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Article

Algorithms for Solving Resolvent of the Sum of Two Maximal Monotone Operators with Finite Family of Nonexpansive Operators

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Abstract

In this research paper, we address a variational problem using maximal monotone operators in conjunction with a finite family of nonexpansive operators. We propose a single-valued mapping whose fixed point allows us to find the solution to the main problem. Subsequently, we propose two algorithms. In the first, we introduce a system of sequences whose limit helps in solving the problem. In the second, we apply the Ishikawa algorithm to our setting using fixed point theory, which enables us to achieve strong convergence. Finally, we provide an illustrative example to demonstrate the applicability of our results.

Keywords: maximal monotone operator; α-strongly monotone; yoshida approximation; fixed point

MSC: 47H05; 47H10; 47J25

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1. Introduction

Convex analysis, monotone operator theory, and the theory of nonexpansive mappings are foundational pillars of nonlinear analysis, intricately linked through their shared mathematical structures. Notably, a wide range of minimization problems encountered in practice can be elegantly reformulated as monotone inclusion problems, highlighting the unifying role these theories play in addressing complex variational challenges. (see [2,4,9,13]).

To begin, let's explore one of the most well-known problems involving maximal monotone operators, which can be stated as follows:

$$\text{find} \quad x \in \mathbb{H} \quad \text{such that} \quad 0 \in A(x),$$

where $\mathbb{H}$ is real Hilbert space with an inner product $\langle .;. \rangle$ and induced norm $\|.\|$, A is maximal monotone operator. there are large number of authors have been interested in this problem one of them Rockafellar in 1976 (see[15]), and they devloped deferent algorithms to find the set of solution of the recent problem, the latter can be written as $S = \{x \in \mathbb{H} | J_\lambda^A(x) = x\}$, where J_λ^A the resolvent of A and one of the most famous of these algorithms is proposed by Mann (see[12]) which defined as follows:

$$\begin{cases} x_0 \in \mathbb{H}, \\ x_{k+1} = (1 - a_k)x_k + a_k J_\lambda^A(x_k), \quad k \geq 0, \end{cases}$$

under suitable conditions on a_k it is proved that the iterative sequence converges strongly to fixed point of J_λ^A, which is equevalent to a solution of the original problem. In this research, we focus on characterizing the set of solutions to the following problem:

Find x in $\mathbb{H}$ such that

$$0 \in A(x) + B(x) + \sum_{i=1}^{n} C_i(x), \tag{1}$$

where $A : \mathbb{H} \rightrightarrows \mathbb{H}$ is α-inverse strongly monotone operator, $B : \mathbb{H} \rightrightarrows \mathbb{H}$ is a β-strongly monotone operator, and $C_{1\leq i\leq n} : \mathbb{H} \to \mathbb{H}$ are a nonexpansive operators.
If $C_i = 0, \quad 1 \leq i \leq n$, and A, B are two maximal monotone operators defined on a real Hilbert space, this case has been studied by many authors (see [2,10,16]) using the Douglas-Rachford iterative algorithm (DRIA) which is defined by the iterative sequence $\{Z_n\}$ as follows:

$$Z_{n+1} = J_\lambda^B(2J_\lambda^A - I)(Z_n) + (I - J_\lambda^B)(Z_n).$$

A.Beddani also proposed a new algorithm to find the set $(A+B)^{-1}(0)$ (see [3,5]). The algorithm defined by the following function:

$$f_\lambda : \mathbb{R}^2 \to \mathbb{R}^2,$$
$$f_\lambda(x,y) = \begin{pmatrix} J_\lambda^A(x) - \frac{x+y}{2} \\ J_\lambda^B(y) - \frac{x+y}{2} \end{pmatrix}, \tag{2}$$

if there exists a pair $(x,y) \in \mathbb{R}^2$ such that $\|f_\lambda(x,y)\| = 0$, then $0 \in A(J_\lambda^A x) + B(J_\lambda^A x)$.

In our work, we focus on one of the most well-known approaches: the Douglas-Rachford Splitting Algorithm (DRSA) in the case of two maximal monotone operators (see [7]). In the subsequent sections, we propose an operator defined as :

$$\Psi_\lambda(x) = J_\lambda^B(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A x) - x + 2J_\lambda^A x) + \lambda A_\lambda(x),$$

we propose several algorithms, one of which is the Ishikawa iterative sequence (see [11]), which guarantees strong convergence under apropriat conditions.
In order to build these algorithms, we need some a preliminary concepts from convex analysis and monotone operator theory.

2. Preliminaries

2.1. Operators and Monotonicity

Let $\mathbb{H}$ be a real Hilbert space, and let $A : \mathbb{H} \rightrightarrows \mathbb{H}$ be a set-valued operator. We denote the domain of A by $\mathrm{dom}(A)$, i.e.,

$$\mathrm{dom}(A) = \{x \in \mathbb{H} : A(x) \neq \emptyset\}.$$

We say that A has full domain if $\mathrm{dom}(A) = \mathbb{H}$. The range of A is defined as

$$\mathrm{Im}(A) = \{y \in \mathbb{H} : \exists x \in \mathbb{H},\ y \in A(x)\}.$$

The graph of A is given by:

$$\mathrm{gph}(A) := \{(x,y) \in \mathbb{H} \times \mathbb{H} : x \in \mathrm{dom}(A),\ y \in A(x)\}.$$

Let $\{A_i\}_{i=1}^n$ be a finite family of operators. Then the sum operator is defined as:

$$\left(\sum_{i=1}^{n} A_i\right)(x) := \left\{\sum_{i=1}^{n} y_i : y_i \in A_i(x),\ i = 1,\dots,n\right\}.$$

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Definition 1 ([6]). *The operator A is said to be monotone if:*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \quad \text{for all} \quad (x_i, y_i) \in \operatorname{gph}(A),\ i = 1, 2.$$

Definition 2 ([6]). *The operator A is said to be α-*strongly monotone *with $\alpha > 0$ if:*

$$\langle y_1 - y_2,\ x_1 - x_2 \rangle \geq \alpha \|x_1 - x_2\|^2 \quad \text{for all } (x_i, y_i) \in \operatorname{gph}(A),\ i = 1, 2.$$

For any operator A and $\lambda > 0$, the *resolvent* of A is defined as:

$$J_\lambda^A := (I + \lambda A)^{-1}.$$

An operator A is said to be *nonexpansive* if:

$$\|y_1 - y_2\| \leq \|x_1 - x_2\| \quad \text{for all } (x_i, y_i) \in \operatorname{gph}(A),\ i = 1, 2.$$

Proposition 1. *For all $\alpha > 0$ and $\lambda > 0$:*

1) *If A is α-strongly monotone, then J_λ^A is $\frac{1}{\alpha\lambda+1}$-Lipschitz continuous.*
2) *If A is α-inverse strongly monotone, then the Yosida approximation A_λ is $\frac{1}{\alpha+\lambda}$-Lipschitz continuous.*

Proposition 2 ([1]). $A_\lambda(y) \in A\big(J_\lambda^A(y)\big), \quad \forall y \in \mathbb{H}.$

Proposition 3 ([1]). *For any $\lambda > 0$, $\delta > 0$, we have*

$$J_\lambda^A(x) = J_\delta^A\left(\frac{\delta}{\lambda}x + \left(1 - \frac{\delta}{\lambda}\right) J_\lambda^A(x)\right).$$

2.2. Maximal Monotone Operators and Convex Functions

An operator A is said to be maximal monotone if it satisfies the following:

1. A is monotone.
2. If B is another monotone operator such that the graph of A (i.e., the set of all pairs $(x, A(x))$) is contained in the graph of B, then $B = A$.

Let $X, Y \subseteq \mathbb{H}$ be convex subsets of a Hilbert space $\mathbb{H}$, and let $f : X \to \mathbb{R}$ be a function.

Definition 3 ([7]). *A function f is said to be convex if*

$$\forall x, y \in X \quad \text{and for all} \quad \lambda \in [0, 1], \quad f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y).$$

If the inequality is strict for all $x \neq y$, then f is called strictly convex. Moreover, f is said to be α-strongly convex if:

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y) - \frac{\alpha\lambda(1-\lambda)}{2}\|x - y\|^2.$$

Definition 4 ([7]). *Let $f : \mathbb{H} \to \mathbb{R}$ be a convex function. The set*

$$\partial f(x) := \{g \in \mathbb{H} : \langle g, y - x \rangle \leq f(y) - f(x) \quad \text{for all} \quad y \in \mathbb{H}\}$$

is called the subdifferential of f at x. The function f is said to be subdifferentiable at x if $\partial f(x) \neq \emptyset$. An element of the subdifferential is called a subgradient.

A famous example of a maximal monotone operator is the subgradient of the function $f(x) = |x|$ on $\mathbb{R}$:

$$\partial f(x) = \begin{cases} -1, & x < 0, \\ [-1, 1], & x = 0, \\ 1, & x > 0. \end{cases}$$

We observe that no operator can contain ∂f on $\mathbb{R}$, hence it is a maximal monotone operator.

Lemma 1 ([8]). *Given any maximal monotone operator A, a real number $\lambda > 0$, and $x \in \mathbb{H}$, we have $0 \in A(x)$ if and only if $J_\lambda^A(x) = x$.*

Lemma 2 ([11]). *Let a real sequence $\{x_k\}_{k=1}^{\infty}$ satisfy the following condition:*

$$x_{k+1} \leq \sigma x_k + \rho_k$$

where $x_k \geq 0$, $\rho_k \geq 0$, and $\lim_{k\to\infty} \rho_k = 0$, $0 \leq \sigma < 1$. Then, $\lim_{k\to\infty} x_k = 0$.

3. Main Result

In this section, we present our main results related to the problem under consideration. Our objective is to address and solve various cases of the following monotone inclusion problem.

$$0 \in A(x) + B(x) + \sum_{i=1}^{n} C_i(x), \tag{3}$$

where $A : \mathbb{H} \to 2^{\mathbb{H}}$ is α-inverse strongly monotone operator, $B : \mathbb{H} \to 2^{\mathbb{H}}$ is β-strongly monotone, and $\{C_i\}_{1\leq i\leq n}$ are finite family of nonexpansive operators $C_i : \mathbb{H} \to \mathbb{H}$.

Let us define the operator $F(\Psi_\lambda) = \{x^* \in \mathbb{H} \mid \Psi_\lambda(x^*) = x^*\}$, and S defined as:

$$S(x) = \left\{ x \in \mathbb{H} \mid 0 \in A(x) + B(x) + \sum_{i=1}^{n} C_i(x) \right\}.$$

First we aim to study the problem defined by the sum of two maximal monotone opertaors A and B, both defined on a Hilbert space $\mathbb{H}$, this problem is formulated as: Find element $x \in \mathbb{H}$ such that,

$$0 \in A(x) + B(x). \tag{4}$$

So let Propose simple algorithm by using the approximation of yoshida which can solve (3.2).

Proposition 4. *For any $\delta > 0$, $\lambda > 0$, we have $A_\lambda(x) = A_\delta(x + \delta A_\lambda(x))$.*

Proof. let $\delta > 0$, $\lambda > 0$, we have

$$J_\lambda^A(x) = J_\delta^A\left(\frac{\delta}{\lambda}x + \left(1 - \frac{\delta}{\lambda}\right)J_\lambda^A(x)\right),$$

this equevalent to,

$$x - \lambda A_\lambda(x) = \frac{\delta}{\lambda}x + \left(1 - \frac{\delta}{\lambda}\right)(x - \lambda A_\lambda(x)) - \delta A_\delta\left(\frac{\delta}{\lambda}x + \left(1 - \frac{\delta}{\lambda}\right)(x - \lambda A_\lambda(x))\right),$$

therefore,

$$-\lambda A_\lambda(x) = -\lambda A_\lambda(x) + \delta A_\delta - \delta A_\delta\left(\frac{\delta}{\lambda}x + \left(1 - \frac{\delta}{\lambda}\right)(x - \lambda A_\lambda(x))\right),$$

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so we conclude,

$$A_\lambda(x) = A_\delta((x + \delta A_\lambda(x))).$$

This complete the proof. □

Proposition 5. *Let $\lambda > 0$ and define the operator $\theta_\lambda : \mathbb{H} \to \mathbb{H}$ by*

$$\theta_\lambda(x) = x + A_\lambda(x) + B_\lambda(x - 2\lambda A_\lambda(x)).$$

If $x^\star$ is a fixed point of θ_λ, then $x^\star - \lambda A_\lambda(x^\star)$ is a solution of the monotone inclusion problem (3.2)*.*

Proof. Assume that x^* is a fixed point of θ_λ. By definition, this implies:

$$x^* + A_\lambda(x^*) + B_\lambda(x^* - 2\lambda A_\lambda(x^*)) = x^*.$$

Subtracting x^* from both sides, we obtain:

$$A_\lambda(x^*) + B_\lambda(x^* - 2\lambda A_\lambda(x^*)) = 0.$$

Let us define $z = x^* - \lambda A_\lambda(x^*)$. Then:

$$x^* = z + \lambda A_\lambda(x^*), \quad \text{and} \quad x^* - 2\lambda A_\lambda(x^*) = z - \lambda A_\lambda(x^*).$$

Substituting these into the previous equation gives:

$$A_\lambda(x^*) + B_\lambda(z - \lambda A_\lambda(x^*)) = 0.$$

Thus, we have:

$$0 \in A(z) + B(z),$$

which means that z is a solution of equation (3.2), as claimed. □

Theorem 1. *for all $\lambda > 0$, if the sequence $\{x_k\}$ which defined as:*

$$\begin{cases} x_0, x_1 \in \mathbb{H}, \\ x_{k+1} = (1-\alpha)x_k + \alpha\theta_\lambda(x_k) + \epsilon_k(x_k - x_{k-1}), & k \geq 0. \end{cases}$$

Where $0 < \alpha < 1$, $\epsilon_k \in]0, 1[$ and $\sum_{i=1}^{\infty} \epsilon_k < \infty$. If $\{x_k\}$ converge to l then $l - \lambda A_\lambda(l)$ solve (3.2).

We now turn to the study of the principal problem (3.1).

Theorem 2. *For all $\lambda > 0$, if $\Psi_\lambda(x) = J_\lambda^B(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x)) - x + 2J_\lambda^A(x)) + \lambda A_\lambda(x)$ if $S \neq \emptyset$, then $\{J_\lambda^A(F(\Psi_\lambda))\} \subseteq S$.*

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Proof. Let x^* be a fixed point of Ψ_λ, so:

$$\begin{aligned} x^* \in F(\Psi_\lambda) &\implies \Psi_\lambda(x^*) = x^* \\ &\implies J_\lambda^B(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x^*)) - x^* + 2J_\lambda^A(x^*)) + \lambda A_\lambda(x^*) = x^* \\ &\implies x^* - \lambda A_\lambda(x^*) = J_\lambda^B(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x^*)) - x^* + 2J_\lambda^A(x^*)) \\ &\implies J_\lambda^A(x^*) = J_\lambda^B(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x^*)) - x^* + 2J_\lambda^A(x^*)) \\ &\implies -\lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x^*)) - x^* + 2J_\lambda^A(x^*) \in \lambda B(J_\lambda^A(x^*)) + J_\lambda^A(x^*) \\ &\implies -x^* + J_\lambda^A(x^*) \in \lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x^*)) + \lambda B(J_\lambda^A(x^*)) \\ &\quad \text{and } x^* - J_\lambda^A(x^*) \in \lambda A(J_\lambda^A(x^*)) \\ &\implies 0 \in \lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x^*)) + \lambda B(J_\lambda^A(x^*)) + \lambda A(J_\lambda^A(x^*)) \\ &\implies J_\lambda^A(x^*) \in \text{zer}(\sum_{i=1}^{n} C_i + A + B). \end{aligned}$$

This complete the proof. □

3.1. Algorithm 1

In this algorithm, we impose an additional condition on the family $\{C_i\}_{1\le i\le n}$, which is that the operators $I + \lambda C_i$ must be bijective.

Proposition 6. *for all $\lambda > 0$, if $F(\Psi_\lambda) \neq \emptyset$, then the system defined as follow:*

$$\begin{cases} J_\lambda^A(x) = \dfrac{x + y + \sum_{i=1}^{i=n} z_i}{n+2} \\ J_\lambda^B(y) = \dfrac{x + y + \sum_{i=1}^{i=n} z_i}{n+2} \\ J_\lambda^{C_1}(z_1) = \dfrac{x + y + \sum_{i=1}^{i=n} z_i}{n+2} \\ . \\ . \\ . \\ J_\lambda^{C_n}(z_n) = \dfrac{x + y + \sum_{i=1}^{i=n} z_i}{n+2}. \end{cases} \tag{5}$$

has a solution $(x, y, z_1, ..., z_n) \in \mathbb{H}^{n+2}$.

Proof. let $F(\Psi_\lambda) \neq \emptyset$, then exist $x^\star \in \mathbb{H}$ such that $\Psi_\lambda(x^\star) = x^\star$, this implies,

$$J_\lambda^A(x^\star) = J_\lambda^B(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x^\star)) - x^\star + 2J_\lambda^A(x^\star)).$$

let we pose,

$$\begin{cases} x^\star = x, \\ -\lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x^\star)) - x^\star + 2J_\lambda^A(x^\star)) = y, \\ J_\lambda^A(x^\star) + \lambda C_i(J_\lambda^A(x^\star)) = z_i. \end{cases}$$

Therfore,

$$\begin{cases} J_\lambda^A(x) = J_\lambda^B(y), \\ J_\lambda^A(x) = J_\lambda^{C_i}(z_i), \\ x + y + \sum_{i=1}^{i=n} z_i = (n+2)J_\lambda^A(x). \end{cases}$$

Consequently,

$$\begin{cases} J_\lambda^A(x) = \dfrac{x + y + \sum_{i=1}^{i=n} z_i}{n+2}, \\ J_\lambda^B(y) = \dfrac{x + y + \sum_{i=1}^{i=n} z_i}{n+2}, \\ J_\lambda^{C_1}(z_1) = \dfrac{x + y + \sum_{i=1}^{i=n} z_i}{n+2}, \\ . \\ . \\ . \\ J_\lambda^{C_n}(z_n) = \dfrac{x + y + \sum_{i=1}^{i=n} z_i}{n+2}. \end{cases}$$

This implies that $(x, y, z_1, ..., z_n)$ is solution of (3.2). □

Theorem 3. *for all $\lambda > 0$, if the system of sequences defined as:*

$$\begin{cases} (x_0, y_0, z_{10}, ..., z_{n0}) \in \mathbb{H}^{n+2}, \\ x_{k+1} = (n+2)J_\lambda^A(x_k) - y_k - \sum_{i=1}^{i=n} z_{ik}, \\ y_{k+1} = (n+2)J_\lambda^A(y_k) - x_k - \sum_{i=1}^{i=n} z_{ik}, \\ z_{ik+1} = J_\lambda^A(x_k) + \lambda C_i(J_\lambda^A(x_k)), \end{cases} \qquad k \geq 0.$$

Converge to $(x^\star, y^\star, z_1^\star, ..., z_n^\star)$ then $J_\lambda^A(x^\star)$ is solution of (1.1).

Proof. let assume that the last system converge to $(x^\star, y^\star, z_1^\star, ..., z_n^\star)$ in $\mathbb{H}^{n+2}$ so, we have

$$\begin{cases} x^\star = (n+2)J_\lambda^A(x^\star) - y_k - \sum_{i=1}^{i=n} z_i^\star, \\ y^\star = (n+2)J_\lambda^A(y^\star) - x^\star - \sum_{i=1}^{i=n} z_i^\star, \\ z_i^\star = J_\lambda^A(x^\star) + \lambda C_i(J_\lambda^A(x^\star)). \end{cases}$$

Therefore,

$$\begin{cases} x^\star + y^\star + \sum_{i=1}^{i=n} z_i^\star = (n+2)J_\lambda^A(x^\star), \\ y^\star + x^\star + \sum_{i=1}^{i=n} z_i^\star = (n+2)J_\lambda^A(y^\star), \\ z_i^\star = J_\lambda^A x^\star + \lambda C_i(J_\lambda^A(x^\star)). \end{cases}$$

Then,

$$x^\star + y^\star + nJ_\lambda^A x^\star + \lambda \sum_{i=1}^{i=n} C_i(J_\lambda^A x^\star) = (n+2)J_\lambda^A(x^\star),$$

After simplify,

$$y^\star = 2J_\lambda^A x^\star - \lambda \sum_{i=1}^{i=n} C_i(J_\lambda^A x^\star) - x^\star$$

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We have also, $J_\lambda^A(x^\star) = J_\lambda^B(y^\star)$.

Consequently,

$$J_\lambda^A(x^\star) = J_\lambda^B(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A(x^\star)) - x^\star + 2J_\lambda^A x^\star).$$

So we conclude that $x^\star$ is fixed point of Ψ_λ, which prove that $J_\lambda^A(x^\star)$ is solution of (1.1). □

Below we will examine the case when $n = 1$, so the problem (1.1) will define as: Find an element x in the Hilbert space $\mathbb{H}$ such that,

$$0 \in A(x) + B(x) + C(x). \tag{6}$$

Where A and B are two maximal monotone operators defined on Hilbert space $\mathbb{H}$ and C is nonexpansive single valued mapping defined also on $\mathbb{H}$. Then the algorithm will defined as:

$$\begin{cases} (x_0, y_0, z_0) \in \mathbb{H}^3, \\ x_{k+1} = 3J_\lambda^A(x_k) - y_k - z_k, \\ y_{k+1} = 3J_\lambda^A(y_k) - x_k - z_k, \\ z_{k+1} = J_\lambda^A(x_k) + \lambda C(J_\lambda^A(x_k)), \end{cases} \qquad k \geq 0.$$

3.2. *Algorithm 2*

Proposition 7. *Let $\{C_i\}_{n\leq i\leq 1}$ be finite family of nonexpansive operators defined on H, A is α-inverse strongly monotone operator, B be β-strongly monotone operator defined on a real Hilbert space. For all $\alpha > 0$, $\beta > 0$ and $\lambda > 0$, Ψ_λ is a L-lipschitzian operator where*

$$L = \frac{n\lambda^2 + (\alpha n + 2)\lambda + \alpha}{(\beta\lambda + 1)(\lambda + \alpha)} + \frac{\lambda}{\lambda + \alpha}.$$

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Proof.

$$
\begin{aligned}
\|\Psi_\lambda(x) - \Psi_\lambda(y)\| &= \left\| J_\lambda^B\left(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A x) - x + 2J_\lambda^A x\right)\right. \\
&\quad \left. - J_\lambda^B\left(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A y) - y + 2J_\lambda^A y\right) - \lambda A_\lambda(x) + \lambda A_\lambda(y)\right\| \\
&< \left\| J_\lambda^B\left(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A x) - (x - J_\lambda^A x)\right)\right. \\
&\quad \left. - J_\lambda^B\left(-\lambda \sum_{i=1}^{n} C_i(J_\lambda^A y) - (y - J_\lambda^A y)\right)\right\| + \frac{\lambda}{\lambda+\alpha}\|x-y\| \\
&< \frac{1}{\beta\lambda+1}\left\| -\lambda \sum_{i=1}^{n} C_i(J_\lambda^A x) - \lambda A_\lambda(x) - J_\lambda^A x \right. \\
&\quad \left. + \lambda \sum_{i=1}^{n} C_i(J_\lambda^A y) + \lambda A_\lambda(y) + J_\lambda^A y \right\| + \frac{\lambda}{\lambda+\alpha}\|x-y\| \\
&< \frac{1}{\beta\lambda+1}\left[\lambda \sum_{i=1}^{n} \|C_i(J_\lambda^A x) - C_i(J_\lambda^A y)\| + \left(\frac{\lambda}{\lambda+\alpha}+1\right)\|x-y\|\right] \\
&\quad + \frac{\lambda}{\lambda+\alpha}\|x-y\| \\
&< \frac{1}{\beta\lambda+1}\left[\lambda \sum_{i=1}^{n} \|J_\lambda^A x - J_\lambda^A y\| + \left(\frac{\lambda}{\lambda+\alpha}+1\right)\|x-y\|\right] \\
&\quad + \frac{\lambda}{\lambda+\alpha}\|x-y\| \\
&< \frac{1}{\beta\lambda+1}\left(\lambda n + \frac{2\lambda+\alpha}{\lambda+\alpha}\right)\|x-y\| + \frac{\lambda}{\lambda+\alpha}\|x-y\| \\
&< \left[\frac{n\lambda^2 + (\alpha n + 2)\lambda + \alpha}{(\beta\lambda+1)(\lambda+\alpha)} + \frac{\lambda}{\lambda+\alpha}\right]\|x-y\|.
\end{aligned}
$$

Theorem 4. *For all λ, α and β positive real numbers: if $\beta > n$, $\alpha > \frac{2}{\beta - n}$ and $0 < \lambda < \frac{(\beta-n)\alpha-2}{n}$, where $n \in \mathbb{N}^*$. Then Ψ_λ is a contractive mapping.*

Proof. If Ψ_λ is contractive, then the inequality satisfies:

$$
\begin{aligned}
\frac{n\lambda^2 + (\alpha n + 2)\lambda + \alpha}{(\beta\lambda+1)(\lambda+\alpha)} + \frac{\lambda}{\lambda+\alpha} < 1 &\iff \frac{(n+\beta)\lambda^2 + (3+\alpha n)\lambda + \alpha}{(\beta\lambda+1)(\alpha+\lambda)} - 1 < 0 \\
&\iff \frac{n\lambda^2 + (2+\alpha n - \alpha\beta)\lambda}{(\beta\lambda+1)(\alpha+\lambda)} < 0. \quad \square
\end{aligned}
$$

After simplification, the next step is to solve the resulting polynomial inequality involving parameters α, n, and β:

$$n\lambda^2 + (2 + \alpha n - \alpha\beta)\lambda < 0,$$

we have:

$$n\lambda^2 + (2 + \alpha n - \alpha\beta)\lambda < 0 \quad \iff \quad \lambda(n\lambda + 2 + n\alpha - \alpha\beta) < 0.$$

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This implies:

$$\begin{cases} \lambda > 0, \\ n\lambda + 2 + n\alpha - \alpha\beta < 0, \quad n > 0. \end{cases}$$

Hence, we deduce the following conditions:

$$\begin{cases} \beta > n, \\ \alpha > \frac{2}{\beta - n}, \\ 0 < \lambda < \frac{\alpha\beta - 2 - n\alpha}{n}, \end{cases} \qquad n > 0.$$

This completes the proof.

In this part, we modify the Ishikawa algorithm to acheive faster convergence of aour sequence $\{\Psi_\lambda x_k\}$. Accordingly, we present the following theorem that defines the modified algorithm:

Theorem 5. *Let $\mathbb{H}$ be a real Hilbert space and $\mathbb{C}$ be a closed convex subspace of $\mathbb{H}$. Let $\Psi_\lambda : \mathbb{C} \to \mathbb{C}$ be a contractive mapping. Let $\{x_k\}$ be a sequence defined iteratively for each integer $k \geq 0$ by*

$$\begin{cases} x_0 \in \mathbb{H}, \\ x_{k+1} = a_k x_k + b_k \Psi_\lambda y_k, \qquad k \geq 0, \\ y_k = c_k x_k + d_k \Psi_\lambda x_k, \end{cases}$$

where $\{a_k\}$, $\{b_k\}$ are sequences of positive numbers satisfying the following conditions:

1. $0 \leq d_k \leq b_k < 1$,
2. $b_k + a_k = 1$,
3. $c_k + d_k = 1$.

If $\{x_k\}$ converges, then it converges to a unique fixed point of Ψ_λ.

3.3. Convergence Analysis

Ishikawa has shown that for any points x, y, z in a Hilbert space and any real number λ:

$$||\lambda x + (1-\lambda)y - z||^2 = \lambda||x - z||^2 + (1-\lambda)||y - z||^2 - \lambda(1-\lambda)||x - y||^2.$$

Let x^* be a fixed point of Ψ_λ, then we have

$$\begin{aligned} ||x_{k+1} - x^*||^2 &= ||a_k x_k + b_k \Psi_\lambda y_k - x^*||^2 \\ &= b_k||\Psi_\lambda y_k - x^*||^2 + a_k||x_k - x^*||^2 - b_k a_k||x_k - \Psi_\lambda y_k||^2 \end{aligned} \tag{7}$$

From contraction condition we have:

$$\|\Psi_\lambda y_k - x^*\|^2 = ||\Psi_\lambda y_k - \Psi_\lambda x^*||^2 \leq \|y_k - x^*\|^2 + h\|y_k - \Psi_\lambda y_k\|^2, \quad \text{where} \quad h = L^2. \tag{8}$$

On the other hand,

$$\|y_k - x^*\|^2 = \|c_k x_k + d_k \Psi_\lambda x_k - x^*\|^2,$$

which expands to:

$$\|y_k - x^*\|^2 = d_k\|\Psi_\lambda x_k - x^*\|^2 + c_k\|x_k - x^*\|^2 - d_k c_k\|x_k - \Psi_\lambda x_k\|^2. \tag{9}$$

Similarly, we can express:

$$\|y_k - \Psi_\lambda y_k\|^2 = d_k\|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2 + c_k\|x_k - \Psi_\lambda y_k\|^2 - d_k c_k\|x_k - \Psi_\lambda x_k\|^2. \tag{10}$$

Moreover, we have the following inequality:

$$\|\Psi_\lambda x_k - x^*\|^2 \leq \|x_k - x^*\|^2 + h\|x_k - \Psi_\lambda x_k\|^2. \tag{11}$$

By introducing equations (11), (10), and (9) into (8), we obtain:

$$\|\Psi_\lambda y_k - x^*\|^2 \leq c_k\|x_k - x^*\|^2 + hc_k\|x_k - \Psi_\lambda x_k\|^2 + d_k\|x_k - x^*\|^2$$
$$-d_k c_k\|x_k - \Psi_\lambda x_k\|^2 + hd_k\|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2 + hc_k\|x_k - \Psi_\lambda y_k\|^2.$$

Thus,

$$\|\Psi_\lambda y_k - x^*\|^2 \leq \|x_k - x^*\|^2 - d_k(c_k - hd_k)\|x_k - \Psi_\lambda x_k\|^2$$
$$+hd_k\|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2 + hc_k\|x_k - \Psi_\lambda y_k\|^2. \tag{12}$$

Substituting equation (12) into equation (7), we get:

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 + hb_k\|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2$$
$$-b_k d_k(c_k - hd_k)\|x_k - \Psi_\lambda x_k\|^2 - b_k(a_k - h + hd_k)\|x_k - \Psi_\lambda y_k\|^2.$$

This shows that $\{\|x_k - x^*\|^2\}$ is decreasing for all sufficiently large k. Since conditions (2) and (3) are satisfied, there exists a subsequence $\{x_{k_m}\}$ of $\{x_k\}$ such that:

$$\lim_{m\to\infty} \|x_{k_m} - \Psi_\lambda x_{k_m}\| = 0.$$

Now, we show that $\{\Psi_\lambda x_k\}$ is a Cauchy sequence. Indeed,

$$\|\Psi_\lambda x_{k_m} - \Psi_\lambda x_{k_\ell}\| \leq \|x_{k_\ell} - \Psi_\lambda x_{k_m}\| + \|x_{k_\ell} - \Psi_\lambda x_{k_\ell}\|$$

Taking the limit as $m, \ell \to \infty$, we have:

$$\|\Psi_\lambda x_{k_m} - \Psi_\lambda x_{k_\ell}\| \to 0.$$

Thus, $\{\Psi_\lambda x_k\}$ is a Cauchy sequence, hence convergent.
Call the limit x^*. Then:

$$\lim_{m\to\infty} \Psi_\lambda x_{k_m} = \lim_{m\to\infty} x_{k_m} = x^*.$$

Using the contraction of Ψ_λ , we have:

$$\|\Psi_\lambda x^* - \Psi_\lambda x_{k_m}\| \leq \|x^* - x_{k_m}\| + \|x_{k_m} - \Psi_\lambda x_{k_m}\| + \|\Psi_\lambda x^* - \Psi_\lambda x_{k_m}\|.$$

Taking the limit as $m \to \infty$, we obtain:

$$\lim_{m\to\infty} \|\Psi_\lambda x^* - \Psi_\lambda x_{k_m}\| = 0.$$

Hence, we conclude that:

$$\|x^* - \Psi_\lambda x^*\| \leq \|x^* - x_{k_m}\| + \|x_{k_m} - \Psi_\lambda x_{k_m}\| + \|\Psi_\lambda x_{k_m} - \Psi_\lambda x^*\|.$$

Taking the limit as $m \to \infty$, we deduce that $\|x^* - \Psi_\lambda x^*\| = 0$, i.e., $x^* = \Psi_\lambda x^*$. Now, we aim to prove that the sequence $\{x_k\}$ converges to the unique fixed point of Ψ_λ.

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$$\begin{aligned} \|x_{k+1} - x^*\|^2 &= \|a_k x_k + b_k \Psi_\lambda y_k - x^*\|^2 \\ &= b_k \|\Psi_\lambda y_k - x^*\|^2 + a_k \|x_k - x^*\|^2 - b_k a_k \|\Psi_\lambda y_k - x^*\|^2. \end{aligned} \tag{13}$$

we know that:

$$\|\Psi_\lambda y_k - x^*\|^2 \le L^2 \|y_k - x^*\|^2 + L^2 \|y_k - \Psi_\lambda y_k\|^2.$$

Suppose that $L^2 = h$, then:

$$\|\Psi_\lambda y_k - x^*\|^2 \le h \|y_k - x^*\|^2 + h \|y_k - \Psi_\lambda y_k\|^2.$$

On the other hand:

$$\begin{aligned} \|y_k - x^*\|^2 &= \|d_k \Psi_\lambda x_k + c_k x_k - x^*\|^2 \\ &= d_k \|\Psi_\lambda x_k - x^*\|^2 + c_k \|x_k - x^*\|^2 - d_k c_k \|\Psi_\lambda x_k - x_k\|^2. \end{aligned} \tag{14}$$

And similarly:

$$\begin{aligned} \|y_k - \Psi_\lambda y_k\|^2 &= \|d_k \Psi_\lambda x_k + c_k x_k - \Psi_\lambda y_k\|^2 \\ &= c_k \|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2 + c_k \|x_k - \Psi_\lambda y_k\|^2 - d_k c_k \|\Psi_\lambda x_k - x_k\|^2. \end{aligned} \tag{15}$$

Hence (15) can be rewritten as follows:

$$\begin{aligned} \|\Psi_\lambda y_k - x^*\|^2 \le\ & h d_k \|\Psi_\lambda x_k - x^*\|^2 + h c_k \|x_k - x^*\|^2 \\ & - h d_k c_k \|\Psi_\lambda x_k - x_k\|^2 + h d_k \|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2 \\ & + h c_k \|x_k - \Psi_\lambda y_k\|^2 - h d_k c_k \|\Psi_\lambda x_k - x_k\|^2. \end{aligned} \tag{16}$$

However, we also have

$$\|\Psi_\lambda x_k - x^*\|^2 \le h \|x_k - x^*\|^2 + h \|\Psi_\lambda x_k - x_k\|^2. \tag{17}$$

By substituting (17) into (16), we obtain:

$$\begin{aligned} \|\Psi_\lambda y_k - x^*\|^2 \le\ & h^2 d_k \|x_k - x^*\|^2 + h^2 \|\Psi_\lambda x_k - x_k\|^2 + h c_k \|x_k - x^*\|^2 \\ & - h d_k c_k \|\Psi_\lambda x_k - x_k\|^2 + h d_k c_k \|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2 \\ & + h c_k \|x_k - \Psi_\lambda y_k\|^2 - h d_k c_k \|\Psi_\lambda x_k - x_k\|^2 \\ \le\ & (h c_k + h d_k) \|x_k - x^*\|^2 - h d_k (2 - 2d_k) \|\Psi_\lambda x_k - x_k\|^2 \\ & + h c_k \|x_k - \Psi_\lambda y_k\|^2 + h d_k \|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2. \end{aligned} \tag{18}$$

Incorporating (18) into (13) yields:

$$\begin{aligned} \|x_{k+1} - x^*\|^2 \le\ & b_k h \|x_k - x^*\|^2 - b_k d_k h (2 - h - 2d_k) \|\Psi_\lambda x_k - x_k\|^2 \\ & + b_k c_k h \|x_k - \Psi_\lambda y_k\|^2 + b_k d_k h \|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2 \\ & + a_k \|x_k - x^*\|^2 - b_k a_k h \|x_k - \Psi_\lambda y_k\|^2 \\ \le\ & a_k (1 - h) \|x_k - x^*\|^2 - b_k d_k h (2 - h - 2d_k) \|\Psi_\lambda x_k - x_k\|^2 \\ & + b_k d_k h \|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2 - b_k a_k (1 - h + h d_k)) \|x_k - \Psi_\lambda y_k\|^2. \end{aligned} \tag{19}$$

Given that $\frac{1-h}{2} \le b_k \le 1 - h$, $0 < h < 1$, $d_k \ge 0$, and $\lim d_k = 0$, there exists a natural number N such that for $k > N$:

$$2 - h - 2d_k \ge 0 \quad \text{and} \quad a_k - h + h d_k \ge 0.$$

Thus, for $k \ge N$, we have

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$$\|x_{k+1} - x^*\|^2 \le \tilde{h}\|x_k - x^*\|^2 + b_k d_k h\|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2,$$

where $0 < \tilde{h} = 1 - \frac{(1-h)^2}{2}$.

From the boundedness of C, it follows that $\|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2$ is bounded. Therefore, we conclude that

$$\lim_{k\to\infty} b_k d_k h\|\Psi_\lambda x_k - \Psi_\lambda y_k\|^2 = 0.$$

From Lemma 2.7, we conclude that $\lim_{k\to\infty} x_k = x^*$. This completes the proof.

3.4. Maximal Monotone Operators and Minimization Problem

We consider the following composite convex optimization problem:

$$\min_{x\in\mathbb{R}^n}[f(x) + G(x) + H(x)], \tag{20}$$

where:

- $f : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function with a Lipschitz continuous gradient, i.e., ∇f is 1-Lipschitz,
- G and H are convex, closed, and proper functions.

Proposition 8. *Let $A = \partial G$, $B = \partial H$ and $C = \nabla f$. Then, the minimization problem (20) is equivalent to finding a zero of the sum of maximal monotone operators, that is:*

$$\textit{Find } x \in \mathbb{R}^n \textit{ such that } 0 \in A(x) + B(x) + C(x). \tag{21}$$

3.5. Example

Let f, G, and H be three real-valued functions defined on $\mathbb{R}$ as follows:

$$f(x) = \frac{1}{2}x^2 + 2, \quad G(x) = \frac{1}{10}x^2, \quad H(x) = 2x^2.$$

We consider the following minimization problem:

$$\min_{x\in\mathbb{R}}(f(x) + G(x) + H(x)).$$

Let us define the following monotone operators corresponding to the gradients of G, H, and f:

$$A(x) = \{\nabla G(x)\} = \left\{\frac{1}{5}x\right\}, \quad B(x) = \{\nabla H(x)\} = \{4x\}, \quad C(x) = \nabla f(x) = x.$$

Then, the minimization problem above is equivalent to the inclusion problem:

$$\text{Find } x \in \mathbb{R} \text{ such that } 0 \in A(x) + B(x) + C(x).$$

We know the resolvents of the operators are given by:

$$J_{\lambda A}(x) = \frac{5}{5+\lambda}x, \quad J_{\lambda B}(x) = \frac{1}{1+4\lambda}x,$$

and hence,

$$\Psi_\lambda(x) = \frac{4\lambda^2 - 5\lambda + 10}{(4\lambda+1)(\lambda+5)}x.$$

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According to Theorem 5, we proceed by choosing the parameters:

$$\lambda = 2, \quad b_n = \frac{1}{(n+1)^2}, \quad d_n = \frac{1}{n+1}.$$

3.5.1. Application of the Algorithm 2

We have the sigle-valued mapping:

$$\Psi_\lambda(x) = \frac{4\lambda^2 - 5\lambda + 10}{(4\lambda+1)(\lambda+5)}x.$$

For $\lambda = 2$, this becomes:

$$\Psi_2(x) = \frac{16}{63}x.$$

We initialize the process as:

$$x_0 = y_0 = 1,$$

and define:

$$d_n = \frac{1}{(n+1)^2}, \quad b_n = \frac{1}{n+1}, \quad a_n = 1 - b_n, \quad c_n = 1 - d_n.$$

Iteration steps

At each iteration $n \geq 0$, we update:

$$y_n = c_n x_n + d_n \Psi_2(x_n) = \left(1 - \frac{1}{(n+1)^2}\right)x_n + \frac{16}{63(n+1)^2}x_n,$$
$$x_{n+1} = a_n x_n + b_n \Psi_2(y_n) = \left(1 - \frac{1}{n+1}\right)x_n + \frac{16}{63(n+1)}y_n.$$

These choices still satisfy the assumptions of Theorem 3.3 and ensure that the sequence $\{x_n\}$ converges to the unique fixed point of Ψ_2, which is:

$$x^* = \Psi_2(x^*) \Rightarrow x^* = 0.$$

4. Conclusion

In conclusion, we propose some algorithms for solving the principle problem. We also supported this work with a set of simple examples and observed the convergence of the sequences proposed in this paper to the same solution of the problem. Nevertheless, the development of alternative algorithms under appropriate conditions that effectively address this class of problems remains an open area of research, offering valuable opportunities for further investigation and advancement.

References

1. Attouch, H. and Baillon, J. and Théra, M. Variational Sum of Monotone Operators. *Journal of Convex Analysis* **1994**, *1*, 1–29.
2. Bauschke, H. and Combettes, P. L. Convex Analysis and Monotone Operator Theory in Hilbert Spaces; Springer; New York, 2011.
3. Beddani, A. Finding a Zero of the Sum of Two Maximal Monotone Operators with Minimization Problem. *Nonlinear Functional Analysis and Applications* **2022** *27* pp. 895–902.
4. Beddani, A.Finding a Zero of the Sum of Three Maximal Monotone Operators. *Journal of Science and Arts* **2022** *22* pp. 795–802.
5. Beddani, A. and Berrailes, A. Zeros of the Sum of a Finite Family of Maximal Monotone Operators. *Journal of Optimization Theory and Applications* **2025** *205* pp. 59.
6. Brézis, H. 1973. *Opérateurs Maximaux Monotones et Semi-Groupes de Contractions dans les Espaces de Hilbert*. Amsterdam: North-Holland, 1973.

 doi:10.20944/preprints202510.0598.v1

7. Cegielski, A. 2012.*Iterative Methods for Fixed Point Problems in Hilbert Spaces* Berlin, Heidelberg: Springer, vol. 2057.
8. Eckstein, A. and Bertsekas, D. P. On the Douglas-Rachford Splitting Method and the Proximal Point Algorithm for Maximal Monotone Operators. *Mathematical Programming* **1992** *55* pp. 293–318.
9. Martinez-Legaz, J. Enrique. Monotone Operators Representable by l.s.c Convex Functions. **2022** *27* pp. 895–902.
10. Ibaraki, T. Approximation of a Zero Point of Monotone Operators with Nonsummable Errors. *Fixed Point Theory and Applications* **2016** pp. 53.
11. Qihou, Liu. A Convergence Theorem of the Sequence of Ishikawa Iterates for Quasi-Contractive Mappings. *Journal of Mathematical Analysis and Applications* **1990** *146* pp. 301–305.
12. Mann, R. Mean Value Methods in Iteration. *Proceedings of the American Mathematical Society* **1953** *4* pp. 506–510.
13. Moudafi, A. and Théra, M. Finding a Zero of the Sum of Two Maximal Monotone Operators. *Journal of Optimization Theory and Applications* **1997** *94* pp. 425–448.
14. Nammanee, K. and Suantai, V. and Cholamjiak, P. Convergence Theorems for Maximal Monotone Operators, Weak Relatively Nonexpansive Mappings and Equilibrium Problems. *Journal of Applied Mathematics* **2012** pp. 16.
15. Rockafellar, R. T. Monotone Operators and the Proximal Point Algorithm. *SIAM Journal on Control and Optimization* **1976** *14* pp. 877–898.
16. Shehu, Y. Convergence Results of Forward-Backward Algorithms for Sum of Monotone Operators in Banach Spaces. *Results in Mathematics* **2019** *74* pp. 138.