the Hubble term with respect to T. This way, the vaccum density is not a specific constant and varies through time. As a result, in the large scale structures, vacuum gets "thiner and "thiner", a process that we like to call as "vacuum entropies". The vacuum from quantum theories is not identical with the vacuum of large scale structures, but rather connected entities, as in quantum theories we can not take in account the "vacuum entropy" process. In section 16, where we have connected Higg's boson with Cosmology and the relation of Hubble's value with coupling constants, this process might be clearer. It is peculiar, but this process could be resempled, in a risky way, as an inverse black hole procedure. In black holes, gravity results to a star to become thicker and thicker, while Cosmos through dark energy become thiner and thiner and as entropy is connected with black hole's horizon and mass, the "vacuum entropy" is connected with Cosmos vacuum density (white hole?). In the same spirit, we can also include the terms for dark matter, radiation, curvature and ordinary matter and form their contributions to Hubble parameter $\mathcal{H}$.

13.3 Similarities and differences between d.matter, dark energy, ordinary matter and radiation

In this paragraph we want to emphasise about the similarities and differences between the main components in Cosmos, as they look to 4-d real observers. First of all, we will refer to ordinary matter and radiation as white part and to dark matter and dark energy as dark part. The key characteristic for white part, is the velocity $\frac{dr_g}{dt}$, while for dark part is $k = \frac{\partial T}{\partial t}$. At the same time radiation, aslo share the velocity k with the dark part, which means that radiation plays an important link that must be deeply understood and connects the dark part with the white part. On the other hand, dark matter, the most mysterious to us entity, reveal a lot of interesting characteristics. In the general case Eq. (169) dark matter part looks very much with the term for ordinary matter, but D_{ij} seems that has less information compared to ordinary mass term. In the simplest model Eq. (171), dark matter shere λ with ordinary matter and k with dark energy. We can also write that dark matter term can be also written as

$$d.matter \rightarrow \lambda ck \rightarrow \lambda c\sqrt{d.energy} \tag{185}$$

Dark matter looks like a hybrid between ordinary matter and dark energy or as a "mediator" that compromises a repulsive and an attractive field. At this time, we do not have a positive answer about the type of interaction of dark matter, as it might be a hybrid of repulsive-attractive interaction. For instance, if dark matter has also negative pressure, as dark energy, it will be seen as attractive to dark energy and repulsive to ordinary matter, but if it has a positive pressure, we will have the opposite behaviour. Definately, we can make some interesting, as they raise naturally from our equations, about the behaviour of dark matter in galaxies. The amount of dark matter in a galaxy it is not affected by the ordinary matter of the galaxy, instead it is affected by λ that coexists in the term for dark matter and ordinary matter. On the other hand, the interaction between dark matter and ordinary matter is affected by the amount of ordinary matter! If λ^2 contibutes up to a portion to ordinary matter, then the amount for dark matter will be $\sqrt{\lambda^2} = \lambda$. Moreover, the amount of dark matter does not follow the behaviour of ordinary matter, as we don not have a "pure" mass term, as dark matter part does not include a v_g term as it exists in ordinary matter and radiation. This last observation, means that there is no a Schwarzschild radius for dark matter. On the other side, in the general case as it is expressed by Eq. (169) where we consider a metric tensor in its general form, we can observe that dark matter will be characterised by a Schwarzschild radius, which automatically means that dark matter can not be identified locally but rather has a large object identity. This comment could explain the problem in detection dark matter. Dark matter follows dark energy, which automatically means that the amount of dark matter follows the law $\lambda c\sqrt{d.energy}$, which means that there is different amount of dark enrgy in early and later galaxies. As a consequence, two identical galaxies, an early and a later one, would have different amount of dark matter and different streched halos. This is a very crucial observation, this prediction rewrites everything that we know about galaxies as concerned dark matter. All the above mentioned characteristics will also change the way that we see and process clusters. Different aged galaxies participate in clusters, which means that each one of them will participate with different amount of dark matter, different halos diameters and all these are not depended on the mass of ordinary matter of the galaxies in the cluster. There is much more information in clusters than we have thought, which is needed in order to answer about a cluster's characteristics. As concerned radiation, in the case that we have only photons, instead of Schwarzschild radius and its velocity, we should use instead the wavelenght λ (not confused with λ of the embedding proccedure), and as this term contains the velocity k, we must expect a stretching behaviour as it happens with CMB. Especially, in the case of CMB, we must admit that it looks that is the only objective way to measure the velocity that Cosmos is expanding due to dark energy. Finally, for the part of ordinary matter, if we want to compare it with our usual term, we can have

$$\lambda^2 \frac{v^2}{c^2} = \frac{G^2}{c^6}\Big(\frac{dm^k}{dt}\Big)^2 \tag{186}$$

or

$$M(t) = \frac{1}{2}\frac{G}{c^4}R(t)\Big(\frac{dm^k}{dt}\Big)^2 \tag{187}$$

where M is the mass term for ordinary matter as it envolves over time.

14 Lagrangian formalism with free endpoint-Real representation

Let us remind us some preliminaries on the subject of free endpoint as they exist in the context of Lagrangian mechanics. Let us consider an action with respect to a Lagrangian that describes a physical system as

$$S = \int_a^b L(x, \dot{x}, t)dt \tag{188}$$

where a,b fixed points of the usual 4-d space-time. The principle of least action $\delta S = 0$ leads to the usual Euler-Lagrange equations. Now, if we change a little bit the "history" by varying the endpoint by $\mathbf{\Delta x} = \delta x - \dot{x}\delta t$ (for more details see "Gravitation" of C.Missner, K. Thorne, J.Wheeler) the variation of the action becomes

$$\delta S = L(x, \dot{x}, t)\delta t + \int_a^{b+\mathbf{\Delta x}} \delta L dt \tag{189}$$

$$= L\delta + \int_a^{b+\Delta\mathbf{x}} \big(\frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial x}\delta x\big)dt \tag{190}$$

$$= L\delta t + \frac{\partial L}{\partial \dot{x}}\Delta\mathbf{x} + \int_a^{b+\Delta\mathbf{x}} \big(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\big)\delta x dt \tag{191}$$

the quantity inside the integral is zero, because it represents the usual geodesic equation. By substituting $\Delta\mathbf{x}$ we have

$$\delta S = \frac{\partial L}{\partial \dot{x}}\delta x - \big(\dot{x}\frac{\partial L}{\partial \dot{x}} - L\big)\delta t \tag{192}$$

this way the dispersion relation is

$$\delta S = p\delta x - E\delta t \tag{193}$$

$$p = \frac{\partial L}{\partial \dot{x}} \qquad E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L$$

the second term is the usual Hamiltonian of the system (Lagrange transformation) which can be read also as

$$-\frac{\partial S}{\partial t} = H\big(\frac{\partial S}{\partial x}, x, t\big) \tag{194}$$

In this spirit, if we begin with the ordinary action of a charge in a gravitational field as an application, we begin with an action of the form

$$S = -\int_a^b mcds - \frac{q}{c}A_i dx^i \tag{195}$$

The term that was vanished at the limits is

$$mcu_i + \frac{q}{c}A_i \tag{196}$$

But, if we vary the endpoint this term does not vanish(the geodesic equation still holds) and leads to the dispersion relation

$$\delta S = -(mcu_i + \frac{q}{c}A_i)\delta x^i \tag{197}$$

and afterwards to the well known relation

$$\frac{\partial S}{\partial x^i} = mcu_i + \frac{q}{c}A_i = p_i + \frac{q}{c}A_i \tag{198}$$

this relation is the starting point to form the covariant derivative of electrodynamics in quantum field theories. The quantity $\frac{\partial S}{\partial x^i}$ is the four vector of the generalised momentum of the particle of charge q, where p_i is the ordinary momentum as it exists in the context of mechanics. The most surprising element is that the term $\frac{q}{c}A_i$ defines also a momentum term, that is formed directly from the electromagnetic field. Adding the usual momentum p_i with the "electromagnetic momentum" we form a generalised momentum. But, through our consideration we have already seen that the electromagnetic field (preferably the generalised K_μ field) is not just an ad-hoc quantity added to the usual geometry of the usual space-time, but rather a geometric term introduced by the antisymmetric tensor $I_{\mu\nu}$. If we put the pieces into order, and start with the elementary length of the extended space-time in the symplectic R^8

$$ds^2 = g_{ij}dx^i dx^j + g_{ij}dy^i dy^j + I_{ij}(dx^i dy^j - dy^j dx^i) \tag{199}$$

and the variation of the action

$$\delta S = \delta \int \big(g_{ij}u^i u^j + g_{ij}v^i v^j + 2I_{ij}u^i v^j\big)ds \tag{200}$$

the free endpoint will lead us to the pair of equations

$$\frac{\partial S}{\partial x^i} = g_{ij}u^i + I_{ij}v^i \tag{201}$$

$$\frac{\partial S}{\partial y^i} = g_{ij}v^i - I_{ij}u^i \tag{202}$$

or in matrix form

$$\begin{pmatrix} \frac{\partial S}{\partial x^i} \\ \frac{\partial S}{\partial y^i} \end{pmatrix} = \begin{pmatrix} g_{ij} & I_{ij} \\ -I_{ij} & g_{ij} \end{pmatrix} \begin{pmatrix} u^i \\ v^i \end{pmatrix}$$

If we set $(p_g)^x_i = g_{ij}u^i$, $(p_g)^y_i = g_{ij}v^i$, $(p_I)^x_i = I_{ij}v^i$, $(p_I)^y_i = I_{ij}u^i$, Eq. (201), (202) becomes

$$\frac{\partial S}{\partial x^i} = (p_g)^x_i + (p_I)^x_i = (P)^x_i \tag{203}$$

$$\frac{\partial S}{\partial y^i} = (p_g)^y_i - (p_I)^y_i = (P)^y_i \tag{204}$$

and the Hamilton-Jacobi equations will be

$$P^i P_i = G^{ij} P_j P_i = D^2 \tag{205}$$

where

$$P_i = \begin{pmatrix} (P)^x_i \\ \\ (P)^y_i \end{pmatrix}$$

It is clear now, that the generalised electromagnetic field K_i is nothing else than a momentum defined directly from the symplectic geometry and associated or born from the symplectic form I_{ij} and on the other hand, our common momenta (plus something more as we have seen in previous sections) are defined and associated by the symmetric form g_{ij}. But, as we have set $K^x_i = I_{ij}v^i = (p_I)^x$ and $K^y_i = I_{ij}u^i = (p_I)^y$, nothing can stop us to set $B^x_i = g_{ij}u^i = (p_g)^x$ and $B^y_i = g_{ij}y^i = (p_g)^y$. As a consequence, the pair Eq. (203), (204) can be written in "field mode" as

$$\frac{\partial S}{\partial x^i} = (B_i)^x + (Ki)^x \tag{206}$$

$$\frac{\partial S}{\partial y^i} = (B_i)^y - (Ki)^y \tag{207}$$

This way, the field B_i is the unified field with respect to the symmetric form g_{ij} in the same spirit as K_i is the unified field with respect with the antisymmetric form I_{ij}. Moreover, B_i as we have seen it is related to gravity and dark field! It is important to note that g_{ij} is invariant under rotations of 180^0 while I_{ij} is invariant under rotations of 360^0. Now, we have the opportunity to separate the two pictures as they are presented. For the first picture, the geometric field is the metric tensor G_{ij}, which is separated to the symmetric g_{ij} and antisymmetric I_{ij} geometric fields. This picture is similar to the picture of general relativity or the way we work in general relativity, which is a geometrical theory. The second picture is to work with ordinary fields, a unified one that we will call Ω_i, which splits to B_i (is produced by g_{ij}) and K_i (is produced by I_{ij}). These two pictures, are organising the "mess" between existing theories and can provide a concrete framework to work with. A beautiful example as we have previously mentioned , that shows this "mess" are Kaluza-Klein theories , where we have put the geometrical field g_{ij} together with the field A_i. As we have shown g_{ij} and A_i are incompatible, and that is the reason that Kaluza-Klein theories finally failed to lead us to a unified theory. It is obvious though, that these two pictures are equivalent, it is just a matter of convenience which one we will choose to work with.

15 Lagrangian formalism with free endpoint-Complex representation

Next step is to repeat the formalism of the previous section directly to C^4. The free endpoind will lead us to equations of the form

$$\frac{\partial S}{\partial z^i} = \frac{\partial S}{\partial x^i} - i\frac{\partial S}{\partial y^i} = G_{ij}U^i = P_i \tag{208}$$

where S is a complex functional of z : $S = S(z)$ due to variation of the free endpoint, G_{ij} is the Hermitian metric tensor, $U^i = \frac{dz^i}{ds}$ a complex velocity and P^i the generalised complex momentum. We can also write this expression as

$$\frac{\partial S}{\partial z^i} = G_{ij}U^i = (g_{ij} + iI_{ij})U^i = g_{ij}U^i + iI_{ij}U^i \tag{209}$$

and the conjugate part

$$(\frac{\partial S}{\partial z^i})^* = (G_{ij}U^i)^* == g_{ij}(U^i)^* - iI_{ij}(U^i)^* \tag{210}$$

We have again the split of the generalised complex momentum into a momentum defined by g_{ij} and a momentum defined by I_{ij}. We can set as previous

$$(p_g)_i = g_{ij}U^i = B_i \tag{211}$$

$$(p_I)_i = I_{ij}U^i = K_i \tag{212}$$

The physical interpretation remains as it was in the previous section. But, we have the appearance of "i" in front of K_i, which is of major significance. Let us restrict to the case that K_i is just the usual A_i, Eq. (209) can be read as

$$\frac{\partial S}{\partial z_i} = (p_g)_i + iA_i = p_i^x + ip_i^y + iA_i \tag{213}$$

and let us remove for now the term p_i^y(which means as we will see that we have remove the mass property), then Eq. (203) becomes

$$\frac{\partial S}{\partial x_i} + i\frac{\partial S}{\partial y_i} = p_i^x + iA_i \tag{214}$$

the term p_i^x is the ordinary momentum of the usual 4-d space-time and as a consequence we can set it by p. Moreover, let us again remove the term $\frac{\partial S}{\partial y_i}$ and we will have then

$$\frac{\partial S}{\partial x_i} = p_i + iA_i \tag{215}$$

and if we solve for p_i

$$p_i = \frac{\partial S}{\partial x_i} - iA_i \tag{216}$$

and now let us abstract S from this expression in order to move into operator form (remember that S is a complex functional), we will have

$$\widehat{p_i} = \frac{\partial}{\partial x_i} - i\widehat{A_i} \tag{217}$$

It is obvious that through our consideration, we have managed to define the usual "covariant derivative" of electromagnetism, which will be better call it Hamilton-Jacodi (H-J) derivative. The only new element is that A_i, now is an operator and not just a classic field. As a result, for all these years the "covariant derivative" is just the operator $\widehat{p_i}$. But, as we have seen, the momentum p_i is defined through the metric tensor g_{ij} which is associated to mass, gravity and dark field. By solving the expression Eq. (215) to p_i, we have lost the information concerning g_{ij}, thus we have lost the possibility to describe mass, gravity and dark field. This is why usual quantum theories failed to include gravity and we needed an ad hoc mechanism (Higg's mechanism) to describe the mass property, which in reality, mass should be connected to gravity as we can "see" it as the "charge" of gravity. We must explain ourselves at this point, we do not want to cancel Higg's mechanism or to cancel Standard Model (SM), but rather to reproduce the already known theories in a well defined framework and moreover, to explore new areas of physics. If our consideration is true , then in order to solve only for gravity and dark field ,we should solve Eq. (209) with respect to I_{ij} as

$$P_i^y = K_i = -i\frac{\partial S}{\partial z_i} + iB_i \tag{218}$$

or in operator form

$$\widehat{P_i^y} = \widehat{K_i} = -i\frac{\partial}{\partial z_i} + i\widehat{B_i} \tag{219}$$

Now, if

$$|P_i| = |P^i| = G_{ij}P_j\overline{P_i} = G^{ij}\frac{\partial S}{\partial z_j}(\frac{\partial S}{\partial z_i})^* = G^{ij}\frac{\partial S}{\partial z_j}\frac{\partial S^*}{\partial \overline{z}_i} \tag{220}$$

the H-J equations are

$$G^{ij}\frac{\partial S}{\partial z_j}\frac{\partial S^*}{\partial \overline{z}_i} = D^2 \tag{221}$$

where D is some constant. If we subject D to another extremum, we can call as a new Lagrangian

$$L = D^2 = G^{ij}\frac{\partial S}{\partial z_j}\frac{\partial S^*}{\partial \overline{z}_i} \tag{222}$$

and a new action

$$\mathcal{S} = \int L = \int G^{ij}\frac{\partial S}{\partial z_j}\frac{\partial S^*}{\partial \overline{z}_i} \tag{223}$$

the extremum will be obtained by varying $\mathcal{S}$

$$\delta\mathcal{S} = 0 \tag{224}$$

and the equations of motion will be

$$(G^{ij}\frac{\partial}{\partial z_j}\frac{\partial}{\partial \overline{z}_i})\mathcal{S} = 0 \tag{225}$$

This equation represents the H-J one in the curved C^4 space. Let us now write the L in the flat case and see if will remind us something in the existing literature. The flat case for signature (4,4) is just

$$L = \frac{\partial\mathcal{S}}{\partial x^i}\frac{\partial\mathcal{S}^*}{\partial x_i} - \frac{\partial\mathcal{S}}{\partial y^i}\frac{\partial\mathcal{S}^*}{\partial y_i} \tag{226}$$

We can restrict the problem to a subspace of C^4 by considering that $x_o = T$ vanishes. This can happen because as we have mentioned T is referred to cosmic time, thus we can consider that it changes slowly and will not affect the motion in the subspace. Actually, that should mean also that the scale is fixed in this subspace. Instead of embed spaces, let us try something new by setting

$$\frac{\partial \mathcal{S}}{\partial y^i} = \frac{m}{\hbar}\mathcal{S} \qquad \frac{\partial \mathcal{S}^*}{\partial y^i} = \frac{m^*}{\hbar}\mathcal{S}^*$$

where m is our ordinary mass value and is actually the momentum p_y in Y space (that is the reason that we have called him mass space). This relation suggests that masses are eigenvalues of the operator $\frac{\partial}{\partial y^i}$. Eq. (226) becomes

$$L = \frac{\partial \mathcal{S}}{\partial x^i}\frac{\partial \mathcal{S}^*}{\partial x^i} - \frac{mm^*}{\hbar^2}\mathcal{S}\mathcal{S}^* \tag{227}$$

Now we are ready to make the most surprising observation. If we set

$$\mathcal{S} = \hbar\varphi \tag{228}$$

and if $m^* \in R$, we have the Klein-Gordon equation for a complex scalar field. The link between the action and the field as it exists in quantum theories is not something new. In [20], the writers, argue on this subject continuously, except the fact that "i" can not be embedded in the equations naturally. Moreover, in the case of the flat G_{ij} the antisymmetric part I_{ij} vanishes, which means that there is no "charge". But we have to remember from section (5) that there was a scalar field E_{ij} for which Eq. (175) also holds. We will show later that this scalar is Higg's field. This way we have finally succeed to define as a quantum field related with an elementary field the $\mathcal{S}^i = \mathcal{S}(z^i) = \hbar\varphi(z^i)$ which is the solution of Eq. (175). The Lie brackets for this mechanics directly in C^4 will be

$$[\mathcal{S}_i, \mathcal{S}_j] = 0 \qquad [P_i, P_j] = 0 \qquad [\mathcal{S}_i, P_j] = \delta_{ij}$$

but if we restrict to the subspace of C^4 by excluding T from the equation and replace the eigenvalues the Lie brackets give us the usual Lie brackets of quantum theory! In next section we will define properly the Lie brackets and their physical meaning. All existing axioms and considerations of the usual quantum theory, are just properties of this treatment. From the waves defined in Y, we observe their momenta, as it happens for every wave. Their momenta is just what we call mass value and this way we have managed to define properly the property of mass and De Broglie's suggestion that all masses have wave properties (matter and wave–particle duality). The probability that is inserted in quantum theories is just our effort to identify a specific point in a wave. We believe that this way the interpretation of quantum theory, now is closed. After all, maybe the hidden variables that many suggested was a reality. The free endpoint that we have used in order to define the particle can be visualised as: imagine an electrified wire where its endpoints are stuck in points a,b, let the endpoint at b, or even both, free and imagine the picture, the electrified wire will start sweeping or scanning all possible positions in space, allowed by its fixed length, trying to find the best path, that through the end-point procedure, still holds! But, this picture happens in the invisible to us space Y, and this behaviour comes from the fact that in Y we have a usual wave. By releasing

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the endpoint we have created integrals with an open endpoint, that behave as "path integrals" because it varies along all possible trajectories in space Y. We can repeat all the context of usual mechanics, concerning propagators, for Y space, giving us this way, the opportunity to write actions of the form Wheeler- De Witt, for the space C^4. Are there many histories this way? We think that we do not need them in the way that they where interpreted till now. We must keep in mind that through all the procedure of the free endpoint the usual geodesic still holds and is fulfilled.

16 Solving for Higg's field

The extended equation of energy as it is presented in section (7) is

$$E^2 = D^2c^2 + p_r^2c^2 - p_m^2c^2 + p_T^2c^2 \tag{229}$$

where p_r the usual momentum in R^3 (length space), p_m the mass-momentum in M^3 (mass space), E_t the usual energy with respect to t and $p_T = E_T$ a new type of energy with respect to cosmological time T and D a constant with momentum units that must be identified. Let us try to introduce in this equation the constants $c, G, \hbar, G_F$. In order to achieve fine tuning in energy scales we introduce the dimensionless constant A, which we call unified constant

$$A = \frac{1}{6}\sqrt{\frac{2}{3}}\sqrt{\frac{G}{G_F}\frac{h}{c}} = 3.26297x10^{-18} \tag{230}$$

where G_F is Fermi's constant.The factor

$$\frac{1}{6}\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{2}\frac{1}{\sqrt{27}} = \frac{8\pi}{3}\frac{1}{8}\frac{\sqrt{6}}{6} = 8\pi\frac{1}{24}\frac{\sqrt{6}}{6} \tag{231}$$

Eq. (229) can be written now as

$$-\hbar^2\frac{\partial^2\Psi}{\partial t^2} + \hbar^2c^2\frac{\partial^2\Psi}{\partial T^2} = D^2c^2 - \hbar^2\nabla_r^2\Psi + A^2m_p^4c^2\nabla_m^2\Psi \tag{232}$$

if we set

$$\Psi(\vec{r}, t, \vec{m}, T) = \sigma(t, T)\psi(\vec{r}, \vec{m}) \tag{233}$$

$Eq. \sim 232$ becomes

$$-\hbar^2\frac{1}{\sigma}\frac{\partial^2\sigma}{\partial t^2} + \frac{1}{\sigma}\hbar^2c^2\frac{\partial^2\sigma}{\partial T^2} = D^2c^2 - \frac{1}{\psi}\hbar^2\nabla_r^2\psi + \frac{1}{\psi}A^2m_p^4c^2\nabla_m^2\psi = \omega^2 \tag{234}$$

$Eq. \sim 234$ splits in two equations as

$$-\hbar^2 \frac{\partial^2 \sigma}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2 \sigma}{\partial T^2} = \omega^2 \sigma \tag{235}$$

$$D^2 c^2 - \hbar^2 \nabla_r^2 \psi + A^2 m_p^4 c^2 \nabla_m^2 \psi = \omega^2 \psi \tag{236}$$

we split $\sigma(t,T) = \varphi(t) g(T)$ and $Eq. \sim 235$ becomes

$$c^2 \frac{1}{g} \frac{\partial^2 g}{\partial T^2} - \frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial t^2} = \left(\frac{\omega}{\hbar}\right)^2 \tag{237}$$

and if we call $\frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial t^2} = k^2$

$$\frac{1}{g} \frac{\partial^2 g}{\partial T^2} = \left(\left(\frac{\omega}{\hbar c}\right)^2 + \left(\frac{k}{c}\right)^2\right) \tag{238}$$

and if we call $\rho^2 = \left(\left(\frac{\omega}{\hbar c}\right)^2 + \left(\frac{k}{c}\right)^2\right)$

$$\frac{1}{g} \frac{\partial^2 g}{\partial T^2} = \rho^2 \tag{239}$$

so we get two equations

$$\frac{1}{\varphi} \frac{\partial^2 \varphi}{\partial t^2} = k^2 \tag{240}$$

$$\frac{1}{g} \frac{\partial^2 g}{\partial T^2} = \rho^2 \tag{241}$$

with general solutions $\varphi(t) = a_1 e^{-kt} + a_2 e^{kt}$ and $g(t) = b_1 e^{-\rho T} + b_2 e^{\rho T}$. If we restrict the solutions to $t, T \geq 0$ we have

$$\sigma(t,T) = N e^{-\rho T} e^{kt} \tag{242}$$

where N constant. Moreover $e^{-\rho T}$ can be normalised if we set $\int_0^\infty N^2 e^{2\rho t} dT = 1$ which means $N = \sqrt{2\rho}$. Furthermore we can calculate the mean value of T as

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$$< T >_t = \int_0^\infty T(\sqrt{2\rho} e^{-\rho T} e^{kt})^2 = \frac{1}{2\rho} e^{2kt} = \frac{1}{2\sqrt{\left(\left(\frac{\omega}{\hbar c}\right)^2 + \left(\frac{k}{c}\right)^2\right)}} e^{2kt} \quad (243)$$

Eq. ∼ 243 resembles the equation of the universe of De-Sitter in a cosmos with vacuum domination

$$R(t) = \frac{\sqrt{2}}{2} \frac{c}{H} e^{Ht} \quad (244)$$

As we solve directly in the 4-d complex space-time, without embeddind, T is not R(t), but rather connected to R(t). On the other hand the time independent *Eq.* ∼ 236 splits in two equations if we set

$$\psi(\vec{r}, \vec{m}) = \zeta(\vec{r})\xi(\vec{m}) \quad (245)$$

$$D^2 c^2 - \frac{1}{\zeta} \hbar^2 \nabla_r^2 \zeta = \mu^2 \quad (246)$$

$$\omega^2 - \frac{1}{\xi} A^2 m_p^4 c^2 \nabla_m^2 \xi = \mu^2 \quad (247)$$

we will solve *Eq.* ∼ 246 using spherical coordinates (r, θ, ϕ). If we set $\zeta(r, \theta, \phi) = R(r)\Upsilon_1(\theta, \phi)$, where $\Upsilon_1(\theta, \phi)$ our usual spherical harmonic functions and $q^2 = \left(\frac{\mu^2 - D^2 c^2}{\hbar^2 c^2}\right) r^2$ and $\lambda_1 = \frac{1}{\Upsilon_1} L_r^2 \Upsilon_1$ where L_r the angular-momentum operator *Eq.* ∼ 246 becomes

$$q^2 \frac{dR}{d\phi^2} + 2q \frac{dR}{dq} + (q^2 - \lambda_1) R = 0 \quad (248)$$

which is a Bessel's differential equation. As a conclusion, the solution of Eq.∼ 246 for a spherical infinite well of radius r_0 becomes

$$\mu^2 - D^2 c^2 = \frac{q_{l_1,k}^2 \hbar^2 c^2}{r_0^2} \quad (249)$$

where $q_{l,k}$ the roots of Bessel's function. This solution in the case that $l_1 = 0$, the roots are $q_{0,k} = k\pi$ for $k = 1, 2, \ldots$. On the other hand the equation *Eq.* ∼ 247 is mathematically equivalent. As a result we can set spherical coordinates (m, Θ, Φ). If we set $\xi(m, \Theta, \Phi) = M(m)\Upsilon_2(\Theta, \Phi)$, where $\Upsilon_2(\Theta, \Phi)$ our usual spherical harmonic functions in M^3 and $Q^2 = \left(\frac{\mu^2 - \omega^2}{A^2 m_P^4 c^2}\right) m^2$ and $\lambda_1 = \frac{1}{\Upsilon_1} L_r^2 \Upsilon_1$ where L_m the angular-momentum operator in mass-space M^3 the equation becomes as

$$Q^2 \frac{dM}{d\Phi^2} + 2Q\frac{dM}{dQ} + (q^2 - \lambda_2)M = 0 \tag{250}$$

where $\lambda_2 = l_2(l_2 + 1)$. The solution in the same spirit is

$$\mu^2 - \omega^2 = \frac{q^2_{l_2,k} A^2 m_p^4 c^2}{m_0^2} \tag{251}$$

where m_0 is the radius in mass-space. For the case $l_2 = 0$ the roots are $q_{0,k} = k\pi$ for $k = 1, 2, ...$ and the solution becomes

$$\mu^2 - \omega^2 = \left(\frac{k\pi A^2 m_P^2 c^2}{m_0}\right)^2 \tag{252}$$

Afterwards if we consider that $m_0 = m_p$ we can have

$$\sqrt{\mu^2 - \omega^2} = k\pi A m_P c^2 \tag{253}$$

in addition, if we set $k = 1$ which is the ground state ($k \neq 0$ from the solution of the differential equation) we have a mass eigenvalue

$$m = \pi A m_P = 125,173945 Gev/c^2 \tag{254}$$

Moreover, if we set equivalently $r_0 = l_P$ in $Eq. \sim 249$ we can have

$$\sqrt{\mu^2 - D^2c^2} = k\pi m_P c^2 \tag{255}$$

from the comparison of $Eq. \sim 253$ with $Eq. \sim 255$ we can get

$$\omega = ADc = E_H \tag{256}$$

Moreover the above mentioned assumption combined with $Eq. \sim 195$ explains the transformation $y^{\alpha^{'}} = \lambda \delta_{\varrho}^{\alpha^{'}} x^{\varrho}$ for $\alpha^{'} = 1, 2, 3$ and $y^0 = y^0(x^0)$ as it was proposed in our first paper. We have to mention that this transformation is now a prediction of our original consideration.

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17 Remarks from the solution

1. We have seen that $< R(t) >_t = < T >_t = \frac{1}{2\rho}e^{2kt}$ give us an exponentially behaviour for T=R(T). If we want to see the nature of t, we must write

$$ln < T >_t = ln\Big(\frac{1}{2\rho}e^{2kt}\Big) \tag{257}$$

and if we solve with respect to t

$$t = \frac{ln < T > +ln\varrho}{2k} \tag{258}$$

It is obvious, that t and T are in different scale and of different nature and function, as it was originally hypothesised. Moreover, if we examine the solution

$$\Psi(\vec{r}, t, \vec{m}, T) = \Psi_r\Psi_m e^{-kt}e^{\rho T} \tag{259}$$

we can figure that t decreases exponentially while T increase exponentially. This assumption gives us in some sense that t is a "local" or "internal" time, while T is a global or "external" one. This effect seems natural if we remember that proper time τ is combined with the time of a mass body propagating in the usual space-time. The only problem that seems to exist, is that a particle described by a wave function has the form in the usual space-time

$$\Psi(\vec{r}, t) = \Psi_r e^{\pm ikt} \tag{260}$$

while in the expression described by the solution of the general equation , the time depended term has the form e^{-kt}. But in previous section we have note that
"m_0 ($m_0 \in R$) describes a mass moving in the usual space-time originating from the sub-space M^3. Different subspaces of R^7 express different m_1, m_2, ... that move inside different subspaces of the usual space-time, forming different "cosmic lines" for different masses m_i, which are connected through usual Lorentz transformations."
Thus, if we consider that local measurements do not be affected by the motion of Cosmos or that $\Delta T \to 0$, we can ignore the parameter T. This way, if we look back to the solution of the general equation, the solution becomes for the time depended part $e^{\pm ikt}$ as it should be for a particle propagating in the usual space-time. Moreover, we can see that there is a big difference under local measurements (t) and global measurements (T). In order to compromise the different measurements we need to abstract the information through the logarithm.

2. The mass value of Higg's boson as was mentioned is

$$m = \pi A m_P = \frac{1}{6}\sqrt{\frac{2}{3}}\pi\sqrt{\frac{\hbar^3}{G_F c}} \tag{261}$$

where we can see that finally its mass value is independent by G. Moreover we have to mention that we have not included any corrections such as Darwin correction due to spin 0. That means that the final value would be little less than $125,17394(5)$ Gev/c^2.

3. As concerned the constant A, we could avoid to include by the beginning. But as is well accepted, a Klein-Gordon equation describes particles with spin 0. Moreover, as it was presented, we expected that the extended Klein-Gordon equation, which was derived directly from the geometry of the flat $R^8 \simeq C^4$, should be linked with the vacuum i.e the Higg's boson (in the case of spherically infinite well). As a conclusion, we have seen that a dimensionless constant with value around 10^{-17} should be added. Due to this observation, combined with the empirical Fermi's relation about G_F, we suspected that this constant should be the above mentioned constant A. Moreover, it looks attractive the fact that the four basic constants in Physics (G_F g, $\hbar$, c) are unified in a single dimensionless form. As concerned the factor $= \frac{1}{6}\sqrt{\frac{2}{3}}$, it was derived by the expression of the general (curved) metric tensor $G \in GL(4,C)$ in the Gell-Mann basis, where the factor $\frac{1}{12}\frac{\sqrt{6}}{6}$ has appeared in the G_{44} component. An interesting fact, that should be interpreted in the future, is that this factor appears in the literature of black holes' horizon, if someone started from the relation of the density and set $R = r_g$

$$d = \frac{M}{V} = \frac{3M}{4\pi R^3} \rightarrow R = \left(\frac{3M}{4\pi d}\right)^{\frac{1}{3}} \tag{262}$$

$$R = r_g \rightarrow \frac{1}{d} = \frac{32\pi}{3}\frac{G^3}{c^6}M^2 = \left(24\frac{1}{6}\sqrt{\frac{2}{3}}\right)^2 \pi \frac{G^3}{c^6}M^2 \tag{263}$$

Additionally, we can examine which is the scale involved with the constant A

$$ln\frac{1}{A} = -lnA = 40,263894 \rightarrow \left(ln\frac{1}{A}\right)^{-1} = \frac{1}{40,263894} \tag{264}$$

while the pure constant $A^{'}$ (without the factor) is

$$ln\frac{1}{A'} = \frac{1}{38,2694} \tag{265}$$

At this part, we imagine that the constant A, should be interpreted as initial condition of Cosmos. In that way, Cosmos could begin by only one Higg's boson (excitation of vacuum), where this boson under tunnelling effect raised to the step E_P. The most crucial element of the model , are the Planck units or Planck epoch. Speciffically, the Planck units indicates that they serve as a critical point for the model. The Planck units can be derived under the hypothesis that the Swhartchild's radius r_g is equal to Compton's wavelength (l_C). If we consider that Cosmos had been in that state (Planck epoch), we can see that exactly after this epoch, we have a disconnection of r_g and l_c and they present two different physical phenomena. An alternative scenario in order to explain the jump from Higg's vacuum to Planck energy avoiding tunnelling, is to hypothesize that an external cause made that jump to exactly Planck energy. Eventually, whatever scenario one chooses to propose or to accept, it seems that Cosmos seems to drain energy all the time by a "storage" unit, which is succeeded through "local" time t to "global" time T.

4. Let us consider two different states of Cosmos which for some reasons, are interesting in order to examine. State A where Cosmos has energy, radius and mass $(E_1, T_1 = R_1(t), M_1)$ and state B with (E_2, T_2, M_2) respectively. If we write the formulas about T_1, T_2

$$< R_1(t) >= T_1 = \frac{c}{H_1(t)} e^{H_1(t)t_1} \tag{266}$$

$$< R_2(t) >= T_2 = \frac{c}{H_2(t)} e^{H_2(t)t_2} \tag{267}$$

from these equations we can find also

$$H_2 t_2 - H_1 t_1 = ln\Big(\frac{H_2 t_2}{H_1 t_1}\Big) \tag{268}$$

We can observe that the quantities $H_i t_i$ are dimensionless. On the other hand, in the context of high energy physics, we have the renormalisation group equations that lead to

$$\frac{1}{a_i(Q^2)} - \frac{1}{a_i(\mu^2)} = \frac{b_i}{4\pi} ln\Big(\frac{Q^2}{\mu^2}\Big) \tag{269}$$

where Q the mass energy scale which we use in order to measure the coupling constants a_i and μ the scale in which we measure the coupling constant and b_i coefficients that come from the groups of invariance. Even if we examined only the flat case, while as we have seen the electromagnetic, nuclear and strong nuclear fields will appear in the curved space, we can assume that equations $Eq. \sim 268, 269$ are equivalent. The difference lies only in the presentation. In cosmology we are mostly interesting in the physical quantities of radius and Hubble's constant while in high energy physics in the energy scales. Thus high energy physics and cosmology are equivalent sectors of physics.

5. We can think the manifold C^4 as a C^3 manifold consisting of vectors $\vec{r}$, $\vec{m}$ and a C manifold consisting of t and T. Afterwards, the "time" manifold C is embedded in the C^3 where the vectors becomes $\vec{r}(t,T)$ and $\vec{m}(t,T)$. Meanwhile, let us consider a "sphere" where the bundle is the C^3 plus the time t (forming a 7-d manifold), while the time T is the radius of this "sphere". The evolutionary behaviour of T makes this "sphere" to expand. The inner "vacuum" which acts as a "trompe" reflects on the expansion of the bundle and can be seen as an expanding manifold governed by a Ricci flow equation. Our effort to go fron "now" to the beginning of Cosmos, is the case of a shrinking manifold. It seems that Poincare's conjecture is not just a mathematical game in our minds, it has also physical meaning and connection. In the beginning t and T are comparable, but after Planck's epoch t and T follow different scale.

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6. It is important to note that, vacuum as a real zero (really empty space) has no meaning at all. An actual zero, should mean that we do not have solutions for the H-J equation. Even in flat space, there is geodesic equation, which gives us a H-J equation, which has a solution $\mathcal{S} = \hbar\varphi$, which has at least an eigenvalue. Moreover, the symplectic R^8 is locally flat, which suggests that we always find the flat metric tensor to work with and that we always find the Higg's field there to wait as the minimum case of a scalar field theory. We can not avoid the Higg's field.

18 Symmetries in C^4

The Lagrangian Eq. (222) must present the unified theory of fields in the curved space C^4, where curved means that all possible fields are included in this expression. We will try to investigate what happens if we embed the usual space-time R^4 in the C^4 using the language of groups, which is more familiar to those that are investigating grand-unified theories, from the point of view of fields. The group of the relation $G_{ij} = g_{ij} + iI_{ij}$, locally comes as

$$GL(4, C) = SO(4, 4) \cap U(4) \tag{270}$$

This group after we embed will come as

$$GL(1, 3, R) \times \frac{GL(4, C)}{GL(1, 3, R)} = \Big(SO(1, 3) \times \frac{SO(4, 4)}{SO(1, 3, R)}\Big) \cap \Big(SU(2) \times \frac{SU(4)}{SU(2)} \times U(1)\Big) \tag{271}$$

or

$$\rightarrow \Big(SO(1, 3) \cap (U(4)\Big) \times \Big(\frac{SO(4, 4)}{SO(1, 3, R)} \cap U(4)\Big) \tag{272}$$

where U(4) breaks as in $Eq. \sim 271$. From our usual 4-d space-time we "see" the $SO(1, 3)$ for Minkowski metric tensor of special relativity and $SU(2)$ But, $SU(2)$ is only locally isomorphic to $SO(3)$ which suggests that we have a local symmetry. This will effect all fields that come from the I_{ij} form. Everything, except these two symmetries comes as an external cause. This is why we have to add additional symmetries in all our efforts of a grand unified theory. The breaking of the original symmetry which is dictated by $Eq. \sim 270$ to the symmetry dictated by $Eq. \sim 271$ is exactly what we call spontaneous symmetry breaking and it represents our effort to observe the C^4 from the usual R^4 space. properties. The physics underlying the part $SO(4, 4)$ is still mysterious and under consideration. If the first part of $Eq. \sim 272$ is the correlation of Higg's boson and the extended Standard Model (as we shall see in section 18), the second part must be interpeted as the dark section. At this point we must present some scenarios
Scenario A: Dark matter follows the patern of ordinary matter, which means that dark matter seems as an extra copy. There exist three dark "fields" as "dark electromagnetism", "dark weak" and "dark strong" (dark bosons) but the associatet dark fermions present only one generation, which means that there exist one "dark electron", one "dark neutrino" and two "dark gluons".

Scenario B: U(4) does not break as it happens with ordinary matter, thus there exists only one field, which means that we have one dark boson and one or four dark "fermions" associated with metric tensor g_{ij}.
Scenario 1: There exist three bosons, a graviton, a "dark matteron" and a "dark energon".
Scenario 2: There exists only one "big boson" associated with metric tensor g_{ij}, that has three different actions $\to$ gravity+dark matter field+dark energy field.
Scenario 3:There is no existence of a particle associated with g_{ij}, but instead the source of gravity, dark matter field and dark energy field is Higg's boson. For instanve on the case of gravity, we have a set of Higg's bosons that are connected point by point of the space-time.
Which combined scenarios are correct is still something under consideration. We must clarify that "particle" for dark matter, it will not be the same as we use it in general in the current context of physics. When we refer to "particles" for dark matter, we are reffering just to the "smalest" entity that characterises dark matter and can be huge in comparison to the ordinary matter's particles. Another one interesting comment, is whereas the break of SO(4,4) suggests that we have differentt Higg's bosons of different scale, which should mean that we have a "white Higg's boson" and a "dark" one. But, we must proceed and conclude with the properties of the "field" $\mathcal{S} = \hbar\varphi$ which satisfies Eq. (225) . This "field" $\mathcal{S}$ must be invariant under transformations of the group $GL(4, C)$ as Eq. (226) suggests. This group is a Lie complex one, of complex dimension n^2, connected but not simple, with its maximal compact Lie subgroup is $U(n)$ and has a fundamental group isomorphic to $\mathcal{Z}$. Moreover, we can write

$$\mathcal{S} \in GL(4, C) \longrightarrow SO(4, 4) \cap U(4) \tag{273}$$

which means that $\mathcal{S}$ is invariant under transformations both of the extended Lorentz group $S0(4, 4)$ and $U(4)$. After the embedding, the original group breaks further and must be invariant under transformations imposed by Eq. (224). The cap $\cap$ suggests that the generators of the group will be correlated or mixed, combining this way all the generators. At this point, we must remember that in the standard formulation of the standard model, we must start by a symmetry group imposed directly in the covariant derivative, further we introduce a symmetry for the field φ (i.e Higg's $SU()2)$ doublet) and afterwards we seek the transformation rule for the fields A_μ, W_μ, G_μ. In our consideration, we have by the beginning the transformation rule for the "field" $\mathcal{S} = \hbar\varphi$ and afterwards we seek how $\dfrac{\partial \mathcal{S}}{\partial z}$ transforms, which will tell us about the invariance of the Lagrangian. In the case that we want to re-introduce the standard model we must set Eq. (216) in operator form and consider the generised field K_i as

$$\widehat{p_i} = \frac{\partial}{\partial x_i} - i\widehat{K_i} \tag{274}$$

and then seek the invariance of the quantity

$$\widehat{p_i}\mathcal{S} = \frac{\partial}{\partial x_i}\mathcal{S} - i\widehat{K_i}\mathcal{S} \tag{275}$$

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But, by solving with respect to p_g, we will lose the information of Lorentz invariance and we will left only with $U(4)$ invariance. This way we should add by hand the Lorentz invariance. The same picture, comes with the formulation of the standard model, where we impose both Lorentz and $U(1)$, $SU(2)$, $SU(3)$ invariance, plus the ad-hoc addition of Higg's field. The Lagrangian described by Eq. (225) is also invariant under transformations of the group $GL(4,C)$. Now, if λ_i are the generators of $GL(4,C)$, the λ_i coincides with the generators of $U(4) \simeq SU(4) \times U(1)$ (the difference between $U(4)$ and $GL(4,C)$ is the different factors, in the case of $U(4)$ the factors are all real, while in the case of $GL(4,C)$ are all complex). Moreover, $GL(4,C)$ locally falls naturally to $U(4)$ as the maximal compact subgroup and therefore we can write for $\mathcal{S}$

$$\mathcal{S}' \longrightarrow \mathcal{S}' = U\mathcal{S} = e^{i\theta_\alpha H_\alpha}\mathcal{S} \tag{276}$$

where θ_α the parameters and H_α the generators of the group and $\alpha = 1, ...16$. For infinitesimal parameters $\mid \theta_\alpha \mid << 1$ we can have

$$\mathcal{S}'(z) \simeq \big(I_{16} + i\theta_\alpha H_\alpha\big)\mathcal{S}(z) \tag{277}$$

$$\delta\mathcal{S}(z) = i\theta_\alpha H_\alpha \epsilon \mathcal{S}(z) \tag{278}$$

$$\delta\mathcal{S}^*(z) = -i\theta_\alpha H_\alpha \epsilon \mathcal{S}(z) \tag{279}$$

But, after embedding R^4 in C^4, the group $GL(4,C)$ will break according to Eq. (271) and each group as it is presented in Eq. (272), will be accompanied by a different scale, as it is was originally indicated in our first paper [1]. Moreover, the original generators of $GL(4,C)$, will break into different generators of the broken groups that are presented in Eq. (272). An extensive study on Eq. (272),and if there are no mistakes in our treatment of Eq. (271), should be able to provide us, with an extension of standard model, which could be the desired unified theory, by the point of view of symmetry group study.

19 A C^4 model for an extended Standard Model

Actually SM is a combination of two successful theories, the Glashow-Weinberg-Salam theory and QCD, in a semi-unified scheme, because QCD in not broken. There is no mixing between the Lagrangians for Electro-weak and Strong interactions and therefore we have no unification of these interactions. The biggest problem, in order to break the symmtry SU(3)which describes strong interactions is that we have to consider a triplet Higg's field, $\varphi \in C^3$, coupled to gauge field, which seems unatural and inconsistent. As a coscequence, gluons do not interact with Higg's field, as it happens with B_μ and W_μ and thus, they remain massless as a pure Yang-Mills theory dictates. On the other hand, quarks become massive, as the quark fields interact with Higg's field (even if we need an adjustment under the consideration of φ^c). We are still lack of a theory, which could answer whereas gluons are massive (which should be a modification of SM) or maslless in a same way as photons remain massless in GWS theory. In a similar way, SM predicts that neutrinos are massless, while experiments lead to an oppsite assumption and several modifications or ad-hoc mechanisms to SM are proposed, such as flip-flop mechanism, in order to overcome this peculiar and akward situation. In addition, quark confinment and asymptotic freedom, can not be answered within the SM model. In this section, we will use the findings of our approach, in order to propose a possible exit to the above mentioned problems. In section (18), we have seen that the unified scheme that involves electromagnetism and nuclear fields (the generalized electromagnetism as we like to call it), is

$$\frac{SU(4)}{SU(2)} \times SU(2) \times U(1) \tag{280}$$

Let us give some remarks,

(i) The original symmetry group is U(4) and it is connected with the antisymmetric part I_{ij} of the complex metric tensor G_{ij} of the C^4 space-time.

(ii) The U(4) breaks as Eq. (272) as a consequence or cause of the embedding. U(4) is not a compact group as it was expected from other theories, but the scale problem can be resolved in a flip flop mechanism, where just only one scale will be inserted.

(iii) The symmetry groups U(4) and the ones described by Eq. (272), are not anymore ad-hoc or phenomenological symmetries. On the contrary, they came directly from the consideration of the C^4 space-time.

(iv) $\frac{SU(4)}{SU(2)}$ is proposed by our approach, instead of SU(3). This quotient is not a goup at all. It is a coset and it belongs in orbit space. Such cosets that involves SU(n) and SO(n) groups are called in the literature as Stiefel manifolds.

(v) Definition: The Stiefel manifold $V_k(F^n)$ is the set of all orthonormal frames in F^n or the homogeneous space for the action of a classical group in a natural manner. For $F = C^n$ it is isomorphical to

$$V_k(C^n) = \frac{SU(n)}{SU(n-k)} \tag{281}$$

and the dimension is

$$V_k(C^n) = 2nk - k^2 \tag{282}$$

(vi) The coset $\frac{SU(4)}{SU(2)}$ is isomorphical, locally, to $S^7 \times S^5$. Moreover, $SU(2) \simeq S^3$, $U(1) \simeq S^1$. As a consequence our extended SM is locally isomorphical to

$$\frac{SU(4)}{SU(2)} \times SU(2) \times U(1) \simeq S^7 \times S^5 \times S^3 \times S^1 \tag{283}$$

(vii) Strong nuclear field is different to weak nuclear and electromagnetic field, as the strong one is described by a coset, while the other two by groups. Locally they obtain similar structure, through spheres. Strong nuclear field seems to be composite, compared to the other two. This scheme, explains the difference between strong nuclear field and the electroweak one. Furthermore, our theory flavors firstly the unification of the nuclear fields and afterwards the nuclear field, alltogether, with electromagnetic field as the unified nuclear field share the symmetry group

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$$\frac{SU(4)}{SU(2)} \times SU(2) \simeq SU(4) \tag{284}$$

(viii) The breaking of U(4) to Eq. (283), is mathematically unique and has a special place in the literature of Stiefel manifolds, as it is the only one that breaks as a direct product of first order spheres

(ix) A triplet Higg's field is no longer needed. A quatraplet Higg's field is inserted as a natural consequence of the whole proccedure. An su(3) algebra is included in the coset $\frac{SU(4)}{SU(2)}$, which means that our usual SU(3) description still almost exists and redefined, as we can still consider 8 gluons, plus the fact that we have more room to resolve problems such as quark confinment and asymptotic freedom. The properties of the coset, as we would see below, does not lead to the consideration of more than 8 gluons, as the extra dimension of the Stiefel manifolds from 12 to 8, can serve us to include the QCD scale. The extra 4 will be seen as auxiliary fields.

(x) The coset $\frac{SU(4)}{SU(2)}$ breaks as

$$\frac{SU(4)}{SU(3)} \times \frac{SU(3)}{SU(2)} \simeq S^7 \times S^5 \tag{285}$$

as S^7 is the isotropy group of $\frac{SU(4)}{SU(3)}$ and S^5 is the isotropy group of $\frac{SU(3)}{SU(2)}$. In addition all these cosets have simple structure. Specifically, for the cosets we could also write
• SU(4) acts transitively on S^7 ,with isotropy group SU(3)
• SU(4) acts transitively on S^5 via double covering SU(4) $\rightarrow$ S0(6) with isotropy group under the coverimg of the preimage of SO(5), which can be identified with Sp(2)
• SU(3) acts transitively on S^5, with isotropy gtroup SU(2)
• $S^7 \times S^5$ is the homogeneous space of SU(4),with isotropy group Sp(2)$\cap$SU(3)=SU(2) $\rightarrow \frac{SU(4)}{SU(2)} \rightarrow S^7 \times S^5$

(xi) We could also insert coupling constants as

$$g\frac{SU(4)}{SU(2)} \times g_2 SU(2) \times g_1 U(1) \rightarrow \tag{286}$$

$$g^{'} S^7 \times g^{''} S^5 \times g_2 S^3 \times g_1 S^1 \tag{287}$$

where g breaks to $g^{'}$ and $g^{''}$. Our usual g_3 coupling constant of strong nuclear field will arise as a mix of $g^{'}$ and $g^{''}$

In order to procced further with the generators of the coset $\frac{SU(4)}{SU(2}$, we should start with SU(4)). The generators of SU(4) are λ_i, $i = 1, 2, ...16$. From these 15 matrices, in order to procced with the coset, we must exclude the matrices $[\lambda_1, \lambda_2, \lambda_3]$ and the coset will have

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$$\frac{SU(4)}{SU(2} \to [\lambda_4, \lambda_5, ..., \lambda_{15}] \tag{288}$$

We break λ_8 and λ_{15} as

$$\lambda_8 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \lambda_8^{'} + \lambda_8^{''}$$

$$\lambda_{15} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \lambda_{15}^{'} + \lambda_{15}^{''}$$

Then $[\lambda_6, \lambda_7, \lambda_8^{'}]$ consist an su(2) algebra as

$$[\lambda_6, \lambda_7, \lambda_8^{'}] \to \begin{pmatrix} 0 & \\ & su(2) \end{pmatrix}$$

and $[\lambda_6, \lambda_7, \lambda_8^{'}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}^{'}]$ consist an su(3) algebra as

$$[\lambda_6, \lambda_7, \lambda_8^{'}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}^{'}] \to \begin{pmatrix} 0 & \\ & su(3) \end{pmatrix}$$

As concerned the existence of $\lambda_8^{'}$, $\lambda_8^{''}$, $\lambda_{15}^{'}$, $\lambda_{15}^{''}$, either we can keep as above mentioned, where $\lambda_8^{''}$, $\lambda_{15}^{''}$) will be additional auxiliary fields or by the beginning we can change basis for $[\lambda_3, \lambda_8, \lambda_{15}] \to [\lambda_3^{'}, \lambda_8^{'}, \lambda_{15}^{'}]$, where the new basis will be expressed as linear combination of the components of the old basis. Furthermore, using appropriate coefficients, we can double the su(2) algebra and form an su(2)× su(3) algebra. As concerned, the rest generators of the coset that do not participate in the su(3) algebra i.e $[\lambda_4, \lambda_5, \lambda_9, \lambda_{10}]$, we have 4 auxiliary fields that do not count as natural degrees of freedom, but can be served to create "mass scales" such as QCD scale. In the case that we include $\lambda_8^{''}, \lambda_{15}^{''}$ we wlii have 6 auxiliary fields (at this point we do not know if it is necessery or not). An interesting point of this analysis, is that we can have a fresh look, as it comes from the spheres. The full coset has 12 dimensions as

$$\frac{SU(4)}{SU(2)} \simeq S^7 \times S^5 \longrightarrow 12 \to 7 + 5 \tag{289}$$

If we analyse these dimensions, the 12 generators will break as follows
• Some of the generators of the su(3) algebra that represent gluons, would be assigned to S^7 and some others to S^5
• Some of the remaining generators that do not produce the su(3) algebra and represent the auxiliary fields would be assigned to S^7 and some others to S^5

As a consequence we face two possible interpretations
a) If the algebra su(3) represents gluons, then there are two types of gluons, the ones that comes from S^7 and the others to S^5, with differences among them.
b)The gluons are same among them, but "come" as a linear combination of two different field arising from S^7 and S^5
Which case is valid, will tell us how to treat the coupling constants. Moreover, the auxiliary fields, could help us to explain in a different way the residual nuclear field. In this spirit, a new covariant derivative should be formed to be connected to the coset $\frac{SU(4)}{SU(2)}$ as

$$D_\mu = \partial_\mu - iG_\mu - i[\quad , \quad] \tag{290}$$

where in the Lie bracket, we should find the auxiliary fields in order to close the su(4) algebra, in the form $[S_i, S_j]$ or $[\partial_i, S_j]$ where S_i are the auxiliary fields. In order, to answer to the type of the gluons, we should further break the avove mentioned covariant derivative, with respect to Eq. (288). The Lie bracket, automatically suggests that the propagator connected to the covariant derivative, will be affected by the auxiliary fields (we can found a similarity with Gribov and Faddeev–Popov ghost theories). All the above mentioned analysis, leads to a prediction of our approach, that we should consider a $\varphi \in C^4$ model, where this field is not anymore an ad-hoc consideration but ruther is indicated by the geometry, where φ will be written as

$$\varphi = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \\ \varphi_5 + i\varphi_6 \\ \varphi_7 + i\varphi_8 \end{pmatrix}$$

This C^4 model, allows the strong nuclear field to fully participate in the Higg's mechanism. In addition, the quatraplet Higg's field φ, can be seen as "doublet doublet". In this sense, if in the well known C^2 G. W. S model we consider the field φ as

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^o \end{pmatrix}$$

where we denote this way the "charged" and "neutral " part, in the C^4 model, we should incude new symbols to denote whereas the components of the C^4 field interacts "charged" or "netral" in the sense of the strong nuclear "charge". In the case of the full covariant derivative, including nuclear and electromagnetic fields, there will be a full hypercharge $Q^{'}$ with $Q^{'}\varphi = 0$ derived from the combination $[I, T_3, T_8, T_{15}]$, which will break as

$$Q^{'}\varphi \to Q\varphi + Q^{''}\varphi \tag{291}$$

where Q is the usual hypercharge and $Q^{''}$ the hypercahrge corresponded to the coset. Moreover, the choice of a quatraplet field, automatically means for the case of fermions, that we have to consider a unified fermion quatraplet as

$$f = \begin{pmatrix} l_i \\ \nu_i \\ u_i \\ d_i \end{pmatrix}_x \rightarrow \frac{\begin{pmatrix} l_i \\ \nu_i \end{pmatrix}_L}{\begin{pmatrix} u_i \\ d_i \end{pmatrix}_B}$$

where $i = 1, 2, 3$ as $l_{1,2,3} = (e, \mu, \tau)$, $n_{1,2,3} = (\nu_e, \nu_\mu, \nu_\tau)$, $u_i = (up, charm, top)$, $d_i = down, strange, bottom)$ and x is a new partivle number that unifies L and B. Moreover, in this spirit, we will be able to reduce the existing particle numbners $(Q, T_3, I_3, S, C, B^{'}, T, L, B)$ to just six, which seems logical, due to the fact that there are six "charges" in SM, one for elecrtomagnetism, 2 "charges" for weak nuclear and three "charges" for strong nuclear field. Six "charges" means six particle numbers. These three particle numbers can be seen as

$$\begin{aligned} up &\rightarrow A \\ charm &\rightarrow B \longrightarrow T_3 \uparrow \\ top &\rightarrow C \\ down &\rightarrow A \\ strange &\rightarrow B \longrightarrow T_3 \downarrow \\ bottom &\rightarrow C \end{aligned}$$

$$\begin{aligned} e &\rightarrow A \\ \mu &\rightarrow B \longrightarrow T_3 \uparrow \\ \tau &\rightarrow C \\ \nu_e &\rightarrow A \\ \nu_\mu &\rightarrow B \longrightarrow T_3 \downarrow \\ \nu_\tau &\rightarrow C \end{aligned}$$

where A, B, C are new particle numbers, and the quarks and leptons are distinguished by L, B. But as x unifies L, B, we just need six particle numbers as

$$\big(Q, A, B, C, T_3, x\big)$$

where A, B, C are new assigned paricle numbers that form a 3-d representation and present the kind of identification such as

$$\begin{aligned} e &\to (A, O, O) \\ \nu_e &\to (A, 0, O) \\ u &\to (A, O, O) \\ d &\to (A, O, O) \end{aligned}$$

and the final identity will be defined from T_3 and thye breaking of x to L, B.

20 Extended Dirac's equation-fermionic geoemtry-Quantisation

Until this part of the manuscript, we have identified, that only bosons have appeared in our consideration and that they were appeared or constructed by the usual geometry of quadratic forms, leaving no room to introduce fermions by the geometry itself. But in general, in the usual context of physics, spinors can be defined and introduced mathematically by the geometric algebra, which is actually the Clifford algebra of a defined vector space equipped with a quadratic form. The most famous example is the Dirac spinor denoted as ψ which is the solution of Dirac equation. In this paper we want to investigate several ways to introduce fermions within our consideration. We would like also to add a crucial comment of M. Atiyah, which show us the difficult situation that spinors have put us.

"No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the "square root" of geometry and, just as understanding the square root of -1 took centuries, the same might be true of spinors"

20.1 Dirac's approach

Let us start by exploring Dirac's approach by the beginning. In the Minkowski space we have the elementary length

$$ds^2 = n_{\mu\nu} dx^\mu dx^\nu \tag{292}$$

where n_{ij} is the Minkowski metric tensor. The action

$$S = mc \int ds \tag{293}$$

lead us to the geodesic and Hamilton-Jacobi equations. From Hamilton-Jacobi equation we derive the energy-momentum 4-vector

$$P = (\frac{E}{c}, p) \tag{294}$$

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and the squared P give us the energy-momentum relation

$$E^2 = p^2c^2 + m^2c^4 \tag{295}$$

Afterwards, we pass to classic quantum physics by introducing the operators $\widehat{p}$ and $\widehat{E}$, which lead us to the Klein-Gordon equation in natural units

$$\left(n^{\mu\nu}\partial_\mu\partial_\nu - m^2\right)\varphi = 0 \tag{296}$$

At this point, everything started by an elementary length defined by a quadratic form in the flat Minkowski space and we were led to bosons. This is mainly the "road" or way of thinking that we have used in our consideration, in order to derive a bosonic equation in the curved C^4 space, as he wave already seen . Afterwards, Dirac managed to derive the square root of the Hamiltonian by introducing a new Hamiltonian of first order derivatives in time and space as

$$H = \alpha p + \beta m \tag{297}$$

where $\alpha^2 = \beta^2 = 1$ and $\{\alpha, \beta\} = 0$ and finally has given us the Dirac equation

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0 \tag{298}$$

and the Dirac matrices satisfy the algebra formed by

$$\{\gamma^\mu, \gamma^\nu\} = n^{\mu\nu}I_4 \tag{299}$$

As a natural step, we can repeat the whole procedure if we start by our Klein-Gordon type equation in the flat C^4 or R^8 given by the Lagrangian of section 16

$$L = \frac{\partial\mathcal{S}}{\partial x^\mu}\frac{\partial\mathcal{S}^*}{\partial x_\mu} - \frac{\partial\mathcal{S}}{\partial y^\mu}\frac{\partial\mathcal{S}^*}{\partial y_\mu} \tag{300}$$

or by introducing, as we have seen, $\mathcal{S} = \varphi$ in natural units

$$L = \frac{\partial\varphi}{\partial x^\mu}\frac{\partial\varphi^*}{\partial x_\mu} - \frac{\partial\varphi}{\partial y^\mu}\frac{\partial\varphi^*}{\partial y_\mu} \tag{301}$$

This way, a Dirac equation type will be

$$i\Gamma^\mu\partial^c_\mu\Psi = 0 \tag{302}$$

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where ∂^c_μ is the Cauchy derivative and Γ^μ are the Gell-Mann matrices in the flat space R^8 given by the relation

$$\{\Gamma^\mu, \Gamma^\nu\} = n^{\mu\nu} I_8 \tag{303}$$

and $n^{\mu\nu}$ is the flat metric tensor in R^8 with signature $(4, 4)$. The only left is to identify the type of the spinor Ψ which accompanies this signature. In general, every spinor in any space is defined by its signature. As a consequence we must give the following informations [1] [3] [26]

1. in an even d=p+q dimensions metric tensor

   $n^{\mu v} = diag(++++, ----)$gamma matrices Γ^μ satisfies the Clifford algebra

   $\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\psi = 2n^{\mu\nu}$

2. In $p - q = 0mod8$ we have a Clifford algebra $C(4, 4)$ and because $p + q = 2 \times 4$ we have a real $2^4 = 16$ representation.

3. For $p - q = 0mod8$, $p + q = 0mod8$ we have a unique irreducible representation which is a real Majoran-Weyl one.

4. The M-W spinors satisfy both of the following conditions:

   $\gamma^{(d+1)}\psi = \psi$, $\overline{\psi} = \psi^+ C_\pm$

   and exist only if $p - q = 0mod8$ for us $d = p + q = 8$

5. If $d = p+q = 0mod8$ we have only kinetic terms in the Lagrangian of the form K_{xy}.

   Where K_{xy}:

   $K_{xy} = \Psi^T_R C\Gamma^\mu \partial_\mu \Psi_L + \lambda \Psi^T_L C\Gamma^\mu \partial_\mu \Psi_R$

6. For the $(4, 4)$ signature the (4_s+4_A)- representation of Γ matrices has to be employed for both values on $n = \pm 1$ in order to provide a M-W basis.

7. The gamma matrices are given by (3)

$$\Gamma^9 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix},$$

$$\Gamma^i = \begin{pmatrix} 0 & \sigma^c \\ \overline{\sigma}^c & 0 \end{pmatrix} i = 1...8$$

   (as presented in appendix 2 in [3])

   Where $\overline{\sigma}_i = -\overline{\sigma}_i^T$, $i = 1...4$ $(antisymmetric)$

   $\overline{\sigma}_i = \overline{\sigma}_i^T$, $i = 5, 6, 7, 8$ $(symmetric)$ and the diagonal charge-conjucation matrices are given by:

$$C^{-1} = 1_4 \oplus -1_4$$

$$\overline{C}^{-1} = -nC^{-1}$$

the terms in the Lagrangian will have the general form

$$i\overline{\Psi}C\Gamma^{\mu}\widehat{P_{\mu}}\Psi = i\overline{\Psi}C\Gamma^{\mu}\big((\widehat{p_{\mu}})_g + i(\widehat{p_{\mu}})_I\big)\Psi \tag{304}$$

and solving to p_g, in the case of flat space

$$i\overline{\Psi}C\Gamma^{\mu}\frac{\partial}{\partial z^{\mu}}\Psi = i\overline{\Psi}C\Gamma^{\mu}\big(\frac{\partial}{\partial x^{\mu}}\Psi + i\frac{\partial}{\partial y^{\mu}}\big)\Psi \tag{305}$$

and if

$$\Gamma^{\mu}\frac{\partial}{\partial y^{\mu}}\Psi = m\Psi \tag{306}$$

we will have the usual Dirac term

$$i\overline{\Psi}\gamma^{\mu}\frac{\partial}{\partial y^{\mu}}\Psi - m\overline{\Psi}\Psi = 0 \tag{307}$$

The question that must be answered is how the original M-W representation, breaks after embedding. In order to get a straight-full answer, we should study extensively the broken symmetry groups as they were presented in section 19. But even if we manage such a task, there will remain two big problems

1. Within our consideration, in the case of bosons, we have started by a ds^2, we have found the geodesic equation, we have formed the H-J equation and after the second extremisation, we have identified the action $\mathcal{S}$ as the field. But, in Dirac approach we cannot identify the original definition of the spinor Ψ as we have done with φ. We do not have a same procedure for fermions, similar to the case of bosons. The only thing left is to guess that

$$\mathcal{S}_f = \sqrt{\hbar}\Psi \tag{308}$$

   In our opinion we should seek a better more geometrically way

2. The second problem, is related with the proper definition of a unified Lagrangian. In standard model we have three different Lagrangians, L_f for fermions, L_b for bosons and L_{int} for interactions of the fields $F_{\mu\nu}F^{\mu\nu}$, $W_{\mu\nu}W^{\mu\nu}$, $G_{\mu\nu}G^{\mu\nu}$ forming a unified Lagrangian as

$$L = L_f + L_b + L_{int} \tag{309}$$

   plus the Higg's mechanism. The question is how we can add these Lagrangians mathematically or geometrically, in the sense of a bigger geometry that can describe the unified Lagrangian. There should exist some geometric form d, where the Lagrangian of this form $L(d)$, should break .

In the two following paragraphs, we will try to investigate and propose some answers in a preliminary base.

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20.2 Cartan's property of triality

We have seen in the first paper, about Cartan's property or principle of triality, concerning a triality that the three signatures $(4, 4)$, $(8, 0)$, $(0, 8)$ share. We will refer to this, as signature triality. But, there is another one triality that exists, as it was illustrated in [1] [3] [26], that we will refer as spinor-vector triality.
"Let us conclude that Triality can be seen not only as a source of duality-mappings, but as an invariance property. In the original Cartan's formulation this is seen as follows. At first, a group G of invariance is introduced as the group of linear homogeneous transformations acting on the $8 \times 3 = 24$ dimensional space, leaving invariant, separately, the bilinears B_V, B_{S^+}, B_{S^-} for vectors, chiral and antichiral spinors respectively (the spinors are assumed commuting in this case) plus a trilinear term T. Next, the Triality group G_{Tr} is defined by relaxing one condition as the group of linear homogeneous transformations leaving invariant T and the total bilinear B_{sum}:

$$B_{sum} = B_V + B_{S^+} + B_{S^-} \tag{310}$$

it can be proven that G_{tr} is given by the semidirect product of G and S_3

$$G_{tr} = G \otimes S_3 \tag{311}$$

Let us consider $V = R^8 \equiv C^4$ then the signatures $(4, 4) \leftrightarrow (0, 8) \leftrightarrow (8, 0)$ concludes a Majorana-Weyl representation for S^+, S^-. Moreover, S^+, S^- are necessary 8 dimensional real spaces. As a result B_V, B_{S^+}, B_{S^-} are each one invariant under $SO(8)$ creating the product $SO(8) \times SO(8) \times SO(8)$ for the B_{sum}. Consequently, in our case $G = SO(8)$ and the group that leaves invariant T is $SO(8) \otimes S_3$ or $Spin(8)$. Let us consider as p the number of plus(+) in signature, q the number of minus(-) and d the dimension of the space

$$(p, q) = (4, 4) \quad d = p + q = 8 \tag{312}$$

Then $d = 0mod8$ and $p - q = 0mod8$. From d and p-q we can conclude that we have a real 8 dimensional Majorana-Weyl representation for the spinor spaces and that the group of automorphisms is $SO(8)$. Let us consider $(V, G), (S^+, s), (S^-, s)$ where $V = C^4 \equiv R^8, G$ the hermitian metric tensor S^+, S^- 8 dimensional spinor spaces and s spin invariant inner-product

$$s = \overline{(\psi, \varphi)} = (\psi^c, \varphi^c) \qquad \forall \varphi, \psi \in \tag{313}$$

Moreover (V, G) , (S^+, s) , (S^-, s) are isomorphical as orthogonal spaces and the Triality B ensures the isometry (because of S_3) between the spaces. Once again, s will be the charge-conjugation operator C, which preserves the spinor spaces and it can be used to raise and low indices. In order to unify the scheme between V, S^+, S^- we could "bosonise" S^+, S^- or "fermionise" V

$$B_V = V_m^T (g^{-1})^{mn} V_n \tag{314}$$

$$B_{S^+} = \Psi^T C^{-1} \Psi \tag{315}$$

$$B_{S^-} = X^T C^{-1} X \tag{316}$$

$$T_{Tr} = \Psi^T C \Gamma^m \Psi = 2(\Psi^T C^{-1} \sigma^m X V_m) \tag{317}$$

where $\Psi = \begin{pmatrix} \Psi_a \\ X_a \end{pmatrix}$, $a = 1, ..., 8$. This trilinear form can serve us, as the geometric form d whose Lagrangian $L(tr)$ is the unified one. Moreover, this trilinear form, breaks into the three above mentioned spaces , where each one can form as associated Lagrangian as, L_{S^+}, L_{S^-} and L_V. Thus, through Cartan's spinor-vector triality, we have a more consistent way to present the addition of L_F and L_b. In addition, we have the original symmetry of the trilinear form, given by the group $Spin(8)$. This way, we can propose an answer, concerning the second above mentioned question. But, the first one, still cannot be answered.

20.3 New geometric structure-Quantisation

We will try, to propose a new way, in order to give some satisfactory answers in both questions, in a preliminary basis. We will take Atiyah's expression literally about the square root of the geometry. Let us start with an R^n space in the beginning and an elementary length as

$$ds^2 = g_{ij} dx^i dx^j \tag{318}$$

where g_{ij} is a bilinear or 2-linear form, which is the usual symmetric metric tensor. We van also, define the length

$$ds = \sqrt{g_{ij} dx^i dx^j} \tag{319}$$

as it exists in the usual context of geometry. Furthermore, we will impose the following question; Can we define the elementary length ds in a different way using a 1-linear form instead? Let us consider that there exists a 1-liner form g_i, so that the elementary length can be written as

$$ds = g_i dx^i \tag{320}$$

If this expression is valid, it must satisfy the relation

$$ds * ds = \left(g_i dx^i\right) * \left(g_j dx^j\right) \equiv g_{ij} dx^i dx^j \tag{321}$$

where(*) is a product that must be defined. But, it must also satisfy a second relation also as

$$g_i dx^i \equiv \sqrt{g_{ij} dx^i dx^j} \tag{322}$$

If this consideration could be proved, we would have the advantage to study a new geometric structure , provided by the 1-linear form. This way, we could define an action $S(g_i)$, which would driven us to the derivation of geodesic equation with respect to g_i. Afterwards, we could continue by expressing the associated Hamilton-Jacobi and finally by a second extremisation, we should be able to find a "Dirac type "equation in R^n. Now, let us consider, that all these can be accomplished(we will give some proposals about this task below) and define a new geometric quantity $f(d_2)$ as

$$d_2 = ds + ds^2 \tag{323}$$

and equip the space R^n by the geometry that $f(d_2)$ defines as

$$\big(R^n, f(d_2)\big) \equiv \big(R^n, g_i, g_{ij}\big) \tag{324}$$

This consideration, leads us to the combination of functional analysis and differential geometry. This combination could be nicely expanded by defining series of different equipped geometries as

$$d_n = \sum a_i ds^i = a_0 + a_1 ds + a_2 ds^2 + ... + a_n ds^n \tag{325}$$

and equip R^n with $f(d_n)$, where we can define n-linear forms as

$$ds = g_i dx^i$$

$$ds^2 = g_{ij} dx^i dx^j$$

$$ds^3 = g_{ijk} dx^i dx^j dx^k$$

$$.$$

$$.$$

$$ds^n = g_{ij...n} dx^i dx^j .....dx^n$$

We will return now, in the case of ds. If

$$ds^2 = g_{ij} dx^i dx^j = (a_i dx^i)(b_j dx^j) \tag{326}$$

it must satisfy $a * b = (a_i b_j)$. If g_{ij} symmetric

$$g_{ij} = a_i b_j = a_j b_i \longrightarrow a_i b_j - a_j b_i = 0 \longrightarrow a * b = 0 \tag{327}$$

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But, then the matrix

$$C = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ . & . \\ . & . \\ a_n & b_n \end{pmatrix}$$

must be of rank 1, which means

$$b = \lambda a \longrightarrow a_i b_j = \lambda a_i a_j \longrightarrow g_{ij} = a_i a_j \tag{328}$$

But the relation $g_{ii} = a_i^2$ must hold, which means

$$g_{ii} = a_i^2 \longrightarrow a_i = \pm\sqrt{| g_{ii} |} \tag{329}$$

but this way we will have

$$g_{ij} = \left(\pm \sqrt{| g_{ii} |}\right)\left(\pm \sqrt{| g_{jj} |}\right) \tag{330}$$

which are the extra conditions. In the case of a Hermitian space we have an elementary length

$$ds^2 = G_{ij} dz^i d\overline{z}^j = a_i \overline{b}_j dz^i d\overline{z}^j = \left(a_i dz^i\right)\left(\overline{b_i dz^j}\right) \tag{331}$$

A natural generalisation leads us to the relation $a * b = (a_i \overline{b}_j)$. If, $G_{ij} = a_i \overline{b}_j$, then $G_{ji} = a_j \overline{b}_i$ and $\overline{G}_{ji} = \overline{a}_j b_i$ because it must be $\overline{G}_{ji} = G_{ij} \longrightarrow a_i \overline{b}_j = \overline{a}_j b_i$. If, we set

$$b_i = \lambda a_i \longrightarrow \overline{\lambda} a_i \overline{a}_j - \lambda \overline{a}_j a_i = 0 \longrightarrow (\overline{\lambda} - \lambda) a_i \overline{a}_j = 0 \longrightarrow \lambda \in R \tag{332}$$

but then $G_{ij} = \overline{\lambda} a_i \overline{a}_j$ or $G_{ij} = a_i \overline{a}_j$ and

$$G_{ji} = a_j \overline{a}_i \longrightarrow \overline{G}_{ji} = \overline{a}_j a_i = G_{ij} \tag{333}$$

which means that G_{ij} is Hermitian as it should be and the consideration holds. As a consequence, we can write

$$ds^2 = \left(a_i dz^i\right)\left(\overline{a_j dz^j}\right) \tag{334}$$

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and it must be also be

$$G_{ii} = a_i\overline{a} = |a_i|^2 \longrightarrow |a_i| = \sqrt{G_{ii}} \tag{335}$$

but because $G_{ij} = a_i\overline{a}_j$ we can have

$$|G_{ij}| = |a_i||\overline{a}_j| = \sqrt{G_{ii}}\sqrt{G_{jj}} \tag{336}$$

which expresses the extra conditions. It seems that G_{ij} is $\sqrt{G_{ii}}\sqrt{G_{jj}}$ multiplied by an amplitude φ! If we return in Eq. (325), this power series could be presented by a function $f(ds)$. For example, if this power series represents a geometric one, the elementary length of this geometry will be

$$\frac{1}{1-ds} \tag{337}$$

or if it is presented by an exponential function will be

$$e^{ds} \tag{338}$$

The next step is to define series of the form of Eq. (325) in complex spaces C^n. The interesting part for physics, is to pass from an elementary length ds^n which form the series d_n, to the Lagrangian which is expressed by d_n and let us symbolize it as $L(d_n$. Then, it is natural to investigate how $L(d_n$ breaks into several pieces as $L(d_1$, $L(d_2$, ... In the most simple case we could imagine a picture as

$$L(d_n) = L(d_1) + L(d_2) + ... \tag{339}$$

where $L(d_1)$ could be linked with fermions, $L(d_2$ with bosons and the other terms with interaction terms. This way, we will meet our consideration of a physical theory in C^4. An interesting task, would be to investigate whereas the action

$$S_\pi = \int Z\sqrt{G}d\Omega \tag{340}$$

can be derived naturally, by an action presented by the defined geometry with elementary length of ds^4 where the variation should be with respect to the length and not with respect to the metric tensor. If the dimension of the space can be written as $n = 2k$, there will be k elementary physical objects defined by $ds, ds^2, ...ds^k$, while the rest will be composite objects. Accordingly, in C^4 there will be only two elementary objects, one presented by ds (fermions, $f\overline{f}$) and the second presented by ds^2 (bosons, $b\bar{b}$ and $b \longrightarrow f\overline{f}$). The other powers will present composite objects. For instance, ds^3, will present the interaction of $b\bar{b}f\overline{f}$ and ds^4 $b\bar{b}b\bar{b} \longrightarrow f\overline{f}f\overline{f}f\overline{f}f\overline{f}$ and we can work the same way with the other powers. But finally, the whole series could present by a function $f(ds)$, where all the interactions could at once presented by this function. This function eventually, could give us all the information needed.

21 Conclusion

We have formulated a theory in C^4 space-time, in order to present a new framework which give us enough room to incorporate gravity, electromagnetism and nuclear fields in a geometrical way. An extended special and general relativity is a result of the hypothesis of C^4 space-time, which lead us to new phenomena and new point of view. An embedding procedure is performed which connects our usual 4-d space-time with the 4-d complex space-time. Through the embedding procedure, we are able to get in contact with observables. Our findings can be summarised as

1. Possible definitions for dark matter and dark energy
2. Similarities, connections between dark matter, dark energy, ordinary matter and radiation
3. Dark energy is not constant, but rather dynamic entity that varies with time. It give us the opportunity to give explanation to the recent akward measurements for Hubble parameter
4. A new Hubble law is derived, as a result of the extended special relativity
5. A new second constant is revealed, besides speed of light, which changes the concept of propagating information and could give answers to EPR problem
6. Quantum theories are identified as a classic mechanics theory in C^4 space-time. The Hamilton-Jacobi equation for $S = \hbar\varphi$, gave us the extended Klein-Gordon equation, which connects Higg's boson with Cosmology.
7. An extension to the standard model of particles
8. It is crucial to evaluate λ as it exists in the embedding functions

All the presented context of this manuscript, is a continuous effort for consistency, between theories existing in Physics. This effort, in our oppinion, seems promising, as we have managed to connect different areas of Physics such as general relativity, quantum theories. cosmology under the same framework, plus enough "room" for new phenomena. Nevertheless, much more must be accomplished in the future, in order to finally test this effort as

1. Analytical presentation of the extended general relativity part by part
2. An information theory in C^4, which will give us the chance to define etities such as entropy, temperature and pressure
3. Density terms for dark matter and dark energy must be derived
4. A final particle theory for dark matter, in accordance with the scenarios presented
5. An investigation, whereas the extended standard model can explain quark confinment and asyptotic freedom
6. A coupling constant for dark matter must be identified

These "must do " points are not matters that our theory can not solve, the necessary material, already exists in this manuscript and further calculation and processing is just required.

22 Appendix A

The full Poincare noncovariant form for signature (1,3) are

1. $[J_m, P_n] = i\varepsilon_{mnk}P_k$

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2. $[J_i, P_o] = 0$
3. $[K_i, P_k] = in_{ik}P_o$
4. $[K_i.P_o] = -iP_i$
5. $[J_m, J_n] = i\varepsilon_{mnk}J_k$
6. $[J_m, K_n] = i\varepsilon_{mnk}K_k$
7. $[K_m, K_n] = -i\varepsilon_{mnk}J_k$

On the other hand the Poincare group for the Galilean case in 4 dimensions are

1. $[J_m, P_n] = i\varepsilon_{mnk}P_k$
2. $[J_i, P_o] = 0$
3. $[K_i, P_k] = 0$
4. $[K_i.P_o] = iP_i$
5. $[J_m, J_n] = i\varepsilon_{mnk}J_k$
6. $[J_m, K_n] = i\varepsilon_{mnk}K_k$
7. $[K_m, K_n] = 0$

This way the Poincare group for (3+1,1+3) is represented

1. $[J_{Rm}, J_{Rn}] = i\varepsilon_{mnk}J_{Rk}$
2. $[J_{Rm}, K_{Rn}] = i\varepsilon_{mnk}K_{Rk}$
3. $[K_{Rm}, K_{Rn}] = -i\varepsilon_{mnk}J_{Rk}$
4. $[J_{Mm}, J_{Mn}] = i\varepsilon_{mnk}J_{Mk}$
5. $[J_{Mm}, K_{Mn}] = i\varepsilon_{mnk}K_{Mk}$
6. $[K_{Mm}, K_{Mn}] = -i\varepsilon_{mnk}J_{Mk}$
7. $[J_{Mm}, J_{Rn}] = i\varepsilon_{mnk}J_{MRk}$
8. $[J_{Rm}, K_{Mn}] = i\varepsilon_{mnk}K_{MRk}$
9. $[J_{Mm}, K_{Rn}] = i\varepsilon_{mnk}J_{MRk}$
10. $[K_{Mm}, K_{Rn}] = 0$
11. $[J_{Rm}, P_{Rn}] = i\varepsilon_{mnk}P_{Rk}$
12. $[J_{Ri}, P_{Ro}] = 0$
13. $[K_{Ri}, P_{Rk}] = in_{ik}P_{Ro}$
14. $[K_{Ri}.P_{Ro}] = -iP_{Ri}$
15. $[J_{Mm}, P_{Mn}] = i\varepsilon_{mnk}P_{Mk}$
16. $[J_{Mi}, P_{Mo}] = 0$
17. $[K_{Mi}, P_{Mk}] = in_{ik}P_{Mo}$
18. $[K_{Mi}.P_{Mo}] = -iP_{Mi}$

while for the (4,4) case:

1. $[J_{Rm}, J_{Rn}] = i\varepsilon_{mnk}J_{Rk}$
2. $[J_{Rm}, K_{Rn}] = i\varepsilon_{mnk}K_{Rk}$
3. $[K_{Rm}, K_{Rn}] = 0$
4. $[J_{Mm}, J_{Mn}] = i\varepsilon_{mnk}J_{Mk}$
5. $[J_{Mm}, K_{Mn}] = i\varepsilon_{mnk}K_{Mk}$

6. $[K_{Mm}, K_{Mn}] = 0$
7. $[J_{Mm}, J_{Rn}] = i\varepsilon_{mnk} J_{MRk}$
8. $[J_{Rm}, K_{Mn}] = i\varepsilon_{mnk} K_{MRk}$
9. $[J_{Mm}, K_{Rn}] = i\varepsilon_{mnk} J_{MRk}$
10. $[K_{Mm}, K_{Rn}] = -i\varepsilon_{mnk} J_{MRk}$
11. $[J_{Rm}, P_{Rn}] = i\varepsilon_{mnk} P_{Rk}$
12. $[J_{Ri}, P_{Ro}] = 0$
13. $[K_{Ri}, P_{Rk}] = 0$
14. $[K_{Ri}.P_{Ro}] = iP_{Ri}$
15. $[J_{Mm}, P_{Mn}] = i\varepsilon_{mnk} P_{Mk}$
16. $[J_{Mi}, P_{Mo}] = 0$
17. $[K_{Mi}, P_{Mk}] = 0$
18. $[K_{Mi}.P_{Mo}] = iP_{Mi}$

23 Appendix B

In 1925 E.Cartan in his original paper "Le principe de dualite et la theorie des groupes simples et semi-simples" discovered that 8-d space has a unique property. Cartan's original statement [9] is:
"Given an element A of SO(8) then there exist elements B and C of SO(8), unique up to sign, such that for any two Cayley numbers x and y in R^8, $A(x)B(y) = C(xy)$ where $A(x)$ denotes the action of A on the vector x and $A(x)B(y)$ denotes the product of the Cayley numbers $A(x)$, $B(y)$. The passage from A to B is induced by an explicit outer automorphism of order 3 of the Lie algebra so(8) of SO(8) and the passage from A to C is induced by an explicit outer automorphism of order 2 of so(8). These outer automorphisms leave fixed each element of the Lie subalgebra g_2 of the exceptional Lie group G_2 of all automorphism of the Cayley algebra."
But the automorphisms of the Lie algebra so(8) (which is the first algebra of the series $D_4, D_5, ..$) lifts to an automorphism of the Lie group Spin(8) which is the universal cover of SO(8). The fixed point of that automorphism is the exceptional group G_2. The definition of Spin(n) group is:
Definition: The group Spin(n) is

$$Spin(Q) = s \in CL(Q)_O : ss^* = 1, sVs^* \subseteq V$$

where V is a vector space and CL(Q) the Clifford geometric algebra of Q. Thus Cartan's statement can be generalised as it is presented in [8]:
"From the theory of Clifford algebras one obtains two non equivalent real spin representations, $\Delta_i : Spin(8k) \longrightarrow SO(2^{4K-1})$, $I = 1,2$ for $K \geq 1$. The vector representation is by definition the universal covering homomorphism $\Delta_0 : Spin(8k) \longrightarrow SO(8K)$ determined up to an homomorphism of $SO(8k)$. The center of Spin(8) is $Z_2 \oplus Z_2$ which has three elements of order two ω_0, ω_1, ω_2 such that ω_i generates the kernel of Δ_i for $i = 0, 1, 2$. Any automorphism of Spin(8k) that is induced by an outer automorphism of SO(8k) fixes Δ_0 and interchanges Δ_1 and Δ_2. If $k = 1$ then each Δ_i maps Spin(8) onto SO(8) hence each Δ_i may be viewed as a covering homomorphism. Furthermore, the group of homomorphisms of Spin(8) modulo the subgroup of inner automorphisms is isomorphic to S_3, the permutation group of $0, 1, 2$ and permutes the Δ_i as can be seen from the Dynkin diagram of SO(8). In this case, for each permutation ijk of $0, 1, 2$ there is an embedding

$$Spin(8) \longmapsto SO(8) \times SO(8) \times SO(8)$$

defined by the correspondence $\xi \mapsto (\Delta_i(\xi), \Delta_j(\xi), \Delta_k(\xi))$, $\xi \in Spin(8)$. This statement is essentially the principle of triality". The proofs were presented in [8] [10].

It is time to see what kind of "structures" have have this triality property. We can find two different uses of triality [1].

1. We have a triality property between $(4,4)$, $(8,0)$, $(0,8)$ signatures that we will call signature's triality, which is a S_3 symmetry (every two of the signatures automatically concludes the other) and has a symmetrical Dykin diagram D_4. So from the signature's triality the three signatures $(4,4)$, $(8,0)$, $(0,8)$ are all correlated and equivalent, which means that the three ones can be "unified" and can be seen as one. This is the most extraordinary and useful fact in order to find an independed signature framework to work. The $(8,0)$, $(0,8)$ signatures provide us a pure octonionioc structure while the (4,4) a real one. We have to mention that only those three signatures share the triality property.
2. We have a triality property between vector and spinor spaces that we will call internal triality. Let us consider a vector space V, S^+ chiral spinor space and S^- antichiral spinor space, then we have the internal triality which unifies V, S^+, S^- to one form giving us the ability to define representations from one space to another; every two of them automatically concludes the other (S_3 symmetry and a D_4 Lie algebra with a Dykin diagram D_4). Each one of V, S^+, S^- is invariant under SO(8) and the unified form under Spin(8). Of course V is 8 dimensional space. Specifically if we define (V, g), (S^+, s^+), (S^-, s^-) vector space and spinor spaces respectively, where g is the metric tensor and s^+, s^- analogous tensors (charge conjugate matrices) in order to lower-raise spinor indices we can define a trilinear form which is the unification of those three spaces. This is why the principle of triality is used in M-theory, due to the ability to unify bosonic-fermionic structures (more details see in [1] [3]). In the context of classical dynamics considered here, this interpretation of triality is not used. However, it is relevant for the study of quantum mechanical behaviour.

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