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Article

The Generalized Mehler-Fock Transform over Lebesgue Spaces

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Abstract: This paper focuses on establishing boundedness properties and Parseval-Goldstein type relations for the generalized Mehler-Fock transform initially introduced by B. L. J. Braaksma and B. M. Meulenbeld (Compositio Math., 18(3):235-287, 1967). Also, we derive an inversion formula for this transform over Lebesgue spaces.

Keywords: generalized Mehler-Fock transform; Lebesgue spaces; inversion formula; boundedness properties; Parseval-Goldstein type relations

MSC: 44A15, 46E30, 47G1

1. Introduction and Preliminaries

The concept of the Mehler-Fock transform has its roots in the groundbreaking contributions of F. G. Mehler [5] and V. A. Fock [2]. Evolving from their seminal work, it emerged as a distinct integral transform, finding applications across a broad spectrum of mathematical physics problems. Extensive research by various scholars has led into its properties and applications. For an in-depth exploration, interested readers can consult the following references, among others [3,4,7–11,13].

We consider the generalized Mehler-Fock transform of a suitable complex-valued function f on $\mathbb{R}_+$ analyzed by B. L. J. Braaksma and B. M. Meulenbeld [11] and R. S. Pathak [13, chapter 11] given by

$$(\mathcal{B}_{m,n}f)(\tau) = \int_0^\infty f(x) P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x \, dx, \quad \tau > 0, \quad (1.1)$$

where m, n are complex numbers with $\Re m < \frac{1}{2}$ and $P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x)$ is the associated Legendre function of the first kind [1, Chapter 3] defined by

$$\begin{aligned} P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) &= \frac{(1 + \cosh x)^{\frac{n}{2}}}{\Gamma(1-m)(\cosh x - 1)^{\frac{m}{2}}} \\ &\times {}_2F_1\left(\frac{1}{2} + i\tau + \frac{n-m}{2}, \frac{1}{2} - i\tau + \frac{n-m}{2}; 1-m; \frac{1 - \cosh x}{2}\right), \end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function [1, p. 57].

A corresponding inversion formula of (1.1) for suitable f is given by

$$f(x) = \int_0^\infty \chi(\tau) P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) (\mathcal{B}_{m,n}f)(\tau) d\tau, \quad x > 0, \quad (1.2)$$

where

$$\begin{aligned}\chi(\tau) &= \Gamma\left(\frac{1-m+n}{2} + i\tau\right) \Gamma\left(\frac{1-m+n}{2} - i\tau\right) \Gamma\left(\frac{1-m-n}{2} + i\tau\right) \\ &\times \Gamma\left(\frac{1-m-n}{2} - i\tau\right) \left[\Gamma(2i\tau)\Gamma(-2i\tau)\pi 2^{n-m+2}\right]^{-1}.\end{aligned}\quad (1.3)$$

The conditions of validity of (1.1) and (1.2) are provided by Braaksma and Meulenbeld [11].

Notice that for $m = n = 0$ one obtains the Mehler-Fock transform of zero order [12, §7-6, p.390] and [11, Theorem 7, p.247]

$$F(\tau) = \int_0^\infty f(x) P_{-\frac{1}{2}+i\tau}(\cosh x) \sinh x \, dx, \quad \tau > 0. \quad (1.4)$$

and a corresponding inversion formula given by

$$f(x) = \int_0^\infty \tau \tanh(\pi\tau) P_{-\frac{1}{2}+i\tau}(\cosh x) F(\tau) d\tau, \quad x > 0. \quad (1.5)$$

The case $m = n$, $\Re m < \frac{1}{2}$ is contemplated in [11, Theorem 7, p.247]

$$F_m(\tau) = \int_0^\infty f(x) P_{-\frac{1}{2}+i\tau}^m(\cosh x) \sinh x \, dx, \quad \tau > 0. \quad (1.6)$$

and a corresponding inversion formula given by

$$\begin{aligned}f(x) &= \frac{1}{\pi} \int_0^\infty \tau \sinh(\pi\tau) \Gamma\left(\frac{1}{2} - m + i\tau\right) \Gamma\left(\frac{1}{2} - m - i\tau\right) \\ &\times P_{-\frac{1}{2}+i\tau}^m(\cosh x) F_m(\tau) d\tau, \quad x > 0.\end{aligned}\quad (1.7)$$

From [13, formula (11.1.8), p.345]

$$P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) = O(x^{-\Re m}) \text{ as } x \rightarrow 0^+. \quad (1.8)$$

From [13, formula (11.1.9), p.345]

$$P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) = O(e^{-\frac{x}{2}}) \text{ as } x \rightarrow +\infty. \quad (1.9)$$

Thus one has

$$P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x = O(x^{1-\Re m}) \text{ as } x \rightarrow 0^+, \quad (1.10)$$

$$P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x = O(e^{\frac{x}{2}}) \text{ as } x \rightarrow +\infty. \quad (1.11)$$

From [13, formula (11.3.2), p.347]

$$\left| P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \right| \leq C \sqrt{\frac{\pi}{2}} \Gamma\left(\frac{1}{2} - \Re m\right) P_{-\frac{1}{2}}^{\Re m,0}(\cosh x), \quad (1.12)$$

where C is a constant independent of x and τ , $\Re m < \frac{1}{2}$, $x > 0$, $\tau > 0$.

Now, observe that from (1.11) the convergence of the integral

$$\int_0^\infty P_{-\frac{1}{2}}^{\Re m,0}(\cosh x) \sinh x \, dx$$

is not assured in ∞ .

For $\gamma \in \mathbb{R}$ we consider the vector space $\mathcal{E}_\gamma$ consisting of all complex-valued measurable functions f on $\mathbb{R}_+$ such that $e^{\gamma x} f(x) \in L^\infty(\mathbb{R}_+)$. In their recent study [6], Maan and Negrín made extensive use of spaces of type $\mathcal{E}_\gamma$.

A norm $\|\cdot\|_\gamma$ on $\mathcal{E}_\gamma$ is given by

$$\|f\|_\gamma = \|e^{\gamma x} f(x)\|_{L^\infty(\mathbb{R}_+)}.$$

With this norm, the map

$$T_\gamma : \mathcal{E}_\gamma \rightarrow L^\infty(\mathbb{R}_+)$$

where for any $f \in \mathcal{E}_\gamma$:

$$(T_\gamma f)(x) = e^{\gamma x} f(x), \quad x \in \mathbb{R}_+,$$

is an isometric isomorphism from $\mathcal{E}_\gamma$ to $L^\infty(\mathbb{R}_+)$. Thus, since $L^\infty(\mathbb{R}_+)$ is complete, then the space $\mathcal{E}_\gamma$ becomes a Banach space.

The $C_c^k(\mathbb{R}_+)$, $k \in \mathbb{N}$, denotes as it is usual the space of compactly supported functions on $\mathbb{R}_+$ which are k -times differentiable with continuity.

The content of this paper is as follows: Section 1 is concerned with the definitions and useful results which are used in the entire sequel. Section 2 provides an inversion formula for the generalized Mehler-Fock transform given by (1.1). Section 3 deals with the continuity features over Lebesgue spaces and Parseval-Goldstein type relations for the generalized Mehler-Fock transform given by (1.1). Section 4 gives concluding remarks.

2. An Inversion Formula over the Spaces $\mathcal{E}_\gamma$

By means of [11, Theorem 5, p.245] one obtains the next inversion formula for the transform (1.1) over the spaces $\mathcal{E}_\gamma$, $\gamma > \frac{1}{2}$.

Theorem 1 (Inversion formula). *Assume $f \in \mathcal{E}_\gamma$, $\gamma > \frac{1}{2}$, and let m, n be complex numbers with $|\Re n| < 1 - \Re m$ and $-\frac{3}{2} < \Re m < \frac{1}{2}$. Suppose that the function f is continuous at the point x and of bounded variation in the neighbourhood of the point x . Then*

$$\int_0^\infty \chi(\tau) P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) (\mathcal{B}_{m,n} f)(\tau) d\tau = f(x),$$

where $\chi(\tau)$ is given by (1.3).

Proof. Observe that being $f \in \mathcal{E}_\gamma$, $\gamma > \frac{1}{2}$, the function $g(t) = f(\log(t + \sqrt{t^2 - 1}))$, $t > 1$, satisfies the condition (0.6) of p. 236 of [11].

Also the function g is continuous at the point $\cosh x$ and of bounded variation in the neighbourhood of the point $\cosh x$.

Thus from Theorem 5 of [11] one has

$$\int_0^\infty \chi(\tau) P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \int_1^\infty g(t) P_{-\frac{1}{2}+i\tau}^{m,n}(t) dt d\tau = g(\cosh x). \quad (2.1)$$

Now, taking in (2.1) $t = \cosh u$, $u > 0$, and taking into account that $g(\cosh u) = f(u)$, one obtains

$$\int_0^\infty \chi(\tau) P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \int_0^\infty f(u) P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh u) \sinh u du d\tau = f(x).$$

□

As a consequence of Theorem 1 one obtains the next result.

Corollary 2 (Injectivity). Assume that the functions $f, g \in \mathcal{E}_\gamma$, $\gamma > \frac{1}{2}$, and let m, n be complex numbers with $|\Re n| < 1 - \Re m$ and $-\frac{3}{2} < \Re m < \frac{1}{2}$. Suppose that the function f and g are continuous on $\mathbb{R}_+$ and of bounded variation in the neighbourhood of each point of $\mathbb{R}_+$. Then, if $\mathcal{B}_{m,n}f = \mathcal{B}_{m,n}g$ on $\mathbb{R}_+$ it follows that $f = g$ on $\mathbb{R}_+$.

3. Boundedness Properties and Parseval-Goldstein Type Relations Over the Spaces $\mathcal{E}_\gamma$

Observe that for m, n being complex numbers with $\Re m < \frac{1}{2}$, $f \in \mathcal{E}_\gamma$, $\gamma > \frac{1}{2}$, $\tau > 0$, one obtains from (1.12)

$$\begin{aligned} & |(\mathcal{B}_{m,n}f)(\tau)| \\ & \leq \|f\|_\gamma \int_0^\infty e^{-\gamma x} f(x) |P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x)| \sinh x \, dx \\ & \leq \|f\|_\gamma C \sqrt{\frac{\pi}{2}} \Gamma\left(\frac{1}{2} - \Re m\right) \int_0^\infty e^{-\gamma x} P_{-\frac{1}{2}}^{\Re m,0}(\cosh x) \sinh x \, dx. \end{aligned} \quad (3.1)$$

Now, from (1.10) and (1.11) the integral

$$\int_0^\infty e^{-\gamma x} P_{-\frac{1}{2}}^{\Re m,0}(\cosh x) \sinh x \, dx$$

converges for $\gamma > \frac{1}{2}$.

Thus the next result holds.

Proposition 3. Set m, n be complex numbers with $\Re m < \frac{1}{2}$. The generalized Mehler-Fock transform $\mathcal{B}_{m,n}$ given by (1.1) is a bounded linear operator from $\mathcal{E}_\gamma$, $\gamma > \frac{1}{2}$, into $L^\infty(\mathbb{R}_+)$. If $f \in \mathcal{E}_\gamma$, $\gamma > \frac{1}{2}$, then

$$\|\mathcal{B}_{m,n}f\|_{L^\infty(\mathbb{R}_+)} \leq M \|f\|_\gamma, \text{ for some } M > 0,$$

and $\mathcal{B}_{m,n}f$ is a continuous function on $\mathbb{R}_+$. Moreover, the generalized Mehler-Fock transform $\mathcal{B}_{m,n}$ is a continuous map from $\mathcal{E}_\gamma$, $\gamma > \frac{1}{2}$, to the Banach space of bounded continuous functions on $\mathbb{R}_+$.

Proof. Let $\tau_0 > 0$ be arbitrary. Since the map

$$\tau \rightarrow P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x$$

is continuous for each fixed $x > 0$, we have

$$P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x \rightarrow P_{-\frac{1}{2}+i\tau_0}^{m,n}(\cosh x) \sinh x \text{ as } \tau \rightarrow \tau_0.$$

Further, we have that

$$\left| P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x - P_{-\frac{1}{2}+i\tau_0}^{m,n}(\cosh x) \sinh x \right| |f(x)|$$

is dominated by the integrable function

$$2C \sqrt{\frac{\pi}{2}} \Gamma\left(\frac{1}{2} - \Re m\right) P_{-\frac{1}{2}}^{\Re m,0}(\cosh x) \sinh x |f(x)|.$$

Therefore, by using dominated convergence theorem, we get

$$\begin{aligned} & |(\mathcal{B}_{m,n}f)(\tau) - (\mathcal{B}_{m,n}f)(\tau_0)| \\ & \leq \int_0^\infty \left| P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x - P_{-\frac{1}{2}+i\tau_0}^{m,n}(\cosh x) \sinh x \right| |f(x)| dx, \end{aligned}$$

which tends to 0 as $\tau \rightarrow \tau_0$.

Thus, $\mathcal{B}_{m,n}f$ is a continuous function on $\mathbb{R}_+$.

Since for each $\tau > 0$ and from (3.1) one has

$$\begin{aligned} & |(\mathcal{B}_{m,n}f)(\tau)| \\ & \leq \|f\|_\gamma C \sqrt{\frac{\pi}{2}} \Gamma\left(\frac{1}{2} - \Re m\right) \int_0^\infty e^{-\gamma x} P_{-\frac{1}{2}}^{\Re m, 0}(\cosh x) \sinh x \, dx \\ & = M \|f\|_\gamma, \text{ for some } M > 0, \end{aligned} \tag{3.2}$$

then $\mathcal{B}_{m,n}f$ is a bounded function.

The linearity of the integral operator implies that the $\mathcal{B}_{m,n}$ transform is linear. Also from (3.2) we get that

$$\|\mathcal{B}_{m,n}f\|_{L^\infty(\mathbb{R}_+)} \leq M \|f\|_\gamma$$

and hence

$$\mathcal{B}_{m,n} : \mathcal{E}_\gamma \rightarrow L^\infty(\mathbb{R}_+), \quad \gamma > \frac{1}{2},$$

is a continuous linear map. $\square$

Also, the next result holds.

Proposition 4. Set m, n be complex numbers with $\Re m < \frac{1}{2}$, $\gamma > \frac{1}{2}$, w be a measurable function on $\mathbb{R}_+$ such that $w > 0$ a.e. on $\mathbb{R}_+$ and $\int_0^\infty w(x) dx < \infty$. Then, for $0 < q < \infty$,

$$\mathcal{B}_{m,n} : \mathcal{E}_\gamma \rightarrow L^q(\mathbb{R}_+, w(x) dx)$$

is a bounded linear operator.

Example 1. Examples of weights w for Proposition 4 are:

- (i) $w(x) = (1+x)^r$, for $r < -1$.
- (ii) $w(x) = e^{rx}$, for $r < 0$.

For a suitable function g we denote

$$(\mathcal{B}_{m,n}^* g)(x) = \sinh x \int_0^\infty g(\tau) P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) d\tau, \quad x > 0, \tag{3.3}$$

where m and n are complex numbers with $\Re m < \frac{1}{2}$.

The $\mathcal{B}_{m,n}^* g$ exists for $g \in L^1(\mathbb{R}_+)$.

In fact, from (1.12)

$$\begin{aligned} & |(\mathcal{B}_{m,n}^* g)(x)| \\ & \leq \sinh x \int_0^\infty |g(\tau)| |P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x)| d\tau \\ & \leq \sinh x C \sqrt{\frac{\pi}{2}} \Gamma\left(\frac{1}{2} - \Re m\right) P_{-\frac{1}{2}}^{\Re m,0}(\cosh x) \int_0^\infty |g(\tau)| d\tau < \infty, \end{aligned} \quad (3.4)$$

for each $x \in \mathbb{R}_+$.

Now, observe that for $\gamma \leq -\frac{1}{2}$ and having into account (1.10) and (1.11) the expression

$$e^{\gamma x} P_{-\frac{1}{2}}^{\Re m,0}(\cosh x) \sinh x \leq A, \text{ for all } x > 0 \text{ and some } A > 0.$$

Thus, for $g \in L^1(\mathbb{R}_+)$ one has

$$\|\mathcal{B}_{m,n}^* g\|_\gamma \leq A \cdot \|g\|_{L^1(\mathbb{R}_+)}, \quad \gamma \leq -\frac{1}{2}.$$

So one obtains the next result.

Proposition 5. Set m, n be complex numbers with $\Re m < \frac{1}{2}$. The linear operator $\mathcal{B}_{m,n}^*$ given by (3.3) is a bounded linear operator from $L^1(\mathbb{R}_+)$ into $\mathcal{E}_\gamma$, $\gamma \leq -\frac{1}{2}$.

The next result holds.

Proposition 6. Set m, n be complex numbers with $\Re m < \frac{1}{2}$ and w be a measurable function on $\mathbb{R}_+$ such that $w > 0$ a.e. on $\mathbb{R}_+$ and $0 < q < \infty$. If $\int_0^\infty \left(P_{-\frac{1}{2}}^{\Re m,0}(\cosh x) \sinh x \right)^q w(x) dx < \infty$. Then

$$\mathcal{B}_{m,n}^* : L^1(\mathbb{R}_+) \rightarrow L^q(\mathbb{R}_+, w(x) dx)$$

is a bounded linear operator.

Proof. From (3.4) one has

$$\begin{aligned} & \left(\int_0^\infty |(\mathcal{B}_{m,n}^* g)(x)|^q w(x) dx \right)^{\frac{1}{q}} \\ & \leq M \cdot \|g\|_{L^1(\mathbb{R}_+)} \cdot \left(\int_0^\infty \left(P_{-\frac{1}{2}}^{\Re m,0}(\cosh x) \sinh x \right)^q w(x) dx \right)^{\frac{1}{q}}, \end{aligned}$$

for some $M > 0$, and so the result holds. $\square$

Example 2. Examples of weights w for Proposition 6 are:

- (i) $w(x) = e^{rx}$, for $r < -\frac{q}{2}$.
- (ii) $w(x) = e^{rx^2}$, for $r < 0$.

The next Parseval-Goldstein type relation holds.

Theorem 7. Set m, n be complex numbers with $\Re m < \frac{1}{2}$, $f \in \mathcal{E}_\gamma$, $\gamma > \frac{1}{2}$ and $g \in L^1(\mathbb{R}_+)$, then the following Parseval-Goldstein type relation holds

$$\int_0^\infty (\mathcal{B}_{m,n}f)(x)g(x)dx = \int_0^\infty f(x)(\mathcal{B}_{m,n}^*g)(x)dx. \quad (3.5)$$

Proof. Applying Fubini's theorem in the following we get

$$\begin{aligned} \int_0^\infty (\mathcal{B}_{m,n}f)(\tau)g(\tau)d\tau &= \int_0^\infty \int_0^\infty f(x)P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x \, dx \, g(\tau)d\tau \\ &= \int_0^\infty \sinh x \int_0^\infty g(\tau)P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x)d\tau f(x)dx \\ &= \int_0^\infty f(x)(\mathcal{B}_{m,n}^*g)(x)dx. \end{aligned}$$

Observe that from (3.4) the $\mathcal{B}_{m,n}^*g$ exists for $g \in L^1(\mathbb{R}_+)$. $\square$

Consider the differential operator

$$L_{m,n,x} = D_x \sinh x D_x (\sinh x)^{-1} + \left[\frac{m^2}{2(1 - \cosh x)} + \frac{n^2}{2(1 + \cosh x)} \right].$$

From [13, formula (11.2.4), p.346]

$$L_{m,n,x} \left(P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x \right) = - \left(\tau^2 + \frac{1}{4} \right) P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x, \quad (3.6)$$

and then for $k \in \mathbb{N}$

$$L_{m,n,x}^k \left(P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x \right) = (-1)^k \left(\tau^2 + \frac{1}{4} \right)^k P_{-\frac{1}{2}+i\tau}^{m,n}(\cosh x) \sinh x. \quad (3.7)$$

Denote

$$L_{m,n,x}^* = (\sinh x)^{-1} D_x \sinh x D_x + \left[\frac{m^2}{2(1 - \cosh x)} + \frac{n^2}{2(1 + \cosh x)} \right].$$

From (3.7) and $f \in C_c^{2k}(\mathbb{R}_+)$, $k \in \mathbb{N}$, it follows

$$\left(\mathcal{B}_{m,n} \left(L_{m,n,x}^{*k} f \right) \right) (\tau) = (-1)^k \left(\tau^2 + \frac{1}{4} \right)^k (\mathcal{B}_{m,n}f)(\tau), \quad \tau > 0. \quad (3.8)$$

Then from Theorem 7 one has the next result.

Theorem 8. If $f \in C_c^{2k}(\mathbb{R}_+)$, $k \in \mathbb{N}$, m, n be complex numbers with $\Re m < \frac{1}{2}$, and $g \in L^1(\mathbb{R}_+)$, then the following Parseval-Goldstein type relation holds

$$(-1)^k \int_0^\infty (\mathcal{B}_{m,n}f)(x)g(x) \left(x^2 + \frac{1}{4} \right)^k dx = \int_0^\infty (L_{m,n,x}^{*k}f)(x)(\mathcal{B}_{m,n}^*g)(x)dx. \quad (3.9)$$

4. Conclusions

In conclusion, this paper has been dedicated to the establishment of boundedness properties and Parseval-Goldstein type relations for the generalized Mehler-Fock transform, as first introduced by B. L. J. Braaksma and B. Meulenbeld in 1967 and subsequently by R. S. Pathak in 1997. Moreover, we

have derived an inversion formula for this transform across Lebesgue spaces. These results not only contribute to a deeper understanding of the generalized Mehler-Fock transform but also pave the way for future investigations into analogous properties within the spaces of type $\mathcal{E}_\gamma$ for diverse integral transforms.

Declarations

Note: The manuscript has no associated data.

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