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Pointparticle Systems on the Prototypes of Zeeman Manifolds and Zeeman Spacetimes

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Abstract: Zeeman manifolds and their relativistic extensions - the Zeeman spacetimes - are new nonstandard unification models for exploring the quantum physics of point-like and also of extended multiparticle systems. Zeeman manifolds still carry Riemannian metrics on which the unification is realized so that the Hamilton operators, for both type of particle systems, are derived from the very same operator - the Riemannian Laplacian given on Zeeman manifolds. That's why the latter is called Monistic Hamilton Operator. The relativistic Wave Mechanics is established on Zeeman spacetimes carrying Lorentzian pseudo Riemannian metrics obtained by static resp. accelerating extensions of Zeeman manifolds into the time direction. Their canonical Laplacian is the Monistic Wave Operator from which the wave operators of specific multiparticle systems are derived. Zeeman manifolds and Zeeman spacetimes cover a wide range of examples. The prototypical ones are established on H-type groups and their relativistic extensions, while the most generic ones arise on HyperKähler-Zeeman manifolds and their relativistic extensions. This selfcontained paper explores a unified quantum theory for pointparticle systems to be defined on prototypical Zeeman manifolds and Zeeman spacetimes.

Keywords: Hamilton operators; dark matter; dark energy; ordinary matter; non-standard unification models

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1. Introduction

The fundamental concepts of classical quantum theory had been explored in strong relation to those of Hamiltonian mechanics. This is clearly shown by the quantum Hamilton operators that were created from relativistic Hamilton functions describing the physical laws in the macroscopic world, but which, nonetheless, also provided access to the microscopic world via the correspondence principle associating the desired relativistic quantum Hamilton operators to relativistic Hamilton functions by well known mathematical procedures corresponding operators to functions. The most basic correspondences are carried out by the Heisenberg group representations that associate operators to the canonical coordinates, and hereby, to all functions built up by the canonical coordinates.

This method is called Hamilton formalism in which the correspondence principle is implemented only in the very last step. Prior to that, one should find a relativistic Lagrange function, in the first place, which is transformed then to a relativistic Hamilton function from which the desired quantum operator is created by using the well known L^2-representation of the Heisenberg group. Despite the great successes of Dirac's relativistic electron operator, the Hamilton formalism became a less effective tool, in the 1940's, when quantum theory faced the great challenge of the infinity puzzle. It was much less complicated to deal only with the Lagrangian formalism describing the particles, in terms of Lagrange functions, as point-like objects moving according to Lagrange equations derived from the Lagrange function by the principle of least action. The motion is ultimately described, by scattering matrices, in much more coarse but manageable ways, which nonetheless overcome many

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of the difficulties arising from the complete Hamilton formalism working with wave-operators. Due to this shift in technicalities, Dirac's original Hamiltonian formalism had gradually been forgotten, despite the great effort he put into taking it back to the main stream of physical researches.

Though the new framework developed in this paper works with matter waves and wave operators, it breaks off not just from the Lagrangian but Dirac's complete Hamiltonian formalism, also. Namely, the classical Hamilton operators immediately emerge as Laplace Beltrami operators to be defined on certain Riemannian manifolds without resorting to any of the Lagrangian or full Hamiltonian formalisms. Furthermore, the manifolds exhibiting this profound relation to quantum physics have non-zero Riemannian curvature that even more emphatically points to the disconnect existing between the new framework and any of the classical formalisms. To see this, it is enough to point out that the complete Hamiltonian formalism produces wave operators on a flat Minkowski spacetime, and the Lagrangian formalism describes the laws of motion in quantum physics by Yang Mills equations and Yang-Mills connections to be defined on $SU(n)$ bundles whose base is a flat Minkowski 4D space.

The ubiquitous flat space in classical theories is the main reason for quantum theory and general relativity are doomed to be two irreconcilable disciplines. Contrary to this, the new framework does not face such an invincible obstacle, which - by describing the well known realistic physical wave operators as Laplacians canonically given on properly curved general relativistic spacetimes - naturally connects quantum physics with general relativity.

On Zeeman manifolds, the unification is realized so that the Hamilton operators for the distinct particle systems, which consists of either point-like or extended particles, are derived from the very same operator - the Laplacian Δ canonically given on the Riemannian Zeeman manifolds. A particular system is associated with a particular subspace of wave functions that happens to be invariant under the action of the Laplacian. Then, the particular Hamilton operator for a particular system is defined by restricting the Δ onto the specific invariant subspace associated with the system. This is why the $-\Delta$ is called "Monistic Hamilton Operator", and the Δ as "Monistic Laplacian".

The pseudo Riemannian metrics of Lorentz signature on Zeeman spacetimes are introduced by static resp. accelerating extension of the Riemannian Zeeman manifolds into the time direction. Since the Zeeman manifolds themselves have non-zero Riemannian and Ricci curvatures, so do the spacetimes obtained by such extensions. The Wave-Laplacian on the static spacetime exhibits several surprising features showing how deeply is rooted in quantum physics. It appears as sum of the classical Schrödinger, the non-relativistic Yukawa, and the gravitational operators among which the first two are well known from classical quantum theory. The name of the third operator is validated by pointing out that a non-relativistic limiting takes it to the Poisson equation known with regard to Newton gravitation. The decomposition in the completely developed theory implies the strong relations among the electromagnetic, electroweak, strong, and gravitational interactions.

Each of these operators is non-relativistic but together they sum up to a relativistic operator. The decomposition also puts Schrödinger's equation - what he obtained by exchanging the the second order differentiation $\frac{\partial^2}{\partial t^2}$ in the Klein-Gordon equation for the first order one $\frac{\partial}{\partial t}$ - in a much larger theoretical context. He implemented this exchange because the original version was not in concert with experiments and realized that the new version was relatively accurate for low speed $v << c$. On Zeeman spacetimes, however, Schrödinger's renovation is put in a whole new light. Namely, the decomposition proves oneself to be the exact mathematical procedure with which the above exchange should be carried out. It still remains there to be involved into the non-relativistic Yukawa operator whose physical importance is shown by playing fundamental role in the spectral mass assignment procedure. The appearance of the Schrödinger, the non-relativistic Yukawa, and the gravitational operators in a single relativistic wave operator clearly exhibits that quantum physics and general relativity are mutually interrelated disciplines.

The Laplacian Δ describes particle systems which only have angular momentum and orbiting spin. The proper spin operators of fermions and bosons, by their actions on spinors, enrich the thery with much more complex features.

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The prototypes of Zeeman manifolds are carried out on Heisenberg type groups whose static resp. solvable extensions define the static resp. accelerating Zeeman spacetimes. Despite their specific appearance, they exhibit the most important features of a generic framework. This paper only describes the Hamilton and Wave Operators for pointparticle systems on these particular Zeeman manifolds and spacetimes. The operators are derived, according to the main idea of the theory, from the Monistic Hamiltonian and Monistic Wave Operators. Most remarkably, they appear in the same form as those explored, in the classical theory, by the Dirac-Hamilton formalism. This coincidence strongly indicates that the new framework describes physics in the strongest possible relation with reality.

The paper is arranged as follows. After introducing the Monistic Hamilton Operator on H-type groups, the Hamilton waves of point particle systems are introduced in terms of discrete Z-Fourier transforms. The Hamilton operators of point particle systems are defined by the action of the Monistic Hamiltonian on the invariant subspace spanned by these wave functions. Surprisingly, it appears, in literally the same form, as the Landau-Zeeman operator of charged particles orbiting in constant magnetic fields perpendicular to the plane of orbiting. The complete Zeeman operator, which involves generic electromagnetic fields, will be studied on generic Zeeman manifolds, separately.

Then, the investigation continues with explicitly computing the spectrum of the Hamilton operator. The explicit formula for the Hamilton waves allows to distinguish the positively oriented pointparticles from those negatively oriented. Although the waves of point particles can not carry charges, their orientation determines the sign of charge the wave packets pick up during the action of the angular momentum operator. A remarkable feature of the spectrum is that it is parity violating, namely, it depends only on the index of positiveness counting the positively oriented (charged) particles in the system. The negatively oriented (charged) particles have the smallest possible eigenvalues which is a strong indication for why the protons must be described in terms of positively oriented waves - namely because they imply bigger eigenvalues, and, ultimately bigger masses - which is contrary to the waves of electrons built up by negatively oriented waves implying significantly lesser masses. The description of the Hamilton operators is concluded by introducing proper spin operators and computing their spectra explicitly.

Wave Mechanics is worked out on relativistic Zeeman spacetimes to be defined by static resp. solvable (accelerating) extensions of Zeeman manifolds into the time direction. This part includes the establishment of the spectral mass assignment procedure on the static extension (which can be seen as a substitute for the Lagrangian mass-assignment by the Higgs mechanism), and determining the wave operators of the Ordinary Matter; Dark Matter; and Dark Energy, on the accelerating Zeeman spacetimes. Their participation ratios $(5\%, 25\%, 70\%)$ - computed on Zeeman spacetimes solely by the Wave Mechanics evolved by the new framework - surprisingly accurately agree with those obtained by the original computations [RS] underlined by the Standard Model and Einstein's gravitational equation modified with a particular value for the cosmological constant. The accelerating Zeeman spacetimes also suggest a scenario for how the evolution of Universe has been taking place. It is a revised version of the Inflation Theory, but with the marked difference that this version describes not just as to what have been taking place after the Big Bang but also that as to what had had been happening prior to the Big Bang.

Part I

Hamiltonians on Zeeman Manifolds

2. Metric Heisenberg-Type Lie Groups

The Zeeman manifold prototypes are established on Riemannian Heisenberg-type Lie groups endowed with the natural leftinvariant metric. They form a subclass of two-step nilpotent Lie groups whose constructions, in two interrelating ways, can be carried out as follows.

The Lie algebra is defined on the Euclidean space $\mathcal{N} = \mathbb{R}^k \times \mathbb{R}^l = \mathcal{X} \times \mathcal{Z}$ - that is, on the direct sum of the X-space $\mathbb{R}^k$ and the Z-space $\mathbb{R}^l$ - where the natural Euclidean inner product is denoted by $\langle\, ,\rangle$. The Lie bracket is defined with regard to a linear space $E(\mathcal{X})$ of skew endomorphisms acting on the X-space, so that the Z-space $\mathcal{Z}$ appears to be the center of the Lie algebra. The definition is first carried out by an endomorphism space that appears as the range of a one to one linear map $\mathbb{J} : \mathcal{Z} \to \mathrm{SkewEnd}(\mathcal{X})$ associating with each $Z \in \mathcal{Z}$ a skew endomorphisms $J_Z : \mathcal{X} \to \mathcal{X}$ acting on the X-space. Then, the desired Lie bracket is defined by $\langle [X,Y], Z\rangle = \langle J_Z(X), Y\rangle$, for all $X, Y \in \mathcal{X}$. This partial Lie bracket $[\mathcal{X},\mathcal{X}] \subset \mathcal{Z}$ together with $[\mathcal{N},\mathcal{Z}] = 0$ clearly determines the complete Lie algebra.

If the Lie bracket is created by a given linear space $E(\mathcal{X})$ of skew endomorphisms acting on the X-space, then both the Z-space and map $\mathbb{J} : \mathcal{Z} \to \mathrm{SkewEnd}(\mathcal{X})$ are to be defined before proceeding with the above definition. For introducing the Z-space, consider the endomorphism space as an abstract vector space and denote it by $\mathcal{Z}$. Then, there is defined a natural one-to-one linear map $\mathbb{J} : \mathcal{Z} \to \mathrm{SkewEnd}(\mathcal{X})$ associating with $Z \in \mathcal{Z}$ the skew endomorphism represented by Z. The so defined $\mathbb{J}$ obviously is a bijective linear map between $\mathcal{Z}$ and $E(\mathcal{X})$. The Z-space also carries the natural inner-product $\langle Z_1, Z_2\rangle = -\frac{1}{k}\mathrm{Tr}(J_{Z_1} \circ J_{Z_2})$, thus the direct sum defines an inner product on $\mathcal{X} \times \mathcal{Z}$, as well. After all these preparations, the construction of the sought Lie-bracket is to be completed in the way as described above.

The particular Heisenberg-type Lie algebras are defined by endomorphism spaces satisfying the Clifford condition $J_Z^2 = -|Z|^2\mathrm{id}$, for all $Z \in \mathcal{Z}$. Polarization makes this to be equivalent to the Clifford-Dirac condition $J_{Z_1} \circ J_{Z_2} + J_{Z_2} \circ J_{Z_1} = -2\langle Z_1, Z_2\rangle\mathrm{id}$. Since the representations of Clifford algebras are carried out by endomorphisms satisfying the Clifford condition, the Heisenberg type Lie algebras can naturally be identified with Clifford modules. Thus, the well known classification of Clifford modules provides classification for the Heisenberg type Lie algebras, also. These Lie algebras and groups were introduced and extensively investigated by Kaplan [K].

The Lie group $N = e^{\mathcal{N}}$ can be described as such to be defined on the (X,Z)-space $\mathbb{R}^k \times \mathbb{R}^l$, also. To that end, transport the above Lie algebra to the tangent space $T_0(\mathbb{R}^k \times \mathbb{R}^l)$, where 0 is the origin, by the natural linear map existing between the (X,Z)-space and $T_0(\mathbb{R}^k \times \mathbb{R}^l)$. This map also lifts $\langle\, ,\rangle$ defined on $\mathbb{R}^k \times \mathbb{R}^l$ to $T_0(\mathbb{R}^k \times \mathbb{R}^l)$, where it is denoted by $\langle .,.\rangle_0 = g_0(.,.)$.

The technical construction of the Lie group can be carried out by the exponential map $\exp_N$, which is well known to be one-to-one on simply connected 2-step nilpotent Lie groups. Thus, it necessarily identifies the Lie group N with its tangent space $T_0(N)$. By (124), the group-product on N is given by

$$(X,Z)(X',Z') = (X + X', Z + Z' + \frac{1}{2}[X,X']) \tag{1}$$

and the left-invariant extensions of the vectors $V \in T_0(N)$ with this group multiplication define the invariant vector fields (3) whose Lie algebra structure is isomorphic to that of $\mathcal{N}$. The left-invariant extension of g_0 onto the group defines the sought Riemann metric g which is well known to be of non-zero Riemannian curvature.

3. The Monistic Hamilton Operator $\mathcal{MHO} = -\Delta$

Theorem 1. *The Laplacian appears on a metric H-type group in the form:*

$$\Delta = \Delta_X + (1 + \frac{1}{4}|X|^2)\Delta_Z + \sum_{\alpha=1}^{l} \partial_\alpha D_\alpha \bullet, \tag{2}$$

where Δ_X and Δ_Z are the respective Euclidean Laplacians on the X- and Z-space, vector system $\{e_\alpha | \alpha = 1, \dots, l\}$ is an orthonormal basis for the Z-space, and $D_\alpha\bullet$ denotes directional derivative regarding the Hopf vector field $X \to J_{e_\alpha}(X)$ to be defined on the X-space.

This theorem is established below by the invariant vector fields written up in terms of the coordinate systems $\{x^1; \dots; x^k\}$ and $\{z^1; \dots; z^l\}$ defined for the orthonormal bases $\{E_1; \dots; E_k\}$ and

$\{e_1; \ldots ; e_l\}$ chosen for the X- and Z-space, respectively. Vectors E_i and e_α extend to the left-invariant vector fields:

$$\mathbf{X}_i = \partial_i + \frac{1}{2}\sum_\alpha \langle [X, E_i], e_\alpha\rangle \partial_\alpha = \partial_i + \frac{1}{2}\sum_\alpha \langle J_\alpha(X), E_i\rangle \partial_\alpha, \quad \mathbf{Z}_\alpha = \partial_\alpha, \tag{3}$$

where $\partial_i = \partial/\partial x^i$, $\partial_\alpha = \partial/\partial z^\alpha$ and $J_\alpha = J_{e_\alpha}$. Then, (2) follows from

$$\Delta = \sum_i (\mathbf{X}_i^2 - \nabla_{\mathbf{X}_i}\mathbf{X}_i) + \sum_\alpha (\mathbf{Z}_\alpha^2 - \nabla_{\mathbf{Z}_\alpha}\mathbf{Z}_\alpha) \tag{4}$$

and the well known (see [Sz1,Sz2])) formulae:

$$\nabla_X X^* = \frac{1}{2}[X, X^*]; \quad \nabla_X Z = \nabla_Z X = -\frac{1}{2}J_Z(X); \quad \nabla_Z Z^* = 0, \tag{5}$$

describing the covariant derivative on H-type groups in terms of the Lie algebra. The negative sign in $\mathcal{MHO} = -\Delta$ is needed to define an elliptic Hamilton operator having positive eigenvalues, for the L^2 functions, that will correspond to the positive energy levels the system can take on.

On Zeeman manifolds, the unification is realized by deriving the Hamilton operators of different type of particle systems from the very same operator - the Laplacian Δ canonically existing on these Riemannian manifolds. On a generic setting, one can distinguish two type of systems, called pointparticle systems and extended particle systems, respectively. On the Zeeman manifold prototypes, they are introduced as follows:

Pointparticle systems are defined by torus bundles obtained by factoring the Z-space by a full Z-lattice $\Gamma_Z = \{Z_\gamma\}$ picked up in the Z-space. The Cartesian product of the X- and Z-space is considered so that the Z-space defines a vector bundle over the X-space, thus Γ_Z defines a torus in the Z-space Z_p, over each point p of the X-space. The so defined torus bundle over the X-space is topologically trivial and both the base and the torus-fibers carry flat metrics. Nonetheless, the Riemannian metric g on the total space does not appear as the Cartesian product of the two flat metrics given on the base and the torus but which has non-zero Riemann and Ricci curvatures. The functions to be defined on the torus bundle can be lifted up to the universal covering space - that is, to the non-factorized N - where they appear as Γ_Z-periodic functions, forming a function space, which, as it is pointed out below, is invariant under the action of Δ. Then, the restriction of the Monistic Hamiltonian $-\Delta$ onto this invariant subspace defines the Hamilton operator of pointparticle systems.

The theory of extended particle systems - which will be detailed in a separate paper - is carried out on Z-ball bundles to be defined by balls $B_\delta(p)$, of radius δ and center 0, picked up in the Z-space Z_p over each point p of the X-space. As compared with the Z-torus bundles, the very same Laplace operator exhibits completely different features when it acts on functions defined on Z-ball bundles. On torus bundles, it appears in the form of the classical Zeeman operator describing charged pointparticles orbiting in constant magnetic fields, while on Z-ball bundles, it shows up as the Hamilton operator of extended particle systems exhibiting very similar features as those described by QFD and QCD with regard to protons, neutrons, and quarks.

Let it also be mentioned that the interior structure of such fundamental particles have been explored, insofar, only by Lagrangian means and the charged pointparticle systems are the only ones which had been investigated, in classical quantum theory, with wave operators as well as with Lagrangian means. Contrary to this, operator-based methods have never been applied to elementary particles described in QFD or QCD. On Zeeman manifolds and spacetimes, these latter type of particles appear as extended particle systems and their wave operators - missing in the literature insofar - are established by deriving them from the Monistic Operator.

The derivation of all Hamiltonians from the very same operator is the principal idea with which the unification is accomplished, both on Zeeman manifolds and Zeeman spacetimes. To emphasize this point, the Laplacian given there is called to be “Monistic", suggesting that it gives rise to the Hamilton

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operators of all type of particle systems. The Hamilton operators of different type of particle systems are differentiated from each other by the particular wave-spaces representing the specific particle systems. They are invariant under the action of the Monistic Hamiltonian and the specific Hamilton operator is defined by restricting the Monistic Operator onto the invariant subspace associated with the specific system. The specific characteristic features are exhibited by the actions of the Monistic Hamiltonian on the waves belonging to the specific invariant subspace. This idea is demonstrated in this paper only for the prototypical pointparticle systems when the Hamilton operator - derived from the Monistic Laplacian - literarily agrees with that established in classical quantum theory by Dirac's Hamilton formalism and the correspondence principle.

4. Hamilton Operators for Pointparticle Systems

4.1. Hamilton Waves and Operators of Pointparticles

On a Z-torus bundle $\mathcal{X} \times (\mathcal{Z}/\Gamma_Z)$, the Hamilton waves of pointparticle systems are introduced by complex valued L^2 functions of the form

$$\Psi(X,Z) = \sum_{\gamma} \psi_{\gamma}(X) e^{2\pi \mathbf{i} \langle \frac{1}{c} Z_{\gamma}, Z \rangle} \tag{6}$$

exhibiting the waves in terms of regular discrete Z-Fourier transform to be defined for the Z-lattice $\Gamma_Z = \{Z_\gamma\}$. In order to comply with the L^2-condition, the complex valued functions $\psi_\gamma(X)$ must be of class L^2 on the X-space, for any Z_γ.

The sub-spaces W_γ, spanned by functions of the form $\psi_\gamma(X)e^{2\pi \mathbf{i}\langle \frac{1}{c}Z_\gamma, Z\rangle}$, define the Fourier-Weierstrass decomposition $\sum_\gamma W_\gamma$ of the Γ_Z-periodic functions, that yield:

Theorem 2. *Each W_γ is invariant under the action of the Laplacian Δ and the action reduces to the functions $\psi_\gamma(X)$ according to the formula*

$$\Delta(\sum_{\gamma} \psi(X) e^{2\pi \mathbf{i} \langle \frac{1}{c} Z_{\gamma}, Z \rangle}) = \sum_{\gamma} \lhd_{\gamma}(\psi(X)) e^{2\pi \mathbf{i} \langle \frac{1}{c} Z_{\gamma}, Z \rangle}, \tag{7}$$

where $\lhd_\gamma$, defined for each lattice point Z_γ separately, is nothing but the Landau-Zeeman operator

$$\lhd_{\gamma} = \Delta_X + \frac{2\mathbf{i}}{c}\pi |Z_{\gamma}| \mathbf{i} D_{\gamma u} \bullet - \frac{4}{c^2}\pi^2 |Z_{\gamma}|^2 (1 + \frac{1}{4}|X|^2) \tag{8}$$

of charged particles orbiting - on complex planes - in constant magnetic fields standing perpendicular to the planes.

To see the indicated relation to the Landau operator, recall that the 2D-Landau-Zeeman operator of a particle, whose electric charge is e and which is orbiting in the (x,y)-plane in constant magnetic field directed toward the z-axis, appears in the form [B,LL,P]:

$$-\frac{\hbar^2}{2m}\Delta_{(x,y)} - \mathbf{i}\frac{\hbar e B}{2mc} D_z \bullet + \frac{e^2 B^2}{8mc^2}(x^2 + y^2) = \tag{9}$$

$$= -\frac{\hbar^2}{2m}(\Delta_{(x,y)} + \frac{2\mathbf{i}}{c}\frac{eB}{2\hbar} D_{zu} \bullet - \frac{4}{c^2}\frac{e^2 B^2}{4\hbar^2}\frac{1}{4}(x^2 + y^2)) \tag{10}$$

where $D_{zu}\bullet = x\partial_y - y\partial_x$.

On a 3D Heisenberg group - which is a H-type group having a 2D X-space $\mathbb{R}^2 = \{(x,y)|x,y \in \mathbb{R}\}$ and a 1D Z-space $\mathbb{R}$ satisfying $Z_\gamma \in \mathbb{R}$ and $J_{\gamma u}(\partial_x) = \partial_y$ - the above formulae clearly determine how the $\lhd_\gamma$ should be turned to be the 2D-Landau-Zeeman operator. Namely, the Z_γ determines the magnitude B_γ of the constant magnetic field by the equation $\pi|Z_\gamma| = (eB_\gamma)/(2\hbar)$. Then, plugging this expression for $\pi|Z_\gamma|$, and multiplying the whole operator with $-\hbar^2/2m$, the $\lhd_\gamma$ is transformed to the Landau-

Zeeman operator, indeed. When described in terms of the parameter $\mu_\gamma = (\pi/c)|Z_\gamma| = (eB_\gamma)/(2c\hbar)$, it is denoted by $\lhd_{\mu_\gamma}$, or simply $\lhd_\mu$ when there is no danger of confusion.

To be more precise, there is an extra constant $4\pi^2|Z_\gamma|^2/c^2 = 4\mu_\gamma^2$ in $\lhd_\mu$ which does not show up in the original Landau-Zeeman operator. It represents the energy density of the constant magnetic field in which the charged particle is orbiting. Since the total energy operator must contain this term, also, (8) should be considered as being the complete Landau-Zeeman operator. As it turns out later, this term plays a major role in establishing the spectral mass-assignment procedure on Zeeman spacetimes.

The relations of $|Z_\gamma|$ resp. μ_γ to B_γ and the magnetic flux quantum $\Phi_0 = \frac{h}{2e} = \frac{2\pi\hbar}{2e} \approx 2.067833848... \times 10^{-15} Wb$ are exhibited by the computations:

$$\pi|Z_\gamma| = c\mu_\gamma = \frac{eB_\gamma}{2\hbar} = \frac{1}{2}\frac{\pi B_\gamma}{h/2e} = \frac{1}{2}\frac{\pi B_\gamma}{\Phi_0}, \quad |Z_\gamma| = \frac{1}{2}\frac{B_\gamma}{\Phi_0}, \quad \mu_\gamma = \frac{1}{2}\frac{\pi B_\gamma}{c\Phi_0}. \tag{11}$$

The above arguments establish the desired identification of $\lhd_\gamma$ with the Landau-Zeeman operator on a 3D Heisenberg group where the $\lhd_\gamma$ acts on a 2D X-space. On a generic $(k+l)$-dimensional H-type group consider a fixed lattice point Z_γ which determines the unit Z-vector $Z_{\gamma u} = Z_\gamma/|Z_\gamma|$ and the complex structure $J_{\gamma u}$ corresponded to $Z_{\gamma u}$. Also consider a complex orthonormal basis $\{Q_1, \dots, Q_\kappa\}$ for the X-space, regarding the complex structure $J_{\gamma u}$. Then, for each i, there is determined a 3D Heisenberg subgroup spanned the vectors $\{Q_i, J_{\gamma u}(Q_i), Z_{\gamma u}\}$, for which the 2D X-space parameterized by (x_i, y_i) is spanned by Q_i and $J_{\gamma u}(Q_i)$, and the 1D Z-space is spanned by $Z_{\gamma u}$. The identification of the $(k = 2\kappa)$D-operator $\lhd_\gamma$ with the $(k = 2\kappa)$D Landau-Zeeman operator is carried out so that one performs the above identification on each 3D Heisenberg subgroup to be defined on the span of vectors $\{Q_i, J_{\gamma u}(Q_i), Z_{\gamma u}\}$. It becomes the Hamilton operator for a system of charged particles that are orbiting in a common constant magnetic field standing perpendicular to each of the complex planes where the individual particles are orbiting.

In the i^{th} Heisenberg subgroup, the rotation is said to be anticlockwise if it appears to be such in the 3-space oriented by the base $(Q_i, J_{\gamma u}(Q_i), Z_{\gamma u})$. In this very same 3-space, complex structure $-J_{\gamma u}$ defines clockwise orbiting. The waves of particles are expressed as polynomials of the holomorphic resp. antiholomorphic coordinates $z_i = x_i + \mathbf{i}y_i$ rep. $\overline{z}_i = x_i - \mathbf{i}y_i$ that indicate anticlockwise resp. clockwise orbiting. These arguments clearly show that there are two ways for defining right- resp. left-handed orbiting of particles: Either one chooses $\mathcal{Z} \to J_\mathcal{Z}$ or $\mathcal{Z} \to -J_\mathcal{Z}$ when it comes to define a H-type group. These two options come to play in defining Heisenberg type Lie groups $H_l^{(a,b)}$ to be described in Section 4.8.

The above interpretations exhibit a fundamental property of the Zeeman manifold models. Although they are multiparticle models, for each individual particle there is assigned an individual 3D Heisenberg group in the X-space of which the individual particle is orbiting It should also be mentioned that, with regard to the curved Riemannian metric g, the X-space is perpendicular to the complete Z-space only at the origin 0_X of the X-space. Thus, in order to establish an exact matching with the Landau-Zeeman operator, the charged particles should be considered to be rotating in the X-space around 0_X, and no particles are considered to be rotating in subspaces perpendicular to $\mathcal{Z}_{\tilde{X}}$ for which $\tilde{X} \neq 0_X$.

Even though it operates on functions depending only on the X-variable, the $\lhd_\gamma$ is not a sub-Laplacian arising from submerging the total space (torus bundle) onto the base (X-space). Its dependence on Z_γ is clearly exhibited by the presence of J_{Z_γ} in the angular momentum operator. The $\lhd_\gamma$ actually is the restriction of the total Laplacian Δ onto the invariant subspace W_γ, and the total spectrum is determined by the spectra of operators $\lhd_\gamma$ computed for each W_γ separately.

The identification of $\lhd_\gamma$ with the Landau-Zeeman operator shows that the Zeeman manifold prototypes have profound connections to quantum physics. Nonetheless, these models only have a limited grip on the generic electromagnetic phenomena. Notice for instance that the magnetism is involved just by the constant magnetic field representing the little magnet in the interior of particles with which the orbiting spin has been established in the classical theory. By the presence of this

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physical entity, one was able to explain the splitting of spectral lines - called Zeeman effect - however, there still remained the question as to why are these lines doubled. This incompleteness was fixed by the relativistic Dirac operator which will also be introduced into the new framework by a few sections later.

The Landau-Zeeman operator refers to electricity only by the charge e, which, together with the magnitude B of the constant magnetic field and some other universal constants determine the magnitude $|Z_\gamma|$. There is clearly shown that this version - having not in its scope electric and non-constant magnetic fields - can not be the operator capable to describing generic electromagnetic phenomena on the microscopic level. Generic Zeeman manifolds involving electromagnetic fields in consent with the Maxwell equations will be introduced in a separate article so that the electromagnetic fields show up in the Laplacian as inbuilt objects. This implementation differs from those seen in todays literature where an electric potential function is just formally added to a primarily established Landau-Zeeman Hamiltonian, without considering the question whether or not the so obtained Hamiltonian is a Riemannian Laplace operator.

This section is concluded with an important remark referring to the identification of the Laplacian (8) with the physically realistic Landau-Zeeman operator whose 2D version is described by (9)-(10). The Laplacian (8) is obtained from the physical operator (9)-(10) by taking out and then ignoring the term $-\hbar^2/2m$. But then, the rest does not contains the mass m at all, thus the identification is far from being complete. There still remain physical entities - like the electric charge e, and the magnitude B of the constant magnetic field - but none of which is attached to mass but only to geometric data points such as the length of Z-vectors. Since only mass can carry electric charge, there arises the important question as to where are the masses which must be there for carrying the electric charges?

This problem is going to be solved by the spectral mass assignment procedure established on Zeeman spacetimes in the second half of this paper. According to those explanations, masses are created, in the first place, as eigenvalues of the Yukawa operator, involved into the Laplacian of the Zeeman spacetime, which, in the second step, are identified with the eigen values of (8) that is also present in the wave-Laplacian of Zeeman spacetime as a part-operator. In the end, charge e appears in the eigenvalues of (8) as an object carried by the mass the eigenvalue represents by the spectral mass assignment procedure. Due to these considerations, (8) is called massless or symbolic Landau-Zeeman operator. The name also refers to such parts, like the angular momentum or magnetic dipole momentum operators, which involve the mass only after the mass-assignment is completed.

If the $\lhd_\gamma$ is defined by the action of Δ on the waves $\psi_\gamma(X)e^{2\pi\mathbf{i}\langle\frac{1}{c}Z_\gamma,Z\rangle}$, then it can be identified with the physical operator established in (9)-(10). However, if the computations are carried out with waves of the form

$$\Psi(X,Z)=\sum_\gamma\psi_\gamma(X)e^{2\pi\mathbf{i}\langle Z_\gamma,Z\rangle}, \tag{12}$$

in which the Fourier function misses $\frac{1}{c}$, then, the latter term is missing also from $\lhd_\gamma$, which, otherwise, appears in the same form as (8). Strictly speaking, it can be identified with the massless Landau-Zeeman operator appearing as in (9)-(10) only under the assumption $c=1$ which implies the relation $\mu=\pi|Z_\gamma|=(eB)/(2\hbar)$. If it is necessary to indicate it, the operator introduced in the latter way will be denoted by $\lhd_{\gamma,c=1}$ or $\lhd_{\mu,c=1}$.

In what follows, quantities μ and B will appear without the index γ, but there shall always be understood that they are defined with regard to a fixed lattice point Z_γ. The simplification is introduced in order to avoid over complicated denotations.

4.2. Explicit Eigenfunctions and Eigenvalues for $\lhd_\mu$

Below, the spectral computations are carried out in two different ways - by using spherical harmonics resp. by Itô's polynomials. Both cases will be demonstrated on the non-compact manifold when the X-space is the complete $\mathbb{R}^k$. Computing with spherical harmonics, developed by Schrödinger,

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is a well known technique, however, Itô's polynomials seem to be never used for determining the eigenfunctions of natural quantum physical operators.

Itô [I] originally introduced his polynomials in context with complex Markov processes and ever since they have been widely used in statistical physics. Their involvement into physical spectral computations helps to discover new features that are out of the scope of Schrödinger's standard method. For instance, the spectrum computed in terms of Itô's polynomials exhibits P-symmetry violation while obeys C-symmetry.

4.2.1. Itô's Hermite Polynomials

According to their original definition [I], Itô's polynomials are introduced on the complex plane $\mathbb{C}$ for the indexes $p, q = 0, 1, 2, \dots$, by

$$H_{pq}(z,\overline{z}) = \sum_{s=0}^{min(p,q)} (-1)^s \frac{p!q!}{s!(p-s)!(q-s)!} z^{p-s}\overline{z}^{q-s}. \tag{13}$$

They form an orthogonal basis in the L^2-function space of complex valued functions. Depending on whether $p \geq q$ or $q \geq p$ is satisfied, they appear in the form $f_c(r^2)z^{p-q}$, or $f_c(r^2)\overline{z}^{q-p}$, where $f_c(t)$ is a polynomial of order $c = min(p,q)$, describable in terms of the Laguerre polynomials:

$$L_c^{(v)}(t) = \sum_{i=0}^{c} \binom{c+v}{c-i} \frac{(-t)^i}{i!}, \tag{14}$$

in the form

$$(-1)^c c! L_c^{(v)}(r^2) z^v \quad \text{resp.} \quad (-1)^c c! L_c^{(v)}(r^2)\overline{z}^v. \tag{15}$$

Thus, substitution $r^2 = z\overline{z}$ converts (15) to (13).

Itô's polynomials are introduced, for arbitrary even dimension $k = 2\kappa$, by products of the polynomials (13) written up in terms of the coordinates z_i to be defined for a fixed complex orthonormal basis. In this case, too, they are denoted by $H_{p,q}(z,\overline{z})$, where $p = (p_1, \dots, p_\kappa)$ and $q = (q_1, \dots, q_\kappa)$ denote κ-tuples of the holomorphic resp. antiholomorphic degrees p_i resp. q_i of the polynomials $z_i^{p_i}\overline{z}_i^{q_i}$ the $H_{p,q}(z,\overline{z})$ is expressed with. That is,

$$H_{pq}(z,\overline{z}) = \prod_i H_{p_i q_i}(z_i,\overline{z}_i) = \prod_i (-1)^{c_i} c_i! L_{c_i}^{(|m_i|)}(r_i^2) \prod_i \zeta_i^{|m_i|} = \tag{16}$$

$$= C_c^{(|m|)}(r_1^2, \dots, r_\kappa^2) \prod_i \zeta_i^{|m_i|}, \tag{17}$$

where $\zeta_i = z_i$, if $m_i > 0$; $\zeta_i = \overline{z}_i$, if $m_i < 0$; and $\zeta_i = 1$, if $m_i = 0$, furthermore, $|m| = (|m_1|, \dots, |m_\kappa|)$ and $c = (c_1, \dots, c_\kappa)$. They yield

Theorem 3. **A.** *When $\mu \neq 0$, then the L^2 eigenfunctions of $\lhd_\mu$ appear in the form*

$$h_{p,q}(X) = H_{p,q}(X)e^{-\mu|X|^2/2} \tag{18}$$

with the corresponding eigenvalue

$$-((4\mathbf{p} + k)\mu + 4\mu^2), \tag{19}$$

where $\mathbf{p} = p_1 + \cdots + p_\kappa$ denotes the holomorphic degree of $H_{p,q}$.

B. *For the zero lattice point $Z_{\gamma=0}$, the invariant subspace $W_{\gamma=0}$ consists of the L^2 functions which do not depend on Z and are defined on the entire X-space. In this case, the Laplacian $\lhd_{\mu=0} = \Delta_X$ reduces to the Euclidean Laplacian Δ_X which has the continuous spectrum $[0, \infty)$ on the L^2 function space. The wave* *space $W_{\gamma=0}$ represents a system of freely moving particles whose spectrum is called, by this reason, free-particle spectrum.*

Theorem A will be established not just by Itô's polynomials but by spherical harmonics, as well. The whole Theorem exhibits important physical features some of which are needed to be established already in this section in order to prepare its complete proof carried out in the next few sections.

The antiholomorphic degree, which does not appear in the eigenvalue, is defined by $\mathbf{q} = q_1 + \cdots + q_\kappa$. The holomorphic and antiholomorphic degrees obviously relate to the right and left handedness, in other words, to the chirality of particles. But one should not take over concepts of the SM to Zeeman manifolds without appropriate reformulation. Chirality, for example, is originally defined in the 3-space in terms of the orientation reversing isometries, whereas, on Zeeman manifolds, one should work with waves whose orientation can not be traced back to isometries acting on the X-space. This is one of the reasons for why is the below introduced Zeeman-chirality of Hamilton waves different from that introduced in SM.

For a fixed lattice point Z_γ and complex structure $J_{\gamma u}$, where $Z_{\gamma u} = Z_\gamma/|Z_\gamma|$, a system of whole particles is given by a complex orthonormal vector system $\mathbf{Q} = \{Q_1, \ldots, Q_\kappa\}$. The Hamilton waves of the system appear in terms of the Itô polynomials. The angular momentum operator referring to each of the individual particles in the system is defined by the endomorphism J_γ. It portrays the i^{th} particle as an object rotating in the complex coordinate plane spanned by $(Q_i, J_{\gamma u}(Q_i))$ in constant magnetic field represented by Z_γ. The latter Z-vector is also seen as a common axis for the individual particles' angular momenta. If such an individualistic complex plane is extended into the direction of Z_γ, then the particle appears to be rotating in a 3-space whose right and left handedness can be introduced by the holomorphic and anti-holomorphic waves expressed in terms of z_i resp. $\overline{z}_i$.

The concept of parity, originally defined as operation of studying a system or sequence of events reflected in a mirror considered in the 3-space, must also be subjected to modifications. It is clear that mirroring should relate to the complex conjugation available on Zeeman manifolds but which seems to be appropriate to introduce parity, also. Conjugation of a function $\psi(X)e^{\frac{2\pi}{c}\mathbf{i}\langle Z_\gamma, Z\rangle}$ can be considered just on $\psi(X)$, in which case the parity is denoted by P_X, or on the whole function, in which case it is denoted by P_{XZ}. The answer to the question as to how do the operators $\lhd_\mu$ and Δ react to P_X and P_{XZ} is as follows.

Theorem 4. **A.** *The L^2 spectrum of the Landau-Zeeman operator $\lhd_\mu$ satisfying $\mu \neq 0$ is P_X-parity violating, whereas the spectrum of Δ does not violates P_{XZ}. The first statement precisely means that the conjugation performed only on $\psi(X)$ involved to $\psi(X)e^{\frac{2\pi}{c}\mathbf{i}\langle Z_\gamma, Z\rangle}$ transforms an eigenfunction of $\lhd_\mu$ to an eigenfunction but which will have a different eigenvalue. By contrast, the complex conjugation on the whole wave $\psi(X)e^{\frac{2\pi}{c}\mathbf{i}\langle Z_\gamma, Z\rangle}$ produces an eigenfunction of Δ which has the same eigenvalue.*

B. *By contrast, the free particle spectrum $[0, \infty)$ does not violate P_X-parity and $P_{X,Z}$-parity is not even defined for free particle systems.*

Theorem **B.** follows from the facts that the waves of free particles only depend on X and the Hamiltonian Laplacian is Δ_X.

The first statement in **A.** follows from facts such as the spectrum only depends on the holomorphic degree $\mathbf{p}$, furthermore, a conjugated eigenfunction remains eigenfunction but which has different eigenvalue. The second statement in **A.** is due to the fact that the original $\mathbf{q}$ shows up as $\mathbf{p}$ regarding the conjugated function. To see this, observe that the eigenfunctions of Δ are of the form $h_{p,q}e^{2\pi\mathbf{i}\langle Z_\gamma, Z\rangle}$, where the holomorphic functions in $h_{p,q}$ are defined by $J_{Z_\gamma u}$, thus conjugation P_{XZ} transforms it to an eigenfunction defined in terms of $-J_{Z_\gamma u}$ and having the same eigenvalue. This proves the mirror symmetry regarding P_{XZ} completely.

The numbers $\tau = \mathbf{p} + \mathbf{q}$ and $\mathbf{m} = \mathbf{p} - \mathbf{q} = 2\mathbf{p} - \tau$ are called the absolute total and the net chirality numbers, respectively. Notice that the above function also is an eigenfunction of the magnetic dipole moment operator with the eigenvalue

$$\mathbf{m}\mu = \frac{\mathbf{m}eB_\gamma}{2\hbar c} = \frac{qB_\gamma}{2\hbar c}, \tag{20}$$

where $q = \mathbf{m}e$ is the total electric charge ready to be picked up by the probabilistic density $\Psi_\gamma\overline{\Psi}_\gamma$ defined, in (6), by a normalized wave Ψ_γ satisfying $||\Psi_\gamma|| = 1$.

The latter argument says that the waves (alias probabilistic amplitudes) do not carry electric charges but charges are defined by the action of the Laplacian and the magnetic dipole moment operator. The so defined charge is picked up by the probabilistic densities, that depend only on the X-variable, and which is also furnished into the eigenvalues of the operators. There is also stated that the charge defined for antiholomorphic waves is negative while it is positive for the holomorphic waves. This distinction can be made only with the probabilistic amplitudes but not with the probabilistic densities.

For a given τ, the range for $\mathbf{m}$ is $-\tau, -\tau+1, \ldots, \tau-1, \tau$. All contributions to the net chirality number originate from $\prod_i \zeta_i^{|m_i|}$ appearing in (16), whereas the action of the magnetic dipole-momentum operator on $C_c^{(|m|)}(r_1^2, \ldots, r_\kappa^2)$ is strongly trivial, meaning that the action results zero, at each step, when the operator acts on the product according to the Leibniz rule. In r_i^2, a holomorphic linear function is multiplied with an antiholomorphic one - which respectively are associated with $+e$ resp. $-e$, and magnetic vector fields pointing into opposite directions - the condition implies that the magnetic field is completely expelled from such waves and the electric fields generated by the electric charges also neutralize each other. These features are similar to those observed in superconductors. They will be compared with the superconductor-waves (Cooper-waves) later in this paper. Index $|m|$ in $C_c^{(|m|)}$ indicates that it also depends on $\prod_i \zeta_i^{|m_i|}$, c, and $\mathbf{c} = c_1 + \cdots + c_\kappa$. One also has

Theorem 5. *For orbiting particle systems characterized by $\mu \neq 0$, the independence of the eigenvalues from the antiholomorphic index $\mathbf{q}$ implies that each eigenvalue has infinite multiplicity. In fact, the eigenfunctions defined by the same $\mathbf{p}$ but distinct $\mathbf{q}$ have the same eigenvalues.*

4.2.2. Technicalities in the Standard Spectral Computations

Traditionally, the eigenfunctions of $\lhd_\mu$ are sought out among functions of the form $f(|X|^2)G^{(n,m)}(X)$ where $G^{(n,m)}(X)$ is an n^{th}-order complex valued spherical harmonics and m denotes the magnetic quantum number. It is an eigenfunction if and only if $f(t)$ is an eigenfunction of the radial Landau-Zeeman operator

$$(\diamond_{\mu,t}f)(t) = 4f''(t) + (2k+4n)f'(t) - (2m\mu + 4\mu^2(1+\frac{1}{4}t))f(t). \tag{21}$$

As explained earlier, the extra constant $4\mu^2$, missing from the original Landau operator, corresponds to the energy density of the constant magnetic field and rounds up the Landau to be the complete energy operator. Since the boundary conditions can be imposed by radial functions, this is a rather general technique applicable not just to exploring eigenfunctions on the whole X-space but on balls B_R or spheres $S_R = \partial B_R$, also, where R denotes radius and the center is the origin 0 of the X-space.

On the whole X-space, there is available also another technique with which the f is sought out in the form $f(|X|^2) = f_{a(n)}(|X|^2)e^{-\frac{1}{2}\mu|X|^2}$, where $f_{a(n)}(t)$ is an a^{th}-order polynomial, which, after plugging it into (21), leads to a radial operator in terms of $f_{a(n)}$. The latter assumption ensures that one seeks out L^2 eigenfunctions defined on the whole X-space. This method, however, does not provide the eigenfunctions in the above described compact cases because the boundary conditions can't be controlled by the polynomials $f_{a(n)}(t)$.

Degree m is defined such that $G^{(n,m)}(X)$ is simultaneously an eigenfunction of $\mathbf{i}D_\mu\bullet$ with eigenvalue $m\mu$. Such homogeneous harmonic polynomials can be constructed by the operator $\Pi_X^{(n)}$ projecting n^{th}-order homogeneous polynomials of the X-variable onto the space of n^{th}-order homogeneous harmonic polynomials of Δ_X. It appears in the form:

$$\Pi_X^{(n)}(P_n(X)) = \sum_{s=0}^{[n/2]} C_s^{(n)} \langle X, X\rangle^s \Delta_X^s(P_n(X)), \tag{22}$$

where $[n/2]$ denotes the greatest integer $\leq n/2$ and the coefficients $C_s^{(n)}$ yield the recursion $C_0^{(n)} = 1$ and

$$C_{s-1}^{(n)} + \big(2s(2(s-1)+l) + 4s(n-2s)\big)C_s^{(n)} = 0. \tag{23}$$

For validating (23), it is enough to see that the projection (22) produces homogeneous harmonic polynomials from a generic n^{th}-order homogeneous polynomial $P_n(X)$ if and only if the coefficients obey (23). The proof of harmonicity of $\Pi_X^{(n)}(P_n(X))$ - which is the only non-trivial question there - is as follows.

Notice that the term obtained by the action of Δ_X on $\Delta_K^{s-1}(P_n(X))$, that stands in the $(s-1)^{th}$ term of the sum (22), is cancelled out by those produced by the action of $\Delta_X = \sum \partial_\alpha \partial_\alpha$ on $\langle X, X\rangle^s$ and by those arising when one differentiation, the ∂_α, acts on $\langle X, X\rangle^s$ and the other on the $(n-2s)^{th}$-order homogeneous polynomial $\Delta_X^s(P_n(X))$. The first action produces the contribution $2s2(s-1) + 2sl = 2s(2(s-1)+l)$ and the second results $4s(n-2s)$. The latter mixed action must be counted twice, since the first differentiation can act either on $\langle X, X\rangle^s$ or on $\Delta_X^s(P_n(X))$. The total contribution is then the strictly positive number:

$$\big(2s(2(s-1)+l) + 4s(n-2s)\big) = 2s(2n+l-2s-2), \tag{24}$$

where $s \geq 1$. There is also produced $C_s^{(n)}\langle X, X\rangle^s \Delta_X^{(s+1)}(P_n(X))$ which will be cancelled out by the first two terms produced when Δ_X acts on the $(s+1)^{th}$ term.

Operator $\Pi_X^{(n)}$ really is a projection. To see this, consider its action on $HP_n(X) = \Pi_X^{(n)}(P_n(X))$, again. Then, because of the harmonicity of $HP_n(X)$, the relation $\Delta_X^s(HP_n(X) = 0$ holds, for any $s \geq 1$, thus

$$\Pi_X^{(n)}(HP_n(X)) = \sum_{s=0}^{[n/2]} C_s^{(n)} \langle X, X\rangle^s \Delta_X^s(HP_n(X)) = \tag{25}$$

$$= C_0^{(n)} \langle X, X\rangle^0 \Delta_X^0(HP_n(X)) = HP_n(X). \tag{26}$$

In other words, it satisfies the characteristic property $\Pi_X^{(n)}\Pi_X^{(n)} = \Pi_X^{(n)}$ of projection operators.

By summing up, this operator, acting on a homogeneous polynomial $\prod_i z_i^{\tilde{p}_i} \bar{z}_i^{\tilde{q}_i}$, produces a desired harmonic polynomial $G^{(n,m)}(X)$, where $n = \sum_i(\tilde{p}_i + \tilde{q}_i)$ and $m = \sum_i(\tilde{p}_i - \tilde{q}_i)$. The spectrum computations described in the next section on the whole X-space show that the eigenvalues appear in the form $-((4a(n) + 4\tilde{\mathbf{p}} + k)\mu + 4\mu^2)$, where $\tilde{\mathbf{p}} = \tilde{p}_1 + \cdots + \tilde{p}_\kappa$. Since the total holomorphic degree of the polynomial $f_{a(n)}(\langle X, X\rangle)G^{(n,m)}(X)$ is $\mathbf{p} = a(n) + \tilde{\mathbf{p}}$, the eigenvalues appear in the same form as in terms of Itô's polynomials.

The quantum numbers of spectroscopy are defined in terms of the standard eigenfunctions. The azimuthal quantum number is defined by the order n of $G^{(n,m)}$, and $a(n)$ is called radial quantum number. The quantum numbers introduced by Itô's polynomials can be expressed by the standard ones by $\tau = n + 2a(n), p = a(n) + \frac{1}{2}(n+m), q = a(n) + \frac{1}{2}(n-m)$, and $\mathbf{c} = a(n) + \sum_i s_i$. The net chirality quantum number m is the same in both cases, thus $\sum_i |m_i|$ is the order of $\prod_i \zeta_i^{|m_i|}$. This is how

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the quantum numbers arising from the two representations of the eigenfunctions communicate with each other.

4.2.3. The Actual Spectral Computations

A. In the standard case satisfying $\mu \neq 0$, the calculations are traced back to finding the eigenvalues of an ordinary differential operator acting on radial functions of the form $f(\mu\langle X, X\rangle)$. If $\mu = 1$, then $D_\mu \bullet f = 0$, $|Z_\mu|^2 = 1$ and $|J_\mu(X)|^2 = \langle X, X\rangle$, which imply:

$$(\lhd_\mu F)(X) = \big(4\langle X, X\rangle f''(\langle X, X\rangle) + (2k+4n)f'(\langle X, X\rangle) \tag{27}$$

$$-(2m + 4((1+\frac{1}{4}\langle X, X\rangle)f(\langle X, X\rangle))\big)G^{(n,m)}(X), \tag{28}$$

where $F(X) = f(\mu\langle X, X\rangle)G^{(n,m)}(X)$ and $G^{(n,m)}$ is an n^{th}-order spherical harmonics which also is an eigenfunction of $\mathbf{i}D_\mu\bullet$ for the eigenvalue $m\mu$. Thus the eigenvalue problem is reduced to the ordinary differential operator

$$(L_{(\mu=1,n,m)}f)(t) = 4tf''(t) + (2k+4n)f'(t) - (2m + 4(1+\frac{1}{4}t))f(t). \tag{29}$$

Notice that $e^{-\frac{1}{2}t}$ is an eigenfunction of this operator with the eigenvalue $-(4\tilde{p} + k + 4)$. The generic eigenfunctions are sought out in the form:

$$f(t) = u(t)e^{-\frac{1}{2}t}, \tag{30}$$

which is an eigenfunction of $L_{(n,m)}$ if and only if the $u(t)$ is an eigenfunction of the differential operator:

$$P_{(\mu=1,n,m)}u)(t) = 4tu''(t) + (2k+4n-4t)u'(t) - (4\tilde{p} + k + 4)u(t). \tag{31}$$

Also notice that, for any degree s, this operator has the uniquely determined polynomial eigenfunction

$$u_{(\mu=1,s,n,m)}(t) = t^s + a_1 t^{s-1} + a_2 t^{s-2} + \cdots + a_{s-1}t + a_s, \tag{32}$$

where the coefficients satisfy the recursion formulae:

$$a_0 = 1 \quad , \quad a_i = -a_{i-1}(s-i)(s+n+\frac{1}{2}k+1-i)s^{-1}. \tag{33}$$

The corresponding Hamiltonian eigenvalue λ satisfies then the relations

$$-\lambda_{(\mu=1,s,n,m)} = -(4s + 4\tilde{p} + k + 4) \quad , \quad \tilde{p} = \frac{1}{2}(m+n). \tag{34}$$

In order to see that all eigenfunctions can be represented in such form, one should prove yet that the polynomials (32) form a basis in $L^2([0,\infty))$. To this end, they are related to Laguerre polynomials customarily defined as such s^{th}-order polynomial eigenfunctions of the operator

$$\Lambda_\alpha(u)(t) = tu'' + (\alpha + 1 - t)u' \tag{35}$$

which have the eigenvalue $-s$ [Sze]. By

$$P_{(\mu=1,n,m)} = 4\Lambda_{(\frac{1}{2}k+n-1)} - (4\tilde{p} + k + 4), \tag{36}$$

the eigenfunctions of (31) and (35) are the same and the eigenvalues appear in the form (34). According to the theory of Laguerre polynomials, for the fixed values k, n, m (which also imply that $\tilde{p}$, too, is a fixed number), the functions $u_{(\mu=1,s,n,m)}$, $s = 0, 1, \ldots \infty$ form a basis in $L^2([0,\infty))$, indeed [Sze].

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Recall that the multiplicity of each eigenvalue (34) is infinity. The multiplicity of the eigenvalues arising from eigenfunctions defined in terms of the spherical harmonics $G^{(n,m)}(X)$ is the dimension of the space spanned by these spherical harmonics.

In case of $\mu \neq 1$, the eigenfunctions are sought in the form

$$u_{(\mu,s,n,m)}(\langle X,X\rangle)e^{-\frac{1}{2}\mu\langle X,X\rangle}G^{(n,m)}(X), \tag{37}$$

which yield

Theorem 6. *If $\mu \neq 0$, the L^2 eigenfunctions of the radial Landau-Zeeman Laplacian of point particles appear in the form:*

$$u_{(\mu,s,n,m)}(t) = u_{(\mu=1,s,n,m)}(\mu t) \tag{38}$$

with the corresponding Hamiltonian eigenvalues λ satisfying

$$-\lambda_{(\mu,s,n,m)} = -((4s+4\tilde{p}+k)\mu + 4\mu^2). \tag{39}$$

For settling this statement notice that the substitution $Y = \mu^{\frac{1}{2}}X$ transforms the pure Landau-Zeeman operator $\lhd_{\mu,X}$ (missing the constant term $4\pi^2|Z_\gamma|^2$) to $\mu\lhd_{\mu=1,Y}$.

In order to prove the statement by Itô's polynomials, express $\lhd_\mu$ on a complex orthonormal coordinate neighborhood $\{z_1,\ldots,z_\kappa\}$ by means of the complex differentiations ∂_{z_i} and $\partial_{\bar{z}_i}$ in the form:

$$\lhd_\mu = \sum_i \left(4\partial_{z_i}\partial_{\bar{z}_i} + 2\mu(\bar{z}_i\partial_{\bar{z}_i} - z_i\partial_{z_i}) - \mu^2 z_i\bar{z}_i\right) - 4\mu^2. \tag{40}$$

This and (13), (14), (15), (35) prove that the 2D Itô polynomials are eigenfunctions of $\lhd_\mu$ with eigenvalues explicitly described above. The multidimensional Itô polynomials are products of those defined for the coordinates z_i as 2D polynomials. As a result, they are eigenfunctions of (40) with the desired eigenvalue.

These computations reveal that Itô's polynomials draw a much subtler picture about the quantum events than that depicted by spherical harmonics. The holomorphic and antiholomorphic indexes **p** and **q**, appearing in their original definition, show that the spectrum is P-symmetry violating, which statement is not that clear when it comes to the standard computations leaving the indices **p** and **q** in certain obscurity.

B. In physics, the classical example for a continuous spectrum is the part of the spectrum of the light emitted by excited atoms of hydrogen. The emission takes place so that free electrons become to be bound to a hydrogen ion that emits photons smoothly spread over a wide range of wavelengths. This is in contrast to the discrete spectral lines due to the electrons falling from some bound quantum state to a state of lower energy. In view of the relation of the invariant subspace $W_{\gamma=0}$ to those defined for $\gamma \neq 0$, the waves in the former case represent free particle systems which can even be associated with charge if one picks up a complex structure $J_{\gamma 0}$ in order to define holomorphic and antiholomorphic polynomials. In what follows, there is given a sketchy proof for the Euclidean Hamiltonian $-\Delta_X$ has the continuous L^2 spectrum $[0,\infty)$ on $W_{\gamma=0}$.

First remember that, by Plancherel's theorem, the appropriately normalized Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n, m) \to L^2(\mathbb{R}^n, m)$ is unitary. Then, define the function $h : \mathbb{R}^k \to \mathbb{R}$ by $h(X) = |X|^2$. If the domain of Δ_X is considered to be all L^2 functions with two weak derivatives in L^2, then consider the closure of the densely defined operator $\mathcal{F}^{-1}\Delta_X\mathcal{F} = -M_h$, where M_h denotes multiplication with h. This is equivalent to say that an L^2 function f is in the domain if and only if hf is there, also. Using this characterization, first define Δ on $C_c^\infty(\mathbb{R}^n)$ and then take its closure. Either way one gets the same densely defined operator. Since the essential range of h is clearly $[0,\infty)$, that is going to be the spectrum of $-\Delta_X$. Moreover, since $m(h=\lambda) = 0$ for each λ, it is all a continuous spectrum.

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4.3. Dirac-Hamilton Operators for Point Particle Systems

4.3.1. Spin Matrices on Zeeman Manifolds

The Zeeman-Hamilton operators acting on functions only endow angular momentum (orbiting spin) to the particles. Proper spin is established with Dirac operators acting on spinors. The Zeeman-Dirac Hamilton operator is a modified version of that of Dirac who formulated his operator by means of 4×4 spin matrixes acting on 4-spinors defined over the 4D Minkowski space. Contrary to this, on Zeeman manifolds, it is introduced, for each lattice point Z_γ, by the complex structure $J_{\gamma u}$ and Pauli's 2×2 spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{41}$$

which, alike to the 4×4 versions, also yield the Clifford-Dirac relations:

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{42}$$

in a way as follows.

4.3.2. The 2D Spin Operators

The generic versions will be introduced on $\mathbb{C}^\kappa$ as a sum of the 2D operators defined on the coordinate planes of a complex orthonormal coordinate system. Thus, case $\kappa = 1$ should be considered, in the first place, before passing to the generic one when when the dimension is $2\kappa = k$.

The natural parameterization (x^1, x^2) presents $\mathbb{C}$ as being the 2-dimensional X-space $\mathbb{R}^2$, on which the Dirac-Hamilton operator is defined, in terms of the vector potential $\mathbf{a} = \mu(-x^2, x^1)$, by

$$\mathcal{DH}_\mu = \sum_{j=1}^{2}(\mathbf{i}\frac{\partial}{\partial x^j} - \mathbf{a}^j)\sigma_j + 2\mu\sigma_0 = \sum_{j=1}^{2} D^j\sigma_j + 2\mu\sigma_0 = \tag{43}$$

$$= \begin{pmatrix} 2\mu & \mathbf{i}(2\partial_{\bar{z}} - \mu z) \\ \mathbf{i}(2\partial_z + \mu\bar{z}) & -2\mu \end{pmatrix}, \tag{44}$$

where $\mu = \frac{\pi}{c}|Z_\gamma|$. Then, simple calculations yield $D^jD_j = \sum_{j=0}^{2}(D^j)^2 = -\lhd_\mu$, which squaring formula is also yielded by the original Dirac-Hamilton operator.

The $\mathcal{DH}_\mu$ is a spin-$\frac{1}{2}$ operator which acts on doublets (2-spinors) of the form $\begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}$ whose components are defined, in terms of Z_γ, by (6). The squaring formula does not mean that the square $\mathcal{DH}^2$ of the Dirac-Hamilton operator provides $-\lhd_\mu$, but which rather appears in the form

$$\mathcal{DH}_\mu^2 = -\lhd_\mu\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2\mu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{45}$$

as will be seen in (64)-(68). The last term is of spin-1 which turns $\mathcal{DH}^2$ to be a spin-1 operator, also.

The 2D operator (43) can be recovered from Dirac's original 4D-operator H_D - that is defined in terms of the usual parameterization (t, x_1, x_2, x_3) of the 4-space, by

$$-\frac{\hbar}{\mathbf{i}c}\frac{\partial}{\partial t} - eV - \sum_{r=1}^{3}\alpha_r\left(\frac{\hbar}{\mathbf{i}}\frac{\partial}{\partial x_r} + \frac{e}{c}\mathbf{a}^r\right) - \alpha_0 mc = -\frac{\hbar}{\mathbf{i}c}\frac{\partial}{\partial t} - H_D, \tag{46}$$

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where

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -\mathbf{i} \\ 0 & 0 & \mathbf{i} & 0 \\ 0 & -\mathbf{i} & 0 & 0 \\ \mathbf{i} & 0 & 0 & 0 \end{pmatrix}, \tag{47}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{48}$$

The recovery of the 2D operator from this one proceeds as follows.

On the Zeeman manifold prototypes, the charged particles are considered to be rotating in that unique inertial system where the constant electromagnetic field defining the orbiting spin (angular momentum) is associated with vanishing electric field, that is, $E = 0$. The constant magnetic field is associated with the vector potential $\mathbf{a} = \mu(0, -x_2, x_1, 0)$, thus the relativistic 4-potential is equal to $(0, \mathbf{a}^1, \mathbf{a}^2, 0)$. In such situation, the Dirac Hamilton operator is completely determined by the entries defined with regard to the (x_1, x_2)-plane, that is, by the entries appearing in the middle 2×2-matrices of Dirac's 4×4-spin matrices. But these middle matrices are exactly the σ-matrices in terms of which the H_D appears in the form of the Dirac-Hamilton operator $\mathcal{DH}$.

4.3.3. Spin Operators Defined in the $k = 2\kappa$ Dimension

On a $k = 2\kappa$-dimensional X-space, the Dirac-Hamiltonian is still defined in terms of 2×2 matrices by

$$\mathcal{DH}_\mu = \sum_{i=1}^{k/2}\sum_{j=1}^{2}(\mathbf{i}\frac{\partial}{\partial x_i^j} - \mathbf{a}_i^j)\sigma_j + 2\mu\sigma_0 = \sum_{i=1}^{k/2}\sum_{j=1}^{2} D_i^j\sigma_j + 2\mu\sigma_0 = \tag{49}$$
$$= \begin{pmatrix} 2\mu & \mathbf{i}\sum_i(2\partial_{\bar{z}_i} - \mu z_i) \\ \mathbf{i}\sum_i(2\partial_{z_i} + \mu\bar{z}_i) & -2\mu \end{pmatrix},$$

where (x_i^1, x_i^2) is the coordinate system on the i^{th} complex coordinate plane, which implies the squaring formula $\mathcal{DH}_\mu^2 = -\lhd_\mu - 2\mu\sigma_0$. The operators $\mathcal{DH}_\mu$ and $\mathcal{DH}_\mu^2$ - also called Fermionic- and Bose-Hamiltonians - respectively are of spin-$\frac{1}{2}$ and spin-1 operators for $k/2$ number of particles that rotate in their designated individual complex planes going through the origin 0 of the X-space and which are perpendicular to a common magnetic field pointing into the same direction as $Z_\gamma(0)$ and having magnitude B determined by $\pi|Z_\gamma(0)| = (eB)/(2\hbar)$. It is noteworthy that Pauli defined his spin-$\frac{1}{2}$ operator by $-\lhd_\mu - \mu\sigma_0$, that is, adding only the half of the spin-one operator present in the Bose-Hamiltonian to the $-\lhd_\mu$.

4.3.4. Eigenfunctions and Eigenvalues of $\mathcal{DH}_\mu$.

The spectral computations are carried out first for the dimension $k = 2$. As it will turn out, the eigenfunctions and eigenvalues of $\mathcal{DH}_\mu$, when $\mu \neq 0$, can be determined by those of $\mathcal{DH}_\mu^2$ and $\lhd_\mu$. The $\mathcal{DH}_\mu$ acts on $\mathbb{C}^2$-valued functions of class L^2 (also called doublets or 2-spinors) which are written up in the form $\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$. The L^2 eigenfunctions of $-\lhd_\mu$ with eigenvalue λ are spanned by doublets of the form $\phi_1 = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}$ or $\phi_2 = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$, where φ is an eigenfunction for $-\lhd_\mu$ with eigenvalue λ. Such a doublet ϕ_j is an eigen-doublet also of $\mathcal{DH}_\mu^2$ with the eigenvalue $\beta_j = \lambda + (-1)^j 2\mu$.

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Everything that has been said insofar is valid also for $\mu = 0$, when $-\lhd_{\mu=0} = 4\Delta_X$, thus, it has the continuous L^2-spectrum $[0, \infty)$. Moreover, the only eigenfunction having 0 eigenvalue is $\varphi = 0$, and the generic eigenvalues yield $\beta_j = \lambda$.

If $\mu \neq 0$, the eigenvalues of $-\lhd_\mu$ – for functions which may be defined either on the whole X-space in terms of Itô's polynomial or on balls by the Dirichlet or Neumann condition – are the strictly positive numbers $\lambda = (4\mathbf{p} + k)\mu + 4\mu^2$, and, due to $4\mu^2$, the eigenvalue $\beta_j = \lambda + (-1)^j 2\mu$ is strictly positive, also. Thus $\beta_j = 0$ if and only if $\varphi = 0$ which defines the trivial eigenspinor $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Theorem 7. *For $k = 2$, all non-trivial eigen-spinors of $\mathcal{DH}_\mu$ appear in the form*

$$\psi_{j+} = \phi_j + \frac{1}{\sqrt{\beta_j}}\, \mathcal{DH}_\mu(\phi_j)\ , \quad \psi_{j-} = \phi_j - \frac{1}{\sqrt{\beta_j}}\, \mathcal{DH}_\mu(\phi_j), \tag{50}$$

where $\beta_j > 0$, for which the corresponding eigenvalues are determined by the equations

$$\mathcal{DH}_\mu(\psi_{j+}) = \sqrt{\beta_j}\ \psi_{j+}\ , \quad \mathcal{DH}_\mu(\psi_{j-}) = -\sqrt{\beta_j}\ \psi_{j-}\ . \tag{51}$$

In the generic k-dimensional cases, the eigen-spinors are products of $\psi^{(i)}_{j+}$ and $\psi^{(i)}_{j-}$ - of those to be determined for $\mathcal{DH}_i$ on the i^{th} coordinate plane - where the product means multiplication of the first resp. second components in order to determine the components of the sought 2-spinors. The eigenvalue regarding the product is defined by the sum of the eigenvalues to be defined for the component eigenspinors.

When $\mu = 0$, then $-\lhd_\mu = -\Delta_X$ and the operator has the continuous L^2-spectrum $[0, \infty)$. Thus $\beta_j = \lambda$, where $\lambda \in [0, \infty)$. Since $\lambda = 0$ holds true only for the trivial doublet $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, all of the above formulae are well defined, saving the validity of the Theorem for all non-trivial eigenfunctions.

In the 2D case, the components of an eigen-spinor are interrelated by the following explanations. For a scalar eigenfunction ϕ_1, standing in the upper component, the lower one is resulted by the action of the decreation operator $2\partial_z + \mu\bar{z}$ on ϕ_1. Designation "decreation" is justified, since it alternates holomorphic coordinates to antiholomorphic ones and produces eigenfunctions of $-\lhd_\mu$ having smaller eigenvalues. (For an alternative proof of the latter statement see the next paragraph.) To determine the interrelated components of ψ_{2+}, start with an eigenfunction ϕ_2 standing in the lower component, from which, in the upper component, an eigenfunction with a greater eigenvalue is produced by the action of the creation operator $2\partial_{\bar{z}} - \mu z$. The latter, by alternating antiholomorphic coordinates to holomorphic ones, increases the eigenvalue, thus, it really works as a creation operator. In higher dimensions, the interrelating components are defined by products of the interrelated component functions to be defined on the coordinate planes.

The above statements about the higher and lower eigenvalues can be proven by the equations (51). The action of $\mathcal{DH}$ on them results $\mathcal{DH}^2(\psi_{j\pm}) = \beta_j \psi_{j\pm}$. It means that both the upper and lower functions are eigenfunctions of $\mathcal{DH}^2$ with the eigenvalue β_j. The equation $-\lhd_\mu = \mathcal{DH}^2 + 2\mu\sigma_0$ implies then that the functions in both components are eigenfunctions also regarding $-\lhd_\mu$ so that the upper one has a bigger eigenvalue than the lower one.

In quantum field theory, the creation and annihilation (decreation) operators are introduced to add or remove a particle from a multiparticle system. They were introduced in the second quantization formalism to carry out the actions of inserting and deleting a single-particle state from the many-body wave function which can be either symmetric (if the particles are bosons) or antisymmetric (if the particles are fermions) with regard to particle exchanges. They are by no means eigenfunctions of Hamilton operators and the creation and annihilation (decreation) operators are not Hamilton operators.

On Zeeman manifolds, by contrast, the creation resp. decreation operators are the components standing in the $(2, 1)$ resp. $(1, 2)$ positions in the matrix for the Fermionic Hamilton operator that

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corresponds to the eigenfunctions ϕ_j new eigenfunctions of the very same operator $-\lhd_\mu$ but which have greater resp. smaller eigenvalues. They respectively determine, besides ϕ_j, the second components of the eigenspinors $\psi_{j\pm}$. This is seen so that the difference of eigenvalues determines a system of multiparticles which are added resp. deleted from the system being in the eigenstate associated with ϕ_j. The masses are determined by the spectral mass-assignment procedure. The mass difference is interpreted so that a boson is absorbed in the first case and emitted in the second case in order to manage the energy difference when the system leaps from state ϕ_j to $\mathcal{DH}(\phi_j)$. In order not to confound the here established creation resp. decreation operators with those introduced in classical quantum theory, they are called Leap-Up and Leap-Down operators and denoted by U and D, respectively.

4.4. Physical Significance of Waves

A. *Physical significance of spectrum:* On Zeeman manifolds, the scalar; Bose; Fermionic; and anti-Fermionic particles are described by probabilistic amplitudes that, in eigenstates, are normalized eigenwaves of the corresponding scalar, Bose, and Dirac Hamiltonians. A scalar or spinor wave Ψ is normalized if it yields $\int \Psi\overline{\Psi} = ||\Psi||^2 = 1$. In classical quantum theory, a strictly observed view is that the physically realistic quantities are not the waves but the probabilistic densities $\Psi\overline{\Psi}$ which probabilistically determine the position of pointparticles that themselves are considered to be dual formation (wave packet) relative to the waves.

This view no longer can be upheld in such a strict form on Zeeman manifold where the action of the Laplacian on eigenwaves defines physically significant quantities, among them, the eigenvalues - which will be identified with the masses of particles in the spectral mass-assigning procedure -, and the electric charge that appears in the eigenvalues defined for eigenwaves. The appearance of the electric charge in an eigenvalue ensures that the new framework is in consent with the earlier described requirement of that the electric charges can not appear alone but must be carried by masses. Other manifestations of physical significance in waves are as follows.

For $\mu \neq 0$, the Fermionic spectrum doubles the spectral lines of those of the scalar operator. If the functions in the upper and lower components of a Fermionic eigenspinor ψ_{j+} with a fixed $j = 1$ or $j = 2$ are considered to be two neighbor eigenstates, then - by the explanations to be found below Theorem 7 - they are differentiated from each other by the way how they are built up by holomorphic and antiholomorphic coordinates. The eigenspinors ψ_{1+} and ψ_{2+} are differentiated by this regard, too, and also by the eigenvalues $\sqrt{\beta_1}$ and $\sqrt{\beta_2}$. All these arguments show that the Fermionic spectrum exhibits Pauli's exclusion principle, saying: No two different Fermionic particle system can occupy the same eigenstate.

With respect to the Bose operator, by contrast, Theorem 7 says that both the upper and lower functions in ψ_{j+} are eigenfunctions with the same eigenvalue $\beta_j = \lambda + (-1)^j 2\mu$, where $j = 1;2$. This is the manifestation of the fact that $\mathcal{DH}^2_\mu$ really is a Bose operator. It is also noteworthy that all level functions in ψ_{1+} and ψ_{2+} are eigenfunctions for $\lhd_\mu$ with the same eigenvalue λ.

B. *The physical significance of μ:* If a charged particle travels in a field-free region that surrounds another region, in which there is a trapped/threaded/
looped magnetic flux F, then upon completing a closed loop the particle's wave function will acquire an additional phase factor exp $\left(\frac{e\Phi}{\hbar c}\right)$. The wave function must be of single valued, at any point in the space, that can be accomplished only by a magnetic flux F quantized according to $\frac{e\Phi}{\hbar c} = 2\pi n$, where $n \in \mathbb{Z}$. This quantization of the magnetic flux is observed in superconductors. Superconductivity is due to a special correlation between pairs of electrons that extends over the whole body of the superconductor. When a Type I superconductor is placed in a magnetic field and cooled below its critical temperature, it excludes all magnetic flux from its interior. This is called the Meissner effect. If there is a "hole" in the superconductor, then flux can be trapped in this hole. The flux trapped in the hole must be quantized. It has been experimentally verified that the trapped flux is quantized in units of $\Phi_0 = \frac{2\pi\hbar c}{2e}$, thus verifying that the charge carriers in superconductors are indeed correlated electron pairs of charge 2e.

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The equations in (11) relate the Laplacian and its eigenvalues to an other physically significant quantity - the magnetic flux quantum and also describe a relation between the lattice quantizations $Z_\gamma, \mu_\gamma, B_\gamma$ and the magnetic flux quantization. This is particularly important with regard to the quantization $2\mu_\gamma = \frac{eB_\gamma}{\hbar c} = \frac{\pi}{c}\frac{B_\gamma}{\Phi_0}$ which quantity shows up in the Bose operator. The boson emerging from this quantization is considered to be an ordinary particle. It also has a virtual manifestation - that is, can be seen as transient quantum fluctuation exhibiting the characteristics of the actual particle - which interpretation is described in the next section. It emerges there - as the virtual particles must do - from Heisenberg group representations and the uncertainty principle.

C. *Particles separating from their antiparticles:* One of the postulates in Dirac's theory says that, for every species of charged particles, there must be another species, called antiparticles, which have the same mass but opposite electric charge. The replacement of a particle by its antiparticle is called charge conjugation symmetry or simply C symmetry.

Antiparticles - matching those of Dirac - exist also on Zeeman manifolds. They are described, insofar, by their Hamilton waves which visualization is further specified on Zeeman spacetimes, after carrying out the spectral mass assignment procedure. As it turns out there, an antiparticle gains the same positive mass as its particle partner if in the Dirac equation the time derivative is taken with respect to the opposite direction and the twisting functions in their waves are exchanged for their complex conjugates. This is equivalent to say that antiparticles, charged with opposite charge as their particle partners, arrive to us from the future. It will be also shown there that a particle and its corresponding antiparticle have the same mass. That is, this proof places Dirac's postulatum among the exactly established mathematical theorems.

The mathematization rises the question as to how can the C-symmetry be defined on Zeeman manifolds in terms of the Hamilton waves. Even though the eigenvalues $\sqrt{\beta_j}$ and $-\sqrt{\beta_j}$ signal the same mass for them, the antiparticle wave of $\Psi_{\gamma j+} = \psi_{j+}e^{2\pi\mathbf{i}\langle\frac{1}{c}Z_\gamma,Z\rangle}$ can not be $\Psi_{\gamma j-} = \psi_{j-}e^{2\pi\mathbf{i}\langle\frac{1}{c}Z_\gamma,Z\rangle}$. This is because ψ_{j-} is not the complex conjugate of ψ_{j+}, thus the charge defined for the antiparticle wave is not the negative of that determined for the particle wave. Conjugating ψ_{j+} does not help either, because that would, by virtue of Theorem 4, only mean the application of the parity transform P_X that transforms the spinor to an eigenspinor having different eigenvalue.

The very same Theorem makes it clear, however, that the correct transform must to be applied is P_{XZ} that keeps the eigenvalue and implements the desired conjugation. Thus, the C-transform must be defined by the exchange:

$$\Psi_{\gamma j+} = \psi_{j+}e^{2\pi\mathbf{i}\langle\frac{1}{c}Z_\gamma,Z\rangle} \rightarrow \overline{\Psi_{\gamma j-}} = \overline{\psi_{j-}}e^{2\pi\mathbf{i}\langle-\frac{1}{c}Z_\gamma,Z\rangle}, \tag{52}$$

after which, the antiparticle wave of $\Psi_{\gamma j+}$ is to be found among the antiparticle waves defined for the lattice point $-Z_\gamma$.

If $\mu \neq 0$, then $Z_\gamma \neq -Z_\gamma$, and the antiparticle waves are completely separated from their partners. The sum of the two waves does not indicate any interaction between them. In case $\mu = 0$, however, $Z_\gamma = -Z_\gamma = 0$ and the particle and antiparticle wave, being in the same space $W_{\gamma=0}$, sum up to a 2-spinor in which only one of the component is non-zero. It is twice of the function written up in the upper resp. lower component, depending on that as to whether $j = 1$ or $j = 2$. Such spinors clearly represent polarized γ-rays whose polarization is determined by the component the function shows up in. The scalar eigenwaves are probabilistic amplitudes of scalar γ-rays, that give rise to a force (γ-ray pressure) whose carriers are photons - the matching particles of scalar γ-rays.

Meeting of a particular particle with a particular antiparticle can be arranged only in Zeeman spacetime. It takes place if their waves are in the same spinor space at the same time. It does not mean that any two particle and antiparticle have appointment for meeting just because their Hamilton spinors are in the same spinor space. Meeting prefers particular particles, and, if their spinors are in the same spinor space, their meeting can take place only at a particular time.

4.5. Heisenberg's Lie Algebras and Their Representations

The real Heisenberg Lie algebra is defined by restricting the Poisson bracket

$$\{f,g\} = \sum \partial_{q_i}(f)\partial_{p_i}(g) - \partial_{p_i}(f)\partial_{q_i}(g) \tag{53}$$

onto the real linear space spanned by the functions $\{q_i, p_i, 1\}$. The real Heisenberg group is obtained by the exponential map that corresponds to the Lie algebra the desired group. Their representations are carried out on the Hilbert space $\mathcal{H}_q$ of complex valued L^2 functions depending only on the position coordinates q_i. I. e., $\mathcal{H}_q = L^2_{\mathbf{C}}(\mathbf{V}_q)$ on which the complex inner product is defined by $\langle f,g\rangle = \int f\overline{g}dq$. The representation $\rho : f \to f_\rho$ of the real Heisenberg algebra is defined by

$$\rho(q_i)(\psi) = q_i\psi \quad , \quad \rho(p_i)(\psi) = \frac{\hbar}{\mathbf{i}}\partial_{q_i}\psi \quad , \quad \rho(1) = id. \tag{54}$$

The associated operators yield the Heisenberg commutation relations

$$[\rho(q_i),\rho(q_j)] = 0 \quad , \quad [\rho(q_i),\rho(p_j)] = \hbar\mathbf{i}\delta_{ij} \quad , \quad [\rho(p_i),\rho(p_j)] = 0, \tag{55}$$

therefore, the ρ is a Lie algebra representation, indeed, and the exponential map $e^{\mathbf{i}\rho}$ defines a unitary representation of the Heisenberg group. The ρ is called a unitary representation with regard to the Lie algebra, also. Since it is carried out on a complex infinite dimensional Hilbert space, furthermore, being an irreducible and unitary representation, the ρ is uniquely determined upto multiplications with unit complex numbers, by the classical Neumann-Stone theorem.

The so represented Heisenberg group and algebra act on the Hilbert space $\mathcal{H} = L^2_{\mathbf{C}\eta}$ of complex valued functions depending only on q_i, where the inner product $\langle f,g\rangle = \int f\overline{g}\eta dq$ is defined by the density $\eta = e^{-\hbar\sum q_i^2}$.

This Hilbert space, however, is not appropriate for investigating operators formulated in terms of the holomorphic and antiholomorphic coordinates, z_i and $\overline{z}_i$. These coordinates can be defined on $\mathbb{C}^\kappa$, where ∂_{p_i} and ∂_{q_i} satisfy $\partial_{p_i} = J(\partial_{q_i})$, thus $z_i = q_i + \mathbf{i}p_i$, and $\overline{z}_i = q_i - \mathbf{i}p_i$ are correct definitions for the holomorphic and antiholomorphic coordinates.

The sought Hilbert space $\mathcal{H} = L^2_{\mathbf{C}\eta}$ must consist of all complex valued L^2 functions depending both on z_i and $\overline{z}_i$ which form an L^2 Hilbert space with the Hilbert density η_μ to be defined, in terms of μ, by $\eta_\mu = e^{-\mu\sum z_i\overline{z}_i}$. It is spanned - in the L^2-sense - by the polynomials written up in terms of the holomorphic and antiholomorphic coordinates. It is isomorphic to the standard Hilbert space $L^2_{\mathbb{C}}$, defined by the standard Euclidean density $\eta = 1$, by the map:

$$L^2_{\mathbf{C}\eta} \to L^2_{\mathbf{C}} \quad , \quad \psi \to \psi e^{-\frac{1}{2}\mu\sum z_i\overline{z}_i}. \tag{56}$$

The complex differentiations ∂_{z_i} , $\partial_{\overline{z}_i}$ are defined by means of the partial differentiations ∂_{q_i} and ∂_{p_i} by

$$\partial_{z_i} = \frac{1}{2}\partial_{q_i - \mathbf{i}p_i} \quad , \quad \partial_{\overline{z}_i} = \frac{1}{2}\partial_{q_i + \mathbf{i}p_i}. \tag{57}$$

Then, the complex Poisson bracket is introduced by

$$\{f,g\}_{\mathbf{C}} = \sum \partial_{z_i}(f)\partial_{\overline{z}_i}(g) - \partial_{\overline{z}_i}(f)\partial_{z_i}(g). \tag{58}$$

The complex Heisenberg algebra is defined by restricting this bracket onto the linear space spanned by the functions $\{z_i, \overline{z}_i, 1\}$. The real canonical coordinates can be recovered by the formulae

$$q_i = \frac{1}{2}(z_i + \overline{z}_i), \quad \text{and} \quad p_i = -\frac{1}{2}\mathbf{i}(z_i - \overline{z}_i). \tag{59}$$

A representation of the complex Heisenberg algebra can be defined by the leap operators in the form

$$\rho_{\mathbf{c}}(z_i) = U_i = \mathbf{i}(2\partial_{\overline{z}_i} - \mu z_i), \quad \rho_{\mathbf{c}}(\overline{z}_i) = D_i = \mathbf{i}(2\partial_{z_i} + \mu\overline{z}_i), \tag{60}$$

which yield Heisenberg's commutation relations

$$[U_i, U_j] = 0 \quad , \quad [U_i, D_j] = -\mu\delta_{ij} \quad , \quad [D_i, D_j] = 0, \tag{61}$$

however, it is a non-unitary representation.

The validity of the latter statement can be seen from computing the transposed operators, in $\mathcal{H} = L^2_{\mathbb{C}\eta}$, which appear in the following forms

$$U_i^* = 2\partial_{z_i} - \mu\overline{z}_i, \quad D_i^* = 2\partial_{\overline{z}_i} - 3\mu z_i. \tag{62}$$

The non-unitary property also follows from the fact that the leap operators map eigenfunctions to eigenfunctions but which have different eigenvalues. Thus, they change the probabilistic amplitudes and also the probabilistic densities. This is the main reason for why they can not be unitary transformations, that always leave the probabilistic densities invariant. The argument rather shows that the leap operators are appropriate for describing proper leaps in the quantum processes.

4.6. Hamiltonians Expressed in Terms of the Leap Operators

In case of $k = 2$, when the X-space is the complex plane $\mathbb{C}$ and the vector potential appears in the form $\mathbf{a} = \mu(-x^2, x^1)$, the Dirac operator $\mathcal{DH}$ acts on two spinors $\phi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$ according to the relations

$$\mathcal{DH}(\phi) = \begin{pmatrix} 2\mu\varphi_1 + U(\varphi_2) \\ -2\mu\varphi_2 + D(\varphi_1)\end{pmatrix}, \tag{63}$$

where the leap-up and leap-down operators U and D are defined in (60).

Due to $2\mu U - U2\mu = 0$ and $U2\mu - 2\mu U = 0$, the (1,2) and (2,1) components of the operator matrix $\mathcal{DH}^2$ vanish, thus it is a diagonal matrix and the operators in the (1,1) resp. (2,2) position of the main diagonal appear in the form $4\mu^2 + UD$ resp. $4\mu^2 + DU$. By (60) and (40),

$$4\mu^2 + UD = 4\mu^2 - \left(4\partial_z\partial_{\overline{z}} + 2\mu(\overline{z}\partial_{\overline{z}} - z\partial_z) - \mu^2 z\overline{z}\right) - 2\mu = - \lhd_\mu - 2\mu, \tag{64}$$

$$4\mu^2 + DU = 4\mu^2 - \left(4\partial_z\partial_{\overline{z}} + 2\mu(\overline{z}\partial_{\overline{z}} - z\partial_z) - \mu^2 z\overline{z}\right) + 2\mu = - \lhd_\mu + 2\mu. \tag{65}$$

These equations imply

$$4\mu^2 + \frac{1}{2}(UD + DU) = -\lhd_\mu, \quad [U, D] = \frac{1}{2}(UD - DU) = -2\mu, \tag{66}$$

$$\begin{pmatrix} 4\mu^2 + UD \\ 4\mu^2 + DU \end{pmatrix} = \begin{pmatrix} 4\mu^2 + \frac{1}{2}(UD + DU) \\ 4\mu^2 + \frac{1}{2}(UD + DU)\end{pmatrix} + \begin{pmatrix} [U, D] \\ [D, U]\end{pmatrix} = \tag{67}$$

$$= \begin{pmatrix} -\lhd_\mu \\ -\lhd_\mu \end{pmatrix} - 2\mu\begin{pmatrix} 1 \\ -1\end{pmatrix}. \tag{68}$$

They prove (45) and that the Bose term $2\mu\sigma_0$ appearing there is due to Heisenberg's uncertainty relations to be considered with regard to the Leap-operators U and D.

These arguments open up a way to deeper understanding the actions of the Leap-Up, Leap-Down, and Bose operators in relation to the uncertainty principle. By this principle, the energy - that is, the spectrum of the Hamilton operator - and time are complementary variables, thus there is a fundamental limit to the precision with which they can be known simultaneously. On the Zeeman space-time, defined by extending the Zeeman manifolds by a relativistic time-axis, the time is known and the uncertainty, that comes to play when one tries to locate as to where is the energy transported by a boson (that is, is the energy emitted or absorbed?), reveals itself as the main vehicle for carrying out the boson transmission explained earlier in relation to the spin-1 operator $\mu\sigma_0$.

On a higher dimensional X-space, the $\mathcal{DH}$ is written up in terms of a complex orthonormal coordinate system $\{z_1, \ldots, z_\kappa\}$ as the sum of the operators $\mathcal{DH}_i$ to be determined for the complex coordinates z_i. In the Bose operator $\sum_{i=1}^{k/2} \mathcal{DH}_i^2$, the spin-1 term is $-k\mu\sigma_0$, while the constant in $-\lhd_\mu$ is $2k\mu^2$.

4.7. The Monistic Dirac Hamiltonian $\mathcal{MDH}$

The Dirac Hamiltonian $\mathcal{DH}$ acts on doublets $\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ whose discrete Z-Fourier transform

$$\Psi_\gamma = \begin{pmatrix} \varphi_1 e^{2\pi \mathbf{i}\langle \frac{1}{c} Z_\gamma, Z\rangle} \\ \varphi_2 e^{2\pi \mathbf{i}\langle \frac{1}{c} Z_\gamma, Z\rangle} \end{pmatrix} = e^{2\pi \mathbf{i}\langle \frac{1}{c} Z_\gamma, Z\rangle} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} e^{2\pi \mathbf{i}\langle \frac{1}{c} Z_\gamma, Z\rangle} \tag{69}$$

defines the actual Hamilton waves for a lattice point Z_γ. The above formulae describe three equivalent variants for denoting multiplications of doublets with functions. The question arises as to which is the outwardly defined Monistic Dirac Hamiltonian $\mathcal{MDH}$ whose action on the doublet ψ appears in the form $\mathcal{MDH}\,\Psi_\gamma = (\mathcal{DH}_\mu\,\phi)e^{2\pi \mathbf{i}\langle \frac{1}{c} Z_\gamma, Z\rangle}$. By a simple calculation, it must be the operator:

$$\mathcal{MDH} = \begin{pmatrix} 2\sqrt{-\Delta_Z} & \mathbf{i}\sum_i (2\partial_{\overline{\mathbf{z}}^i} - \mathbf{z}^i\sqrt{-\Delta_Z}) \\ \mathbf{i}\sum_i (2\partial_{\mathbf{z}^i} + \overline{\mathbf{z}}^i\sqrt{-\Delta_Z}) & -2\sqrt{-\Delta_Z} \end{pmatrix}, \tag{70}$$

whose eigenspinors, with regard to Z_γ, appear in the form $\phi e^{2\pi \mathbf{i}\langle Z_\gamma, Z\rangle}$, where ϕ is an eigenspinor of $\mathcal{DH}_\mu$.

4.8. The χ-Transform; Left- and Right-Handed Orbiting

4.8.1. Classification of Clifford Modules and H-Type Groups

The Lie algebra of H-type groups has been defined by endomorphism spaces, $J_{\mathcal{Z}}$, satisfying the Clifford condition $J_Z^2 = -|Z|^2 id$ for all $Z \in \mathcal{Z}$. The usage of endomorphism spaces makes the classification of Clifford modules - the representations of Clifford algebras - to be equivalent to the classification of the Cliffordian endomorphism spaces. The latter classification is accomplished in a well known theory proving that, upto equivalence, a generic Cliffordian endomorphism space can be represented in the following forms.

The X-space is defined as an $(a+b)$-times Cartesian product of a smaller space $\mathcal{Y} = \mathbb{R}^{n_l}$ on which a below described l-dimensional endomorphism space $j_{\mathcal{Z}}$ is acting. They define the endomorphisms acting on the whole X-space $\mathcal{X} = \mathcal{Y}^a + \mathcal{Y}^b$ according to the formula $J_Z^{(a,b)} = j_Z \times \cdots \times j_Z \times -j_Z \times \cdots \times -j_Z$, where $j_Z \times \cdots \times j_Z$ acts on $\mathcal{Y}^a$ and $-j_Z \times \cdots \times -j_Z$ on $\mathcal{Y}^b$.

The first part of the classification theorem states the existence of irreducible Clifford endomorphism spaces, for any positive integer l, and also determines the dimension n_l of $\mathcal{Y}$, by a formula depending only on l. By a more concrete description, the possible endomorphism spaces appear, upto isomorphisms, in two different ways, that are differentiated from each other by the relations $l = 3 \mod 4$ and $l \neq 3 \mod 4$.

(**A**) If $l = 3 \mod 4$, then there exist (up to equivalence) exactly two irreducible Clifford endomorphism space, $J_l^{(1,0)}$ and $J_l^{(0,1)}$, acting on $\mathcal{Y} = \mathbb{R}^{n_l}$. The first space can be converted to the second one by $J_Z^{(0,1)} = -J_Z^{(1,0)}$, that is, by switching the endomorphisms to their negatives.

In this category, for a given pair (a, b) of non-negative integers, the H-type group having endomorphisms of the form $J_Z^{(a,b)}$ is denoted by $H_l^{(a,b)}$. Notice that, for a given l and $(a+b)$, the corresponding groups share the same Z- and X-spaces. It is also well known [Sz1,Sz2] that two groups, $H_l^{(a,b)}$ and $H_l^{(a',b')}$ satisfying $a+b = a'+b'$, are locally inequivalent (non-isometric) unless $(a,b) = (a',b')$ upto

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an order. The non-isometry can be proven, for instance, by pointing out that $H_l^{(a,b)}$ and $H_l^{(a',b')}$ have non-isomorphic groups of local isometries. Particularly important examples are the groups $H_3^{(a,b)}$ resp. $H_7^{(a,b)}$, having $\mathcal{Y}$ of dimension $n_3 = 4$ resp. $n_7 = 8$ and created by the quaternionic resp. Cayley numbers.

(**B**) If $l \neq 3 \mod 4$, then there exist (upto equivalence) exactly one irreducible Clifford endomorphism space acting on $\mathcal{Y} = \mathbb{R}^{n_l}$. Although endomorphism spaces $J_l^{(1,0)}$ and $J_l^{(0,1)} = -J_l^{(1,0)}$ define isomorphically isometric groups $H_l^{(a,b)} \cong H_l^{(b,a)}$ the same denotations are used as in (**A**).

With regard to both type of H-type groups, for a given $H_l^{(a,b)}$, the particles fall into two different classes: those living in $\mathcal{X}^{(a,0)} = \mathcal{Y}^a$, and those orbiting in $\mathcal{X}^{(0,b)} = \mathcal{Y}^b$. Both share the same Z-space, but on the X-space, their angular momentum is defined in terms of $J_l^{(1,0)}$ and $J_l^{(0,1)} = -J_l^{(1,0)}$, respectively.

In order to understand this separation, consider two H-type groups, $H_l^{(a+b,0)}$ and $H_l^{(a,b)}$ (generic examples $H_l^{(a,b)}$ and $H_l^{(a',b')}$ satisfying $(a+b) = (a'+b')$ will be described later in this section). They have the same Z-space but their X-space is decomposed differently, as indicated by $(a+b,0)$ and (a,b). Suppose that in both cases the Z-space is factorized with the same lattice $\{Z_\gamma\}$. For a fixed Z_γ and the complex structure $J_{Z_{\gamma u}} = J_{\gamma u}$, consider an orthonormal complex basis - with regard to $J_{\gamma u}$ - in $\mathcal{Y}$ and copy it into each component space in order to create an orthonormal complex basis for the complete X-space, both regarding $J = J_{Z_{\gamma u}}^{(a+b,0)}$ and $J' = J_{Z_{\gamma u}}^{(a,b)}$. The same basis copied into each component $\mathcal{Y}_i$ makes the waves in the components to be comparable. The comparison reveals that the holomorphic and antiholomorphic coordinates defined on $\mathcal{Y}^a$ appear in the opposite way when they are considered to be defined on $\mathcal{Y}^b$, where the holomorphic functions are defined by $-J_Z$ and the antiholomorphic ones by J_Z.

4.8.2. The χ-Transforms

A χ-transform maps a wave $\Psi_\gamma \in W_\gamma$ defined on $H_l^{(a+b,0)}$ to $\Psi'_\gamma \in W'_\gamma$ defined on $H_l^{(a,b)}$. It associates, to any Itô polynomial written up in terms of $J_{\gamma u}$ and a fixed complex orthonormal basis $\mathbf{Q}_\gamma = \{Q_{\gamma 1} \dots Q_{\gamma \kappa}\}$, the polynomial written up - in terms of $J'_{\gamma u}$ but the same Z-lattice $\{Z_\gamma\}$, Fourier function $e^{2\pi \mathbf{i}\langle Z, \frac{1}{c} Z_\gamma\rangle}$, and complex orthonormal basis $\mathbf{Q}_\gamma$ - in the very same form. The only difference is that the waves are formulated on $H_l^{(a+b,0)}$ in terms of the complex coordinate functions z_i and $\overline{z}_i$ to be defined by $J_{\gamma u} = J_{\gamma u}^{(a+b,0)}$, whereas on $H_l^{(a,b)}$, the coordinate functions z'_i and $\overline{z}'_i$ are defined by $J'_{\gamma u} = J_{\gamma u}^{(a,b)}$.

The χ-transform mapping W_γ to W'_γ is denoted by χ_γ, whereas the $\chi = \oplus_\gamma \chi_\gamma$, to be defined with regard to all Z_γ, maps the whole L^2 function space defined on the torus bundle $H_l^{(a+b,0)}/\Gamma_Z$ to that defined on $H_l^{(a,b)}/\Gamma_Z$. It heavily depends on the bases $\mathbf{Q}_\gamma$ chosen independently for the lattice points Z_γ. That is, there are infinitely many χ_γ-transforms defined between W_γ and W'_γ and there are increasingly more χ-transforms defined between the whole L^2 function spaces.

From the point of view of $H_l^{(a+b,0)}$, the map reverses the orbiting direction on $\mathcal{Y}^b$ while leaving it untouched on $\mathcal{Y}^a$. As a result, the holomorphic coordinates on $\mathcal{Y}^b$ are exchanged for antiholomorphic ones and vice versa. This may suggest that processes invariant with regard to χ-transforms should be considered as those obeying mirror symmetry, however, a very important feature having by the χ-transforms is that they act between L^2 function spaces and can not be reproduced as induced maps defined with regard to isometries acting on the H-type groups. Since the symmetries in physics are formulated in terms of transformations acting on space or spacetime, no exact matching exits between χ- and mirror-symmetries.

The association of holomorphic resp. antiholomorphic coordinates with positive resp. negative electric charges does not mean that the χ-transforms could be seen as actual charge exchange operators. Since they exchange only certain holomorphic and antiholomorphic functions with each other they can not be seen as those which actually exchange electric charges with each other. For understanding this argument remember that electric charges are defined by the actions of the Laplacian on the eigenwaves,

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which do not remain with them (with the waves) but are picked up by the probabilistic densities determined by the product of the normalized eigenwaves with their complex conjugates. Since in a probabilistic density, there appear the same number of holomorphic resp. antiholomorphic linear waves, thus there is no way to define charges carried by the densities, either, which (the densities) only probabilistically describe the position of the particles whose mass - determined by the spectral mass assignment procedure - becomes the carrier of electric charges. Also notice that the χ-transforms correspond functions to be defined on different manifolds to each other, which can only have some reminiscences of chirality conjugations that is performed in the space and not on the waves. All these distinguishing features are the basic factors for to call them χ-transforms, which name distinguishes them from C-transforms and other symmetries introduced in physics.

4.8.3. Isospectralties Established by χ-Transforms

If a χ-transform is performed on each invariant subspace W_γ, it defines a map on the whole L^2 function space that intertwines the complete Hamiltonian as well as the Euclidean Laplacians Δ_X and Δ_Z included into the Δ and Δ'. Since the X-radial functions are also invariant, the torus bundles restricted onto the balls B_R and spheres S_R - which have radius R and whose center is at the origin - of the X-space are isospectral, also. This is because the action of the Laplacians Δ resp. Δ' on waves Ψ resp. $\Psi' = \chi(\Psi)$ are described by exactly the same formulae, meaning that a χ is an intertwining operator for Δ and Δ', indeed. These statements - that can also be established by the above explicit spectral computations - are particularly interesting in the case $l = 3 \mod 4$ when the compared metrics have different local geometries, exhibited so that the corresponded metrics have non-equivalent curvatures and non-isomorphic groups of local isometries [Sz1,Sz2]. That is, none of these geometric objects are determined, generically, by the spectra of Riemannian manifolds.

4.8.4. χ-Transforms Defined for Particle Systems

The above χ-transforms depend on the orthonormal complex bases $\mathbf{Q}_\gamma$ considered for the lattice points Z_γ, independently. On the Zeeman manifold prototypes, by contrast, a fixed particle system is represented by a fixed orthonormal system $\mathbf{Q} = \{Q_1, \dots, Q_{k/2}\}$ of X-vectors that does not depend on Z_γ. It represents $k/2$ whole particles orbiting in the complex planes spanned by Q_i and $J_{Z_{\gamma u}}(Q_i)$ in constant magnetic field - defined in terms of Z_γ - that is perpendicular to the planes where the particles are orbiting. In case of $\dim(\mathcal{Z}) > 1$, the system $\mathbf{Q}$ can not form a complex orthonormal basis with respect to all complex structures $J_{\gamma u}$ and it may happen, also, that the vectors in $\mathbf{Q} = \{Q_1, \dots, Q_{k/2}\}$ are not complex linearly independent with regard to some complex structures $J_{\gamma u}$. In both cases, the transform $\chi = \oplus_\gamma \chi_\gamma$ - defined as direct sum of the transforms χ_γ that depend on γ and map W_γ to W'_γ - must be seriously modified so as to create a transform $\chi_{\mathbf{Q}}$ depending only on $\mathbf{Q} = \{Q_1, \dots, Q_{k/2}\}$ but still mapping W_γ to W'_γ and intertwining Δ with Δ' and the Dirac Hamiltonian $\mathcal{DH}$ with $\mathcal{DH}'$.

If the system $\mathbf{Q} = \{Q_1, \dots, Q_{k/2}\}$ is linearly independent, in the complex sense, with regard to $J_{\gamma u}$, then consider the orthonormal complex basis $\mathbf{Q}_\gamma = \{Q_{\gamma 1}, \dots, Q_{\gamma \kappa}\}$ obtained from $\mathbf{Q}$ by the complex Gram-Schmidt orthogonalization process. Then, there is defined, by the above procedure, a uniquely determined transform χ_γ that maps W_γ to W'_γ and intertwines Δ with Δ', and $\mathcal{DH}$ with $\mathcal{DH}'$. As pointed out below, this transform is the same as that directly definable by $\mathbf{Q}$ and $J_{\gamma u}$ without turning the $\mathbf{Q}$ to an orthonormal system. The advantage of describing the same transform in terms of a complex orthonormal basis $\mathbf{Q}_\gamma$ is that one can use it to introduce the Dirac Hamiltonian, by the very same definitions applied earlier, and to prove the indicated intertwining properties.

Let it be mentioned that the complex independence of $\mathbf{Q} = \{Q_1, \dots, Q_\kappa\}$ regarding $J_{\gamma u}$ can be formulated in several different ways. It takes place if and only if the vector system $\mathbf{Q}_{\gamma\mathbb{R}} = \{Q_1, J_{\gamma u}(Q_1) \dots, Q_\kappa, J_{\gamma u}(Q_\kappa)\}$, or the holomorphic and antiholomorphic coordinate functions $z_i(X) = \langle Q_i + \mathbf{i}J_{\gamma u}(Q_i), X\rangle$ and $\bar{z}_i(X) = \langle Q_i - \mathbf{i}J_{\gamma u}(Q_i), X\rangle$, where $i = 1, \dots, \kappa$, form linearly independent systems over $\mathbb{R}$.

If the system $\mathbf{Q} = \{Q_1, \dots, Q_{k/2}\}$ is linearly independent just in the real sense but not in the complex sense - regarding a complex structure $J_{\gamma u}$ - then apply complex Gram-Schmidt orthogonalization

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to $\mathbf{Q}$, in order to figure it out as to which vectors Q_i should be thrown out (omitted) so as the rest forms a complex independent system.

First consider the first two vectors, Q_1 and Q_2, and the real planes P_1 and P_2 spanned by $\{Q_1, J_{\gamma u}(Q_1)\}$ and $\{Q_2, J_{\gamma u}(Q_2)\}$, respectively. The vectors Q_1 and Q_2 determine a complex linearly independent system if and only if $P_1 \cap P_2 = \{0\}$.

Indeed, if this relation is true, then $\{Q_1, J_{\gamma u}(Q_1), Q_2, J_{\gamma u}(Q_2)\}$ is, over $\mathbb{R}$, a linearly independent vector system and $\{Q_1, Q_2\}$ determine complex linearly independent vectors. If, on the other hand, $P_1 \cap P_2 \neq \{0\}$, then the intersection either is of 1D, or of 2D. The first case can not take place, since, if P_1 and P_2 intersect each other in a 1D linear and real vector space spanned by the vector V, then $J_{\gamma u}(V)$ must be both in P_1 and P_2, thus be parallel to V. But this is an assumption contradicting $J_{\gamma u}(V) \perp V$. As a result, $P_1 = P_2$ holds and Q_1, Q_2 determine a complex linearly depending system. To be more precise, the relation $Q_1 \perp Q_2$ implies either $Q_2 = J_{\gamma u}(Q_1)$ and $Q_1 = -J_{\gamma u}(Q_2)$, or $Q_2 = -J_{\gamma u}(Q_1)$ and $Q_1 = J_{\gamma u}(Q_2)$. Either way, the Q_2 complex linearly depends on Q_1 and should be thrown out from the system.

If Q_1 and Q_2 determine a complex linearly independent system, then consider the complex orthonormal system $\{Q_1, \tilde{Q}_2\}$ obtained from it by the complex Gram-Schmidt orthogonalization. To this end, first consider $Q_2^* = Q_2 - \langle Q_2, J_{\gamma u}(Q_1)\rangle J_{\gamma u}(Q_1)$, which trivially yields $Q_2^* \perp J_{\gamma u}(Q_1)$ and provides $\tilde{Q}_2$ by normalization. Then, $\{Q_1, J_{\gamma u}(Q_1), \tilde{Q}_2, J_{\gamma u}(\tilde{Q}_2)\}$ forms a real orthonormal vector system, due to the relations $Q_1 \perp J_{\gamma u}(Q_1)$, $\tilde{Q}_2 \perp J_{\gamma u}(\tilde{Q}_2)$, $\tilde{Q}_2 \perp J_{\gamma u}(Q_1)$, $Q_1 \perp J_{\gamma u}(\tilde{Q}_2)$, $Q_1 \perp \tilde{Q}_2$, $J_{\gamma u}(Q_1) \perp J_{\gamma u}(\tilde{Q}_2)$.

The χ_γ-transform defined for the waves formulated in terms of Q_1 and Q_2 is the same as that defined for the waves formulated in terms of Q_1 and $\tilde{Q}_2$. One should check out this statement only for Q_1 and Q_2 that belong to $\mathcal{Y}^b$, and even there, only for Q_2, because it is the only vector which is exchanged with a different $\tilde{Q}_2$. Regarding the vector Q_2, the transform is defined by mapping $z_2(X) = \langle Q_2 + \mathbf{i}J_{\gamma u}(Q_2), X\rangle$ to $z_2'(X) = \langle Q_2 + \mathbf{i}J'_{\gamma u}(Q_2), X\rangle$ and $\bar{z}_2(X) = \langle Q_2 - \mathbf{i}J_{\gamma u}(Q_2), X\rangle$ to $\bar{z}_2'(X) = \langle Q_2 - \mathbf{i}J'_{\gamma u}(Q_2), X\rangle$. Since on $\mathcal{Y}^b$, the action of $J'_{\gamma u}$ is the same as that of $-J_{\gamma u}$, the χ_γ-transform can be identified there with complex conjugation acting on $z_2(X)$, which statement is also true for $z_1(X)$, $z_1'(X)$ and $\bar{z}_1(X)$, $\bar{z}_1'(X)$ which are defined in terms of Q_1 but they are not exchanged for new ones by the complex Gram-Schmidt orthogonalization.

Next compute as to where are the functions $z_2^*(X)$ and $\bar{z}_2^*(X)$, to be defined in terms of $Q_2^* = Q_2 - \langle Q_2, J_{\gamma u}(Q_1)\rangle J_{\gamma u}(Q_1)$, are mapped by the χ_γ-transform defined for the waves formulated in terms of Q_1 and Q_2. Since the Q_2^* differs from Q_2 by the term $-\langle Q_2, J_{\gamma u}(Q_1)\rangle J_{\gamma u}(Q_1)$, one should only understand as to where is $-\langle Q_2, J_{\gamma u}(Q_1)\rangle\langle J_{\gamma u}(Q_1), X\rangle$ mapped by the transform. Since it is linear over $\mathbb{R}$ and $-\langle Q_2, J_{\gamma u}(Q_1)\rangle \in \mathbb{R}$, this coefficient should not be changed by the transform. The change of $\langle J_{\gamma u}(Q_1), X\rangle$ is to be determined from knowing that $\mathbf{i}\langle J_{\gamma u}(Q_1), X\rangle = (1/2)(z_1(X) - \bar{z}_1(X)$ which transforms to

$$\overline{\mathbf{i}\langle J_{\gamma u}(Q_1), X\rangle} = (1/2)\overline{(z_1(X) - \bar{z}_1(X)} = (1/2)(\bar{z}_1(X) - z_1(X)) = \tag{71}$$

$$-(1/2)(z_1(X) - \bar{z}_1(X) = -\mathbf{i}\langle J_{\gamma u}(Q_1), X\rangle = \bar{\mathbf{i}}\langle J_{\gamma u}(Q_1), X\rangle. \tag{72}$$

Since the $\chi_{\mathbf{Q}}$ is only real linear but not complex linear, the change caused by it is accountable only for the complex conjugation of $\mathbf{i}$ but no effect is implemented on $\langle J_{\gamma u}(Q_1), X\rangle$. This is why $-\langle Q_2, J_{\gamma u}(Q_1)\rangle J_{\gamma u}(Q_1)$ is mapped to itself, but which is equal to $-\langle Q_2, J'_{\gamma u}(Q_1)\rangle J'_{\gamma u}(Q_1)$ - the same expression obtained by exchanging both $J_{\gamma u}$ there for $J'_{\gamma u}$. This proves that the χ_γ transform defined for Q_1 and Q_2 is the same as that defined for Q_1 and $\tilde{Q}_2$ obtained by the Gram Schmidt orthogonalization.

Involving Q_3 to the investigations, repeat the above process for Q_1 and Q_3, if Q_2 is thrown out from the system. Otherwise, investigate the relation of Q_3, $J_{\gamma u}(Q_3)$, and the real plane P_3 spanned by the vectors $\{Q_3, J_{\gamma u}(Q_3)\}$ to the real 4D space $P_1 \oplus P_2 = P_1 \oplus \tilde{P}_2$. By the arguments seen before, the intersection $(P_1 \oplus P_2) \cap P_3$ can not be of 1D, thus either $(P_1 \oplus P_2) \cap P_3 = \{0\}$ or $P_3 \subset P_1 \oplus P_2$. In the latter case, being dependent on $\{Q_1, J_{\gamma u}(Q_1), Q_2, J_{\gamma u}(Q_2)\}$, the Q_3 should be removed and the investigations must move on by considering the system $\{Q_1, Q_2, Q_4\}$. Otherwise, $\{Q_1, Q_2, Q_3\}$

determines a complex linearly independent system and a complex orthonormal system $\{Q_1, \tilde{Q}_2, \tilde{Q}_3\}$, where the creation of $\tilde{Q}_3$ is carried out as follows. First define Q_3^*, by normalizing $Q_3 - \langle Q_3, \tilde{Q}_2\rangle\tilde{Q}_2$, and then $\tilde{Q}_3$, by normalizing $Q_3^* - \langle Q_3^*, J_{\gamma u}(Q_1)\rangle J_{\gamma u}(Q_1) - \langle Q_3^*, J_{\gamma u}(\tilde{Q}_2)\rangle J_{\gamma u}(\tilde{Q}_2)$. Both systems determine the same complex vector space and in it the same χ-transform intertwining Δ with Δ' and $\mathcal{DH}$ with $\mathcal{DH}'$.

The process results a complex independent system $\{Q_1, Q_{\tilde{2}}, Q_{\tilde{3}}, \dots, Q_{\tilde{\kappa}}\}$ and a complex orthonormal system $\{Q_1, \tilde{Q}_{\tilde{2}}, \tilde{Q}_{\tilde{3}}, \dots, \tilde{Q}_{\tilde{\kappa}}\}$ that define the same $\chi_{\tilde{\gamma}}$-transform but which, generically speaking, only maps a subspace $\tilde{W}_\gamma$ of W_γ to a subspace $\tilde{W}'_\gamma$ of W'_γ. In other words, for a fixed particle system $\mathbf{Q}$, one is able to create a $\chi_{\mathbf{Q}}$-transform that is defined not on the whole L^2 space but only on a proper subspace of L^2 functions. To establish isospectrality of the scalar and the proper spin Hamiltonians with regard to the whole L^2 space, one should use multiple particle systems and the χ-transforms defined for them. Their domains - the subspace of waves where the χ-transforms are acting - overlap each other and different χ-transforms generically map the same wave to different waves. Nonetheless, eigenwaves are always mapped to eigenwaves with the same eigenvalue. The procedure - carried out above by considering complex orthonormal vector systems $\mathbf{Q}_\gamma$ for each Z_γ, separately - shows that, by considering distinct particle system $\mathbf{Q}$, one can really establish the isospectrality with regard to the whole L^2 space.

On Zeeman manifolds, the investigations are carried out by uniformly defined functions which appear in the same form in terms of the vectors $\mathbf{Q} = \{Q_1, Q_2, Q_3, \dots, Q_\kappa\}$ and complex structure $J_{\gamma u}$. This gives rise to the question as to how should they be considered for structures $J_{\gamma u}$ for which the system $\mathbf{Q}$ is complex linearly dependent and the independent system $\{Q_1, Q_{\tilde{2}}, Q_{\tilde{3}}, \dots, Q_{\tilde{\kappa}}\}$ is a proper subset of $\mathbf{Q}$. In those cases the functions $z_{\hat{2}}, \bar{z}_{\hat{2}}, \dots, z_{\hat{\kappa}}, \bar{z}_{\hat{\kappa}}$ defined in terms of the thrown out system $Q_{\hat{2}}, Q_{\hat{3}}, \dots, Q_{\hat{\kappa}}\}$ should be expressed as linear combinations of the independent coordinates $z_1, \bar{z}_1, z_{\tilde{2}}, \bar{z}_{\tilde{2}}, \dots, z_{\tilde{\kappa}}, \bar{z}_{\tilde{\kappa}}$ so as to obtain a function depending only on the independent system.

Chirality transforms (conjugations) χ can be defined also between $H_l^{(a,b)}$ and $H_l^{(a',b')}$, where $a + b = a' + b'$. To see, how to do this, suppose $a < a'$, thus $b > b'$ and $\mathcal{X} = \mathcal{Y}^a + \mathcal{Y}^{(b-b')} + \mathcal{Y}^{b'}$. Then a χ non-trivially acts only on those complex valued linear functions that are defined in terms of Q_i satisfying $Q_i \in \mathcal{Y}^{(b-b')}$.

A χ-symmetry does not requires that the waves of each particle is exchanged for that of the oppositely orbiting particle. Non-trivial action is implemented only for waves associated with vectors Q_i included to the subspace $\mathcal{Y}^{(b-b')}$. If all particles waves are exchanged for those of the oppositely orbiting particles, then the χ-transform maps functions defined on $H_l^{(a,b)}$ to those being in $H_l^{(b,a)}$. It intertwines then the Hamiltonians of two isometrically isomorphic H-type groups. In such cases, the isospectrality statement is trivial, still it deserves some credit for the χ being an intertwining operator that is not induced by an isometry acting on the group.

The χ-symmetries act between two locally non-isometric Riemannian metrics $H_l^{(a,b)}$ and $H_l^{(a',b')}$ if and only if $l = 3 \mod 4$, $(a+b) = (a'+b')$, but $(a,b) \neq (a',b')$ upto order. Then, a χ-transform flips - to oppositely orbiting particles - not all particles but only those living in the subspace $\mathcal{Y}^{(b-b')}$ of $\mathcal{Y}^a + \mathcal{Y}^b$. Whereas, those living in the complement subspace, are not effected by the transaction. All these arguments prove

Theorem 8. *The above described χ-transformations - defined between the L^2 function spaces of the factorized H-type groups $H_l^{(a,b)}/\Gamma_Z$ and $H_l^{(a',b')}/\Gamma_Z$, where $a+b = a'+b'$ and the full Z-lattice Γ_Z factorize the common Z-space - are new type of physical symmetries which bear similarities to chirality conjugation but are markedly different from C and P symmetries. Their most important distinguishing feature is that they act between the L^2 function spaces defined on two different manifolds which can not be established as maps induced by those defined between the base manifolds on which the functions are defined. By contrast, the C, P, and mirror symmetries are formulated in terms of transformations acting on the physical space or spacetime which can not be related to transformations acting on H-type groups.*

Each χ-transformation intertwines the corresponding Laplacian, angular momentum operator, and all proper spin operators, including the Dirac operator, to be defined on $H_l^{(a,b)}/\Gamma_Z$ and $H_l^{(a',b')}/\Gamma_Z$, respectively. As a result the two manifolds are isospectral regarding the spectrum of any of these operators.

The just described spectral coincidence is the manifestation of a physical law that can be explained only after establishing the spectral mass assignment procedure carried out in the next sections. As described there, this mechanism not just defines mass for the particles but also identifies the created mass with the elements of the spectrum of the Schrödinger Hamiltonian. In combination with this statement, the isospectrality means that a particle system and its partner system, assigned to it by the χ transform, must have the same mass, even though the exchange is performed not for the whole system but just partially, for some of the elements of the system.

The following closing remarks argue for the indecomposability of H-type groups, namely, they explain as to why doesn't decompose a system of $k/2$ particles represented by a $(k+1)$-dimensional Heisenberg group into a system of independent particles represented by a Cartesian product of $k/2$ number of 3D Heisenberg groups?

The key to understanding this is that, on a $(k+1)$-dimensional Heisenberg group, the complex structures J_i, defined on the complex coordinate planes associated with the coordinates z_i, are identified with the same Z-vector Z_u which contributes only $4\mu^2$ to the spectrum. Whereas, in the Cartesian product, this term appears as the $k/2$-times of $4\mu^2$. Since this is the only difference between the two spectra, a $(k+1)$-dimensional Heisenberg group is always on a lower energy level than that represented by the Cartesian product of $k/2$ number of 3-dimensional Heisenberg groups.

As a result, spontaneous decomposition of a $(k+1)$D Heisenberg group into a Cartesian product of 3D groups is physically impossible and it can be exerted only by force. The same argument clarifies as to why do not the particle systems represented by $H_l^{(a,b)}$ spontaneously decompose into Cartesian products of the smaller groups $H_l^{(1,0)}$ resp. $H_l^{(0,1)}$. The explanation is the same, the systems represented by the Cartesian products are on higher energy levels as those represented by $H_l^{(a,b)}$.

Part II

Zeeman Spacetimes

Zeeman spacetimes are relativistic extensions of Zeeman manifolds into the time direction. Both static and expanding (accelerating) extensions are investigated. In case of point particle systems, there is an option to start with extending the Z-torus bundle, but, in order to emphasize that the new framework derives specific wave operators from the same Monistic Operator, the non-factorized H-type group is going to be extended, and the Wave Laplacian for the point particle system will be obtained by restricting the Monistic Wave Laplacian Δ_{Ext} onto the space of Γ_Z-periodic functions.

The relativity is implemented by extending the Riemannian metric of Zeeman manifolds into Lorentzian pseudo Riemannian spacetime metrics. The Ordinary Matter Operators are established on the static extensions, in concert with the observations verifying that the OM is not expanding. This characteristic feature of OM remains true also in the accelerating models - mathematically defined by the solvable extensions of H-type groups - where the OM-operators show up together with those of the Dark Matter DM and Dark Energy DE. There is mathematically proven that the OM is represented in an accelerating environment as a substance which neither expands nor shrinks which features only refer to the space surrounding OM.

Alike to the Monistic Hamilton Operator, the static Monistic Wave Laplacian also has strong relations to classical quantum operators. It decomposes into three parts - (1) the non-relativistic Yukawa operator, (2) the actual Schrödinger operator, and (3) the gravitation operator - among which the first two are well known in quantum mechanics and the 3rd one will be brought in connection with the Newton's law of gravitation, by non-relativistic limiting, and also with general relativity in several different ways. Each of these operators is non-relativistic, but together they make up a relativistic operator - the relativistic Laplacian of Zeeman spacetimes. The decomposition is highly consequential in the theory. It gives a precise mathematical explanation as to why Schrödinger had to exchange the second order derivative $\frac{\partial^2}{\partial t^2}$ showing up in the Klein Gordon operator of electron for the first order one, $\frac{\partial}{\partial t}$, in order to have a physically realistic operator, moreover, the relation of the Yukawa operator to the de Broglie matter waves gives rise to a spectral mass-assignment procedure that corresponds mass to ordinary particles, by a new method avoiding the Higgs mechanism.

The three matter-energy formations are explored, together, on the accelerating extensions, where the Monistic Accelerating Wave Operator is corresponded to the stress energy tensor. Both the tensor and the corresponded operator naturally decompose into three parts exposing the characteristic features of Ordinary Matter, Dark Matter, and Dark Energy. The above described term $\frac{\partial}{\partial t}$ showing up in the operators allows to define moving directions. OM and DM move into the shrinking while DM into the expanding time direction. Most surprisingly, they accurately exhibit their expected basic features and show up according to the same $(5\%, 25\%, 70\%)$ participation ratios they were observed in Nature. The accelerating models also suggest a cosmological model which recognizes the Big Bang, but, as opposed to the original inflation theory, also describes physical events taking place before the Big Bang.

5. The Static Zeeman Spacetime

5.1. The Static Monistic Wave Laplacian

The static extension of a Zeeman manifold N with the time axis $\mathbb{R}$ is defined by the metric Cartesian product $N \times \mathbb{R}$ where the components are perpendicular and the indefinite metric of Lorentz signature is direct sum of the positive definite metric given on N and that given on $\mathbb{R}$ by $< \partial_t, \partial_t >= -\frac{1}{c^2}$. Because of the Cartesian product, the static Monistic Wave Laplacian Δ_{St} appears in the form

$$\{\Delta_Z + \frac{2m\mathbf{i}}{\hbar}\partial_t - \frac{1}{c^2}\partial^2_{tt}\} + \{\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum \partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_t\}. \tag{73}$$

It involves the trivial operator $\frac{2m\mathbf{i}}{\hbar}\partial_t - \frac{2m\mathbf{i}}{\hbar}\partial_t = 0$ that is there for decomposing the Monistic Wave Laplacian into the sum of two classical wave operators - the non-relativistic Yukawa and the Schrödinger operator. The undecomposed operator is defined by cancelling this trivial term out, after which the non-relativistic Yukawa appears as the original relativistic Yukawa operator. Both play important roles in establishing the spectral mass assignment procedure carried out as follows.

5.2. Mass-Assignment on the Static Zeeman Spacetime

The mass-assignment is completed through multiple steps. In the first one, the main role is played by Yukawa's strong force operator that is present, in original form, in the undecomposed Laplacian. It is well known that its eigenfunctions are de Broglie's classical matter waves that represent heavy matter particles whose masses appear as eigenvalues of the Yukawa operator. This link to de Broglie's mass assignment opens up a whole new way to carry out a spectral mass-assigning mechanism, by only using the Monistic Wave Operator, without referring to the Higgs mechanism. In order to explore their physical contents on Zeeman spacetimes, the exposition starts with reviewing the de Broglie waves and Yukawa's strong force operator so as to see how did they emerge in classical quantum theory.

De Broglie's classical matter waves are defined (cf. [P]. Vol. 5) by:

$$\Psi_{deB}(Z,t) = \int\int\int A(K_1,K_2,K_3)e^{\mathbf{i}(\frac{1}{c}\langle K,Z\rangle - \omega t)}dK_1dK_2dK_3, \tag{74}$$

$$\text{where} \quad \sqrt{|K|^2 + \frac{m^2c^4}{\hbar^2}} = \omega. \tag{75}$$

They had originally been introduced for describing the ordinary matter of non-zero resting mass as matter-waves and the associated matter-particles as wave packets.

When $m = 0$, in the ω, then $\Psi_{deB}(Z,t)$ describes light-waves whose wave packets are photons of zero resting mass that travel in vacuum with the lightspeed c. A single light-ray-wave, emanated in vacuum into the direction of a vector K, is $Ae^{\mathbf{i}(\frac{1}{c}\langle K,Z\rangle-\omega t)}$, where A is constant and $m = 0$. When A depends on K, but still $m = 0$, then the above formula represents superposition of light-ray-waves. De Broglie extended the wave interpretation of light to ordinary matter having non-zero resting mass by simply introducing a non-zero mass-term into the ω. The wave packets matching these waves may or may not have resting mass, still they can not have proper spin. Even angular momentum is non-existing for them. In other words, they are pure scalar bosons at this point.

Yukawa's strong force operator: Waves (74) satisfy the relativistic scalar wave equation:

$$(\Delta_Z - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\Psi_{deB}(Z,t) = \frac{m^2c^2}{\hbar^2}\Psi_{deB}(Z,t) \tag{76}$$

(cf. [2.2] in [P], Vol. 5, pages 3), which serves as wave equation for the strong force. This operator appears in the Laplacian (73) of the static Zeeman spacetime, hereby relating the theory both to the de Broglie waves and the scalar wave equation of the strong force.

Originally, Yukawa used it for exploring the pion - the agent of the strong force - and described it, in vacuum, by a scalar field U satisfying the wave equation:

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \kappa^2)U = 0. \tag{77}$$

According to his theory, the static potential between two nucleons at a distance r is proportional to $\exp(-\mathbf{x}/r)$, where $\mathbf{x}$ is a constant with the dimension of reciprocal length, and the range of forces being given by $1/\mathbf{x}$.

Wave $\tilde{\Psi}(Z,t) = \exp(\mathbf{i}mc^2t/\hbar)\Psi(Z,t)$ satisfies:

$$(\Delta_Z + \mathbf{i}\frac{2m}{\hbar}\frac{\partial}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\tilde{\Psi}(Z,t) = 0 \tag{78}$$

(cf. [2.11] in [P], Vol. 5, pages 4). Notice that, due to $\tilde{\Psi}\tilde{\Psi}^* = \Psi\Psi^*$, where $*$ denotes complex conjugation, the waves $\tilde{\Psi}$ and Ψ determine the same probabilistic density, thus they describe the same quantum state.

Equation (78) will be used to transform the static Monistic Wave Operator, which originally contains the second order term ∂_{tt}, to the parabolic Schrödinger operator containing only first order differentiations regarding the time variable t. Identity $\tilde{\Psi}\tilde{\Psi}^* = \Psi\Psi^*$ only means that the probabilistic density does not change after the mass is turned over to the Schrödinger operator. That is, after the transaction is completed, nothing is left behind for the possession of the Yukawa operator. The transform is technically realized by writing up the Monistic Wave Laplacian in the form (73), where it is decomposed into the sum of (78) and the Schrödinger operator, appearing as the second term in the sum.

Integral formula (74) is primarily used to establish the mass assignment procedure for extended particle systems. For pointparticles, the idea is carried out by a discrete version of the de Broglie waves introduced in the next section. But, in order to see both the similarities and the differences, let the exposition be continued with briefly describing the procedure for extended particle systems. The waves $\Psi(X,Z,t)$ are defined there by the so called twisted Z-Fourier transform:

$$\Psi_{st}(X,Z,t) = \int_{\mathbb{R}^l} e^{\mathbf{i}(\frac{1}{c}\langle Z,K\rangle-\omega t)}\phi_{\mathbf{s}}(|X|,|K|)\Pi^{\mathbf{s}}_{K_u}(\psi(X,K_u))dK. \tag{79}$$

which, over each X-vector, become eigenfunctions of the Yukawa operator that acts in the (Z,t)-space as part of the Monistic Wave Laplacian.

The mass appears, in the first place, as an eigenvalue of this operator. But, at this point, the so introduced mass does not have any relation to the spectrum of the Schrödinger operator that also shows up in the Monistic Hamilton Operator. This missing link is established by the decomposition applied in (73) in which the harmonic functions of the first operator (78) appear in the form $\tilde{\Psi}(X,Z,t) = \exp(\mathbf{i}mc^2t/\hbar)\Psi(Z,t)$. This step is interpreted such that the mass, which appears in the first place as an eigenvalue of the Yukawa operator, is exhausted from the eigen-function $\Psi(X,Z,t)$ by transforming it to the new form $\tilde{\Psi}(X,Z,t)$, on which only the Schrödinger operator acts non-trivially from the whole Monistic Wave Laplacian. The mass is, so to speak, handed over to the Schrödinger operator in order to establish the desired relations between the mass and its spectrum.

But this step is not immediately available. Namely, it turns out, that the Schrödinger operator also decomposes into two parts. One of them is the parabolic Schrödinger operator whose harmonic solutions really establish the sought spectral relations and the other is the gravitational operator mentioned earlier. Because of these features, the second in (73) is called “mammoth" Schrödinger, which, without the gravitation operator, forms the actual Schrödinger operator. The non-relativistic Yukawa operator (78) is also called exhausted Yukawa operator and $\tilde{\Psi}(X,Z,t)$ is the exhausting wave. In the very last step, to each eigenvalue λ of the Hamilton operator appearing in the actual Schrödinger operator, there is corresponded a uniquely determined mass m - which originally appears as an eigenvalue of the Yukawa operator - by an equation that establishes the identification of the eigenvalues of the Schrödinger Hamiltonians with masses and hereby concluding the spectral mass-assigning procedure. Because of the complications arising with regard to extended particles, the last step is described only for pointparticles.

Matter acquiring mass on the static Zeeman spacetime: For pointparticle systems, the mass-assignment is completed in three steps.

Step 1.) The de Broglie matter waves (74), and their adaptation (80) to the extended particle systems suggest that the static matter waves of point particle systems should be introduced by a discrete version of the Z-Fourier transforms. To be more precise, they must be separately formulated for the Fourier-Weierstrass subspaces W_γ, in terms of the complex structure $J_{\gamma 0} = J_\gamma/|Z_\gamma|$, a fixed complex orthonormal vector system $\mathbf{Q} = \{Q_1,\dots,Q_\kappa\}$, and the Itô polynomials $H^\gamma_{pq}(z_i,\overline{z}_i)$ established in terms of $J_{\gamma 0}$ and $\mathbf{Q}$. These conditions determine that, for a fixed (p,q), the correct formula for the desired waves should be:

$$\Psi^{pq}_{\text{st}}(X,Z_\gamma,t) = e^{\mathbf{i}(\frac{1}{c}\langle Z,Z_\gamma\rangle-\omega t)}H^\gamma_{pq}(z_i,\overline{z}_i), \tag{80}$$

$$\text{where} \qquad \omega = \sqrt{|Z_\gamma|^2 + \frac{m^2c^4}{\hbar^2}}. \tag{81}$$

Itô’s polynomials can be used only in the case when the X-space is the complete $\mathbb{C}^\kappa$. When the eigenfunctions of $\lhd_\mu$ are computed over balls of the X-space, then the corresponding wave is defined by exchanging the Itô polynomials for the eigenfunctions to be expressed in the form $f(|X|^2)G^{(n,m)}(X)$. Since there is no other difference between the two cases, in the below discussions, all formulae will be described in terms of Itô’s polynomials.

Lemma 1. *Waves (80) are eigenfunctions of the Yukawa operator*

$$\Delta_Z - \frac{1}{c^2}\partial^2_{tt}, \tag{82}$$

with the eigenvalue $m^2c^2/\hbar^2$. This operator is incorporated into the undecomposed Monistic Wave Laplacian (73) and the mass m - appearing in its eigenvalues in the first place - will be assigned, in the spectral mass

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assignment procedure, as mass for the particle systems being in eigenstates to be defined by the eigenfunctions of the Schrödinger Hamiltonian that is also included into (73).

Proof. Differentiations of Ψ^{pq}_{st} with $-\frac{1}{c^2}\partial^2_{tt}$ resp. Δ_Z result $(\omega^2/c^2)\Psi^{pq}_{st}$ resp. $-(1/c^2)|Z_\gamma|^2\Psi^{pq}_{st}$. Thus,

$$(\Delta_Z - \frac{1}{c^2}\partial^2_{tt})\Psi^{pq}_{st} = \frac{1}{c^2}(|Z_\gamma|^2 + \frac{m^2c^4}{\hbar^2} - |Z_\gamma|^2)\Psi^{pq}_{st} = \frac{m^2c^2}{\hbar^2}\Psi^{pq}_{st}. \tag{83}$$

□

Step 2.) exploits

Lemma 2. *The new wave:*

$$\tilde{\Psi}^{pq}_{st}(X, Z_\gamma, t) = \exp(\frac{\mathbf{i}mc^2t}{\hbar})\Psi^{pq}_{st}(X, Z_\gamma, t) = \tag{84}$$

$$= \exp(\frac{\mathbf{i}mc^2t}{\hbar})\exp(\mathbf{i}(\frac{1}{c}\langle Z, Z_\gamma\rangle - \omega t))H^\gamma_{pq}(z_i, \bar{z}_i) \tag{85}$$

is harmonic with regard to the non-relativistic Yukawa operator

$$\mathbf{G} = \Delta_Z + \frac{2m\mathbf{i}}{\hbar}\partial_t - \frac{1}{c^2}\partial^2_{tt}, \tag{86}$$

to be found as the first operator in the decomposed Laplacian (73).

Proof. Differentiations of $\tilde{\Psi}^{pq}_{st}$ with $\frac{2m\mathbf{i}}{\hbar}\partial_t$, $-\frac{1}{c^2}\partial^2_{tt}$, and Δ_Z result

$$(-2\frac{m^2c^2}{\hbar^2} + 2\frac{m\omega}{\hbar} + \frac{1}{c^2}\frac{m^2c^4}{\hbar^2} - 2\frac{1}{c^2}\frac{mc^2\omega}{\hbar} + \frac{m^2c^2}{\hbar^2})\tilde{\Psi}^{pq}_{st} = 0. \tag{87}$$

□

The mass assignment procedure is continued with this new wave function. Its harmonicity regarding (86) is interpreted so that the mass, generated as an eigenvalue in the first step, is completely exhausted (pulled out) from $\Psi^{pq}_{st}(X, Z_\gamma, t)$ and handed over to $\tilde{\Psi}^{pq}_{st}$ and the action of the “mammoth" Schrödinger operator in order to explore its relations to the eigenfunctions and eigenvalues of the Hamiltonian included to the Schrödinger operator. Hence the name for (86) - the Exhausted Yukawa Operator. The two exponential functions appearing in $\tilde{\Psi}^{pq}_{st}$ are called outer and inside exponential functions, respectively.

Step 3.) In the last step, the action of the mammoth Schrödinger operator, the second in (73), on a wave $\tilde{\Psi}^{pq}_{st}$ is investigated. The action of $-\frac{2m\mathbf{i}}{\hbar}\partial_t$ on $\exp(\mathbf{i}mc^2t/\hbar)$ results multiplication with $\frac{2m^2c^2}{\hbar^2}$. If its action is considered only on this outer exponential function $\exp(\mathbf{i}mc^2t/\hbar)$ but not on $e^{\mathbf{i}(\langle Z,K\rangle - \omega t)}$ appearing inside, then the $-\frac{2m\mathbf{i}}{\hbar}\partial_t$ together with the Monistic Operator:

$$\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum\partial_\alpha D_\alpha\bullet, \tag{88}$$

forms the actual Schrödinger operator. The action on the inside exponential function defines the gravitational operator. Their separation is mathematically described in the Summary Section 5.5

5.3. The Actual Schrödinger Operator

The harmonic functions defined regarding the actual Schrödinger operator can be determined in exactly the same way how Schrödinger found them with regard to his operator. According to those computations, one should find first the eigenfunctions of the Monistic Hamilton Operator (88) whose Hamiltonian eigenvalues λ are identified, then, with the corresponding mass by the equation $-\lambda = -\frac{2m^2c^2}{\hbar^2}$. This is the final step in the spectral mass-assigning procedure. The mass gets involved

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into the eigenvalue λ and only those mass-values m are realized as actual masses of particle systems which can be identified with the elements of the discrete spectrum of the Hamiltonian (88) by the above formula. As a result, the mass can take only discrete positive values for the particle systems satisfying $Z_\gamma \neq 0$.

Also remember that operator (88) is massless, referring only to charges and orbiting spin but not to mass, as explained below (9)-(10). It regains the mass-term only after multiplying it with $-\hbar^2/2m$, effecting that the eigenvalues for the full-fledged operator are identified with mc^2 - the energy defined for m according to the Einstein equation. In other words, the spectral mass-assignment procedure defines mass for particle systems in agreement with the Einstein equation $E = mc^2$.

Explicit computations of the spectrum of the Schrödinger Hamiltonian are established above. The Dirac and Bose operators are defined by exchanging the scalar Hamiltonians for proper spin operators. The time-direction for antiparticles is opposite to that of the corresponding particles. This is in consent with the view that antiparticles arrive to us from the future.

5.4. The Gravitational Operator

The action of $-\frac{2m\mathbf{i}}{\hbar}\partial_t$ on the inside exponential function $e^{\mathbf{i}(\frac{1}{c}\langle Z_\gamma, K\rangle - \omega t)}$, showing up in $\tilde{\Psi}^{pq}_{\mathrm{st}}$, is equivalent to multiplication with the constant $-\frac{2m}{\hbar}\omega$. It is considered to be the result of the action of the gravitation operator Ω on $\tilde{\Psi}^{pq}_{\mathrm{st}}$. Due to the trivial actions of the actual Schrödinger and the non-relativistic Yukawa operators, it is the same as the action of the whole Δ_{st} on $\tilde{\Psi}^{pq}_{\mathrm{st}}$. That is:

$$\Omega(\tilde{\Psi}^{pq}_{\mathrm{st}})(X, Z_\gamma, t) = \Delta_{st}(\tilde{\Psi}^{pq}_{\mathrm{st}})(X, Z_\gamma, t) = -\frac{2m}{\hbar}\omega\tilde{\Psi}^{pq}_{\mathrm{st}}(X, Z_\gamma, t), \tag{89}$$

where ω is defined by (90). This means that $\tilde{\Psi}^{pq}_{\mathrm{st}}$ is an eigenfunction of $\Omega = \Delta_{st}$ with eigenvalue $-\frac{2m}{\hbar}\omega$. For deeper understanding, consider the power series expansion of

$$\omega = \frac{mc^2}{\hbar}\sqrt{1 + \frac{\hbar^2|Z_\gamma|^2}{m^2c^4}}, \tag{90}$$

carried out as

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots, \tag{91}$$

$$\omega = \frac{mc^2}{\hbar}(1 + \frac{1}{2}\frac{\hbar^2}{m^2c^4}|Z_\gamma|^2 - \frac{1}{8}\frac{\hbar^4}{m^4c^4}|Z_\gamma|^8 + \frac{1}{16}\frac{\hbar^6}{m^6c^{12}}|Z_\gamma|^6 - \dots) = \tag{92}$$

$$= (\frac{mc^2}{\hbar} + \frac{\hbar}{2mc^2}|Z_\gamma|^2 - \frac{\hbar^3}{8m^3c^6}|Z_\gamma|^4 + \frac{\hbar^5}{16m^5c^{10}}|Z_\gamma|^6 - \dots) \tag{93}$$

that reveals that this operator describes the gravitational interaction among the heavy particles whose waves are harmonic solutions of the actual Schrödinger and the non-relativistic Yukawa operators. For large m, the term $\hbar^2/m^2c^4 = (\hbar/mc^2)^2$, and those following it are negligible, thus, their cancellation turns the action to multiplication with $-2m^2c^2/\hbar^2$. At this point the gravitation is considered with regard to the massless Hamilton operator. In order to have it regarding the Laplacian in which the mass is reinstalled, the whole gravitation operator is to be multiplied with $-\frac{\hbar^2}{2m}$. This operation changes the above calculation to multiplication with mc^2 - the energy determined for m by the Einstein equation. Also notice that each of the above formulae are expressed by the dimensionless Planck energy $mc^2/\hbar$ if the reinstallment of mass is carried out with multiplication with $-\frac{\hbar^2}{2m}$. The link to Newton's gravitation is established by the following arguments.

Newton's law of gravitation states that two point-like massive bodies attract each other through a force $\mathbf{F}$ whose norm is $||\mathbf{F}|| = Gm_1m_2/r^2$. It can be derived from a local potential, Φ, in terms of which the generic Newton's law appears according to the Poisson equation $\nabla^2\Phi = 4\pi G\rho$, where ρ is the mass density of matter, from which the gravitation is originated.

Einstein traced back his theory of general relativity to that of Newton by the modified field equation:

$$\Box\Phi = -\frac{4\pi G}{c^2}T, \quad \text{where} \quad \Box = -\frac{1}{c^2}\partial_t^2 + \nabla^2. \tag{94}$$

That is, $\Box$ is the wave operator of the flat spacetime, and T is the trace of the stress-energy tensor of matter. Such a scalar theory of gravity obeys the principle of special relativity, and it reproduces Poisson's equation in the non-relativistic limit $c^{-1} \to 0$.

Field equation (94) is well known to be very controversial, failing the probe of all experiments trying to attach to it realistic physical content meaningful in Einstein's general relativity. From the point of view of the new framework, this only means that the flat-spacetime operator $\Box$ can not define realistic matter that is inherently in-woven into the fabric of the Einstein spacetime.

But exactly this misuse suggests the right way for reformulating (94). To this end, exchange $\Box$ for the spacetime Laplacian Δ_{st} in which the mass is reinstalled in the Hamilton operator and consider its action on matter waves. The energy corresponding to the mass is defined only in the eigenstates where it appears to be equal to the eigenvalue of the Hamilton operator. By the decomposition applied in the spectral mass assignment procedure, the action of the wave operator reduces to the action of the gravitation operator. As seen in the above expansion, the action is nothing but multiplication with a real number whose leading term - according to the above expansion - is multiplication with mc^2 which gives a direct relation to the density ρ in the Poisson equation and also to the density T appearing in (94). The correct version also contains additional terms expressed in terms of the dimensionless Planck energy $mc^2/\hbar$. This is the proper quantum mechanical version of (94) which still clearly reveals how this interpretation can be related to Newton's gravitation by non-relativistic limiting.

The matter represented by the eigenfunctions of Δ_{st} and its several versions does not appear to be in-woven into the fabric of the static spacetime. Instead, the mass appears in the eigenvalues for the eigenwaves, showing that the wave operator and its eigenwaves only furnish external matter to the stage-spacetime without changing its inner geometric structure. This quantum physical and general relativistic representation of matter still has obvious relation to (94), and, through it, to the Poisson equation established in relation to Newton's gravitation.

Notice yet that the relation $\pi|Z_\gamma| = \mu = eB/2\hbar c$, established under (9)-(10), suggests relations to other forces, as well. Namely, it says that $|Z_\gamma|$ is proportional to the magnitude of the Lorentz force the charged particle feels when it is orbiting in constant magnetic field. Thus, $|Z_\gamma|^2$ is proportional to the magnitude of the Coulomb force with which the charged system acts on itself. The pure gravitation is defined by the first order approximation in the above expansion. The neglected rest also refers to forces of electromagnetic origin.

5.5. Summary for the Static Scalar Wave Operator Δ_{st}

The static Monistic Wave Laplacian Δ_{St}, to be described on a static prototypical spacetime by (73), decomposes into the sum of the non-relativistic Yukawa operator

$$\Delta_Z + \frac{2m\mathbf{i}}{\hbar}\partial_t - \frac{1}{c^2}\partial^2_{tt} \tag{95}$$

and the non-relativistic mammoth Schrödinger operator

$$\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum \partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_t. \tag{96}$$

The mammoth Schrödinger operator further decomposes into the sum of the gravitation operator, defined for a fixed Z_γ by multiplication with

$$-\frac{2m^2c^2}{\hbar^2}\sqrt{1 + \frac{\hbar^2}{m^2c^4}|Z_\gamma|^2}, \tag{97}$$

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and the actual Schrödinger operator

$$\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum \partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_t + \frac{2m^2c^2}{\hbar^2}\sqrt{1 + \frac{\hbar^2}{m^2c^4}|Z_\gamma|^2}. \tag{98}$$

In this form, both act on the same wave $\tilde{\Psi}^{pq}_{st}(X, Z_\gamma, t)$, but they become completely separated if they are considered to be acting on waves redefined from $\tilde{\Psi}^{pq}_{st}$ as follows. For the actual Schrödinger operator, keep the outer exponential function intact, and cancel ωt from the inside exponential function. For the gravitational operator, cancel the outer but keep the inside exponential function completely, that is, consider the wave $\Psi^{pq}_{st}(X, Z_\gamma, t)$. Then, the so obtained waves are harmonic with regard to the actual Schrödinger

$$\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum \partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_t = 0, \tag{99}$$

and the gravitational operator $\Omega + \frac{2m\mathbf{i}}{\hbar}\partial_t$, respectively. In either ways, the whole Wave Laplacian Δ_{st} is decomposed into a sum of the non-relativistic Yukawa, actual Schrödinger, and the gravitation operators. Each of them is non-relativistic but summing up to the relativistic operator Δ_{st}.

The waves $\tilde{\Psi}^{pq}_{st}(X, Z_\gamma, t)$, defined in (84)-(85), are harmonic with respect to the non-relativistic Yukawa and the actual Schrödinger operators and they are eigenfunctions of the gravitation operator with eigenvalues involving the mass established in the spectral mass-assignment procedure. The harmonicity with regard to the first two operators implies that $\tilde{\Psi}^{pq}_{st}(X, Z_\gamma, t)$ is an eigenfunction of the whole operator Δ_{st}, and (89) can be seen as the quantum mechanical version of the Poisson equation - the fundamental identity for Newton's gravitation. The latter relation should be established not with the massless operator Δ_{st} but with $(-\hbar^2/2m)\Delta_{st}$ which transforms the eigenvalue $-\frac{2m}{\hbar}\omega$ to $\hbar\omega$, making the leading term in its power series expansion to be equal to mc^2.

The world lines of particle systems free-falling in this gravitational field are parameterized with t, indicating that they run into the future time direction. Movement in the opposite direction is implied when the spectral mass assignment procedure is carried out with $-\Delta_{st}$, thus it starts out with the Yukawa operator $-\Delta_Z + \frac{1}{c^2}\partial^2_{tt}$ defining negative eigenvalues, $-m^2c^2/\hbar^2$, for the waves $\Psi^{pq}_{st}(X, Z_\gamma, t)$. Then, the non-relativistic Yukawa operator takes the form

$$-\Delta_Z - \frac{2m\mathbf{i}}{\hbar}\partial_t + \frac{1}{c^2}\partial^2_{tt}, \tag{100}$$

with respect to which the wave $\tilde{\Psi}^{pq}_{st}(X, Z_\gamma, t)$ appears to be harmonic, and $\frac{2m\mathbf{i}}{\hbar}\partial_t$ is passed over to the mammoth Schrödinger operator. The actual Schrödinger is established so that $-2m^2c^2/\hbar^2$ should be identified with an eigenvalue of the negative definite Schrödinger Hamiltonian Δ, which, after being done, also defines a gravitational operator that is negative of that established above when the process starts out from Δ_{st}. That is, the spectral mass assignment procedure assigns negative masses to antiparticle systems free falling into the past time direction $-t$ to be considered relative to the real time t of particle systems. In other words, they appear from the future of OM particle systems.

5.6. *Summary for the Static Dirac Wave Operator*

The Monistic Dirac Wave Operator and its decompositions are established on the static prototypical Zeeman spacetime. By the first version, the Dirac wave operator $\mathcal{DW}_{\mu\pm}$ is introduced only on the (X, t) spacetime by supplementing the first order differential operator $\pm(2\sqrt{m_j/\hbar})\mathbf{i}\partial_t$ to the Dirac Hamiltonian $\mathcal{DH}_\mu$ described in (49). It is not yet the finalized Monistic Dirac wave operator $\mathcal{MDW}_{\mu\pm}$ - which must be defined on the whole (X, Z, t) spacetime - but which helps to find out the finalized version. Signs $\pm$ refer to particles and antiparticles, respectively. It is written up in terms of a mass m_j which - after establishing $\mathcal{MDW}_{\mu\pm}$ and the corresponding spectral mass assignment procedure - will be identified with the eigenvalues $\sqrt{\beta_{j\pm}}$ of the Dirac Hamiltonian $\mathcal{DH}_{\mu\pm}$.

According to these arguments, $\mathcal{DW}_{\mu\pm}$ is introduced by:

$$\mathcal{DW}_{\mu\pm} = -\frac{\pm 2\sqrt{m_j}}{\sqrt{\hbar}}\mathbf{i}\partial_t + \sum_{i=1}^{k/2}\sum_{j=1}^{2}(\mathbf{i}\frac{\partial}{\partial x_i^j} - \mathbf{a}_i^j)\sigma_j + 2\mu\sigma_0 = \tag{101}$$

$$= \begin{pmatrix} -\frac{\pm 2\sqrt{m_j}}{\sqrt{\hbar}}\mathbf{i}\partial_t + 2\mu & \mathbf{i}\sum_i(2\partial_{\bar{z}_i} - \mu z_i) \\ \mathbf{i}\sum_i(2\partial_{z_i} + \mu\bar{z}_i) & -\frac{\pm 2\sqrt{m_j}}{\sqrt{\hbar}}\mathbf{i}\partial_t - 2\mu \end{pmatrix}. \tag{102}$$

It acts on doublets written up in the form

$$\tilde{\Psi}_{j\pm} = e^{\pm\mathbf{i}\sqrt{m_j}c^2 t/\sqrt{\hbar}} e^{\pm\mathbf{i}(\frac{1}{c}\langle Z, Z_\gamma\rangle - \omega_j t)}\,\psi_{j\pm} = e^{\pm\mathbf{i}\sqrt{m_j}c^2 t/\sqrt{\hbar}}\,\Psi_{j\pm}, \tag{103}$$

where the eigenspinors $\psi_{j\pm}$ of $\mathcal{DH}_{\mu\pm}$ are described in Theorem 7, and ω_j is defined by

$$\omega_j = \sqrt{|Z_\gamma|^2 + \frac{m_j c^4}{\hbar}}. \tag{104}$$

The $\mathcal{DW}_{\mu\pm}$ is considered to be the mammoth Dirac operator which decomposes into the actual Dirac operator (when ∂_t acts on the outer exponential function) and the gravitation operator (when ∂_t acts on the inner exponential function). Two-spinor (103) is harmonic with regard to the actual Dirac operator if and only if $\sqrt{\beta_j} = \sqrt{\lambda + (-1)^j 2\mu} = \frac{2m_j c^2}{\hbar}$, where $\sqrt{\beta_j}$ is the eigenvalue of the Dirac Hamiltonian for the eigenspinor $\psi_{j\pm}$. The gravitation operator acts on them in the same way and defines the same eigenvalue for both components of (103).

By adding

$$\mathbf{G}_{\mu\pm} = \Delta_Z + \frac{\pm 2\sqrt{m_j}\mathbf{i}}{\sqrt{\hbar}}\partial_t - \frac{1}{c^2}\partial_{tt}^2, \tag{105}$$

to $\mathcal{DW}_{\mu\pm}$, one extends the partial operator $\mathcal{DW}_{\mu\pm}$ to be defined also on the Z-space and creates the complete Monistic Dirac operator $\mathcal{MDW}_{\mu\pm}$ to be defined on the (X,Z,t)-space. The $\mathbf{G}_{\mu\pm}$ acts, on both components of the 2-spinors, in the same way and emerges as the non-relativistic Yukawa operator to be defined for the Monistic Dirac operator. The spinors (103) are harmonic regarding both $\mathbf{G}_{\mu\pm}$ and the actual Dirac operator, thus the action of the complete Monistic Dirac operator reduces to the gravitation operator. This statement also shows that the relativistic Monistic Dirac Operator decomposes into a sum of three non-relativistic operators.

If the trivial term $\frac{\pm 2\sqrt{m_j}\mathbf{i}}{\sqrt{\hbar}}\partial_t - \frac{\pm 2\sqrt{m_j}\mathbf{i}}{\sqrt{\hbar}}\partial_t = 0$ is cancelled out, there remains the Monistic Dirac operator to be decomposed into the sum of the Yukawa operator $\Delta_Z - \frac{1}{c^2}\partial_{tt}^2$ and the Dirac Hamiltonian $\mathcal{DH}_{\mu\pm}$. That is,

$$\mathcal{MDW}_{\mu\pm} = \Delta_Z - \frac{1}{c^2}\partial_{tt}^2 + \sum_{i=1}^{k/2}\sum_{j=1}^{2}(\mathbf{i}\frac{\partial}{\partial x_i^j} - \mathbf{a}_i^j)\sigma_j + 2\mu\sigma_0 = \tag{106}$$

$$= \begin{pmatrix} \Delta_Z - \frac{1}{c^2}\partial_{tt}^2 + 2\mu & \mathbf{i}\sum_i(2\partial_{\bar{z}_i} - \mu z_i) \\ \mathbf{i}\sum_i(2\partial_{z_i} + \mu\bar{z}_i) & \Delta_Z - \frac{1}{c^2}\partial_{tt}^2 - 2\mu \end{pmatrix}. \tag{107}$$

It does not involves the mass m_j but which has to be established by the spectral mass assignment procedure. The spinor

$$\Psi_{j\pm} = e^{\pm\mathbf{i}(\frac{1}{c}\langle Z, Z_\gamma\rangle - \omega_j t)}\,\psi_{j\pm} \tag{108}$$

is an eigenspinor of the Yukawa operator with eigenvalue $\frac{m_j c^2}{\hbar}$, thus the spectral mass assignment procedure can be established for Fermions in the same way as for scalar particles. The mass appears,

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in the first place, in this eigenvalue which is assigned, then, to an eigenspinor of the Dirac Hamiltonian with eigenvalue $\sqrt{\beta_j} = \sqrt{\lambda + (-1)^j 2\mu} = \frac{2m_jc^2}{\hbar}$.

The process continues with considering the new wave

$$\tilde{\Psi}_{j\pm} = e^{\pm \mathbf{i}\sqrt{m_j}c^2 t/\sqrt{\hbar}}\Psi_{j\pm} = e^{\pm \mathbf{i}\sqrt{m_j}c^2 t/\sqrt{\hbar}} e^{\pm \mathbf{i}(\frac{1}{c}\langle Z, Z_\gamma\rangle - \omega_j t)}\psi_{j\pm} \tag{109}$$

which is harmonic with respect to the non-relativistic Yukawa, and the actual Dirac operator, if m_j is determined by the eigenvalue $\sqrt{\beta_j}$ of $\mathcal{DH}_{\mu\pm}$ according to the equation $\sqrt{\beta_j} = \sqrt{\lambda + (-1)^j 2\mu} = \frac{2m_jc^2}{\hbar}$. This is the point when the mass is identified with an eigenvalue of the Dirac Hamiltonian, which involves - via μ - the electric charge e. This is how the charge will be carried by the mass represented by the eigenvalue.

The change of sign before ∂_t has the affect that antiparticles have positive masses only if they are considered to be moving into the past-time direction. This is interpreted so that antiparticles arrive to us from the future.

The above statements are settled by proving two Lemmas.

Lemma 3. *The spinors (108) are eigenfunctions of the Yukawa operator*

$$\Delta_Z - \frac{1}{c^2}\partial^2_{tt}, \tag{110}$$

with the eigenvalue $m_jc^2/\hbar$. The operator acting on 2-spinors is incorporated into the Monistic Dirac Wave Laplacian (106)-(107).

Proof. Differentiations of $\Psi_{\mu\pm}$ with $-\frac{1}{c^2}\partial^2_{tt}$ resp. Δ_Z result $(\omega_j^2/c^2)\Psi_{\mu\pm}$ resp. $-(1/c^2)|Z_\gamma|^2\Psi_{\mu\pm}$. Thus,

$$(\Delta_Z - \frac{1}{c^2}\partial^2_{tt})\Psi_{\mu\pm} = \frac{1}{c^2}(|Z_\gamma|^2 + \frac{m_jc^4}{\hbar} - |Z_\gamma|^2)\Psi^{pq}_{\mu\pm} = \frac{m_jc^2}{\hbar}\Psi_{\mu\pm}. \tag{111}$$

□

Lemma 4. *The spinor*

$$\tilde{\Psi}_{\mu\pm}(X, Z_\gamma, t) = \exp(\pm\frac{\mathbf{i}\sqrt{m_j}c^2 t}{\sqrt{\hbar}})\Psi_{\mu\pm}(X, Z_\gamma, t) = \tag{112}$$

$$= \exp(\frac{\pm\mathbf{i}\sqrt{m_j}c^2 t}{\sqrt{\hbar}})\exp(\pm\mathbf{i}(\frac{1}{c}\langle Z, Z_\gamma\rangle - \omega_j t))\psi_{j\pm} \tag{113}$$

is harmonic with regard to the non-relativistic Yukawa operator

$$\mathbf{G}_{j\pm} = \Delta_Z + \frac{\pm 2\sqrt{m_j}\mathbf{i}}{\sqrt{\hbar}}\partial_t - \frac{1}{c^2}\partial^2_{tt}. \tag{114}$$

Proof. Differentiations of $\tilde{\Psi}_{\mu\pm}$ with $\frac{\pm 2\sqrt{m_j}\mathbf{i}}{\sqrt{\hbar}}\partial_t$, $-\frac{1}{c^2}\partial^2_{tt}$, and Δ_Z result

$$(-2\frac{m_jc^2}{\hbar} + 2\frac{\sqrt{m_j}\omega_j}{\sqrt{\hbar}} + \frac{1}{c^2}\frac{m_jc^4}{\hbar} - 2\frac{1}{c^2}\frac{\sqrt{m_j}c^2\omega_j}{\sqrt{\hbar}} + \frac{m_jc^2}{\hbar})\tilde{\Psi}_{\mu\pm} = 0. \tag{115}$$

□

The waves $\Psi_{\mu+}$ and $\Psi_{\mu-}$ are not corresponding particle-antiparticle waves, because both define the very same electric charge. Switching the a charge to its opposite charge is performed by applying P_{XZ}-transform to the wave $\Psi_{\mu+}$. It does not change the eigenvalue of eigenwaves, thus, there is defined the same mass by the spectral mass-assignment procedure. As a result, the antiparticle of the particle

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represented by the wave $\Psi_{\mu+}$ is represented by the antiparticle wave $P_{XZ}(\Psi_{\mu-}) = P_{XZ}(\Psi_{\mu+})_{\mu-}$, which can be obtained either by performing P_{XZ}-transform on $\Psi_{\mu-}$, or, by the antiparticle wave $P_{XZ}(\Psi_{\mu+})_{\mu-}$ to be defined from the particle wave $P_{XZ}(\Psi_{\mu+})$.

By assuming $c = 1$, in all of the above formulae, the spectral mass assignment procedure is turned to be a method for computing $P = m/\hbar = (2\pi/c^2)(mc^2/h) = (2\pi/c^2)\nu$ and $P_j = m_j/\hbar = (2\pi/c^2)(m_jc^2/h) = (2\pi/c^2)\nu_j$, where $\nu = mc^2/h$ and $\nu_j = m_jc^2/h$ denote the Planck-Einstein frequencies telling that how many times can the Planck constant be found in the energies mc^2 and m_jc^2. In case of the scalar Laplacian Δ_{st}, one should perform the reinstallation of mass into the massless Hamiltonian by multiplying it with $-\hbar/2m$. The Yukawa and the non-relativistic Yukawa operators appear then in the form

$$\Delta_Z - \partial^2_{tt}, \quad \text{and} \quad \mathbf{G} = \Delta_Z + \mathbf{i}2P\partial_t - \partial^2_{tt}. \tag{116}$$

The mammoth Schrödinger operator is written up in terms of $-\mathbf{i}2P\partial_t$. They act on the waves Ψ and $\tilde{\Psi}$ to be defined by

$$\Psi^{pq}_{\text{st}}(X, Z_\gamma, t) = e^{\mathbf{i}(\langle Z, Z_\gamma\rangle - \omega t)} H^\gamma_{pq}(z_i, \bar{z}_i), \quad \omega = \sqrt{|Z_\gamma|^2 + P^2}, \tag{117}$$

$$\tilde{\Psi}^{pq}_{\text{st}}(X, Z_\gamma, t) = \exp(\mathbf{i}Pt)\Psi^{pq}_{\text{st}}(X, Z_\gamma, t) = \tag{118}$$

$$= \exp(\mathbf{i}Pt)\exp(\mathbf{i}(\langle Z, Z_\gamma\rangle - \omega t))H^\gamma_{pq}(z_i, \bar{z}_i). \tag{119}$$

The spectral mass assigning procedure defines a P that appears as an eigenvalue of the Hamilton operator if the mass is restored by multiplying the massless Hamiltonian with $\hbar/2m$. The action of the gravitation operator is equivalent to multiplying with $\omega = \sqrt{|Z_\gamma|^2 + P^2}$.

With regard to the Dirac operator, the Yukawa and the non-relativistic Yukawa operators - acting on 2-spinors - appear in the form

$$\Delta_Z - \partial^2_{tt}, \quad \text{and} \quad \mathbf{G}_{\mu\pm} = \Delta_Z \pm 2\mathbf{i}\sqrt{P_j}\partial_t - \partial^2_{tt}. \tag{120}$$

The mammoth Dirac operator is written up in terms of $2\mathbf{i}\sqrt{P}\partial_t$. They act on waves $\Psi_{j\pm}$ and $\tilde{\Psi}_{j\pm}$ to be defined by

$$\Psi_{j\pm} = e^{\pm\mathbf{i}(\langle Z, Z_\gamma\rangle - \omega_j t)}\psi_{j\pm}, \quad \omega_j = \sqrt{|Z_\gamma|^2 + P_j}, \tag{121}$$

$$\tilde{\Psi}_{j\pm}(X, Z_\gamma, t) = \exp(\pm\mathbf{i}\sqrt{P_j}t)\Psi_{j\pm}(X, Z_\gamma, t) = \tag{122}$$

$$= \exp(\pm\mathbf{i}\sqrt{P_j}t)\exp(\pm\mathbf{i}(\langle Z, Z_\gamma\rangle - \omega_j t))\psi_{j\pm}. \tag{123}$$

The spectral mass assigning procedure defines P_j such that $2P_j$ is identified with the eigenvalues $\sqrt{\beta_j}$ of the Dirac Hamiltonian. The action of the gravitation operator is equivalent to multiplying with $\pm 2\sqrt{P_j}\omega_j$.

It is noteworthy that the Monistic Dirac operator $\mathcal{MDW}_{\mu\pm}$ decomposes into the sum of the proper Dirac operator $\mathcal{DW}_{\mu\pm}$ and the pure scalar operator $\mathbf{G}_{\mu\pm}$ by which no angular momentum or orbiting spin can be established in the Z-space. It is even more remarkable that the latter operator is used for creating mass and then passing it over to the Fermionic particles. This inference brings the new framework close to the SM in which there is theorized the existence of a boson of 0 spin which is the excitation of the Higgs field and which plays basic role to assign mass to particles by the Higgs mechanism. The analogy clearly suggests that the spectral mass assignment procedure analogously establishes the Higgs mechanism with waves and wave operators, as opposed to the Lagrangian means applied in SM.

The following discussions, carried out on the accelerating models, take the advantage of the simplification offered by the assumption $c = 1$. It can be easily removed and c restored in consent with the above formulae.

6. Solvable Extension of 2-Step Nilpotent Lie Groups

The prototypes of the expanding models are evolved on the solvable extensions of H-type groups. Since the rate of expansion is exponential, they are also called exponentially accelerating or, simply, accelerating models. Any 2-step nilpotent Lie group N extends to a solvable group SN that is defined on the half-space $N \times \mathbb{R}_+$ by the group multiplication

$$(X, Z, t)(X', Z', t') = (X + t^{\frac{1}{2}}X', Z + tZ' + \frac{1}{2}t^{\frac{1}{2}}[X, X'], tt'). \tag{124}$$

The nilpotent subgroup appears to be defined on the level set corresponding to $t = 1$, where the formula describes the group multiplication on N.

The Lie algebra is defined on $\mathcal{S} = \mathcal{N} \oplus \mathcal{T}$ by the Lie brackets:

$$[\partial_t, X] = \frac{1}{2}X; \ [\partial_t, Z] = Z; \ [\mathcal{N}, \mathcal{N}]_{/SN} = [\mathcal{N}, \mathcal{N}]_{/N}, \tag{125}$$

where $X \in \mathcal{X}$ and $Z \in \mathcal{Z}$. Alike to the nilpotent case, this Lie algebra can be considered to be defined in the tangent space at the unity $\mathbf{e} = (0, 0, 1)$. Then, the extension of the tangent vectors $\mathbf{v} \in T_{\mathbf{e}}(SN)$ into left-invariant vector fields identifies this Lie algebra with that of the left-invariant vector fields.

The indefinite metric tensor g is introduced by the left-invariant extension of the indefinite inner product $g_{\mathbf{e}}(.,.) = \langle .,. \rangle$ defined for $T_{\mathbf{e}}(SN)$ by the formulae $\langle \partial_t, \partial_t \rangle = -1$; $\langle \partial_t, \mathcal{N} \rangle = 0$; and $\langle \mathcal{N}, \mathcal{N} \rangle = \langle \mathcal{N}, \mathcal{N} \rangle_{/N}$. The respective left-invariant extensions of the unit vectors $E_i = \partial_i$; $e_\alpha = \partial_\alpha$; $\epsilon = \partial_t$, picked up at $\mathbf{e} = (0, 0, 1)$, appear in the form:

$$\mathbf{Y}_i = t^{\frac{1}{2}}\mathbf{X}_i \quad ; \quad \mathbf{V}_\alpha = t\mathbf{Z}_\alpha \quad ; \quad \mathbf{T} = t\partial_t, \tag{126}$$

where $\mathbf{X}_i$ and $\mathbf{Z}_\alpha$ are left invariant vector fields on N.

According to these formulae, not t but T defined by $\partial_T = \mathbf{T}$ is the time-parameterization which turns the trajectories of t parameter lines into geodesics of SN. Relation: $\partial_T = (dt/dT)\partial_t$ yields: $(dt/dT) = t$; $\ln t = T$; $t = e^T$, thus a t-level set is the same as the $T = \ln t$-level set and subgroup N corresponds both to $t = 1$ and $T = 0$. The reversed time is $\tau = -T$.

Let $c_x(s)$ resp. $c_z(s)$ be integral curves of the finite lengths $||c_x||$ resp. $||c_z||$ of the invariant vector fields $\mathbf{X}$ resp. $\mathbf{Z}$ defined on N. The flow generated by ∂_τ moves them to $c_x^\tau(s)$ resp. $c_z^\tau(s)$ so that the length of $\dot{c}_x(s)$ resp. $\dot{c}_z(s)$, and the corresponding curves, change according to $||c_x^\tau|| = ||c_x||e^{\tau/2}$ resp. $||c_z^\tau|| = ||c_z||e^\tau$. That is, both lengths are increasing, as time τ is passing by, such that the rate of change (derivative with respect to τ) is proportional to the length of the curves. In other words, this space-time represents an expanding universe where the distance between the objects is growing according to a law similar to that of Hubble.

Nevertheless, the law manifested on the Zeeman spacetime is quite different from that of Hubble which had been mathematically established, after Hubble's observations, by the Friedmann cosmological model. Namely, according to the Hubble law, for a given distance d, the speed of expansion is constant, which assumption also implied that there was a time, called Big Bang, when the objects were all at exactly the same place, forming an infinitesimally small universe of infinite density.

On Zeeman spacetimes, by contrast, the expansion is accelerating by a rate proportional to d. Even more so, the acceleration is not constant but also accelerating by a rate which is again proportional to d. This pattern continues up to the infinity, that is, the expansion is accelerating, regarding each order, and each of the higher order accelerations is proportional to the distance between the observer and the watched galaxy. To be more precise, the acceleration is exponential whose higher order rates can be computed by the higher order derivatives of the above arc-length functions. Since the differentiations only act on the exponential term non-trivially, the higher order derivatives really appear to be non-zero and proportional to the arc-length. As a result, the original Hubble law is modified so as to refer to any of the higher order accelerations. That kind of acceleration also implies that the time for Big Bang pushed out to $\tau = -\infty$. A technical description of acceleration - which will also be helpful in understanding the following computations - is as follows.

First notice that, on the half-space $N \times \mathbb{R}_+$, the accelerating spacetime metric g can be considered together with the static spacetime metric g_{st}. Both models have the same time-lines but they are parameterized differently, with T and t, respectively. They are complete lines, in the sense $-\infty < T < \infty$, with regard to g and only half lines, in the sense $0 < t < \infty$, with regard to g_{st}. The two parameterizations are connected with each other by $t = e^T = e^{-\tau}$. They also have common horospheres H - defined as $T = \ln t = -\tau$ level sets - which perpendicularly intersect the time parameter lines regarding both metrics. The horospheres appear to be decomposed into X- and Z-horospheres endowed with Euclidean metrics $g^X; g^Z$ and $g^X_{st}; g^Z_{st}$, respectively. The metrics g^H and g^H_{st} on a horosphere H are not direct sums of those given on the X- and Z-horospheres, but are curved Yang-Mills metrics defined by curved Yang-Mills connections. The expanding metrics on the horospheres conformally relate to the static metrics according to the relations $g^X = e^\tau g^X_{st}$ and $g^Z = e^{2\tau} g^Z_{st}$, which display the accelerating expansion in consent with the earlier description when the expansion was made apparent by comparing the lengths of curves $c_x(s)$ and $c_z(s)$ measured regarding both metrics.

Notice yet that, with respect to the expanding metric, the time lines run into an ideal point, laying at $-\infty$, as $\tau \to -\infty$. This is the idiomorphic point from which the expanding model starts out. Due to $t = e^T = e^{-\tau}$, $\lim_{t\to\infty} d_{st}(c^t_x(s_1), c^t_x(s_2)) = \infty$, and $\lim_{t\to\infty} d_{st}(c^t_z(s_1), c^t_z(s_2)) = \infty$ - where $s_1 \neq s_2$ and d_{st} is the distance measured with regard to g_{st} - this ideal point is not well defined regarding the static metric.

All considerations in the following sections are carried out by assuming $c = 1$. This is only a technical assumption, simplifying the computations, and the investigations in the previous section show how the formulae can be turned into those expressed in terms of the realistic speed c of light.

7. The Accelerating Monistic Wave Laplacian

The Monistic Wave Laplacian Δ_E can explicitly be established on the expanding (accelerating) Zeeman spacetime by the well known formulae

$$\Delta_E = \sum_i (\mathbf{Y}_i^2 - \nabla_{\mathbf{Y}_i}\mathbf{Y}_i) + \sum_\alpha (\mathbf{V}_\alpha^2 - \nabla_{\mathbf{V}_\alpha}\mathbf{V}_\alpha) - \partial^2_{\tau\tau} = \tag{127}$$

$$= (\sum_i \mathbf{Y}_i^2 + \sum_\alpha \mathbf{V}_\alpha^2 - \partial^2_{\tau\tau}) - (\sum_i \nabla_{\mathbf{Y}_i}\mathbf{Y}_i + \sum_\alpha \nabla_{\mathbf{V}_\alpha}\mathbf{V}_\alpha), \tag{128}$$

$$\nabla_{X+Z}(X^* + Z^*) = \nabla^N_{X+Z}(X^* + Z^*) - (\frac{1}{2}\langle X, X^*\rangle + \langle Z, Z^*\rangle)\partial_t, \tag{129}$$

$$\nabla_X \partial_t = -\frac{1}{2}X, \quad \nabla_Z \partial_t = -Z, \quad \nabla_{\partial_t} X = \nabla_{\partial_t} X = \nabla_{\partial_t}\partial_t = 0, \tag{130}$$

in which the covariant derivatives in (129)-(130) are written up in terms of the Lie algebra - to be defined in the tangent space at the identity of the solvable group - and ∇^N denotes the covariant derivative on the H-type group. They equivalently describe the covariant derivative in terms of the invariant vector fields showing up in (127) and (128).

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These identities combined with (5), (4), (3), (126), $T = \ln t$, and $\partial_T = t\partial_t$ reveal Δ_E in the forms as follows.

$$e^T(\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum\partial_\alpha D_\alpha\bullet) + (e^{2T}\Delta_Z - \partial^2_{TT}) - (\frac{k}{2} + l)\partial_T = \tag{131}$$

$$e^T(\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum\partial_\alpha D_\alpha\bullet) + e^{2T}(\Delta_Z - \partial^2_{tt}) - (\frac{k}{2} + l + 1)\partial_T = \tag{132}$$

$$= e^T(\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum\partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_t) + \tag{133}$$

$$+(e^{2T}\Delta_Z - e^{2T}\partial^2_{tt}) + (\frac{2m\mathbf{i}}{\hbar} - (\frac{k}{2} + l + 1))\partial_T = \tag{134}$$

$$e^T(\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum\partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_t) + e^{2T}(\Delta_Z + \frac{2m\mathbf{i}}{\hbar}\partial_t - \partial^2_{tt}) \tag{135}$$

$$+(\frac{2m\mathbf{i}}{\hbar} - (\frac{k}{2} + l + 1) - \frac{2m\mathbf{i}}{\hbar}e^T)\partial_T = \tag{136}$$

$$t(\Delta_X + \frac{1}{4}|X|^2\Delta_Z + \sum\partial_\alpha D_\alpha \bullet - \frac{2m\mathbf{i}}{\hbar}\partial_t) + t^2(\Delta_Z + \frac{2m\mathbf{i}}{\hbar}\partial_t - \partial^2_{tt}) + \tag{137}$$

$$+(\frac{2m\mathbf{i}}{\hbar} - (\frac{k}{2} + l + 1) - \frac{2m\mathbf{i}}{\hbar}t)t\partial_t \tag{138}$$

In (131), the Δ_E is still expressed in terms of T. Then, it is gradually altered so as to see how does it depend on the static time t? In (133), the mammoth Schrödinger is established by implementing $-\frac{2m\mathbf{i}}{\hbar}\partial_t + \frac{2m\mathbf{i}}{\hbar}\partial_t = 0$ so that $-e^T\frac{2m\mathbf{i}}{\hbar}\partial_t = -\frac{2m\mathbf{i}}{\hbar}\partial_T$ is incorporated as the last term in (133) and $\frac{2m\mathbf{i}}{\hbar}e^T\partial_t = \frac{2m\mathbf{i}}{\hbar}\partial_T$ is used to establish, together with $-(\frac{k}{2} + l + 1))\partial_T$, the second operator standing at the end of (134). In (135), the non-relativistic Yukawa is established by implementing $\frac{2m\mathbf{i}}{\hbar}\partial_t - \frac{2m\mathbf{i}}{\hbar}\partial_t = 0$ and putting $-e^{2T}\frac{2m\mathbf{i}}{\hbar}\partial_t = -e^T\frac{2m\mathbf{i}}{\hbar}\partial_T$ into (136).

In (137), there appear the static mammoth Schrödinger and the non-relativistic Yukawa operators multiplied, on the left side, with $t = e^T$ and $t^2 = e^{2T}$, respectively. Due to the trivial action of the actual Schrödinger operator, the action of the whole mammoth operator on static matter waves reduces to the action of the operator that is t-times of the static gravitational operator. To have the total gravitation operator, one should add to it the excess gravitational operator EG appearing in (138).

This appearance clearly indicates that the model represents the Ordinary Matter in an accelerating environment so that the expansion - or shrinking - only refers to the surrounding space but having no effect on the OM itself. This feature is expressed in several different ways. The accelerating actual Schrödinger operator is the t-times of the static actual Schrödinger operator, thus the harmonic solutions are the same with regard to both of them. The same is true with regard to both the relativistic and non-relativistic Yukawa operators, implying that the quantum laws of OM are the same in the static as well as in the accelerating environment and the expansion or shrinking only take place in the ambient space.

To support these statements mathematically, closer look should be taken at the action of the total gravitation operator on the static waves

$$\tilde{\Psi}^{pq}_{\mathrm{st}} = \exp(\mathbf{i}mt/\hbar)\Psi^{pq}_{\mathrm{st}}(X, Z_\gamma, t) = \exp(\mathbf{i}mt/\hbar)e^{\mathbf{i}(\langle Z, Z_\gamma\rangle - \omega t)}H^\gamma_{pq}(z_i, \bar{z}_i). \tag{139}$$

Notice that this formula is written up, alike those in computations (116)-(123), by assuming $c = 1$.

When Δ_E acts on $\tilde{\Psi}^{pq}_{\mathrm{st}}$, then (139) is nullified by the actual Schrödinger and also by the non-relativistic Yukawa operator showing up in (137). The action of the static gravitational operator - computed by the action of $-t\frac{2m\mathbf{i}}{\hbar}\partial_t$ on the inside exponential function that produces the multiplicative factor $-t\frac{2m\omega}{\hbar}$ - is part of the total gravitational action but kept separately, to be attached to the static operator, so as to show that the wave appears, after the action of the static gravitational operator, in an accelerating environment without showing any sign of disturbance. The acceleration - represented by the factor t - only refers to the surrounding space but not to the matter effected by the static gravitation.

Turning to the excess gravitational operator (138), it provides the factor

$$Q(t) = 2(\frac{m^2}{\hbar^2} - \frac{m}{\hbar}\omega)t(t-1) + \mathbf{i}(\frac{k}{2} + l + 1)(\omega - \frac{m}{\hbar})t \tag{140}$$

with which the wave is multiplied in order to express the action with regard to this operator. For any fixed t, the eigenwave of the static operator is multiplied with a complex number that is seen as being the eigenvalue for EG at the time t. The so obtained wave still defines, for any t, the very same probabilistic density as that defined by (139) for the static operator which shows again that the accelerating expansion or shrinking of the ambient space has no effect on the quantum state of OM.

The simplest proof for the OM is not expanding is that the hydrogen atom must had had the same spectrum billions of years ago as of today. To secure this conclusion, it is enough to measure the spectrum of the light emanated by far away galaxies - from where it took billions of years till the light arrived to the earth - and to compare it with the spectrum of the mundane hydrogen atoms.

The model suggest a scenario for the evolution of the Universe. This part of the proposal is only a preparation which is going to be completed at the very end of the last section. The spectral mass assignment procedure carried out on accelerating Zeeman spacetimes by Δ_E makes use only the actual Schrödinger and the gravitational operator of OM without involving the excess gravitational operator. By the arguments to be found in Section 5.5, it exhibits OM particle systems moving (free-falling) into the shrinking time direction toward the idiomorphic point pinned down by $\tau = -\infty$, or $t = infty$. This restricted operator alone, does not indicate that Big Bang had to have happening at some point but it only describes this everlasting accelerating free-fall. In order to see if the excess gravitational operator has something in it which would prove that the Big Bang was to be really happening, rewrite (140), by using

$$\omega = \frac{m}{\hbar}\sqrt{1 + \frac{\hbar^2|Z_\gamma|^2}{m^2}} \tag{141}$$

in the form

$$2\frac{m^2}{\hbar^2}(1 - \sqrt{1 + \frac{\hbar^2|Z_\gamma|^2}{m^2}})t(t-1) + \mathbf{i}(\frac{k}{2} + l + 1)\frac{m}{\hbar}(\sqrt{1 + \frac{\hbar^2|Z_\gamma|^2}{m^2}} - 1)t \tag{142}$$

$$= \frac{\hbar^2|Z_\gamma|^2/m^2}{1 + \sqrt{1 + \hbar^2|Z_\gamma|^2/m^2}}\Big(- 2\frac{m^2}{\hbar^2}t(t-1) + \mathbf{i}(\frac{k}{2} + l + 1)\frac{m}{\hbar}t\Big). \tag{143}$$

It shows that the real function involving $-2\frac{m^2}{\hbar^2}t(t-1)$ changes sign at $t = 1$ (or, $\tau = 0$) which seems to be strongly suggesting that it was the exact time when the Big Bang was happening.

To support this idea, describe the evolution by the spectral mass assignment procedure based on the operator $-\Delta_E$ which works with anti-OM particle systems entering the Universe at the primordial time-point $\tau = -\infty$. Then it is free-falling toward the expanding time direction controlled by an anti-gravitation operator, that is negative of the above OM-gravitation operator. The real function $+2\frac{m^2}{\hbar^2}t(t-1)$ involved into this version has limit $+\infty$ at $t = \infty$ and keeps being positive with regard to the static time t on the interval $(1, t = \infty)$. The positive sign indicates that during this period of time the total gravitation is pushing the complete antimatter together into the expanding time direction with a force that is gradually diminishing, until the excess gravitation becomes zero, at the time $t = 1$, and negative on the interval $(t = 0, t = 1)$. The change of sign means that, on this interval, it starts to push backwards, into the shrinking direction, resulting that some portion of the antimatter keeps going into the expanding direction while others into the shrinking time direction. The latter turnaround switches antiparticles to particles which behave as those described above. The former category - which keeps moving into the expanding direction - determines DE, while those turned around switched to being DM and OE. The pure imaginary number valued functions in $-Q(t)$ only change the amplitudes of

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the waves in which they give rise to phase shifts, but they do not play role in determining the moving direction.

This preparatory description is interrupted, but is to be returned at the very end of this paper, after clarifying some obscured details hindering the continuation. One needs to determine, for instance, the wave operators of OM; DM; and DE, which are rather unclear at this point, since in the decomposition (137) and (138) the first formula describes OM, but nothing is said about as to how are DE and DM represented in (138).

In what follows, this problem is solved by using a quite different approach. In the first place, the separation of OM, DM, and DE is carried out by distinguishing their densities in the time-time component of the accelerating spacetime's stress-energy tensor. There is clearly suggested that the individual gravitations should be explored in terms of the time-directions the individual matter-energy formations are moving into. Namely, geometric arguments will show that moving into the expanding direction gives rise to antigravitation, therefore, it must be the moving-direction for DE. Contrary to this, gravitational effect should arise with regard to DM that must move, therefore, into the shrinking time direction. For OM, the gravitation arises on the static spacetime as described earlier, but, if it is considered to be in the accelerating model, it too prefers the shrinking time direction.

The decomposition of the densities inscribed into the time-time component of stress-energy tensor is followed by decomposing the whole stress-energy tensor and determining the wave operators for OM, DM, and DE by corresponding them to the regarding individual stress-energy tensors. All these operators will be put to test, to see, if they stand the probes of experiments.

8. The Undecomposed Stress-Energy Tensor

For establishing the stress-energy tensor on the accelerating spacetime, one should start with computing the complete curvature. This rather lengthy calculation is manageable by (129) and (130) providing the curvature tensor in terms of the Lie algebra defined in the tangent space $T_{\mathbf{e}}(SN)$ at $\mathbf{e} = (0, 0, T = 0)$. Then, the tensor field can be defined all over the manifold by plugging in for the Lie algebra elements the corresponding left-invariant vector fields. From this lengthy but straightforward technical computation, the Ricci tensor Ri of type $(1, 1)$ emerges in the form:

$$Ri(X) = \rho(X) + (\frac{k}{4} + \frac{l}{2})X; \tag{144}$$

$$Ri(Z) = \rho(Z) + (\frac{k}{2} + l)Z \quad ; \quad Ri(T) = -(\frac{k}{4} + l)T, \tag{145}$$

where the covariant Ricci tensor ρ of type $(0, 2)$ is described on $\mathcal{N}$ in terms of $H(X, X^*, Z, Z^*) := \langle J_Z(X), J_{Z^*}(X^*)\rangle$ by $\rho(X, Z) = 0$ and

$$\rho(X, X^*) = -\frac{1}{2}\sum_{\alpha=1}^{l} H(X, X^*, e_\alpha, e_\alpha) = -\frac{1}{2}H_{\mathcal{X}}(X, X^*) = -\frac{l}{2}\langle X, X^*\rangle, \tag{146}$$

$$\rho(Z, Z^*) = \frac{1}{4}\sum_{i=1}^{k} H(E_i, E_i, Z, Z^*) = \frac{1}{4}H_{\mathcal{Z}}(Z, Z^*) = \frac{k}{4}\langle Z, Z^*\rangle. \tag{147}$$

These formulae imply:

$$Ri(X) = \frac{k}{4}X; \quad Ri(Z) = (\frac{3k}{4} + l)Z; \quad Ri(T) = -(\frac{k}{4} + l)T, \tag{148}$$

$$\mathcal{R} = \frac{k^2}{4} + l(\frac{3k}{4} + l) + (\frac{k}{4} + l) = \mathcal{R}_S + (\frac{k}{2} + l)^2 + (\frac{k}{4} + l), \tag{149}$$

where $\mathcal{R}_S = -kl/4$ is the scalar curvature on the static spacetime.

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As a consequence, the stress-energy tensor $E(A, A^*) = Ri(A, A^*) - \frac{1}{2}\mathcal{R}\langle A, A^*\rangle$, where $A = X + Z + T$ and $A^* = X^* + Z^* + T$, appears in terms of the static stress energy-tensor E_{st} in the form:

$$E(A, A^*) = E_{st}(A, A^*) + (\frac{k}{2} + l)(\frac{1}{2}\langle X, X^*\rangle + \langle Z, Z^*\rangle) \tag{150}$$
$$-\frac{1}{2}(\frac{k}{2} + 2l)\langle T, T\rangle - \frac{1}{2}((\frac{k}{2} + l)^2 + \frac{k}{4} + l)\langle A, A^*\rangle,$$

$$E_{st}(A, A^*) = \rho(A, A^*) - \frac{1}{2}\mathcal{R}_S\langle A, A^*\rangle = \frac{l}{2}(\frac{k}{4} - 1)\langle X, X^*\rangle + \tag{151}$$
$$+\frac{k}{4}(\frac{l}{2} + 1)\langle Z, Z^*\rangle + \frac{kl}{8}\langle T, T\rangle = \frac{kl}{8}\langle A, A^*\rangle - \frac{l}{2}\langle X, X^*\rangle + \frac{k}{4}\langle Z, Z^*\rangle =$$
$$= \frac{kl}{8}(\langle A, A^*\rangle - \frac{4}{k}\langle X, X^*\rangle + \frac{2}{l}\langle Z, Z^*\rangle).$$

In what follows, the main goal is to find the stress-energy tensors and wave operators of the matter-energy formations OM, DM, and DE. In the first few steps, the search is carried out on Zeeman spacetimes without talking about the waves since they can only be introduced after determining the wave operators. The stress-energy tensor of the whole spacetime, and those of the particular formations only describe matter that is inherently in-woven into the fabric of the spacetime. The actual matter will be introduced by waves, which takes part in interactions described by the actions of the wave operators determined for the formations. Nonetheless, since they are corresponded to their stress-energy tensors, the inherent tensor will have a most marked imprint in the actual stress-energy tensors and the matter described by waves and wave-operators.

9. Inherent Densities for OM, DM, and DE

The inherent densities of OM, DM, and DE will be determined by decomposing the total density:

$$-\frac{kl}{8} + \frac{1}{2}((\frac{k}{2} + l)^2 + \frac{3k}{4} + 3l) \tag{152}$$

appearing in the time-time component of the stress-energy tensor $E(A, A^*)$. Since the model is homogeneous, the densities are to be constant and the participation ratio of a particular formation can be computed by the ratio of its density and the total density. The correct ratio is expected to be closely relating to the experimentally known values $(5\%, 25\%, 70\%)$ for the participation ratios of OM, DM, and DE.

Since there is only one term, the $-kl/8$, referring to the non-expanding Ordinary Matter, the corresponding density must be $OM(k, l) = kl/8$. By cancelling $-\frac{kl}{8}$ out from (152), there remains back

$$\frac{1}{2}((\frac{k}{2} + l)^2 + \frac{3k}{4} + 3l). \tag{153}$$

The correct division should be determined by this and $+kl/8$ so that it must agree with the above described experimental data. But there are only a few choices for the decomposition and the right one could be pinned down by checking on the participation ratios. According to the below computations, the latter requirement is satisfied when the densities are defined in the form:

$$OM(k, l) = \frac{kl}{8}, \quad DM(k, l) = \frac{kl}{8} + \frac{1}{4}(\frac{k}{2} + l)^2, \tag{154}$$
$$DE(k, l) = \frac{kl}{8} + \frac{1}{2}(\frac{1}{2}(\frac{k}{2} + l)^2 + \frac{3k}{4} + 3l). \tag{155}$$

If $\frac{kl}{8}$ is not counted in them, the rest parts in $DM(k, l)$ and $DE(k, l)$ sum up to (153).

There are several reasons as to why does $\frac{kl}{8}$ associated with OM appear also in $DM(k, l)$ and $DE(k, l)$. According to later explained arguments, DM and DE can not be described by themselves but by their actions on OM. Also notice that they have opposite moving directions, thus their masses

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are computed to be corresponded to eigenvalues having opposite signs (+ for DM and − for DE), so, they seem to be cancelling out each other in the sum, anyway. But the latter argument is incorrect and should be revised so that the sum of such OM-objects represents meeting of OM-particles with their OM-antiparticles, thus, they disappear in γ-rays whose agents (the photons, dual particles to the rays) move with speed c, therefore, they have no resting mass or resting energy but just radiation energy manifesting in radiation energy. The density of such radiation origin is represented by $\frac{kl}{8}$ in $DM(k,l)$ and $DE(k,l)$.

The decomposition may seem to be rather ad hoc, but this is what independently follows from rigorous physical and mathematical arguments explained in the following sections. This section only focuses on the question as to whether the above densities signal, in any way, the participation ratios $(5\%, 25\%, 70\%)$ known from observations and theoretically corroborated by the Standard Model combined with Einstein's gravitational equation modified with the cosmological constant [Pe1,Pe2,RS]. On Zeeman spacetimes, however, these computations are carried out, rather differently, first on the one-particle models represented by the solvable extensions of 3D Heisenberg groups. In that case, $k = 2$, $l = 1$, and the corresponding densities are:

$$DE(2,1) = \frac{14}{4}, \quad DM(2,1) = \frac{5}{4}, \quad OM(2,1) = \frac{1}{4}. \tag{156}$$

Thus the total density is:

$$TOT(2,1) = \frac{14}{4} + \frac{5}{4} + \frac{1}{4} = \frac{20}{4}, \tag{157}$$

implying the corresponding participation ratios:

$$DER(2,1) = \frac{DE(2,1)}{TOT(2,1)}100\% = \frac{14}{20}100\% = 70\%, \tag{158}$$

$$DMR(2,1) = \frac{DM(2,1)}{TOT(2,1)}100\% = \frac{5}{20}100\% = 25\%, \tag{159}$$

$$OMR(2,1) = \frac{OM(2,1)}{TOT(2,1)}100\% = \frac{1}{20}100\% = 5\%. \tag{160}$$

That is, the participation ratios measured in Nature show surprising agreement with those computed by the one-particle model $SH_1^{(1,0)} \cong SH_1^{(0,1)}$ that are defined by solvable extension of the 3D Heisenberg group $H_1^{(1,0)} \cong H_1^{(0,1)}$.

Does this mean that they are the same on the generic models $SH_l^{(a,b)} \cong SH_l^{(b,a)}$, as well? At first sight, it seems to be discouraging that the above computations repeated for other models provide different ratios. For instance, on $SH_3^{(1,0)}$, where $k = 4$ and $l = 3$, they are:

$$DE(4,3) = \frac{50}{4}, \quad DM(4,3) = \frac{26}{4}, \quad OM(4,3) = \frac{6}{4}, \tag{161}$$

$$TOT(4,3) = \frac{50}{4} + \frac{26}{4} + \frac{6}{4} = \frac{82}{4}, \tag{162}$$

$$DER(4,3) = \frac{DE(4,3)}{TOT(4,3)}100\% = \frac{50}{82}100\% = 60.97\%, \tag{163}$$

$$DMR(4,3) = \frac{DM(4,3)}{TOT(4,3)}100\% = \frac{26}{82}100\% = 31.70\%, \tag{164}$$

$$OMR(4,3) = \frac{OM(4,3)}{TOT(4,3)}100\% = \frac{6}{82}100\% = 7.31\%. \tag{165}$$

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Different densities are defined on $SH_7^{(1,0)}$, where $k = 8$ and $l = 7$, so:

$$DE(8,7) = \frac{176}{4}, \quad DM(8,7) = \frac{122}{4}, \quad OM(8,7) = \frac{28}{4}. \tag{166}$$

$$TOT(8,7) = \frac{176}{4} + \frac{122}{4} + \frac{28}{4} = \frac{326}{4}, \tag{167}$$

$$DER(8,7) = \frac{DE(8,7)}{TOT(8,7)}100\% = \frac{176}{327}100\% = 53.82\%, \tag{168}$$

$$DMR(8,7) = \frac{DM(8,7)}{TOT(8,7)}100\% = \frac{122}{327}100\% = 37.42\%, \tag{169}$$

$$OMR(8,7) = \frac{OM(8,7)}{TOT(8,7)}100\% = \frac{28}{327}100\% = 8.58\%. \tag{170}$$

The densities on $SH_1^{(a,b)}$, $SH_3^{(a,b)}$, $SH_7^{(a,b)}$ differ not just from each other but from the above ones, also, when $ab \neq 0$.

Despite their great variety, all models $SH_l^{(a,b)}$ satisfying $k, l \neq 0$ exhibit the same participation ratios $(DER, DMR, OMR) = (70\%, 25\%, 5\%)$ if one clearly understands how DE, DM, and OM are measured and observed on Zeeman spacetimes. The key point is that each model $SH_l^{(a,b)}$ represents a multiparticle system built up in particular way from the one-particle models $SH_1^{(1,0)} \cong SH_1^{(0,1)}$. The latter models tightly cover $SH_l^{(a,b)}$ forming totalgeodesic submanifolds there.

To describe them mathematically, consider a fixed $Z_\gamma \neq 0$ that determines the complex structure $J_{\gamma 0}$ - corresponded to $Z_{\gamma 0} = (1/|Z_\gamma|)Z_\gamma$ - and a complex plane $\mathbb{C}$, through the origin of the X-space, defined with regard to $J_{\gamma 0}$. Then, the Heisenberg group is defined on $H_1^{(1,0)} = \mathbb{C} \oplus \mathbb{R} \cdot Z_{\gamma 0}$, where $\mathbb{R} \cdot Z_{\gamma 0}$ is the 1D linear subspace spanned by $Z_{\gamma 0}$, and the solvable extension $SH_1^{(1,0)}$ defines a totalgeodesic submanifold in $SH_l^{(a,b)}$. Since $Z_{\gamma 0}$ and the complex plane $\mathbb{C}$ can be chosen arbitrarily, they tightly cover $SH_l^{(a,b)}$, indeed.

The most important distinguishing feature of the spacetime $SH_l^{(a,b)}$ is that it is a multiparticle model, yet it also defines a relativistic 4-spacetime for the individual particles. They are considered to be not just as 1-particle models but also as Observers which are equipped with everything to define their own stress energy tensor, and the corresponding stress energy tensors and densities for DE, DM, and OM. On each of them, the participation ratios are measured by $(DER, DMR, OMR) = (70\%, 25\%, 5\%)$. It is also important to note that the participation ratios in Nature had been computed by assuming a general relativistic 4-spacetime and no observations, computations, or theorizing have been carried out in other dimensions. Thus, there is no other way for observing and measuring the formations than by the totalgeodesic 4-spacetimes $SH_1^{(1,0)} \cong SH_1^{(0,1)}$ which cover the whole spacetime $SH_l^{(a,b)}$ not just tightly but also in multiple ways.

The question arises, then, as to what is measured by the above quantities $DER(k,l), DMR(k,l)$, and $DMR(k,l)$? If $SH_l^{(a,b)}$ were a Cartesian product of the total geodesic 4-spacetimes, then $(DER, DMR, OMR) = (70\%, 25\%, 5\%)$ would be the participation ratios also for the whole spacetime. The above computations establish the ratios on the total spacetime by taking into consideration the particular ways how a particular $SH_l^{(a,b)}$ is built up by 4D-spacetimes. These particularities determine how the DE; DM; and OM defined on the totalgeodesic 4-spacetimes $SH_1^{(1,0)} \cong SH_1^{(0,1)}$ are integrated to define DE; DM; and OM in $SH_l^{(a,b)}$. The most influential particularities are those determining how the building blocks $SH_l^{(1,0)} \cong SH_l^{(0,1)}$ emerge from the Clifford modules. Equally important is the fact that the individual particles have a common Z-space which lowers the eigenvalues and the energy to be inscribed into the fabric of the spacetime, by the arguments described at the very end of Section 4.8.

A major drawback with the above computations is that they refer to OM, DM, and DE that are inherent in the geometric fabric of the Zeeman spacetime. Even there, it is not clearly suggested as to how can the stress-energy tensors and the corresponding wave operators be determined for the individual formations. The three formations should ultimately be represented by waves having basic features explorable by the actions of the individual wave operators on the waves. A Zeeman spacetime only serves as a stage where the quantum events are described by waves and wave operators.

Another concern arising with regard to inherent densities is that they do not explain as to why is a pulling-together force - characteristic for gravitation - is acting with regard to OM and DM and why is this force pushing-away (antigravitational) when it comes to DE? These features will be independently furnished by prescribing the direction in which a particular formation is moving. In what follows, this Moving-Direction is denoted by MD.

In order to furnish the correct characteristic features, DM and OM are supposed to be moving along the T parameter lines, into the shrinking time direction. That is, $MD = T$. Such a movement induces shrinking both on the X- and Z-horospheres that manifests so that the points on the horospheres acceleratingly fall toward each other, due to a pulling-together force acting along the horospheres. The DE, by contrast, is moving into the expanding time direction - i. e., $MD = \tau$ - that gives rise to the characteristic pushing-away force acting along the horospheres.

The pulling-together force defined in relation to DM gives explanation for Zwicky's observations showing that galaxies within clusters were zooming around far quicker than their mass would logically dictate. In consent with the above explanations, on Zeeman spacetimes, the phenomenon is due to the stars' reaction to the compressing (pulling) force the Dark Matter exerts on them. Contrary to this, movement into the expanding direction gives rise to a pushing away force, characteristic for the action of the dark energy. As it is pointed out later, both forces act among particle galaxies but have no effect on the quantum physical structure of the ordinary matter making up the individual galaxies.

The accelerating motion along horospheres gives rise to the possibility for defining inertial mass. The mass arising from the spectral mass-assigning procedure is considered to be neutral, however, if it is considered to be placed into the accelerating spacetime, where its inertia can be tested against the acceleration defined on the horospheres, the neutral mass turns to be an inertial mass. On the other hand, the Zeeman spacetime - being a general relativistic model - inherently represents a gravitational field in which the neutral mass appears as a gravitational mass. The two type of masses give rise to the classical question as to how does the equivalence of the inertial and gravitational masses manifest itself on Zeeman spacetimes? All of these problems are brought to conclusion in the following sections.

10. Rewriting $E(A, A^*)$ and $E_{st}(A, A^*)$

Formulae (150) and (151) for the expanding and static stress-energy tensors $E(A, A^*)$ and $E_{st}(A, A^*)$ are to be rewritten in order to reveal the stress energy tensors for OM, DM, and DE. Like the above considered density, the rearranged tensor appears to be decomposed into three parts - corresponding to the three formations - that are sought out so that each of them must exhibit the basic features of the corresponding formation. The rearrangement must also prepare a correspondence principle associating individual wave operators to the individual stress energy tensors. The desired wave operators are expected to be invariant with regard to the isometries of the homogeneous prototypical Zeeman spacetime and they must also yield the equivalence of gravitational and inertial masses.

These are the criterions that are to be implemented, not at once but step by step. In this preparatory section, there are made only two minor steps. In the first one, $E(A, A^*)$ is rearranged into three groups, denoted by $[...]_1$; $[...]_2$; and $[...]_3$, as follows.

$$E(A, A^*) = [2E_{st}(A, A^*)]_1 + \tag{171}$$

$$+[-\frac{1}{4}(\frac{k}{2}+l)^2\langle A, A^*\rangle + (\frac{k}{2}+l)(\frac{1}{2}\langle X, X^*\rangle + \langle Z, Z^*\rangle)]_2 + \tag{172}$$

$$+[-E_{st}(A, A^*) - \frac{1}{2}(\frac{k}{2}+2l)\langle T, T\rangle - \frac{1}{2}(\frac{1}{2}(\frac{k}{2}+l)^2 + \frac{k}{4}+l)\langle A, A^*\rangle]_3. \tag{173}$$

Notice that $-E_{st}(A, A^*)$ in the third group cancels out one $E_{st}(A, A^*)$ to be found in the first group. Moreover, $-\frac{1}{2}(\frac{k}{2}+l)^2\langle A, A^*\rangle$ in (150) is divided into two equal pieces so as to incorporate them into the second and third group, respectively. This rearranged formula still faithfully represents $E(A, A^*)$.

The next version is expressed in terms of $\langle A, A^*\rangle$, $\langle X, X^*\rangle$, and $\langle Z, Z^*\rangle$, that can be achieved by involving $< T, T >$ into $\langle A, A^*\rangle$. This is done already for $E_{st}(A, A^*)$, in (151). The third group transforms to this form after adding and also subtracting $(\langle X, X^*\rangle + \langle Z, Z^*\rangle)$ from $\langle T, T\rangle$ after which $< T, T >$ appears to be involved into $\langle A, A^*\rangle$. Then one has the decomposition

$$E(A, A^*) = 2[\frac{kl}{8}\langle A, A^*\rangle - \frac{l}{2}\langle X, X^*\rangle + \frac{k}{4}\langle Z, Z^*\rangle]_1 + \tag{174}$$

$$+[-\frac{1}{4}(\frac{k}{2}+l)^2\langle A, A^*\rangle + (\frac{k}{2}+l)(\frac{1}{2}\langle X, X^*\rangle + \langle Z, Z^*\rangle)]_2 + \tag{175}$$

$$+[-\frac{kl}{8} - \frac{1}{2}(\frac{1}{2}(\frac{k}{2}+l)^2 + \frac{3k}{4} + 3l)\langle A, A^*\rangle + \tag{176}$$

$$-(-\frac{l}{2}\langle X, X^*\rangle + \frac{k}{4}\langle Z, Z^*\rangle) + \frac{1}{2}(\frac{k}{2}+2l)(\langle X, X^*\rangle + \langle Z, Z^*\rangle)]_3. \tag{177}$$

The so obtained formulae still do not give account on the formation's Moving Directions MD that is furnished, in the next section, by redirecting the stress-energy tensor. There will also be shown that Δ_E is obtainable from $\langle A, A^*\rangle = \langle X, X^*\rangle + \langle Z, Z^*\rangle - \langle T, T\rangle$ by plugging in for $\langle X, X^*\rangle$, $\langle Z, Z^*\rangle$, and $\langle T, T\rangle$ certain differential operators which method will be applied to obtain the wave operators of individual formations, also. It turns out as well that an individual stress-energy tensor yields the equivalence of gravitational and inertial masses if and only if it appears to be a constant multiple of $\langle A, A^*\rangle$, in which case the corresponding operator appears as constant multiple of the invariant operator Δ_E. This statement clearly suggests that, for furnishing the equivalence of the two type of masses, one should find a redirection that ends up with individual stress-energy tensors appearing as scalar multiples of $\langle A, A^*\rangle$. The search is continued in the next section, in consent with these criterions.

11. Wave Operators of OM, DM, and DE

By (127), Δ_E is obtainable from $\langle A, A^*\rangle = \langle X, X^*\rangle + \langle Z, Z^*\rangle - \langle T, T\rangle$ by substituting certain operators for $\langle T, T\rangle$, $\langle X, X^*\rangle$, and $\langle Z, Z^*\rangle$. These operator-to-function correspondences are defined by

$$\sum_i(\mathbf{Y}_i^2 - \nabla_{\mathbf{Y}_i}\mathbf{Y}_i) \to \langle X, X^*\rangle, \tag{178}$$

$$\sum_\alpha(\mathbf{V}_\alpha^2 - \nabla_{\mathbf{V}_\alpha}\mathbf{V}_\alpha) \to \langle Z, Z^*\rangle, \tag{179}$$

$$\partial^2_{TT} \to \langle T, T\rangle \tag{180}$$

which say that the operators standing on the left side are plugged in for the bilinear functions standing on the right side. Then, the so obtained operator needs to be expanded - by (3), (4), (5), (126) - as done in (127)-(138), so as to get Δ_E in finalized forms.

Correspondence of wave operators to relativistic metric tensors is not a new idea which was also used, for instance, for establishing the Klein-Gordon operator of electron. In that case, substitution meant plugging into the metric tensor of a 4D flat Minkowski space, whereas, in the present case, Δ_E arises from the metric tensor of a curved general relativistic spacetime. Other operators - obtained by plugging into the metric tensors of curved manifolds - are Δ_{st}, and the Monistic Laplacian Δ defining the Hamiltonians on H-type groups.

The curvature endows the operators with new features manifesting only on properly curved manifolds. For understanding as to what is achieved by this, let's remember that Pauli's field theory together with all modern Lagrangian frameworks are based on the Klein-Gordon wave operator, considered on a flat 4D Minkowski space, and all efforts had been defeated trying to exchange it for a curved 4D spacetime. The new features arising on curved spacetimes include the possibility to define mass engaging gravitational interactions in consent with general relativity and also with quantum theory. A precise introduction of OM, DM, and DE together with their wave operators is also possible there by further generalizing Klein-Gordon's original idea by applying it not to metric but to the non-trivial stress-energy tensors defined on properly curved relativistic spacetimes. The technical details are as follows.

If the above substitution is applied to a part $[\dots]_i$ of the stress energy tensor, then the so obtained operator is denoted by $[\dots]_i^+$. Anti-substitutions, defined by:

$$-\sum_i(\mathbf{Y}_i^2-\nabla_{\mathbf{Y}_i}\mathbf{Y}_i) \rightarrow \langle X,X^*\rangle, \tag{181}$$

$$-\sum_\alpha(\mathbf{V}_\alpha^2-\nabla_{\mathbf{V}_\alpha}\mathbf{V}_\alpha) \rightarrow \langle Z,Z^*\rangle, \tag{182}$$

$$-\partial^2_{TT} \rightarrow \langle T,T\rangle, \tag{183}$$

result operators denoted by $[\dots]_i^-$.

These substitutions will be applied to the redirected stress-energy tensor which provides, in the end, the wave operators of particular matter-energy formations in the form $Q\Delta_E$ where the non-zero constant $Q\in\mathbb{R}$ is characteristic for the formation. It's sign, for instance, determines the moving direction MD. If $Q>0$, then the spectral mass assignment procedure is carried out in the same way how it was processed for Δ_E, by the waves

$$\exp(\mathbf{i}mt/\hbar)\Psi^{pq}_{\mathrm{st}}(X,Z_\gamma,t)=\exp(\mathbf{i}mt/\hbar)e^{\mathbf{i}(\langle Z,Z_\gamma\rangle-\omega t)}H^\gamma_{pq}(z_i,\overline{z}_i). \tag{184}$$

The eigenvalue $-(2m\mathbf{i}/\hbar)(\mathbf{i}mt/\hbar)$ obtained by deriving the exterior exponential function $\exp(\mathbf{i}mt/\hbar)$ with $-(2m\mathbf{i}/\hbar)\partial_t$ is identified with a positive eigenvalue λ of the Hamilton operator appearing as part of the actual Schrödinger operator. Since the particle systems having positive masses attract each other gravitationally, they must move into the shrinking time-direction.

In case of $Q<0$, by contrast, the spectral mass assignment procedure is carried out by $-\Delta_E$, in which the Yukawa operator is the negative of that considered above, thus it prepares negative eigenvalues and masses to be transformed to the actual Schrödinger operator. All these operators act on (184) but they are negative of those considered in the previous case. As a result, the negative mass is identified with the negative eigenvalue $-\lambda$ of the Hamiltonian Laplacian Δ. The so described particles have negative masses, thus they are antiparticles of those described previously and move in the opposite expanding time direction.

The sign of constants standing before $\langle A,A^*\rangle$ in $[\dots]_1$ and $[\dots]_3$ are correct, since the wave operators of OM and DE are expected to be emerging from these terms. This is not so for $[\dots]_2$ in which a negative number stands before $\langle A,A^*\rangle$, contrary to the expectation that this term will produce the wave operator of DM. In order to fix this problem, reverse the time-direction for this term by the correspondence $[\dots]_2^-$ and consider the correspondence with regard to the complete stress energy tensor in the form $[\dots]_1^+ + [\dots]_2^- + [\dots]_3^+$.

The first important thing has to be mentioned about this redirected stress-energy tensor is that it only contains terms involving $\langle A,A^*\rangle$ and gets rid of all other terms showing up in the space-space block that make the appearance of the original stress energy tensor rather chaotic. To see this notice that $[(-\frac{l}{2}\langle X,X^*\rangle+\frac{k}{4}\langle Z,Z^*\rangle]_3$ in the third group cancels out a term from the first group $[\dots]_1$, thus, the statement is proven by

$$[-\frac{l}{2}\langle X,X^*\rangle+\frac{k}{4}\langle Z,Z^*\rangle]_1^+ + [(\frac{k}{2}+l)(\frac{1}{2}\langle X,X^*\rangle+\langle Z,Z^*\rangle)]_2^- + \tag{185}$$

$$+[\frac{1}{2}(\frac{k}{2}+2l)(\langle X,X^*\rangle+\langle Z,Z^*\rangle)]_3^+ = \tag{186}$$

$$=[-\frac{l}{2}\langle X,X^*\rangle+\frac{k}{4}\langle Z,Z^*\rangle]_1^+ + [\frac{l}{2}\langle X,X^*\rangle-\frac{k}{4}\langle Z,Z^*\rangle]_{2+3}^+ = 0. \tag{187}$$

Due to this statement, the redirected stress-energy tensor $D(A, A^*)$ appears to be expressed only by $\langle A, A^* \rangle$ so that the first group contains twice of the stress energy tensor corresponded to OM. Since OM is identified with $\frac{kl}{8}$, the excess $\frac{kl}{8}$ is provided to DM. By the final versions, the $D(A, A^*)$ appears to be divided into three groups, to each of which there is corresponded an operator defined by $[\dots]_i^+$, that appear in the form as follows.

$$D(A, A^*) = [\frac{kl}{8}\langle A, A^* \rangle]_1 + [(\frac{kl}{8} + \frac{1}{4}(\frac{k}{2} + l)^2)\langle A, A^* \rangle]_2 + \tag{188}$$

$$+[-(\frac{kl}{8} + \frac{1}{2}(\frac{1}{2}(\frac{k}{2} + l)^2 + \frac{3k}{4} + 3l)\langle A, A^* \rangle]_3, \tag{189}$$

$$WO_{OM} = [\frac{kl}{8}\langle A, A^* \rangle]_1^+ = \frac{kl}{8}\Delta_E, \tag{190}$$

$$WO_{DM} = [\frac{kl}{8} + \frac{1}{4}(\frac{k}{2} + l)^2\langle A, A^* \rangle]_2^+ = (\frac{kl}{8} + \frac{1}{4}(\frac{k}{2} + l)^2)\Delta_E, \tag{191}$$

$$WO_{DE} = [-(\frac{kl}{8} + \frac{1}{2}(\frac{1}{2}(\frac{k}{2} + l)^2 + \frac{3k}{4} + 3l)\langle A, A^* \rangle]_3^+ = \tag{192}$$

$$= -(\frac{kl}{8} + \frac{1}{2}(\frac{1}{2}(\frac{k}{2} + l)^2 + \frac{3k}{4} + 3l)\Delta_E, \tag{193}$$

where WO_{OM}; WO_{DM}; and WO_{DE} denote the Wave Operators of OM; DM; and DE, respectively.

The redirected stress energy tensor and the corresponding wave operators display OM, DM, and DE with the correct characteristic features. The parts of OM and DM are kept together by pulling forces that originate - in both cases - from pushing into the shrinking time direction. In other words, $MD = T$ holds along the geodesics running into the same ideal point to be found at $T = \infty$. Such a push induces accelerating contractions on the paraspheres intersecting these geodesics perpendicularly. The pressure inscribed into the diagonal of the space-space block of the stress-energy tensor is determined by the multiple of the constant standing before $\langle A, A^* \rangle$ and the acceleration measured on the paraspheres. Regarding OM and DM, the pulling together force is identified with the contracting pressure and the constants standing before $\langle X, X^* \rangle$ and $\langle Z, Z^* \rangle$ are interpreted as inertial densities defined for the masses inherent in the Zeeman spacetime. As opposed to these, the same constants before $\langle T, T \rangle$ are considered to be the heavy-densities. Their identity reveals the equivalence of the two type of masses. The same arguments can be said about DE, for which $MD = \tau$, giving rise to a pushing away force, characteristic of antigravitation.

The equivalence of inertial and gravitational masses was first measured by Loránd Eötvös, which became one of the most important corner stones of Einstein's general relativity. On the accelerating Zeeman spacetimes, this law is furnished so that, regarding each energy-matter formation, the redirected stress-energy appears to be a constant multiple of the metric tensor $\langle A, A^* \rangle$. If a Riemann manifold exhibits this feature, it is called Einstein manifold. On a prototypical and accelerating Zeeman spacetime, this so called Eötvös-Einstein property is not true regarding the original but just for the redirected stress-energy tensor. Such manifolds - whose stress energy tensors can be redirected so as to satisfy the Eötvös-Einstein property - are called Einstein-Eötvös spacetimes. The above described acceleration and inertia force, observable on the paraspheres, are named after János Bolyai who discovered - independently from Nikolai Lobachevsky - hyperbolic geometry and studied the geometry of paraspheres (the same as horospheres), extensively.

12. Summary for the Accelerating Wave Operators

The stress-energy tensor (150) defined for the relativistic solvable extensions of H-type groups is rearranged (redirected) and then decomposed into three parts as described in (188)-(189). Each of them is constant multiple of the metric tensor $\langle A, A^* \rangle$ of the accelerating spacetime, which appearance is exhibited only by the redirected tensor but not by the original stress-energy tensor. The decomposition also sets apart the constant density - to be found in the time-time component of the original stress-energy tensor - into constant densities to be defined for OM, DM, and DE, respectively. By (154)-(155)

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and (188)-(193), they agree with those standing before $\langle A, A^* \rangle$ in the decomposition of the redirected stress-energy tensor.

The constant densities allow to compute the participation ratios with which OM, DM, and DE compose the matter. For the one-particle models - defined by solvable extensions of 3D Heisenberg groups - the computations result (5%,25%,70%) which is in strong agreement with those to be observed in Nature. This recognition approved to be helpful not just identifying the densities but the stress-energy tensors of specific matter-energy formations, also. For the generic models $SH_l^{(a,b)}$, the participations ratios are different showing that the participation ratios depend on the particular ways how the one particle models are integrated to model specific multiparticle systems.

The expanding wave Laplacian Δ_E can be defined by plugging in for $\langle T, T \rangle$, $\langle X, X^* \rangle$, and $\langle Z, Z^* \rangle$ showing up in the metric tensor $\langle A, A^* \rangle$ the operators described in (178)-(180). This association extended to the redirected stress-energy tensor corresponds to the stress-energy tensors of particular formations constant multiples of Δ_E - which define the wave operator for each individual formation - in which the constant is nothing but the signed density defined for the formation. This sign is positive for OM and DM, and negative for DE, meaning - as shown by the spectral mass assigning procedure - that positive mass is defined for OM and DM, and negative one for DE.There is also implied that OM and DM are moving toward the shrinking, while DE the expanding time direction. Each of these stress-energy tensors exhibit the Einstein-Eötvös property, meaning that they yield the equivalence of gravitational and inertial masses.

The solvable groups $SH_l^{(a,b)}$ are considered to be only as underlying mathematical structures (stages) on which the matter-energy formations OM, DM, and DE perform interactions describable by the actions of the Laplacian canonically given on these properly curved spacetime stages on the waves representing the matter. This new approach, reconciling quantum theory with general relativity, has never been explored in the literature insofar. Among the three matter-energy formations, the OM is the only actual matter. It is placed into an accelerating environment, but which, as shown by (131)-(138), is not expanding or shrinking. The accelerating environment only has effect on the surrounding space without influencing the quantum physical laws effecting OM through the actions of the non-relativistic Yukawa, Schrödinger, and gravitational operators explored in the static spacetime.

The geometry of $SH_l^{(a,b)}$ (inscribed into its curvature tensor) only determines how to decompose and redirect the non-Einstein stress-energy tensor so as to obtain Einsteinian stress-energy tensors for OM, DM, and DE, respectively. For each of them, there is defined a density, according to (188)-(193), agreeing with those established in (154)-(155) differently. Due to the Einstein-Eötvös property, the wave operators of OM, DM, and DE can be obtained by the same function-to-operator correspondence how Δ_E is corresponded to the metric tensor $\langle A, A^* \rangle$. With these operators, mass-assignment can be carried out for each formation, independently, resulting masses computable so that the mass to be defined for an eigenvalue on the static spacetime is multiplied with the density to be defined for the formation. The participation ratios computed by the inherent densities are the same as those computed by the ratios of the masses defined for the formations by this spectral mass assignment procedure, for each eigenvalue of the static Schrödinger Hamiltonian.

Similar statements hold true for the Dirac operator. According to the new framework, there exist only OM-particles and their Anti-OM-particles. The OM-particles appear by themselves, or under the influence of DM, with positive masses and moving into the shrinking time direction. The DM is not an observable matter. It only manifests itself through its action - exertion of the Zwicki effect - on OM. Anti-OM-particles have negative masses and move into the expanding time-direction. They arrive to us, ordinary people, from the future. The DE is not an observable energy either. It manifests itself only by the antigravitation driving the Anti-OM-particles away from each other without influencing any quantum physical laws obeyed by the OM.

The cancellations in (185)-(187) describe meeting of particles with antiparticles after which they disappear in γ-rays travelling with speed of light. They have no resting mass but just radiation energy. There are γ-rays both in the X- and Z-space, that are distinguished from each other by the denotations

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γ_X and γ_Z, whose physics are describable by the Euclidean Hamilton-Laplacians Δ_X and Δ_Z, as seen earlier for free particles. This is the energy excess that prevents the system from satisfying the Einstein-Eötvös property. Since the γ-rays travel with speed of light, the test regarding the equivalence of gravitational and inertial mass is meaningless. But, once they are purged out and sent to be present among free particles, the rest of the system obeys the equivalence of the two type of masses.

A particularly interesting feature of the Monistic Dirac Operator is that it is the combination of a Dirac operator, defined on the (X, t)-space, and a scalar operator - the non-relativistic Yukawa operator - defined in the (Z, t)-space. This idea is not strange to the Standard Model, either, in which there is hypothesized the existence of the Higgs boson - the only boson of spin 0 - that has been used to define mass for the particles, by Lagrangian means. The very same task is performed by the non-relativistic Yukawa operator in cooperation with the Yukawa operator on Zeeman spacetimes, which coincidence can be seen as a touching point connecting the mass assignment procedures carried out in the two theories, independently. The just described arguments directly relate the non-relativistic Yukawa operator to the Higgs boson, however, the differences in their establishment make it impossible to declare that the Higgs boson is the same as that emerging from the non-relativistic Yukawa operator.

Each physical experiment is theory-depending. In the time when DE was discovered [Pe1,RS], the experimental data was evaluated by the modified general relativity in which the Einstein equation was supplemented by the cosmological constant in order to give explanation for the expansion of the Universe. It is still noteworthy that the observed acceleration and participation ratios were computed by a well determined cosmological constant [RS], and DE has been identified with the vacuum energy. Moreover, there had been no explanations given for the origin of DM. Although there are some similarities, the new framework applies different approach. The most important difference is that the latter models are established on well determined curved spacetimes that exhibit acceleration on their own right, without using cosmological constant, moreover, they have their own stress-energy tensors corresponding to Einstein's original definition. It should also be emphasized that the new framework gives a unified theory for all three formations and explains as to how do they emerge, together, from properly curved relativistic spacetimes.

To conclude the paper, the scenario proposed to describe the evolution of the Universe at the end of Section 7 is brought to a temporary closure. According to those explanations, the Universe came into existence from the idiomorphic point corresponding to $\tau = -\infty$ as such to be composed by anti-OM particle systems. Then, it uneventfully advanced, toward the expanding time direction τ, until the τ-worldlines reached the H-type group N embedded into SN as level set corresponding to $t = 1$ resp. $\tau = 0$. The travel, that took up infinitely long time, was monitored, so as to understand the processes the matter must had been going through, by the action of $-\Delta_E$ on the static waves

$$\exp(\mathbf{i}mt/\hbar)\Psi^{pq}_{\text{st}}(X, Z_\gamma, t) = \exp(\mathbf{i}mt/\hbar)e^{\mathbf{i}(\langle Z,Z_\gamma\rangle - \omega t)}H^\gamma_{pq}(z_i, \bar{z}_i), \tag{194}$$

which also determines a spectral mass assignment procedure defining negative masses to the anti-OM particles.

As explained earlier, the anti-OM was not expanding but only the space surrounding it. The latter effect was due to the action of the gravitational operator composed by that defined for the anti-OM, and also by $-Q(t)$ mixing up those of dark origin. The latter is called excess gravitational operator, which acts on (194) as multiplication with

$$\frac{\hbar^2|Z_\gamma|^2/m^2}{1 + \sqrt{1 + \hbar^2|Z_\gamma|^2/m^2}}\left(2\frac{m^2}{\hbar^2}t(t-1) - \mathbf{i}(\frac{k}{2} + l + 1)\frac{m}{\hbar}t\right). \tag{195}$$

The real part in this function changes sign when $t = 1$ resp. $\tau = 0$, effecting that the expansion taking place during the period $\tau < 0$ was turned, in some extent, backward into the shrinking direction. This change was the ultimate reason for the Big Bang was taking place. The turned around matter became OM and DM, and that kept travelling into the expanding time direction formed DE. An essential part of the Big Bang was also the production of the γ-rays spawned from meeting of particles with antiparticles which must had been taking place due to the relations (185)-(187).

In short, the new framework, too, recognizes the reality of Big Bang taking place on the H-type group $N \subset SN$ at the time $t = 1$ resp. $\tau = 0$. It gives birth to OM, DM - travelling into shrinking time direction -, and also to DE pushing the antimatter further into the expanding time direction. In the spectral mass-assignment procedure, OM and DM are assigned with positive, while DE with negative masses. As a consequence, the Big Bang had also separated the matter having positive masses from the antimatter having negative masses. This explains the underwhelming presence of antimatter among OM-particles, however, those stuck with us - or produced in an OM-environment - move into the shrinking (downward) direction together with the OM particles. Prior to the Big Bang, the different type of matter formations made up a chaotic medley that was pushed toward the expanding time direction, together, by a continually weakening gravitational force, so much so that the excess gravitation vanished on $N \subset SN$ at the time of Big Bang and started with pushing back into the shrinking direction. This turnaround was weakening the antigravitation originated from DE even more till it completely dies away at $t = 0$ resp. $\tau = \infty$.

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