Propagation of a Gaussian-top-hat function

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ABSTRACT

We study the propagation of a particular field that we call Gaussian-top-hat that presents self-focusing and maintains its shape for some propagation distances.

Introduction

During the last two decades, research in optical fields that maintain their shape [1, 2] during propagation has generated a myriad of applications with a great impact in fields such as particle manipulation, medicine, imaging and materials processing, to name just a few. Just as important are fields that present self-focusing effects during propagation, i.e., the field energy concentrates in a small region for some propagation distances [3, 4].

In this contribution, we present a novel field, which we call Gaussian-top-hat, that presents interesting properties during propagation, such as self-focusing and propagation invariance. The field was conceived from the derivation of the Fresnel diffraction integral through the use of the squeeze operators in quantum mechanics and the fractional Fourier transform.

Propagation in terms of the Fractional Fourier transform and the squeeze operators

The propagation of light in free space is described by the paraxial wave equation

$$ik\frac{\partial E(x,y,z)}{\partial z} = -\frac{1}{2}\frac{\partial^2 E(x,y,z)}{\partial x^2} - \frac{1}{2}\frac{\partial^2 E(x,y,z)}{\partial y^2}, \quad (1)$$

where k is the wavevector. We may define the operators $\hat{p}_\alpha = -i\partial/\partial\alpha$, with $\alpha = x, y$ such that we rewrite the above equation as

$$i\frac{\partial E(z)}{\partial z} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2k}E(z), \quad (2)$$

with the simple formal solution

$$E(z) = \exp\left[-i\frac{z}{2k}(\hat{p}_x^2 + \hat{p}_y^2)\right]E(0). \quad (3)$$

We define annihilation and creation operators for the harmonic oscillator,

$$\hat{a}_\alpha = (\hat{\alpha} + i\hat{p}_\alpha)/\sqrt{2}, \qquad \hat{a}_\alpha^\dagger = (\hat{\alpha} - i\hat{p}_\alpha)/\sqrt{2}, \qquad \alpha = x, y \quad (4)$$

to write equation (3) in the form

$$\begin{aligned} E(z) &= \exp\left[-i\frac{z}{2k}\left(\hat{n}_x + \frac{1}{2} - \frac{\hat{a}_x^2}{2} - \frac{\hat{a}_x^{\dagger 2}}{2}\right)\right] \quad (5)\\ &\times \exp\left[-i\frac{z}{2k}\left(\hat{n}_y + \frac{1}{2} - \frac{\hat{a}_y^2}{2} - \frac{\hat{a}_y^{\dagger 2}}{2}\right)\right]E(0), \end{aligned}$$

with $\hat{n}_\alpha = \hat{a}_\alpha^\dagger\hat{a}_\alpha$ the number operator in quantum mechanics. We may use operator algebras [5] to write any of the exponentials above as

$$\begin{aligned} \exp\left(-i\frac{z}{2k}\hat{p}^2\right) &= \hat{S}\left(ire^{-i\omega}\right)\exp\left[-i\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right] \\ &= \hat{S}(\xi)\hat{F}(\omega), \quad (6) \end{aligned}$$

where we have defined $\xi = re^{i\theta}$ and ω such that

$$e^{i\omega} = \frac{1 + i\frac{z}{2k}}{\sqrt{1 + \left(\frac{z}{2k}\right)^2}}, \qquad e^{i\theta} = ie^{i\omega}, \quad (7)$$

with $r = \ln\left(\sqrt{1 + \left(\frac{z}{2k}\right)^2} - \frac{z}{2k}\right)$.

In equation (6) we have used $\hat{S}(\xi)$ which is the squeeze operator [6] and $\hat{F}(\omega)$ is the fractional Fourier transform, such that the solution to the paraxial wave equation is finally given by

$$E(z) = \hat{S}_x(\xi)\hat{S}_y(\xi)\hat{F}_x(\omega)\hat{F}_y(\omega)E(0), \quad (8)$$

i.e., the application of squeeze operators to the two-dimensional fractional Fourier transform of the field at $z = 0$.

Diffraction integral

Given the field at $z = 0$, $E(x, y, 0)$ by its inverse Fourier transform

$$E(x,y,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ixu+ivy}\tilde{E}(u,v)du\,dv, \quad (9)$$

application of the solution (3) renders the well known diffraction integral

$$E(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iz\frac{u^2+v^2}{2k}}e^{ixu+ivy}\tilde{E}(u,v)du\,dv, \quad (10)$$

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where we have used the fact that the exponential is an eigenfunction of the derivative. By writing explicitly the Fourier transform inside the integral (10) we obtain

$$E(x,y,z) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iz\frac{u^2+v^2}{2k}} \times e^{iu(x-x')+iv(y-y')}E(x',y',0)dx'dy'dudv,$$

that we can rearrange to integrate first on the variables u and v, i.e.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iz\frac{u^2+v^2}{2k}} e^{iu(x-x')+iv(y-y')}dudv = -i\frac{2k\pi}{z}e^{i\frac{k(x-x')^2}{2z}}e^{i\frac{k(y-y')^2}{2z}}, \quad (11)$$

yielding the Fresnel integral of diffraction

$$E(x,y,z) = \frac{-ik}{2\pi z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(x,y,0)e^{i\frac{k}{2z}[(x-x')^2+(y-y')^2]}dx'dy', \quad (12)$$

In the last equation the factor e^{ikz} can be omitted because it doesn't contribute to the field intensity.

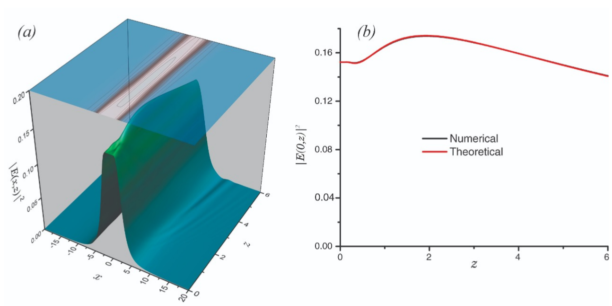

Figure 1: (a) Propagation of the field given by equation 13 for $a = 10$ and $k = 1$. (b) Numerical and analytical comparison of the propagated field $|E(0,x)|^2$.

Propagation of special Gaussian

The squeeze operators in equation (8), that produce a compression in the variables x and y, suggest that, if we start with a function that has them in the denominator, beams that have some interesting properties as self-focusing or are quasi-invariant as they propagate, could be produced.

For simplicity we consider now a one dimensional function to be propagated (obviously, what we consider for the variable x applies for the variable y as well), in particular a *special* Gaussian given by

$$f(x) = \frac{1}{N}\left[1 - e^{-\frac{a}{x^2}}\right], \quad (13)$$

with the normalization constant $N = \sqrt{2\sqrt{\pi a}\left(2-\sqrt{2}\right)}$. The propagated field is then given by

$$E(x,z) = \frac{e^{-i\frac{\pi}{4}}}{N}\sqrt{\frac{k}{2\pi z}}\int_{-\infty}^{\infty}\left[1-e^{-\frac{a}{x'^2}}\right]e^{i\frac{k(x-x')^2}{2z}}dx', \quad (14)$$

that for $x = 0$ propagates along the z-axis as

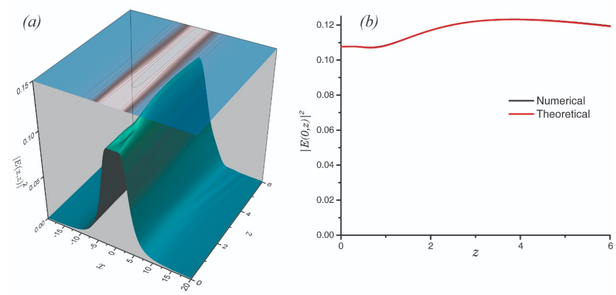

Figure 2: Propagation of the field given by equation 13 for $a = 20$ and $k = 1$. (b) Numerical and analytical comparison of the propagated field $|E(0,x)|^2$.

$$E(0,z) = -2\frac{e^{-i\pi/4}}{N}\sqrt{\frac{1}{2\pi z}}\int_0^{\infty}(e^{-\frac{ka}{u^2}}-1)e^{i\frac{u^2}{2z}}du, \quad (15)$$

or, equivalently [7],

$$E(0,z) = \frac{1}{N}\left(1-\exp\left(-2\sqrt{\frac{ka}{i2z}}.\right)\right) \quad (16)$$

We plot the propagated field in figures 1 and 2 for the values $a = 10$ and $a = 20$, respectively. It may be observed that there is a self-focusing effect as well as the fact that the field is bounded for some propagation distances.

Conclusion

We have shown that the Gaussian-top-hat function presents interesting characteristics while it propagates, namely, it is bounded for some propagation distances and shows self-focusing effects.

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