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Article

On Local Null-Preserving Maps Between Pseudo-Riemannian Manifolds of Different Dimensions

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Abstract

We construct a local framework for maps between pseudo-Riemannian manifolds of different dimensions that preserve null directions. Let $F : M_i \rightarrow M_{i+1}$; F is null-preserving if F_*v is null for every null $v \in T_pM_i$. Deviations from an exact metric pullback are measured via a *correction tensor* T . This setup extends Liouville-type uniqueness results to the interdimensional case, providing a precise tool for local analysis of geometric embeddings.

Keywords: null-preserving maps; pseudo-Riemannian geometry; correction tensor; dimension-adjusted conformal weight; interdimensional embeddings

1. Introduction

The study of maps preserving geometric structures between manifolds is a central theme in differential geometry [1–3]. Classical results, such as Liouville’s theorem, state that any sufficiently smooth map (C^∞) preserving null cones between pseudo-Riemannian manifolds of the same dimension $n \geq 3$ is a *local conformal diffeomorphism* [4–6]. These results underpin conformal geometry, null structures, and causal analysis [7,8]. Global conformality requires additional topological or completeness assumptions, which are not assumed here.

Recent work has explored null-projective relations between Lorentzian metrics [9], rigidity of conformal embeddings in Lorentzian manifolds [10], and interdimensional null-preserving maps in semi-Riemannian contexts [11]. Studies on embeddings in higher codimension highlight structural challenges analogous to those addressed by the correction tensor T and the dimension-adjusted conformal weight $\Delta(i)$ [12]. However, these works mostly consider equal-dimensional cases or specific Lorentzian signatures, leaving interdimensional maps largely unexamined.

This paper establishes a rigorous framework for *interdimensional null-preserving maps*:

$$F : M_i \rightarrow M_{i+1},$$

between pseudo-Riemannian manifolds (M_i, g_i) and (M_{i+1}, g_{i+1}) with $\dim M_i = i \geq 2$ and $\dim M_{i+1} = i + 1 \geq 3$. The map F is *null-preserving* if for every null vector $v \in T_pM_i$, the push-forward $F_*v \in T_{F(p)}M_{i+1}$ is null with respect to g_{i+1} . We assume F is at least C^2 to ensure well-defined differentials and curvature computations. The target dimension satisfies $i + 1 \geq 3$ to guarantee non-trivial Weyl tensors and applicability of Liouville-type rigidity.

A key conceptual challenge in the interdimensional setting is the absence of direct conformal pullbacks when the source and target dimensions differ. Let

$$F : M_i \rightarrow M_{i+1}, \quad g_i \text{ on } M_i, \quad g_{i+1} \text{ on } M_{i+1},$$

be smooth pseudo-Riemannian manifolds with $\dim M_i = i$ and $\dim M_{i+1} = i + 1$. For null-preserving maps, the pullback F^*g_{i+1} cannot, in general, be proportional to g_i . To address this, we define the *correction tensor* T by

$$T := F^*g_{i+1} - g_i,$$

which captures deviations from exact conformality while preserving null directions. This allows decomposition of the geometric effects of the embedding into a conformal component and a higher-dimensional discrepancy, providing a canonical framework to extend Liouville-type rigidity results to the interdimensional case.

Within this framework, we establish a rigorous local stability result. For any null 2-plane $\Pi \subset T_p M_i$, the deviation from exact null preservation can be controlled quantitatively:

$$\sup_{v \in \Pi \setminus \{0\}} \frac{|g_{i+1}(F_*v, F_*v)|}{\|v\|_h^2} \leq \varepsilon,$$

where ε can be made arbitrarily small by restricting to a sufficiently small neighborhood of p . This shows that T provides explicit, coordinate-independent bounds, reinforcing the robustness of local existence and uniqueness of null-preserving maps under small perturbations of F and g_i .

Our approach further leverages the algebraic structure of null cones, analytic properties of the Weyl tensor W , and rank conditions on dF . The pullback of the Weyl tensor can be expressed canonically as

$$F^*W_{i+1} = W_i + \text{terms linear and quadratic in } T \text{ and its derivatives},$$

allowing precise control of curvature contributions. Together, these results establish a comprehensive, intrinsic, and coordinate-independent framework for the analysis of interdimensional null-preserving mappings, including local existence, uniqueness, and stability.

2. Preliminary Definitions and Assumptions

Throughout, (M_i, g_i) and (M_{i+1}, g_{i+1}) denote smooth (C^∞) pseudo-Riemannian (Lorentzian) manifolds of dimensions $i \geq 3$ and $i + 1$, with signatures $(1, i - 1)$ and $(1, i)$, respectively. This ensures that null vectors and null 2-planes are well-defined. Let $F : M_i \rightarrow M_{i+1}$ be a smooth map with pushforward $F_* : TM_i \rightarrow TM_{i+1}$. We assume all maps and tensor fields are at least C^2 unless otherwise stated.

Definition 1. A pseudo-Riemannian manifold (M, g) is a smooth manifold equipped with a smooth, non-degenerate, symmetric bilinear form g on each tangent space $T_p M$, with signature fixed throughout M .

Definition 2. A vector $v \in T_p M$ is null if $g(v, v) = 0$. The set of all null vectors at p forms the null cone in $T_p M$.

Definition 3. A null 2-plane $\Pi \subset T_p M$ is a two-dimensional subspace spanned by null vectors $k, \ell \in T_p M$ such that $g(k, k) = g(\ell, \ell) = 0$ and $g(k, \ell) \neq 0$.

Definition 4. A map $F : M_i \rightarrow M_{i+1}$ is null-preserving if it is C^2 , maps every null vector $v \in T_p M_i$ to a null vector $F_*v \in T_{F(p)} M_{i+1}$ with respect to g_{i+1} , and satisfies the maximal rank condition $\text{rank } F_* = i$ for all $p \in U \subset M_i$.

Definition 5. A null frame at $p \in M$ is a basis $\{k, \ell, e_A\}$ of $T_p M$, with k, ℓ null and e_A orthonormal spacelike vectors, satisfying

$$g(k, k) = g(\ell, \ell) = 0, \quad g(k, \ell) = 1, \quad g(k, e_A) = g(\ell, e_A) = 0, \quad g(e_A, e_B) = \delta_{AB}.$$

Definition 6. A set of transverse potentials $\{\phi_j\}$ are smooth functions on M_i such that their gradients $\{\nabla\phi_j\}$ are linearly independent, guaranteeing maximal rank conditions for local expansions of F .

Definition 7. The classical Liouville theorem states that any C^∞ local map $F : M \rightarrow M$ preserving null directions is a local conformal diffeomorphism: for each $p \in M$, there exists a neighborhood $U \ni p$ and a smooth positive function $\Omega : U \rightarrow \mathbb{R}^+$ such that

$$F^*g = \Omega^2 g \quad \text{on } U.$$

Global conformality may require additional topological assumptions such as simply connectedness or completeness.

Definition 8. The Weyl tensor C of a pseudo-Riemannian manifold (M, g) of dimension $n \geq 4$ is the trace-free part of the Riemann tensor:

$$C_{abcd} = R_{abcd} - \frac{1}{n-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{R}{(n-1)(n-2)}g_{a[c}g_{d]b}.$$

It is conformally invariant and vanishes identically for $n = 3$.

Assumption 1. The manifolds (M_i, g_i) and (M_{i+1}, g_{i+1}) are connected, smooth (C^∞), and possess trivial holonomy. Additionally, M_{i+1} is assumed simply connected to allow global extensions of local conformal maps.

Assumption 2. The Weyl tensor C is assumed to be real-analytic (or at least C^∞ with unique continuation property) to guarantee the applicability of Liouville-type uniqueness results in interdimensional settings.

Assumption 3. All transverse potentials ϕ_j satisfy $\nabla\phi_j \neq 0$ and are linearly independent to ensure maximal rank of associated maps and expansions.

Null-preserving maps can also be generalized to *null-projective equivalence* between semi-Riemannian metrics [11], offering a modern perspective on interdimensional rigidity.

3. Correction Tensor

The central difficulty in the interdimensional setting lies in the fact that a direct conformal relation of the form $F^*g_{i+1} = \Omega^2 g_i$ cannot hold when the dimensions differ. This creates the need to measure precisely how far a null-preserving map deviates from behaving conformally. Since classical Liouville and null-cone rigidity results provide no quantitative tool for such deviations, we introduce a tensor that isolates the conformally compatible component of F^*g_{i+1} from the part responsible for distortion. This leads naturally to the correction tensor :

$$T = F^*g_{i+1} - \Delta(i)^2 g_i \tag{1}$$

where $\Delta(i)$ is the dimension-adjusted conformal weight(see section 4). By construction, T vanishes on all null directions, i.e. $T(v, v) = 0$ for every null vector $v \in TM_i$, and $T \equiv 0$ precisely when F behaves like a conformal embedding modulo dimension shift. Thus, T measures the intrinsic geometric discrepancy produced by interdimensional embedding and identifies the exact obstruction to classical conformal behavior. Nevertheless, the tensor alone does not determine the map; additional smoothness, curvature, and rank conditions remain essential.

The pushforward of the metric under an interdimensional null-preserving map is generally not quadratic in form; hence no direct relation exists between the pullback F^*g_{i+1} and g_i [4,5,9].

Definition 9. Let $F : M_i \rightarrow M_{i+1}$ be a C^2 smooth null-preserving map between pseudo-Riemannian manifolds (M_i, g_i) and (M_{i+1}, g_{i+1}) . The correction tensor T is the symmetric $(0, 2)$ -tensor on M_i defined by :

$$F^* g_{i+1} = g_i + T, \quad (2)$$

and satisfies the null-preserving condition:

$$T(X, X) = 0, \quad \forall X \in T_p M_i \text{ null w.r.t. } g_i. \quad (3)$$

Equivalently, in a local orthonormal or null frame $\{e_a\}$ at $p \in M_i$, we write

$$T(e_a, e_b) = F^* g_{i+1}(e_a, e_b) - g_i(e_a, e_b), \quad (4)$$

independent of any coordinate chart. The smoothness of T ensures well-defined covariant derivatives ∇T and $\nabla^2 T$ for curvature computations.

Proposition 1. T is symmetric as a $(0, 2)$ -tensor:

$$T(X, Y) = T(Y, X), \quad \forall X, Y \in T_p M_i. \quad (5)$$

Proposition 2. For any null vector $k \in T_p M_i$ w.r.t. g_i :

$$T(k, k) = 0. \quad (6)$$

This ensures that T does not alter null directions, consistent with F being null-preserving.

Proposition 3. Let $U \subset M_i$ be an open neighborhood. If F satisfies the rank condition in U (i.e., $\text{rank}(F_*) = i$ at all $p \in U$), then:

$$\text{rank}((g_i + T)|_{\text{null}}) = \text{rank}(g_i|_{\text{null}}), \quad (7)$$

pointwise. This guarantees that T preserves the dimension of null subspaces locally.

Remark 1. In any frame $\{e_a\}$, T can be expressed as at most quadratic in derivatives of F :

$$T(e_a, e_b) = O(F_* \otimes F_*) \quad (8)$$

with linear and quadratic terms in first derivatives of F . This tensorial expression is independent of coordinates and is crucial for expressing pullbacks of curvature tensors and applying Liouville-type rigidity results.

For null hypersurfaces and quasi-conformal structures, see [13,14], which provides relevant results for submanifold embeddings.

4. Dimension-Adjusted Conformal Weight

Let $F : M_i \rightarrow M_{i+1}$ be a C^∞ null-preserving map with correction tensor T defined by

$$F^* g_{i+1} = g_i + T \quad (9)$$

where g_i and g_{i+1} are pseudo-Riemannian metrics on M_i and M_{i+1} , respectively.

Definition 10. For any $p \in M_i$ and null vector $v \in T_p M_i$ ($g_i(v, v) = 0$), the null preservation condition implies:

$$T(v, v) = 0. \quad (10)$$

Consider a null 2-plane $\Pi \subset T_p M_i$ spanned by two linearly independent null vectors v, w with $g_i(v, w) \neq 0$. The dimension-adjusted conformal weight $\Delta(i) : M_i \rightarrow \mathbb{R}^+$ is defined by :

$$(F^* g_{i+1})(v, w) = \Delta(i)^2 g_i(v, w) \quad (11)$$

$$\Delta(i)^2 = 1 + \frac{T(v, w)}{g_i(v, w)}. \quad (12)$$

Lemma 1. Let $\Pi \subset T_p M_i$ be a null 2-plane spanned by non-collinear null vectors v, w with $g_i(v, w) \neq 0$. If $\{v', w'\}$ is any other null basis of Π , related to $\{v, w\}$ by an $\text{SO}(1, 1)$ transformation, then

$$\frac{(F^* g_{i+1})(v, w)}{g_i(v, w)} = \frac{(F^* g_{i+1})(v', w')}{g_i(v', w')}.$$

Hence the dimension-adjusted conformal weight $\Delta(i)^2$ is well defined on Π .

Proof. Let $\Pi \subset T_p M_i$ be a null 2-plane of signature $(1, 1)$, and let $\{v, w\}$ be a null basis with $g_i(v, w) \neq 0$. Any other null basis $\{v', w'\}$ of Π is related by an element of $\text{SO}^+(1, 1)$, so that

$$v' = av, \quad w' = a^{-1}w, \quad a > 0.$$

This follows from the fact that the null cone in a Lorentzian 2-plane consists of the two isotropic lines, and any orientation-preserving linear transformation that maintains nullness acts by reciprocal scaling on the basis vectors.

Since both g_i and $F^* g_{i+1}$ are bilinear,

$$(F^* g_{i+1})(v', w') = (F^* g_{i+1})(av, a^{-1}w) = (F^* g_{i+1})(v, w),$$

and likewise

$$g_i(v', w') = g_i(av, a^{-1}w) = g_i(v, w).$$

Hence, the ratio

$$\frac{(F^* g_{i+1})(v, w)}{g_i(v, w)}$$

is invariant under the action of $\text{SO}^+(1, 1)$ on null bases of Π , and therefore intrinsic to the plane Π itself. $\square$

Remark 2. Since any change of null basis of the 2-plane Π is given by an $\text{SO}(1, 1)$ -rescaling and F is null-preserving, the ratio

$$\frac{(F^* g_{i+1})(v, w)}{g_i(v, w)}$$

is invariant. Hence the dimension-adjusted conformal weight $\Delta(i)$ is well-defined on Π .

Lemma 2. Let $\{k, \ell, e_A\}$ be a local normalized null frame at $p \in M_i$ such that :

$$g_i(k, \ell) = 1, \quad g_i(k, k) = g_i(\ell, \ell) = 0, \quad g_i(k, e_A) = g_i(\ell, e_A) = 0.$$

Then, in this canonical frame,

$$\Delta(i)^2 = 1 + T(k, \ell) \quad (13)$$

so the dimension-adjusted conformal weight depends solely on T and is coordinate-independent.

Remark 3. For any null 2-plane in $T_p M_i$, the pullback of g_{i+1} restricted to that plane satisfies

$$F^* g_{i+1}|_{\Pi} = \Delta(i)^2 g_i|_{\Pi} \quad (14)$$

so $\Delta(i)$ captures the exact scaling of null directions induced by F . It provides a rigorous, coordinate-independent measure of interdimensional scaling, quantifying deviation from classical conformality while respecting the null structure.

5. Pullback of the Weyl Tensor

Throughout this section, we assume $i \geq 3$ so that all denominators in the Weyl tensor formula are well-defined. Let $F : M_i \rightarrow M_{i+1}$ be a C^3 smooth null-preserving map, so that the correction tensor T is C^2 smooth and covariant derivatives ∇T and $\nabla^2 T$ exist classically.

A detailed understanding of F^*g_{i+1} requires decomposing it relative to the null structure of (M_i, g_i) . To achieve this, we work in a normalized null frame $\{k, \ell, e_A\}$, which allows us to separate the contributions along null, mixed, and transverse directions. The key structural identity arises from the component $F^*g_{i+1}(k, \ell)$, which satisfies :

$$F^*g_{i+1}(k, \ell) = \Delta(i)^2, \quad (15)$$

$$\Delta(i)^2 = 1 + T(k, \ell). \quad (16)$$

This expresses the dimension-adjusted conformal weight entirely in terms of the correction tensor and shows that $\Delta(i)$ captures the exact scaling of null directions under F . The decomposition of F^*g_{i+1} obtained in this frame provides a precise local description of how null geodesics and spatial directions transform. However, the analysis remains purely local and may break down when approaching singularities of the null distribution.

Lemma 3. Let C_{i+1} denote the Weyl tensor of (M_{i+1}, g_{i+1}) . Then the pullback under F satisfies

$$F^*C_{i+1} = C_i + P[T, \nabla T, \nabla^2 T], \quad (17)$$

where C_i is the Weyl tensor of (M_i, g_i) and $P[T, \nabla T, \nabla^2 T]$ is a universal tensorial expression depending linearly and quadratically on T and its first and second covariant derivatives with respect to g_i .

In any orthonormal or null frame $\{e_a\}$ at $p \in M_i$, the expression of P is

$$\begin{aligned} P(e_a, e_b, e_c, e_d) &= Q(e_a, e_b, e_c, e_d) \\ &\quad - \frac{2}{i-2} (g_i(e_a, e_c) Q(e_d, e_b) - g_i(e_b, e_c) Q(e_d, e_a)) \\ &\quad + \frac{2}{(i-1)(i-2)} g_i(e_a, e_c) g_i(e_d, e_b) \operatorname{tr}_{g_i}(Q[T]). \end{aligned} \quad (18)$$

Here $Q[T]$ denotes the universal $(0,4)$ -tensor obtained from T , ∇T and $\nabla^2 T$ by contracting all admissible index combinations allowed by the symmetries of the Weyl tensor. In particular, $Q[T]$ is multilinear, satisfies the algebraic Weyl symmetries, and vanishes when $T \equiv 0$.

Proof. Let (M_i, g_i) and (M_{i+1}, g_{i+1}) be smooth pseudo-Riemannian manifolds, and let $F : M_i \rightarrow M^{i+1}$ be a C^3 null-preserving map of maximal rank with pushforward F_* . Denote by T the correction tensor measuring the deviation of F^*g_{i+1} from g_i , so that

$$F^*g_{i+1} = g_i + T.$$

Since $T \in C^2$, its first and second covariant derivatives ∇T and $\nabla^2 T$ exist classically. In a normalized null frame $\{k, \ell, e_A\}$, the pullback satisfies $F^*g_{i+1}(k, \ell) = 1 + T(k, \ell)$, capturing the scaling of null directions under F .

The Levi-Civita connection associated with F^*g_{i+1} expands as

$$\nabla^{F^*g_{i+1}} = \nabla^{g_i} + \Gamma[T],$$

where $\Gamma[T]$ depends linearly on ∇T and quadratically on T . Substituting into the Riemann tensor, we obtain

$$R(F^*g_{i+1}) = R(g_i) + Q[T, \nabla T, \nabla^2 T],$$

where

$$Q_{abcd}[T] = (\nabla^2 T)_{abcd} + (\nabla T \cdot \nabla T)_{abcd} + (T \cdot \nabla T)_{abcd} + (T \cdot T)_{abcd},$$

encodes deviations due to T and its covariant derivatives up to second order. The Weyl tensor, being the traceless projection of the Riemann tensor, satisfies

$$C_{i+1} = R_{i+1} - \frac{1}{i-1} g_{i+1} \odot \text{Ric}_{i+1} + \frac{R_{i+1}}{2i(i-1)} g_{i+1} \odot g_{i+1}.$$

Pulling back this expression and using linearity of the traceless projection yields

$$F^*C_{i+1} = C_i + P[T, \nabla T, \nabla^2 T],$$

where $P[T, \nabla T, \nabla^2 T]$ is universal, coordinate-independent, and vanishes identically if $T \equiv 0$. In a null frame $\{k, \ell, e_A\}$, P naturally decomposes along null, mixed, and transverse directions, with contributions

$$P(k, \ell, k, \ell) \sim (\nabla^2 T + \nabla T \cdot \nabla T + T \cdot \nabla T + T \cdot T)(k, \ell, k, \ell),$$

$$P(k, e_A, \ell, e_B) \sim (\nabla^2 T + \dots)(k, e_A, \ell, e_B),$$

$$P(e_A, e_B, e_C, e_D) \sim (\nabla^2 T + \dots)(e_A, e_B, e_C, e_D),$$

making explicit the local effect of T without enumerating all i^4 components. The construction is local: $P[T, \nabla T, \nabla^2 T]$ is defined pointwise in any neighborhood $U \subset M_i$ where F satisfies rank and smoothness assumptions, and is valid in any smooth frame.

Collecting all contributions, we conclude :

$$F^*C_{i+1} = C_i + P[T, \nabla T, \nabla^2 T],$$

with P universal, local, and vanishing when $T \equiv 0$. This completes the analytical proof.

□

Remark 4. $F \in C^3$ and $T \in C^2$ are required for classical existence of ∇T and $\nabla^2 T$. All expressions are coordinate-independent and valid in any smooth frame. Frame-independent analysis of null directions can be supported by geometric results on the null tangent bundle [15,16].

Remark 5. All statements are local: $P[T, \nabla T, \nabla^2 T]$ is defined pointwise in any open neighborhood $U \subset M_i$ where F satisfies the rank condition and smoothness assumptions.

6. Local Existence of Null-Preserving Maps

The local existence of null-preserving maps between manifolds of differing dimensions is non-trivial, as the defining conditions form an overdetermined system of partial differential equations. Classical existence theorems address only equal-dimensional conformal maps; hence, a new framework is required. Our approach exploits the integrability of the null distribution, the smoothness of the dimension-adjusted conformal weight $\Delta(i)$, and the compatibility constraints linking the correction tensor T to the curvatures of M_i and M_{i+1} . Under these conditions, the system defining a null-preserving map admits local solutions of maximal rank, realizing the geometric structure encoded by T and $\Delta(i)$

in neighborhoods of Σ_i . These solutions, however, may not extend globally, as rank or integrability conditions can fail outside small neighborhoods.

Proposition 4. *Let (M_i, g_i) be a smooth pseudo-Riemannian manifold of dimension $i \geq 3$, and let $\Sigma_i \subset M_i$ be a smooth embedded codimension-1 submanifold. Then there exists an open neighborhood $U \supset \Sigma_i$ and a C^∞ map*

$$F : U \rightarrow M_{i+1}$$

satisfying the following:

1. F preserves null vectors.
2. F has maximal rank: $\text{rank}(F_*) = i$ at every point in U .
3. F admits transverse potentials $\phi_j \in C^\infty(U)$, $j = 1, \dots, m$, parametrizing displacements along extra dimensions, with linearly independent gradients.

Proof. Let $\Sigma_i \subset M_i$ be a smooth codimension-1 submanifold and choose a smooth splitting of the tangent bundle along Σ_i :

$$TM_i|_{\Sigma_i} = T\Sigma_i \oplus N\Sigma_i,$$

where $N\Sigma_i$ is a smooth transverse bundle. Introduce local coordinates $(x^1, \dots, x^i)$ adapted to Σ_i , so that $\Sigma_i = \{x^i = 0\}$ and $(x^1, \dots, x^{i-1})$ parametrize directions tangent to Σ_i .

Let $k \in TM_i$ be a smooth null vector field defined in a neighborhood $U \subset M_i$ of Σ_i , tangent to the null cone of g_i , and extend k to generate a local null foliation $\mathcal{F}$ with leaves N_p through each $p \in U$. Define smooth transverse potentials $\phi_j : U \rightarrow \mathbb{R}$, $j = 1, \dots, m$, with gradients satisfying

$$g_i(k, \nabla \phi_j) = 0 \quad \forall k \in \mathcal{F}_p,$$

so that each ϕ_j varies only in directions transverse to the null foliation. Assume $\{\nabla \phi_j\}$ are linearly independent at each point to ensure maximal rank of the pushforward.

Define the map

$$F : U \rightarrow M_{i+1}, \quad F(x^1, \dots, x^i) = (x^1, \dots, x^i, \phi_1(x), \dots, \phi_m(x)).$$

For any null vector $v \in T_p M_i$, the pushforward is

$$F_* v = (v^1, \dots, v^i, \sum_{\mu} \partial_{\mu} \phi_1 v^{\mu}, \dots, \sum_{\mu} \partial_{\mu} \phi_m v^{\mu}),$$

with squared norm

$$g_{i+1}(F_* v, F_* v) = g_i(v, v) + \sum_j (\partial_{\mu} \phi_j v^{\mu})^2 = 0,$$

showing that F preserves null directions exactly. By Lemma 4.1, the dimension-adjusted conformal weight $\Delta(i)$ depends only on the correction tensor T , making null preservation coordinate-independent.

The differential F_* has rank i due to linear independence of $\nabla \phi_j$, and smoothness follows from that of ϕ_j and the local coordinates. Extending along integral curves of transverse vectors in $N\Sigma_i$ produces an open neighborhood $U \supset \Sigma_i$ where F remains C^∞ , null-preserving, and of maximal rank, completing the construction. $\square$

The construction of the local null-preserving map F around a codimension-1 submanifold $\Sigma_i \subset M_i$ follows standard results in differential geometry concerning tubular neighborhoods and local embeddings. For details, see [10,17].

Remark 6. The construction is purely local and relies only on the smooth structure of (M_i, g_i) and existence of a null foliation with transverse potentials. It provides a rigorous mathematical foundation for local interdimensional null-preserving maps without reference to physical interpretations.

7. Liouville-Type Uniqueness

Lemma 4. Let V be a Lorentzian vector space of dimension $n \geq 3$, and let $L : V \rightarrow V$ be linear and null-cone preserving:

$$g(v, v) = 0 \implies g(Lv, Lv) = 0 \quad \forall v \in V \text{ null}.$$

Then L is conformal: $L^*g = \lambda^2 g$ for some $\lambda \neq 0$.

Proof. Choose a null basis $\{k, \ell, e_A\}$ with $g(k, \ell) = 1$ and $g(e_A, e_B) = \delta_{AB}$. Null-cone preservation implies L maps null lines to null lines. Linear constraints on the coefficients of L enforce a uniform scaling on all cross-terms, yielding $L^*g = \lambda^2 g$ for some $\lambda \neq 0$ (see [11] for details). $\square$

We generalize this classical rigidity to interdimensional maps. In dimensions i and $i + 1$, a null-preserving map F of maximal rank, satisfying Weyl-compatibility and controlled correction tensor T , is uniquely determined by its action on null directions and curvature data. This leads to an interdimensional Liouville-type uniqueness theorem, fundamentally local and dependent on smoothness, rank, and curvature regularity.

Theorem 1. Let $F, G : U \subset M_i \rightarrow M_{i+1}$ be C^∞ null-preserving maps with

$$\text{rank}(F_*) = \text{rank}(G_*) = i \quad \text{on } U.$$

Assume:

1. $i + 1 \geq 3$,
2. $W := F(U) \cap G(U)$ is nonempty and connected,
3. the Weyl tensor C_{i+1} is real-analytic on W (or C^∞ with unique continuation),
4. $F = G$ on a smooth codimension-1 submanifold $\Sigma_i \subset U$, if given.

Then there exists a unique C^∞ conformal diffeomorphism

$$H : W \rightarrow W, \quad H^*g_{i+1} = \Omega^2 g_{i+1},$$

such that

$$G = H \circ F \quad \text{on } U \cap F^{-1}(W),$$

where $\Omega > 0$ is uniquely determined by the frame-independent correction tensors of F and G . If $F = G$ on Σ_i , then $H|_{F(\Sigma_i)} = \text{id}$ up to the conformal factor.

Proof. Fix $p \in U$ with images $q = F(p)$ and $r = G(p)$. Let $L_F = F_*|_p$ and $L_G = G_*|_p$. By null-cone rigidity, L_F and L_G are conformal on their images, i.e., $L_F^*g_{i+1} = \lambda_F^2 g_i$ and $L_G^*g_{i+1} = \lambda_G^2 g_i$ for some $\lambda_F, \lambda_G \neq 0$. Define a linear map

$$H_q := L_G \circ L_F^{-1} \quad \text{on } L_F(T_p M_i) \subset T_q M_{i+1}.$$

H_q preserves null directions, hence is conformal: $H_q^*g_{i+1} = \lambda(q)^2 g_{i+1}$ with smooth $\lambda(q)$.

Extend H to W by $H = G \circ F^{-1}$ locally. Rank and smoothness guarantee local inverses and a smooth global patch over W . The conformal factor is uniquely fixed by the equality of the pullback Weyl tensors and the frame-independent correction tensors. If $F = G$ on Σ_i , the construction yields $H = \text{id}$ on $F(\Sigma_i)$ up to Ω , and connectedness propagates this identity throughout W . $\square$

8. Local Stability of Null 2-Planes

This section presents the local stability theorem for null 2-planes under null-preserving maps, with explicit estimates that express the correction tensor T in terms of the differential dF and the target metric g_{i+1} . We fix an auxiliary Riemannian metric h on U to measure norms; all constants below depend smoothly on the local geometry (metrics and charts) but the geometric conclusion is independent of the particular choice of h .

Lemma 5. *Let $F : U \rightarrow M_{i+1}$ be C^2 and let x^a (resp. y^α) be local coordinates on $U \subset M$ (resp. on $F(U) \subset M_{i+1}$). Then in these coordinates the pullback metric has components*

$$(F^*g_{i+1})_{ab}(x) = \partial_a F^\alpha(x) \partial_b F^\beta(x) (g_{i+1})_{\alpha\beta}(F(x)),$$

and the correction tensor T defined by $F^*g_{i+1} = g_i + T$ has components:

$$T_{ab}(x) = \partial_a F^\alpha(x) \partial_b F^\beta(x) (g_{i+1})_{\alpha\beta}(F(x)) - (g_i)_{ab}(x).$$

Proof. Let (x^a) be coordinates on U and (y^α) coordinates on $F(U)$. By definition of the pullback of a $(0,2)$ -tensor,

$$(F^*g_{i+1})_{ab}(x) = g_{i+1}(dF(\partial_a), dF(\partial_b)).$$

In coordinates, $dF(\partial_a) = \partial_a F^\alpha \partial_\alpha$, hence

$$(F^*g_{i+1})_{ab} = \partial_a F^\alpha \partial_b F^\beta (g_{i+1})_{\alpha\beta}(F(x)).$$

The correction tensor is defined by $T_{ab} = (F^*g_{i+1})_{ab} - (g_i)_{ab}$, giving the stated formula. $\square$

Lemma 6. *Fix a relatively compact neighborhood V with $\bar{V} \subset U$ and choose a smooth Riemannian metric h on U . There exists a constant*

$$C_0 = C_0(g_i, g_{i+1} \circ F, |dF|_h) > 0$$

such that for all $x \in V$ and all vectors $u, w \in T_x M$,

Proof. We expand the coordinate formula of Lemma 5 about x . Write

$$\begin{aligned} T_{ab}(y) - T_{ab}(x) &= (\partial_a F^\alpha(y) \partial_b F^\beta(y) - \partial_a F^\alpha(x) \partial_b F^\beta(x)) g_{\alpha\beta}(F(y)) \\ &\quad + \partial_a F^\alpha(x) \partial_b F^\beta(x) (g_{\alpha\beta}(F(y)) - g_{\alpha\beta}(F(x))) - (g_i)_{ab}(y) + (g_i)_{ab}(x). \end{aligned}$$

Each difference is bounded by products of uniform sup-norms: e.g.

$$|\partial_a F^\alpha(y) \partial_b F^\beta(y) - \partial_a F^\alpha(x) \partial_b F^\beta(x)| \leq C_1 \|dF(y) - dF(x)\|_{op,h},$$

with C_1 depending on $|dF|_h$. Also

$$|g_{\alpha\beta}(F(y)) - g_{\alpha\beta}(F(x))| \leq \|\nabla g_{i+1}\|_{L^\infty(F(V))} \|F(y) - F(x)\|_{h'},$$

where h' is any chart metric on the target; by mean value $\|F(y) - F(x)\| \leq |dF|_h \text{dist}_h(y, x)$, and hence the sup over $y \in V$ yields a bound by $\|g_{i+1} \circ F - g_{i+1} \circ F(x)\|_{C^0(V)}$. Finally the terms involving g_i are controlled by $\|g_i - g_i(x)\|_{C^0(V)}$. Combining these, and contracting $T_{ab}(y)$ with $u^a w^b$, we obtain the claimed inequality with a constant C_0 that is a polynomial in the sup norms indicated. $\square$

Proposition 5. Under the hypotheses above, fix $x = p$. There exist constants $A, B > 0$ depending only on $|dF|_h$, $|d^2F|_h$ and $\|\nabla g_{i+1}\|$ on V such that

$$T \leq A \|dF - dF(p)\|_{op,h} + B \|F - F(p)\|_{C^0(V)} + |g_i - g_i(p)|.$$

In particular, if $\|dF - dF(p)\|_{C^0(V)}$ and $\|F - F(p)\|_{C^0(V)}$ are small then $\|T\|_{C^0(V)}$ is small.

Proof. From Lemma 6 we have a linear control of $|T_x(u, w)|$ by the three sup-terms. The term $\|F - F(p)\|_{C^0(V)}$ controls $g_{i+1} \circ F - g_{i+1} \circ F(p)$ via $\|\nabla g_{i+1}\|_{L^\infty(F(V))}$. The constants A, B are explicit (polynomials) in $|dF|_h$ and $|d^2F|_h$ because Taylor expand $\partial_a F^\alpha(y)$ around p and bound the remainder by $\|d^2F\|_{L^\infty(V)} \cdot \text{diam}(V)$. Thus the stated inequality follows. $\square$

We can now state and prove the strong quantitative local stability theorem.

Theorem 2. Let the hypotheses be as in Lemma 5, and fix $V \Subset U$ with $p \in V$. Choose an auxiliary Riemannian metric h on U and denote by $\cdot$ the corresponding sup norm over V . Let $\Pi \subset T_p M$ be a null 2-plane at p . Then for all nonzero $v \in \Pi$ one has the bound

$$|g_{i+1}(F_*v, F_*v)| \leq \left(A \|dF - dF(p)\| + B \|F - F(p)\| + |g_i - g_i(p)| \right) \|v\|_h^2,$$

where $A, B > 0$ are the constants of Proposition 5. Consequently, if $\|dF - dF(p)\|_{C^0(V)}$ and $\|F - F(p)\|_{C^0(V)}$ (and $\|g_i - g_i(p)\|_{C^0(V)}$) are made arbitrarily small by shrinking V , then the invariance ratio

$$r_h(\Pi) := \sup_{v \in \Pi \setminus \{0\}} \frac{|g_{i+1}(F_*v, F_*v)|}{\|v\|_h^2}$$

satisfies $r_h(\Pi) \rightarrow 0$.

Proof. By definition $g_{i+1}(F_*v, F_*v) = (F^*g_{i+1})(v, v) = g_i(v, v) + T_p(v, v)$. Since $v \in \Pi$ is null w.r.t. g_i , $g_i(v, v) = 0$ and therefore

$$|g_{i+1}(F_*v, F_*v)| = |T_p(v, v)| \leq T \|v\|_h^2.$$

Applying Proposition 5 yields the stated quantitative bound. The convergence statement follows by noting that all sup-norm terms on the right can be made arbitrarily small by taking V sufficiently small (smoothness of F and g guarantees this). This proves the theorem. $\square$

Remark 7. The constants A, B may be written explicitly in terms of polynomials in $|dF|_h$, $|d^2F|_h$ and $\|\nabla g_{i+1}\|$. For applications one may compute these in local coordinates for an explicit F .

9. Global Extension under Topological Constraints

Let (M_i, g_i) and (M_{i+1}, g_{i+1}) be smooth pseudo-Riemannian manifolds and $F : U \subset M^i \rightarrow M_{i+1}$ a smooth null-preserving map of maximal rank with correction tensor T defined by $F^*g_{i+1} = g_i + T$.

Proposition 6. Assume: (i) M_{i+1} is simply connected, (ii) holonomy of M_{i+1} is trivial along $F(U)$, (iii) the local null foliation on U extends smoothly along integral curves. Then there exists a global smooth map $\tilde{F} : M_i \rightarrow M^{i+1}$ and a smooth positive function Ω such that :

$$\tilde{F}^*g_{i+1} = \Omega^2(g_i + T),$$

and $\tilde{F}$ preserves null directions globally.

Proof. Let $\{\varphi_j\}$ be a set of local transverse potentials defined on $U \subset M_i$. Since M_{i+1} is simply connected, any loop in M_{i+1} is contractible, and hence the potentials φ_j can be lifted consistently along

any path in M_{i+1} , making them globally well-defined up to constants. Trivial holonomy along $F(U)$ ensures that parallel transport along any loop preserves the null cone structure in $T_{F(p)}M_{i+1}$ for each $p \in U$. In particular, if $v \in T_pM_i$ is null, then for the pushforward $F_*v \in T_{F(p)}M_{i+1}$, we have F_*v lying in the null cone of g_{i+1} , by the frame-independent null-cone rigidity lemma [11,18].

Let k be the null vector field generating the local foliation. The assumption that the local null foliation extends smoothly along integral curves allows us to define a global foliation $\mathcal{F}$ of M_i . For each $p \in M_i$, let $\gamma : [0, 1] \rightarrow M_i$ be an integral curve of k with $\gamma(1) = p$ and $\gamma(0) \in U$. We define the global map:

$$\tilde{F}(p) := \lim_{t \rightarrow 1^-} F(\gamma(t)).$$

By the smooth dependence of F on the base point and the linear independence of $\{\nabla \varphi_j\}$, this limit exists and defines a smooth map on all of M_i . By construction and trivial holonomy, the pushforward of any null vector $v \in T_pM_i$ remains in the null cone of g_{i+1} , so $\tilde{F}$ is globally null-preserving.

Finally, at each point $p \in M_i$, choose null vectors $v, w \in T_pM_i$ with $g_i(v, w) \neq 0$ and define the conformal factor Ω by:

$$\Omega^2(p) := \frac{g_{i+1}(\tilde{F}_*v, \tilde{F}_*w)}{g_i(v, w) + T(v, w)}.$$

Since F and T are smooth and v, w are linearly independent null vectors, $\Omega^2(p)$ is smooth and strictly positive. Therefore, the relation:

$$\tilde{F}^*g_{i+1} = \Omega^2(g_i + T)$$

holds globally on M_i , completing the proof. $\square$

Remark 8. *If any topological condition fails, global extension may be obstructed: nontrivial $\pi_1(M_{i+1})$ may prevent lifting of transverse potentials, and nontrivial holonomy may alter null directions under transport. Then F may only exist locally.*

10. Discussion

In this work, we have developed a rigorous framework for null-preserving maps between pseudo-Riemannian manifolds of differing dimensions. In contrast to classical equal-dimensional embeddings, interdimensional maps encounter intrinsic geometric obstructions: null cones differ in size and structure, generic null directions are not automatically compatible, and exact conformality cannot be assumed without additional geometric structures. To address these challenges, we introduced the correction tensor $T := F^*g_{i+1} - g_i$, which quantifies deviations from exact preservation of the source metric. The sup-norm of T can be locally controlled in terms of the differential dF , the source metric g_i , and the target metric g_{i+1} , as expressed in Proposition 5, providing precise estimates on the deformation of null directions under F . Complementing this, the dimension-adjusted conformal weight $\Delta(i)$ captures the scaling of null directions between manifolds of dimensions i and $i + 1$, ensuring a coherent comparison of null cones across dimensions. Moreover, the pullback of the Weyl tensor affords analytic control over curvature contributions, which is crucial for uniqueness statements and the determination of conformal factors.

Using these constructs, we establish the local existence of null-preserving maps with transverse potentials. For any null 2-plane $\Pi \subset T_pM_i$, there exists a neighborhood $V \subset M_i$ and a smooth map $F : V \rightarrow M_{i+1}$ embedding Π while controlling the deviation from nullity through T . Explicitly, for $v \in \Pi \setminus \{0\}$, we have

$$|g_{i+1}(F_*v, F_*v)| = |T_p(v, v)| \leq T \|v\|_h^2,$$

and by Proposition 5,

$$T \leq A \|dF - dF(p)\|_{op,h} + B \|F - F(p)\|_{C^0(V)} + |g_i - g_i(p)|,$$

so that the invariance ratio

$$r_h(\Pi) := \sup_{v \in \Pi \setminus \{0\}} \frac{|g_{i+1}(F_*v, F_*v)|}{\|v\|_h^2} \longrightarrow 0$$

as V is shrunk, demonstrating the local stability of null 2-planes under smooth perturbations of F . Furthermore, a Liouville-type uniqueness result asserts that two smooth null-preserving maps of maximal rank coincide on overlapping neighborhoods up to a conformal diffeomorphism, with the conformal factor uniquely determined by the analytic properties of the Weyl tensor pullback.

These results collectively form a self-contained theory of interdimensional null-preserving morphisms, linking existence, uniqueness, and stability through explicit, intrinsic estimates. While the present framework guarantees local embeddings, global extensions may be obstructed by topological considerations, such as nontrivial fundamental groups, holonomy, or singularities, highlighting the interplay between local geometric constructions and global manifold structure. The framework also provides a foundation for further developments, including classification of global null-preserving maps, analysis of embeddings with degenerate null structures, preservation of higher-rank tensorial structures, and explicit computation of stability constants for applications in geometric analysis or mathematical physics. Overall, the theory furnishes a precise, intrinsic, and generalizable approach to interdimensional null-preserving mappings, combining theoretical rigor with quantitative control.

11. Conclusions

In this study, we have developed a self-contained and rigorous framework for interdimensional null-preserving morphisms between pseudo-Riemannian manifolds. The introduction of the *correction tensor* T allows for precise quantification of intrinsic geometric deviations in interdimensional embeddings, while the *dimension-adjusted conformal weight* $\Delta^{(i)}$ captures the exact scaling of null directions across different dimensions. By analyzing the pullback of the Weyl tensor under such maps, we obtain explicit control over curvature contributions and their influence on conformality.

Under natural smoothness and rank assumptions, we establish the local existence and uniqueness of null-preserving maps. Moreover, we provide a rigorous quantitative stability result, showing that the deviation from exact null preservation can be made arbitrarily small by restricting to sufficiently small neighborhoods. This provides precise control over the preservation of null structures and clarifies the interplay between curvature, conformality, and null geometry in an interdimensional setting, combining theoretical rigor with computational applicability.

Future research directions include the classification of global null-preserving embeddings, the treatment of singular or degenerate conformal geometries, and extensions to mappings that preserve higher-order tensorial structures, holonomy, or additional geometric invariants. Overall, the results furnish a robust foundation for further exploration in interdimensional differential geometry and related applications in high-codimension embeddings.

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