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Frank Vega *

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Disproving the Riemann Hypothesis with Primorial Bounds

Frank Vega

Information Physics Institute, 840 W 67th St, Hialeah, FL 33012, USA; vega.frank@gmail.com

Abstract: The Riemann Hypothesis posits that all non-trivial zeros of the Riemann zeta function have a real part of $\frac{1}{2}$. As a pivotal conjecture in pure mathematics, it remains unproven and is equivalent to various statements, including one by Nicolas in 1983 asserting that the hypothesis holds if and only if $\prod_{p \leq x} \frac{p}{p-1} > e^{\gamma} \cdot \log \theta(x)$ for all $x \geq 2$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. Defining $N_n = 2 \cdot \ldots \cdot p_n$ as the n-th primorial, the product of the first n primes, we employ Nicolas' criterion to prove that there exists a prime $p_k > 10^8$ and a prime $p_{k'}$ such that $\theta(p_{k'}) \leq \theta(p_k)^2$ and $p_k^{1.907} \ll p_{k'} < p_k^2$, where $p_k^{1.907} \ll p_{k'}$ implies $p_{k'}$ is significantly larger than $p_k^{1.907}$. This existence leads to $\frac{N_k}{\varphi(N_k)} \leq e^{\gamma} \cdot \log \log N_k$, contradicting Nicolas' condition and confirming the falsity of the Riemann Hypothesis. This result decisively refutes the conjecture, enhancing our insight into prime distribution and the behavior of the zeta function's zeros through analytic number theory.

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1. Introduction

The Riemann Hypothesis, first articulated by Bernhard Riemann in 1859, asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ occur along the critical line where the real part of the complex variable s is $\frac{1}{2}$. Esteemed as the preeminent unsolved problem in pure mathematics, it constitutes a cornerstone of Hilbert's eighth problem from his famed list of twenty-three challenges and is one of the Clay Mathematics Institute's Millennium Prize Problems. In recent years, advances across diverse mathematical domains-such as analytic number theory, algebraic geometry, and non-commutative geometry-have edged us closer to resolving this enduring conjecture [1].

Defined over the complex numbers, the Riemann zeta function $\zeta(s)$ exhibits zeros at the negative even integers, known as trivial zeros, alongside other complex values termed non-trivial zeros. Riemann's conjecture specifically pertains to these non-trivial zeros, positing that their real part universally equals $\frac{1}{2}$. This hypothesis is not merely an abstract curiosity; its significance derives from its profound implications for the distribution of prime numbers-a fundamental aspect of mathematics with far-reaching applications in computation and theory. A deeper grasp of prime number distribution promises to enhance algorithm efficiency and illuminate the intrinsic architecture of numerical systems.

Beyond its technical ramifications, the Riemann Hypothesis embodies the elegance and mystery of mathematical exploration. It probes the limits of our comprehension of numbers, galvanizing mathematicians to transcend conventional boundaries and pursue transformative insights into the mathematical cosmos. As such, it remains a beacon of intellectual ambition, driving the relentless quest for knowledge at the heart of the discipline.

In this paper, we prove the Riemann Hypothesis false by establishing the existence of a prime $p_k > 10^8$ and a corresponding prime $p_{k'}$ that satisfy the conditions $\theta(p_{k'}) \leq \theta(p_k)^2$ and $p_k^{1.907} \ll p_{k'} < p_k^2$, where $\theta(x)$ is the Chebyshev function. Leveraging Nicolas' criterion, which asserts that the hypothesis holds if and only if $\frac{N_k}{\varphi(N_k)} > e^{\gamma} \cdot \log \log N_k$ for all primorials N_k, we demonstrate that these bounds on $p_{k'}$ relative to p_k lead to $\frac{N_k}{\varphi(N_k)} \leq e^{\gamma} \cdot \log \log N_k$, thus contradicting the criterion. Our

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proof combines analytic number theory tools, including Mertens' theorem and primorial estimates, to rigorously confirm this result. This resolution of a central conjecture in mathematics offers profound insights into prime distribution and challenges long-held assumptions about the zeros of the Riemann zeta function.

2. Background and Ancillary Results

In mathematical number theory, the Chebyshev function $\theta(x)$ is defined as

$$\theta(x) = \sum_{p \leq x} \log p,$$

where the summation includes all prime numbers p less than or equal to x, and log denotes the natural logarithm. In contrast, the prime counting function $\pi(x)$, which tallies the number of primes up to x, is expressed as

$$\pi(x) = \sum_{p \leq x} 1,$$

with the sum similarly ranging over all primes $p \leq x$. Together, these functions furnish essential tools for exploring the distribution of primes and related functions, bridging elementary definitions to deeper analytical insights.

In 1734, Leonhard Euler made a seminal contribution to mathematics by evaluating the Riemann zeta function at $s = 2$, a result tied to his resolution of the Basel problem [2]. This work not only showcased his ingenuity but also laid foundational insights into number theory.

Proposition 1. *The value of the zeta function at 2 is defined as [2, (1) pp. 1070]:*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{p_k^2}{p_k^2 - 1} = \frac{\pi^2}{6},$$

where p_k denotes the k-th prime number (often written as p_n for the n-th prime), n is a natural number, and $\pi \approx 3.14159$ is the ubiquitous mathematical constant bridging number theory, geometry, and beyond. Euler's proof elegantly unifies the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ with the infinite product over primes, culminating in the exact value $\frac{\pi^2}{6}$.

Another constant of profound significance, the Euler-Mascheroni constant $\gamma \approx 0.57721$, emerges in analytic number theory and is defined through two equivalent expressions:

$$\gamma = \lim_{n \to \infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right) = \int_{1}^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor} \right) dx,$$

where $\lfloor x \rfloor$ denotes the floor function, yielding the greatest integer less than or equal to x. This constant frequently appears in studies of harmonic sums and integral approximations.

Definition 1. *We say that the condition* $\mathsf{Nicolas}(x)$ *holds if:*

$$\prod_{p \leq x} \frac{p}{p-1} > e^{\gamma} \cdot \log \theta(x),$$

where p ranges over all primes less than or equal to x, $e \approx 2.71828$ is the base of the natural logarithm, and $\theta(x) = \sum_{p \leq x} \log p$ is the Chebyshev function.

Finally, a primorial number of order n, denoted N_n, is the product of the first n prime numbers:

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$$N_n = \prod_{k=1}^{n} p_k.$$

For example, $N_3 = 2 \cdot 3 \cdot 5 = 30$. This construction is pivotal in exploring properties of primes and their distributions, often intersecting with conjectures like the Riemann Hypothesis. Together, these concepts weave a rich tapestry of mathematical relationships, illuminating the intricate structure of numbers.

In number theory, the Dedekind psi function is defined as $\Psi(n) = n \cdot \prod_{p|n}\left(1 + \frac{1}{p}\right)$, where the product is taken over all distinct prime numbers p dividing n. Similarly, Euler's totient function, which counts the integers up to n that are coprime to n, is given by $\varphi(n) = n \cdot \prod_{p|n}\left(1 - \frac{1}{p}\right)$. These functions play a crucial role in analyzing arithmetic properties of numbers, particularly primorials-products of the first k primes, denoted $N_k = \prod_{i=1}^{k} p_i$.

Proposition 2. *For all natural numbers $k \geq 4$, as established by Choie et al. [3], the following inequality holds:*

$$\frac{\Psi(N_k)}{N_k} < e^{\gamma} \cdot \log\log N_k,$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Furthermore, we can relate Ψ and φ through the primorial N_k as follows:

$$\frac{N_k}{\varphi(N_k)} = \frac{\Psi(N_k)}{N_k} \cdot \prod_{p|N_k} \frac{p^2}{p^2 - 1}.$$

Since N_k is the product of the first k primes, and the infinite product over all primes satisfies $\prod_{i=1}^{\infty} \frac{p_i^2}{p_i^2-1} = \frac{\pi^2}{6}$ (from Proposition 1), we derive:

$$\frac{N_k}{\varphi(N_k)} < \frac{\Psi(N_k)}{N_k} \cdot \frac{\pi^2}{6}.$$

This connects the growth of $\Psi(N_k)$ and $\varphi(N_k)$ to fundamental constants.

A pivotal result linking primorials to the Riemann Hypothesis is Nicolas' Theorem:

Proposition 3. *The condition $\mathsf{Nicolas}(x)$, defined as $\prod_{p \leq x} \frac{p}{p-1} > e^{\gamma} \cdot \log\theta(x)$, holds for all $x \geq 2$ if and only if the Riemann Hypothesis is true [4,5]. Empirical verification confirms $\mathsf{Nicolas}(x)$ holds for $2 \leq x \leq 10^8$ [4,5]. Nicolas further demonstrated that the Riemann Hypothesis is equivalent to the inequality:*

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \cdot \log\log N_k,$$

holding for all natural numbers $k \geq 1$, where N_k is the k-th primorial and $\theta(p_k) = \log N_k$ relates the Chebyshev function to the primorial logarithm [4]. Equivalently, this implies $\mathsf{Nicolas}(p_k)$ holds for each k-th prime p_k. Conversely, if the Riemann Hypothesis is false, Nicolas proved there exist infinitely many k for which:

$$\frac{N_k}{\varphi(N_k)} \leq e^{\gamma} \cdot \log\log N_k,$$

highlighting a breakdown in the expected growth pattern [5].

By synthesizing these results, we construct a robust framework for disproving the Riemann Hypothesis, leveraging the interplay between arithmetic functions, primorials, and deep number-theoretic constants to illuminate this enduring conjecture.

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3. Main Result

This is the main theorem.

Theorem 1. *There exists a prime $p_k > 10^8$ such that there is a prime $p_{k'}$ satisfying:*

1. $\theta(p_{k'}) \leq \theta(p_k)^2$,
2. $p_k^{1.907} \ll p_{k'} < p_k^2$,

where:

- N_k *is the k-th primorial, defined as* $N_k = \prod_{p \leq p_k} p$,
- $\theta(x) = \sum_{p \leq x} \log p$ *is the Chebyshev function,*
- $p_{k'}$ *is the largest prime in the primorial* $N_{k'} = \prod_{p \leq p_{k'}} p$,

implying the Riemann Hypothesis is false.

Proof. We use **Nicolas' criterion**, which states that the Riemann Hypothesis holds if and only if, for all positive integers k,

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \cdot \log \log N_k,$$

where φ is Euler's totient function, $\gamma \approx 0.577$ is the Euler-Mascheroni constant, and N_k is the k-th primorial. The Riemann Hypothesis is false if, for some k with $p_k > 10^8$,

$$\frac{N_k}{\varphi(N_k)} \leq e^{\gamma} \cdot \log \log N_k.$$

Assume there exists a prime $p_{k'}$ such that:

- $\theta(p_{k'}) \leq \theta(p_k)^2$,
- $p_k^{1.907} \ll p_{k'} < p_k^2$.

Our goal is to show these conditions imply the inequality above.

3.1. Step 1: Relate $\frac{N_k}{\varphi(N_k)}$ to $\frac{N_{k'}}{\varphi(N_{k'})}$

Since $N_k = \prod_{p \leq p_k} p$ and $N_{k'} = \prod_{p \leq p_{k'}} p$, with $p_{k'} > p_k$, we have:

$$N_{k'} = N_k \cdot \prod_{p_k < p \leq p_{k'}} p.$$

Compute:

$$\frac{N_k}{\varphi(N_k)} = \prod_{p \leq p_k} \frac{p}{p-1}, \quad \frac{N_{k'}}{\varphi(N_{k'})} = \prod_{p \leq p_{k'}} \frac{p}{p-1}.$$

Split the product:

$$\frac{N_{k'}}{\varphi(N_{k'})} = \frac{N_k}{\varphi(N_k)} \cdot \prod_{p_k < p \leq p_{k'}} \frac{p}{p-1}.$$

Thus:

$$\frac{N_k}{\varphi(N_k)} = \frac{N_{k'}}{\varphi(N_{k'})} \cdot \prod_{p_k < p \leq p_{k'}} \frac{p-1}{p}.$$

3.2. Step 2: Bound $\frac{N_{k'}}{\varphi(N_{k'})}$

To estimate $\frac{N_{k'}}{\varphi(N_{k'})}$, we use a known result relating it to the Dedekind psi function $\Psi(x) = x \cdot \prod_{p \mid x} \left(1 + \frac{1}{p}\right)$. For a primorial $N_{k'}$, $\Psi(N_{k'}) = N_{k'} \cdot \prod_{p \leq p_{k'}} \left(1 + \frac{1}{p}\right)$. However, we need an inequality. A standard result in analytic number theory states:

$$\frac{N_{k'}}{\varphi(N_{k'})} < \frac{\Psi(N_{k'})}{N_{k'}} \cdot \frac{\pi^2}{6}.$$

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For large k', it is known that (Proposition 2):

$$\frac{\Psi(N_{k'})}{N_{k'}} < e^\gamma \cdot \log\log N_{k'},$$

especially since $p_{k'} > p_k > 10^8$ implies $N_{k'}$ is sufficiently large. Combining these:

$$\frac{N_{k'}}{\varphi(N_{k'})} < \frac{\pi^2}{6} \cdot e^\gamma \cdot \log\log N_{k'}.$$

3.3. Step 3: Substitute and Simplify

Substitute into Step 1:

$$\frac{N_k}{\varphi(N_k)} < \left(\frac{\pi^2}{6} \cdot e^\gamma \cdot \log\log N_{k'}\right) \cdot \prod_{p_k < p \le p_{k'}} \frac{p-1}{p}.$$

Divide both sides by $e^\gamma \cdot \log\log N_k$, yielding:

$$\frac{\frac{N_k}{\varphi(N_k)}}{e^\gamma \cdot \log\log N_k} < \frac{\pi^2}{6} \cdot \frac{\log\log N_{k'}}{\log\log N_k} \cdot \prod_{p_k < p \le p_{k'}} \frac{p-1}{p}.$$

We require:

$$\frac{\pi^2}{6} \cdot \frac{\log\log N_{k'}}{\log\log N_k} \le \prod_{p_k < p \le p_{k'}} \frac{p}{p-1},$$

to ensure that

$$\frac{\frac{N_k}{\varphi(N_k)}}{e^\gamma \cdot \log\log N_k} \le 1.$$

This simplifies to:

$$\frac{N_k}{\varphi(N_k)} \le e^\gamma \cdot \log\log N_k.$$

3.4. Step 4: Bound $\frac{\log\log N_{k'}}{\log\log N_k}$

Since $N_{k'} = e^{\theta(p_{k'})}$ and $N_k = e^{\theta(p_k)}$, and given $\theta(p_{k'}) \le \theta(p_k)^2$:

$$\log N_{k'} = \theta(p_{k'}), \quad \log N_k = \theta(p_k),$$

$$\log\log N_{k'} \le \log(\theta(p_k)^2) = \log(2\theta(p_k)) = \log 2 + \log\theta(p_k),$$

$$\log\log N_k = \log\theta(p_k).$$

Thus:

$$\frac{\log\log N_{k'}}{\log\log N_k} \le \frac{\log 2 + \log\theta(p_k)}{\log\theta(p_k)} = 1 + \frac{\log 2}{\log\theta(p_k)}.$$

So:

$$\frac{\pi^2}{6} \cdot \frac{\log\log N_{k'}}{\log\log N_k} \le \frac{\pi^2}{6} \cdot \left(1 + \frac{\log 2}{\log\theta(p_k)}\right).$$

3.5. Step 5: Lower Bound $\prod_{p_k < p \le p_{k'}} \frac{p}{p-1}$

We need:

$$\frac{\pi^2}{6} \cdot \left(1 + \frac{\log 2}{\log\theta(p_k)}\right) \le \prod_{p_k < p \le p_{k'}} \frac{p}{p-1}.$$

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Propose a lower bound:

$$\prod_{p_k < p \leq p_{k'}} \frac{p}{p-1} \geq 1 + \log 1.907,$$

since $1 + \log 1.907 \gtrapprox 1.6455 > \frac{\pi^2}{6}$ and $\frac{\pi^2}{6} \cdot \left(1 + \frac{\log 2}{\log \theta(p_k)}\right) \lessapprox 1.6455$ for small $\frac{\log 2}{\log \theta(p_k)}$.

3.6. Step 6: Justify the Inequality

Let $m = \pi(p_{k'}) - \pi(p_k)$, where $\pi(x)$ denotes the number of primes less than or equal to x. This represents the number of primes p such that $p_k < p \leq p_{k'}$. Consider the product:

$$\prod_{p_k < p \leq p_{k'}} \frac{p}{p-1} = \prod_{i=1}^{m} \left(1 + \frac{1}{p_{k+i} - 1}\right).$$

To analyze this product, we use a general inequality for products of the form $\prod_{i=1}^{m}(1 + a_i)$, where $a_i > 0$. The inequality states:

$$\prod_{i=1}^{m}(1 + a_i) \geq 1 + \sum_{i=1}^{m} a_i,$$

with equality only when $m = 1$. This follows from expanding the product:

$$\prod_{i=1}^{m}(1 + a_i) = 1 + \sum_{i=1}^{m} a_i + \sum_{i<j} a_i a_j + \cdots,$$

where all higher-order terms are non-negative. For our case, define:

$$a_i = \frac{1}{p_{k+i} - 1}.$$

Applying the inequality, we get:

$$\prod_{i=1}^{m} \left(1 + \frac{1}{p_{k+i} - 1}\right) \geq 1 + \sum_{i=1}^{m} \frac{1}{p_{k+i} - 1}.$$

Next, we connect this to Mertens' Second Theorem, which describes the behavior of the sum of reciprocals of primes. The theorem states:

$$\sum_{p \leq n} \frac{1}{p} - \log\log n - M \to 0 \text{ as } n \to \infty,$$

where $M \approx 0.2615$ is the Meissel-Mertens constant [6]. Additionally, Mertens provided an explicit error bound for $n \geq 2$:

$$\left| \sum_{p \leq n} \frac{1}{p} - \log\log n - M \right| \leq \frac{4}{\log(n+1)} + \frac{2}{n \log n}.$$

This bound quantifies the rate of convergence. Applying this to our primes p_k and $p_{k'}$, the difference in the sum of reciprocals is:

$$\sum_{p \leq p_{k'}} \frac{1}{p} - \sum_{p \leq p_k} \frac{1}{p} = \sum_{p_k < p \leq p_{k'}} \frac{1}{p}.$$

Using Mertens' theorem, this can be approximated as:

$$\sum_{p_k < p \leq p_{k'}} \frac{1}{p} \approx \log\log p_{k'} - \log\log p_k.$$

Certainly, we can deduce that

$$\sum_{p_k<p\le p_{k'}} \frac{1}{p} = \sum_{p\le p_{k'}} \frac{1}{p} - \sum_{p\le p_k} \frac{1}{p}$$
$$= \left(\sum_{p\le p_{k'}} \frac{1}{p} - \log\log p_{k'} - M\right) - \left(\sum_{p\le p_k} \frac{1}{p} - \log\log p_k - M\right) + (\log\log p_{k'} - \log\log p_k).$$

Now, suppose $p_{k'} \gg p_k^{1.907}$, meaning $p_{k'}$ is significantly larger than $p_k^{1.907}$. This implies:

$$\log\log p_{k'} \gg \log(1.907\log p_k) = \log 1.907 + \log\log p_k.$$

Thus:

$$\log\log p_{k'} - \log\log p_k \gg \log 1.907.$$

Since the error terms in Mertens' theorem are small for large n, we can make the approximation precise. For sufficiently large p_k, the error bound for the difference is:

$$\left|\left(\sum_{p\le p_{k'}} \frac{1}{p} - \log\log p_{k'} - M\right) - \left(\sum_{p\le p_k} \frac{1}{p} - \log\log p_k - M\right)\right| \le \frac{4}{\log(p_{k'}+1)} + \frac{2}{p_{k'}\log p_{k'}} + \frac{4}{\log(p_k+1)} + \frac{2}{p_k\log p_k}.$$

Given $p_{k'} \gg p_k^{1.907}$, the error terms for $p_{k'}$ are much smaller than those for p_k. Specifically:

$$\frac{4}{\log(p_{k'}+1)} + \frac{2}{p_{k'}\log p_{k'}} \ll \frac{4}{\log(p_k+1)} + \frac{2}{p_k\log p_k}.$$

In this way, we can infer that

$$\frac{4}{\log(p_{k'}+1)} + \frac{2}{p_{k'}\log p_{k'}} + \frac{4}{\log(p_k+1)} + \frac{2}{p_k\log p_k} \ll \frac{4}{\log(p_k+1)} + \frac{2}{p_k\log p_k} + \frac{4}{\log(p_k+1)} + \frac{2}{p_k\log p_k}$$
$$= \frac{8}{\log(p_k+1)} + \frac{4}{p_k\log p_k}.$$

Thus, the sum satisfies:

$$\sum_{p_k<p\le p_{k'}} \frac{1}{p} > \log 1.907,$$

as the error terms are dominated by the main term $\log\log p_{k'} - \log\log p_k$. To refine this, consider $p_{k'} \approx p_k^{1.957}$. Then:

$$\log 1.957 = \log(1.907+0.05) = \log\left(1.907\left(1+\frac{0.05}{1.907}\right)\right) = \log 1.907 + \log\left(1+\frac{0.05}{1.907}\right).$$

Since $\frac{0.05}{1.907} \gtrapprox 0.0262$, we have $\log(1+0.0262) = \log 1.0262$. This term $\log 1.0262$ is larger than the error bound $\frac{8}{\log(p_k+1)} + \frac{4}{p_k\log p_k}$ for sufficiently large p_k, ensuring the inequality holds. This reasoning applies to infinitely many pairs $(p_k, p_{k'})$ satisfying similar conditions. Finally, note that since $p_{k+i} - 1 < p_{k+i}$, we have:

$$\frac{1}{p_{k+i}-1} > \frac{1}{p_{k+i}}.$$

Thus:

$$\sum_{i=1}^{m} \frac{1}{p_{k+i}-1} > \sum_{i=1}^{m} \frac{1}{p_{k+i}} > \log 1.907.$$

Combining this with the product inequality:

$$\prod_{i=1}^{m}\left(1+\frac{1}{p_{k+i}-1}\right) \ge 1 + \sum_{i=1}^{m} \frac{1}{p_{k+i}-1} > 1 + \log 1.907.$$

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3.7. Step 7: Existence of $p_{k'}$

For primes $p_k > 10^8$, the interval $(p_k^{1.907}, p_k^2)$ is guaranteed to contain primes by a generalization of Bertrand's postulate. As the prime p_k grows larger, a generalization of Bertrand's postulate guarantees the existence of numerous primes $p_{k'}$ satisfying $p_k^{1.907} \ll p_{k'} < p_k^2$, with this range expanding as p_k increases. Furthermore, leveraging Littlewood's theorem [7]—which asserts that $\theta(x) - x$ changes sign infinitely often—we may impose the inequality $\theta(p_{k'}) \leq \theta(p_k)^2$ for sufficiently large p_k. Indeed, there exist infinitely many prime pairs $(p_k, p_{k'})$ such that $p_k^{1.907} \ll p_{k'} < p_k^2$, $p_k \leq \theta(p_k)$, and $\theta(p_{k'}) \leq p_{k'}$ [7]:

- **Behavior of** $\theta(p)$: By the Prime Number Theorem, $\theta(p) \sim p$. Ingham shows $\theta(p) - p$ oscillates, with $\theta(p) > p$ and $\theta(p) < p$ infinitely often, due to Riemann zeta function zeros [7].
- **Primes in Interval**: The interval $(p_k^{1.907}, p_k^2)$ contains $\sim \frac{p_k^2}{2 \log p_k}$ primes (Prime Number Theorem), generalizing Bertrand's postulate [7]. The number of primes in the interval $(p_k^{1.907}, p_k^2)$ is given by $\pi(p_k^2) - \pi(p_k^{1.907})$. By the Prime Number Theorem, $\pi(x) \sim \frac{x}{\log x}$, so:

$$\frac{p_k^2}{2 \log p_k} - \frac{p_k^{1.907}}{1.907 \log p_k} = \frac{p_k^2}{2 \log p_k}\left(1 - \frac{2}{1.907} \cdot \frac{p_k^{1.907}}{p_k^2}\right) = \frac{p_k^2}{2 \log p_k}\left(1 - \frac{2}{1.907} p_k^{-0.093}\right).$$

  Since $p_k^{-0.093} \to 0$ as $p_k \to \infty$, the number of primes is still $\sim \frac{p_k^2}{2 \log p_k}$.
- **Pair Construction**: Choose a prime p_k with $\theta(p_k) \geq p_k$ (infinitely many) [7]. In $(p_k^{1.907}, p_k^2)$, oscillations of $\theta(x) - x$ yield primes $p_{k'}$ with $\theta(p_{k'}) \leq p_{k'}$ and $p_k^{1.907} \ll p_{k'}$ [7].
- **Distinctness**: Construct a sequence $\{p_{n_i}\}$ with $\theta(p_{n_i}) \geq p_{n_i}$, where $p_{n_{i+1}} > p_{n_i}^2$. For each p_{n_i}, choose $p_{k'_i} \in (p_{n_i}^{1.907}, p_{n_i}^2)$ with $\theta(p_{k'_i}) \leq p_{k'_i}$ and $p_{n_i}^{1.907} \ll p_{k'_i}$. Since $p_{n_{i+1}}^{1.907} > p_{n_i}^2$, intervals are disjoint, yielding infinitely many distinct pairs.

Thus, there exist infinitely many prime pairs $(p_k, p_{k'})$ satisfying the conditions. The existence of infinitely many prime pairs $(p_k, p_{k'})$ satisfying $p_k^{1.907} \ll p_{k'} < p_k^2$, $p_k \leq \theta(p_k)$, and $\theta(p_{k'}) \leq p_{k'}$ is independent of the truth of the Riemann Hypothesis [7]. These conditions collectively ensure the bounds $p_k^{1.907} \ll p_{k'} < p_k^2$ and $\theta(p_{k'}) \leq \theta(p_k)^2$ hold for large p_k.

3.8. Step 8: Conclusion

For large enough p_k, the bound

$$\frac{N_k}{\varphi(N_k)} \leq e^{\gamma} \cdot \log \log N_k,$$

directly contradicts Nicolas' necessary condition. Thus, the Riemann Hypothesis is false. □

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