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Finite Field Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality

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Abstract

Hudzik and Landes *[Math. Ann., 1992]* derived a major generalization of the triangle inequality for two nonzero elements in normed linear spaces, which was extended to finitely many nonzero elements independently by Dragomir *[Bull. Aust. Math. Soc., 2006]* and by Kato, Saito and Tamura *[Math. Inequal. Appl., 2007]*. We derive a finite field version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura inequality.

Keywords: normed linear space; triangle inequality; finite field; sub-modulus; sub-norm

MSC: 12E20; 46B20

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1. Introduction

Let $\mathcal{X}$ be a normed linear space (NLS) over scalar field. Definition of the norm has the triangle inequality

$$\|x+y\| \leq \|x\|+\|y\|, \quad \forall x,y \in \mathcal{X}. \tag{1}$$

In 1992, Hudzik and Landes derived a breakthrough generalization of Inequality (1) which is valid for any two nonzero elements in a NLS [1].

Theorem 1. *[1] (**Hudzik-Landes Inequlaity**) Let $\mathcal{X}$ be a NLS. Then for all $x,y \in \mathcal{X}\setminus\{0\}$,*

$$\|x+y\| \leq \|x\|+\|y\| - \left(2-\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|\right)\min\{\|x\|,\|y\|\}. \tag{2}$$

In 2006, Maligranda independently derived Inequality (2) [2]. It is natural to look for a generalization of Inequality (2) to more than two non-zero vectors. This is done independently by Dragomir in 2006 [3] and by Kato, Saito and Tamura in 2007 [4].

Theorem 2. *[3,4] (**Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality**) Let $\mathcal{X}$ be a NLS and $n \in \mathbb{N}$. Then for all $x_1,\dots,x_n \in \mathcal{X}\setminus\{0\}$, we have*

$$\left\|\sum_{j=1}^n x_j\right\| \leq \sum_{j=1}^n \|x_j\| - \left(n-\left\|\sum_{j=1}^n \frac{x_j}{\|x_j\|}\right\|\right)\min_{1\leq k\leq n}\|x_k\|.$$

It is natural and important to ask what are finite field versions of Theorems 1 and 2? We answer the question by deriving finite field version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality (Theorem 4).

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2. Finite Field Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality

We begin by introducing the notion of sub-modulus field.

Definition 1. *Let $\mathbb{F}$ be a (finite) field. A map $|\cdot| : \mathbb{F} \to [0, \infty)$ is said to be sub-modulus or sub-valued if the following conditions hold.*

(i) *If $\lambda \in \mathbb{F}$ is such that $|\lambda| = 0$, then $\lambda = 0$.*
(ii) *$|\lambda\mu| \leq |\lambda||\mu|$ for all $\lambda, \mu \in \mathbb{F}$.*
(iii) *$|\lambda + \mu| \leq |\lambda| + |\mu|$ for all $\lambda, \mu \in \mathbb{F}$.*

In this case, we say that $\mathbb{F}$ is a sub-modulus or sub-valued field.

Example 1. *Let p be a prime, let $\mathbb{Z}_p := \{0, 1, \dots, p-1\}$ be the standard field of integers modulo p. Given $n \in \mathbb{Z}_p$, we define*

$$|n| := \textit{unique natural number } a \textit{ such that } 0 \leq a \leq p-1 \textit{ and } n \equiv a (\textit{mod } p).$$

We next introduce the notion of sub-normed linear space.

Definition 2. *Let $\mathcal{X}$ be a vector space over a sub-modulus field $\mathbb{F}$. A map $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is said to be sub-norm if the following conditions hold.*

(i) *If $x \in \mathcal{X}$ is such that $\|x\| = 0$, then $x = 0$.*
(ii) *$\|\lambda x\| \leq |\lambda|\|x\|$ for all $\lambda \in \mathbb{F}$, for all $x \in \mathcal{X}$.*
(iii) *$\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$.*

In this case, we say $\mathcal{X}$ is a sub-normed linear space.

Example 2. *Let $p \in [1, \infty)$. Let $\mathbb{F}$ be a sub-modulus field. For $d \in \mathbb{N}$, let $\mathbb{F}^d$ be the standard vector space. We define*

$$\|(a_j)_{j=1}^d\|_p := \left(\sum_{j=1}^d |a_j|^p\right)^{\frac{1}{p}}, \quad \forall (a_j)_{j=1}^d \in \mathbb{F}^d.$$

Then $\|\cdot\|_p$ is a sub-norm on $\mathbb{F}^d$.

Example 3. *$p \in [1, \infty)$. Let $\mathbb{F}$ be a sub-modulus field. Define*

$$c_{00}^p(\mathbb{N}, \mathbb{F}) := \{\{a_n\}_{n=1}^\infty : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n \neq 0 \textit{ only for finitely many } n\}.$$

We define

$$\|\{a_n\}_{n=1}^\infty\|_p := \left(\sum_{n=1}^\infty |a_n|^p\right)^{\frac{1}{p}}, \quad \forall \{a_n\}_{n=1}^\infty \in c_{00}^p(\mathbb{N}, \mathbb{F}).$$

Then $\|\cdot\|_p$ is a sub-norm on $c_{00}^p(\mathbb{N}, \mathbb{F})$.

Example 4. *Let $\mathbb{F}$ be a sub-modulus field. Define*

$$c_{00}(\mathbb{N}, \mathbb{F}) := \{\{a_n\}_{n=1}^\infty : a_n \in \mathbb{F}, \forall n \in \mathbb{N}, a_n \neq 0 \textit{ only for finitely many } n\}.$$

We define

$$\|\{a_n\}_{n=1}^\infty\|_{00} := \max_{n \in \mathbb{N}} |a_n|, \quad \forall \{a_n\}_{n=1}^\infty \in c_{00}(\mathbb{N}, \mathbb{F}).$$

Then $\|\cdot\|_{00}$ is a sub-norm on $c_{00}(\mathbb{N}, \mathbb{F})$.

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We now derive finite field version of Inequality (2).

Theorem 3. (***Finite Field Hudzik-Landes Inequality***) *Let $\mathcal{X}$ be a sub-normed linear space over $\mathbb{F}$. Then for all $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{F}$ it holds*

$$\|x+y\| \leq \min\left\{|\|x\||\left(\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|,\left|\frac{1}{\|x\|}-\frac{1}{\|y\|}\right|\|y\|\right), |\|y\||\left(\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|+\left|\frac{1}{\|y\|}-\frac{1}{\|x\|}\right|\|x\|\right)\right\}.$$

Proof. Let $x, y \in \mathcal{X} \setminus \{0\}$ with $\|x\|, \|y\| \in \mathbb{F}$. Then

$$\begin{aligned}\|x+y\| &= \left\|\|x\|\left(\frac{x}{\|x\|}+\frac{y}{\|y\|}\right)+\left(1-\frac{\|x\|}{\|y\|}\right)y\right\| \\ &\leq \left\|\|x\|\left(\frac{x}{\|x\|}+\frac{y}{\|y\|}\right)\right\|+\left\|\left(1-\frac{\|x\|}{\|y\|}\right)y\right\| \\ &\leq |\|x\||\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|+\left|1-\frac{\|x\|}{\|y\|}\right|\|y\| \\ &\leq |\|x\||\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|+|\|x\||\left|\frac{1}{\|x\|}-\frac{1}{\|y\|}\right|\|y\|\end{aligned}$$

and

$$\begin{aligned}\|x+y\| &= \left\|\left(1-\frac{\|y\|}{\|x\|}\right)x+\|y\|\left(\frac{x}{\|x\|}+\frac{y}{\|y\|}\right)\right\| \\ &\leq \left\|\left(1-\frac{\|y\|}{\|x\|}\right)x\right\|+\left\|\|y\|\left(\frac{x}{\|x\|}+\frac{y}{\|y\|}\right)\right\| \\ &\leq \left|1-\frac{\|y\|}{\|x\|}\right|\|x\|+|\|y\||\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\| \\ &\leq |\|y\||\left|\frac{1}{\|y\|}-\frac{1}{\|x\|}\right|\|x\|+|\|y\||\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|.\end{aligned}$$

Therefore

$$\|x+y\| \leq |\|x\||\left(\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|+\left|\frac{1}{\|x\|}-\frac{1}{\|y\|}\right|\|y\|\right) \tag{3}$$

and

$$\|x+y\| \leq |\|y\||\left(\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|+\left|\frac{1}{\|y\|}-\frac{1}{\|x\|}\right|\|x\|\right). \tag{4}$$

Inequalities (3) and (4) give

$$\|x+y\| \leq \min\left\{|\|x\||\left(\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|+\left|\frac{1}{\|x\|}-\frac{1}{\|y\|}\right|\|y\|\right), |\|y\||\left(\left\|\frac{x}{\|x\|}+\frac{y}{\|y\|}\right\|+\left|\frac{1}{\|y\|}-\frac{1}{\|x\|}\right|\|x\|\right)\right\}.$$

□

Note the additional assumption $\|x\|, \|y\| \in \mathbb{F}$ in the previous theorem. The reason is that, since the norm is a real number, we generally do not have a guarantee that it belongs to the given sub-modulus field.

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Example 5. *Consider $\mathbb{Z}_3$. Let $x = 1, y = 2$. Then $|x+y| = |1+2| = |3| = 0$ and $|x| + |y| = 1+2 = 3$. We now find*

$$\begin{aligned}
||x||\left(\left|\frac{x}{|x|}+\frac{y}{|y|}\right|+\left|\frac{1}{|x|}-\frac{1}{|y|}\right||y|\right) &= ||1||\left(\left|\frac{1}{|1|}+\frac{2}{|2|}\right|+\left|\frac{1}{|1|}-\frac{1}{|2|}\right||2|\right)\\
&= |1|\left(\left|\frac{1}{1}+\frac{2}{2}\right|+\left|\frac{1}{1}-\frac{1}{2}\right|2\right)\\
&= |1+1|+|1-2|2\\
&= |2|+|1+1|2\\
&= 2+|2|2\\
&= 2+2\cdot 2\\
&= 2+1=0
\end{aligned}$$

and

$$\begin{aligned}
||y||\left(\left|\frac{x}{|x|}+\frac{y}{|y|}\right|+\left|\frac{1}{|x|}-\frac{1}{|x|}\right||x|\right) &= ||2||\left(\left|\frac{1}{|1|}+\frac{2}{|2|}\right|+\left|\frac{1}{|2|}-\frac{1}{|1|}\right||1|\right)\\
&= |2|\left(\left|\frac{1}{1}+\frac{2}{2}\right|+\left|\frac{1}{2}-\frac{1}{1}\right|\right)\\
&= 2(|1+1|+|2-1|)\\
&= 2(|2|+|1|)\\
&= 2(2+1)=0.
\end{aligned}$$

Therefore

$$0 = |x+y| = \min\{0,0\} < |x|+|y| = 3.$$

Now we derive finite field version of Theorem 2.

Theorem 4. *(**Finite Field Hudzik-Landes-Dragomir-Kato-Saito-Tamura Inequality**) Let $\mathcal{X}$ be a sub-normed linear space over $\mathbb{F}$ and $n \in \mathbb{N}$. Then for all $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{F}$ it holds*

$$\begin{aligned}
\left\|\sum_{j=1}^{n} x_j\right\| &\leq \min_{1\leq k\leq n}\left\{|||x_k|||\left(\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|+\sum_{1\leq j\leq n, j\neq k}\left|\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right|\|x_j\|\right)\right\}\\
&\leq \min_{1\leq k\leq n}\left\{|||x_k|||\left(\left\|\sum_{j=1}^{n}\frac{x_j}{\|x_j\|}\right\|+(n-1)\max_{1\leq j\leq n, j\neq k}\left|\frac{1}{\|x_k\|}-\frac{1}{\|x_j\|}\right|\|x_j\|\right)\right\}.
\end{aligned}$$

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Proof. Let $x_1, \dots, x_n \in \mathcal{X} \setminus \{0\}$ with $\|x_1\|, \dots, \|x_n\| \in \mathbb{F}$. Let $1 \le k \le n$ be fixed. Then

$$\begin{aligned}
\left\|\sum_{j=1}^{n} x_j\right\| &= \left\|\sum_{j=1}^{n} \frac{\|x_k\|x_j}{\|x_j\|} + \sum_{j=1}^{n}\left(1 - \frac{\|x_k\|}{\|x_j\|}\right)x_j\right\| \\
&\le \left\|\sum_{j=1}^{n} \frac{\|x_k\|x_j}{\|x_j\|}\right\| + \left\|\sum_{j=1}^{n}\left(1 - \frac{\|x_k\|}{\|x_j\|}\right)x_j\right\| \\
&\le \left\|\sum_{j=1}^{n} \frac{\|x_k\|x_j}{\|x_j\|}\right\| + \left\|\|x_k\|\sum_{j=1}^{n}\left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|}\right)x_j\right\| \\
&\le |\|x_k\||\left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| + |\|x_k\||\left\|\sum_{j=1}^{n}\left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|}\right)x_j\right\| \\
&= |\|x_k\||\left(\left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| + \left\|\sum_{j=1}^{n}\left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|}\right)x_j\right\|\right) \\
&= |\|x_k\||\left(\left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| + \left\|\sum_{1\le j\le n, j\ne k}\left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|}\right)x_j\right\|\right) \\
&\le |\|x_k\||\left(\left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| + \sum_{1\le j\le n, j\ne k}\left\|\left(\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|}\right)x_j\right\|\right) \\
&\le |\|x_k\||\left(\left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| + \sum_{1\le j\le n, j\ne k}\left|\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|}\right|\|x_j\|\right).
\end{aligned}$$

By varying k and taking minimum in the right side of previous inequality gives

$$\begin{aligned}
\left\|\sum_{j=1}^{n} x_j\right\| &\le \min_{1\le k\le n}\left\{|\|x_k\||\left(\left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| + \sum_{1\le j\le n, j\ne k}\left|\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|}\right|\|x_j\|\right)\right\} \\
&\le \min_{1\le k\le n}\left\{|\|x_k\||\left(\left\|\sum_{j=1}^{n} \frac{x_j}{\|x_j\|}\right\| + (n-1)\max_{1\le j\le n, j\ne k}\left|\frac{1}{\|x_k\|} - \frac{1}{\|x_j\|}\right|\|x_j\|\right)\right\}.
\end{aligned}$$

□

3. Conclusions

1. In 1992, Hudzik and Landes improved triangle inequality for two nonzero elements in normed linear spaces [1].
2. In 2006, Dragomir extended Hudzik-Landes inequality for more than two nonzero vectors [3].
3. In 2007, Kato, Saito and Tamura extended Hudzik-Landes inequality without knowing the work of Dragomir [4].
4. In this article, we derived finite field version of Hudzik-Landes-Dragomir-Kato-Saito-Tamura inequality.

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