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Cosmography of the Minimally Extended Varying Speed of Light Model

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Abstract: Cosmography, as an integral branch of cosmology, strives to characterize the Universe without relying on pre-determined cosmological models. This model-independent approach utilizes Taylor series expansions around the current epoch, providing a direct correlation with cosmological observations and the potential to constrain theoretical models. Various observable quantities in cosmology can be described as different combinations of cosmographic parameters. Furthermore, one can apply cosmography to models with a varying speed of light. In this case, the Hubble parameter can be expressed by the same combination of cosmographic parameters for both the standard model and varying speed of light models. However, for the luminosity distance, the two models are represented by different combinations of cosmographic parameters. Hence, luminosity distance might provide a method to constrain the parameters in varying speed-of-light models.

Keywords: cosmography; varying speed of light models; standard cosmological model; Hubble parameter

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1. Introduction

One can characterize the expansion of the Universe by the dimensionless scale factor $a(t)$. The evolution of the scale factor is determined by the Friedmann equations of general relativity, specifically for a spatially homogeneous and isotropic Universe. Recent empirical research indicates an accelerated expansion of the Universe. Essentially, this implies that the second derivative of the scale factor, denoted as $\ddot{a}(t)$, is positive, or equivalently, that the first derivative $\dot{a}(t)$ progressively increases over time. Moreover, theoretically, the scale factor can be Taylor expanded from the current epoch to a nearby time, and the dimensionless coefficients of these expansions are referred to as cosmographic parameters. Although the scale factor is not directly observable, it is possible to constrain cosmographic parameters based on cosmological observations. This approach, utilizing the kinematics of the scale factor, is called cosmography, allowing for the study of the cosmological evolution of the scale factor and, conversely, determining other physical quantities that govern the dynamics of the scale factor [1–30].

Relying solely on the cosmological principle, the method of cosmography serves as a kinematic description of the Universe's evolution, specifically focusing on the dynamics of cosmological expansion. Cosmography provides a versatile framework for handling cosmological parameters, relying solely on the kinematics of the Universe. This model-independent approach eliminates the need for dependence on any specific theoretical model, enabling a more generalized analysis. Notably, it abstains from defining any model a priori, thereby avoiding the necessity to postulate gravity for determining the Universe's dynamics. The effectiveness of cosmography is demonstrated by its ability to choose models consistent with cosmological observations. Cosmography primarily concentrates on the later phases of the Universe's development. The Taylor expansions associated with cosmography typically pertain to the observable domain where $z \ll 1$, facilitating the establishment of constraints on the current Universe. In our discussion, we explore the modification of late-time cosmography to apply the varying speed of light models.

As evident from the above equations, the effects of the meVSL model can manifest in each cosmographic parameter. It can potentially influence the values of the density contrasts in the ΛCDM

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model. In the subsequent follow-up paper, we will discuss the impact of this varying speed-of-light theoretical model on the values of density contrasts, comparing it with the standard model using several cosmological observational datasets. In the following section, we compare the representations of cosmography in the meVSL model for each observable with those in the standard model, highlighting differences. In Section 3, we investigate the expression of the luminosity distance within the framework of the meVSL model. Section 4 describes how cosmographic parameters are represented by density contrasts when applying the LambdaCDM model. The final section concludes with a discussion.

2. Cosmolography of varying speed of light models

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric of varying speed of light (VSL) models is given by

$$ds^2 = -c(t)^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right] , \tag{1}$$

where k is a constant, related to curvature of space and $a(t)$ is the scale factor. It describes the evolution of the Universe. The coordinates (t, r, θ, ϕ) of the RW metric are called comoving coordinates. The spatial coordinates of objects remain the same, but their physical (proper) distance grows with time as space expands. The cosmic time gives the time a comoving observer measures at $(r, \theta, \phi) =$ constant. Let the photon be emitted at time t_* with $r = 0$ and absorbed by an observer located at $r = r_0$ at the present epoch t_0. One can express this time as $t_* = t_0 - \delta t$, where the time difference δt is significantly smaller than the current cosmic time t_0. This relationship implies that the emitting galaxy is close to our location, and the time it takes for the photon to reach us is negligible compared to the current cosmic time. Thus, one can Taylor expand the scale factor about t_0

$$\begin{aligned} a(t_*) &= a(t_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n a(t)}{dt^n}\bigg|_{t=t_0} (t_* - t_0) = a_0 - \dot{a}_0 \delta t + \frac{1}{2!}\ddot{a}_0 \delta t^2 - \frac{1}{3!}\dddot{a}_0 \delta t^3 + \frac{1}{4!} a_0^{(4)} \delta t^4 + \cdots \\ &\equiv 1 - H_0 \delta t - \frac{1}{2} q_0 H_0^2 \delta t^2 - \frac{1}{3!} j_0 H_0^3 \delta t^3 + \frac{1}{4!} s_0 H_0^4 \delta t^4 + \cdots , \end{aligned} \tag{2}$$

where $a(t_0) \equiv a_0 = 1$ denotes the value of the scale factor at the present epoch. The so-called cosmographic parameters H_0, q_0, j_0, and s_0 are Hubble, deceleration, jerk, and snap parameters, respectively. They are dimensionless. In cosmology, the Hubble, deceleration, jerk, and snap parameters characterize the first four-time derivatives of the scale factor. These parameters elucidate the rate and acceleration/deceleration of cosmic expansion, serving as valuable tools for comprehending the behavior and expansion history of the Universe. Widely employed in observational data analysis and cosmological research, they aid in probing cosmological models and measuring significant quantities such as the density of matter in the Universe and the equation of state for dark energy. While the scale factor a is an unobservable quantity, the cosmological redshift z is an observable physical quantity. Moreover, one can also define the cosmological redshift in the vicinity of small values for $\delta z = z_* - z_0 = z_*$ using these parameters as

$$\begin{aligned} z_* = \frac{a_0}{a(t_0 - \delta t)} - 1 \simeq H_0 \delta t + \frac{1}{2!}(2 + q_0)(H_0 \delta t)^2 + \frac{1}{3!}(6 + 6q_0 + j_0)(H_0 \delta t)^3 + \frac{1}{4!} \\ \left(24 + 36 q_0 + 6 q_0^2 + 8 j_0 - s_0\right)(H_0 \delta t)^4 + \cdots , \end{aligned} \tag{3}$$

where z_0 is the present cosmological redshift and equals 0.

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Now, let's apply cosmographic parameters to observable quantities, starting with the Hubble parameter. The Hubble parameter in the Taylor series is

$$H(\delta z) \simeq H(z_0) + \left.\frac{dH}{dz}\right|_{z_0} \delta z + \frac{1}{2!}\left.\frac{d^2 H}{dz^2}\right|_{z_0} \delta z^2 + \frac{1}{3!}\left.\frac{d^3 H}{dz^3}\right|_{z_0} \delta z^3 + \cdots$$
$$\simeq H_0 \left[1 + (1+q_0)\,\delta z + \frac{1}{2!}\left(j_0 - q_0^2\right)\delta z^2 + \frac{1}{3!}\left(3q_0^2 + 3q_0^3 - 4q_0 j_0 - 3j_0 - s_0\right)\delta z^3 + \cdots \right]. \quad (4)$$

This result holds consistently for both the standard cosmology model (SCM) and VSL models.

Considering a null geodesic in an FLRW metric along the line of sight (LOS), the total LOS comoving distance can be obtained using Eq. (1)

$$D_c(r_0) \equiv \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} \equiv S_k^{-1}[r_0] = \left\{ \begin{array}{ll} \frac{1}{\sqrt{k}} \sin^{-1}\left[\sqrt{k} r_0\right] & , k > 0 \\ r_0 & , k = 0 \\ \frac{1}{\sqrt{|k|}} \sinh^{-1}\left[\sqrt{|k|} r_0\right] & , k < 0 \end{array} \right\} = \int_{t_*}^{t_0} \frac{c(t)}{a(t)} dt = \int_{z_0}^{z_*} \frac{c(z)}{H(z)} dz\,. \quad (5)$$

Notably, in contrast to the SCM, the comoving distance D_c is influenced by the z-dependent speed of light. Consequently, the transverse comoving distance r_0 can be derived from Equation (5)

$$r_0(z_*) \equiv S_k[D_c(r_0)] = \left\{ \begin{array}{ll} \frac{1}{\sqrt{k}} \sin\left[\sqrt{k} \int_{z_0}^{z_*} \frac{cdz}{H(z)}\right] & , k > 0 \\ \int_{z_0}^{z_*} \frac{cdz}{H(z)} = D_c & , k = 0 \\ \frac{1}{\sqrt{|k|}} \sinh\left[\sqrt{|k|} \int_{z_0}^{z_*} \frac{cdz}{H(z)}\right] & , k < 0 \end{array} \right. . \quad (6)$$

Given that the acceleration of the Universe is a relatively recent phenomenon, we can focus our analysis on the vicinity of small values for the redshift interval $\delta z = z_* - z_0 = z_*$ in Equation (6). For a short redshift interval $\delta z \equiv z_* \simeq 0$, one can expand the Taylor series of r_0 around z_0

$$r_0(\delta z) \simeq \left\{ \begin{array}{ll} \int_{z_0}^{z_0+\delta z} \frac{c(z)dz}{H(z)} - \frac{k}{3!}\left(\int_{z_0}^{z_0+\delta z} \frac{c(z)dz}{H(z)}\right)^3 & , k > 0 \\ \int_{z_0}^{z_0+\delta z} \frac{c(z)dz}{H(z)} = r_0 & , k = 0 \\ \int_{z_0}^{z_0+\delta z} \frac{c(z)dz}{H(z)} - \frac{k}{3!}\left(\int_{z_0}^{z_0+\delta z} \frac{c(z)dz}{H(z)}\right)^3 & , k < 0 \end{array} \right. . \quad (7)$$

Measuring distances to cosmological objects stands as the primary method for probing the cosmic metric and deciphering the expansion history of the Universe. In the meVSL model, the comoving distance at redshift z_* is given by

$$D_c(r_0) = \int_{z_0}^{z_0+\delta z} \frac{c(z)dz}{H(z)} \simeq \int_{z_0}^{z_0+\delta z} \frac{c_0}{H_0} dz \left[1 + B_0\delta z + C_0 \delta z^2 + D_0 \delta z^3 + \cdots\right] = \frac{c_0}{H_0}\delta z \left[1 + \frac{B_0}{2}\delta z + \frac{C_0}{3}\delta z^2 + \frac{D_0}{4}\delta z^3 + \cdots\right], \quad (8)$$

where

$$B_0 = -\frac{H_0'}{H_0} + \frac{c_0'}{c_0} = -\left(q_0 + 1 + \frac{b}{4}\right), \tag{9}$$

$$C_0 = \frac{1}{2!}\left[-\frac{H_0''}{H_0} + 2\left(\frac{H_0'}{H_0}\right)^2 + \frac{c_0''}{c_0} - 2\frac{c_0'}{c_0}\frac{H_0'}{H_0}\right] = \frac{1}{2}\left[-j_0 + 3q_0^2 + 4q_0 + 2 + \frac{b}{4}\left(\frac{b}{4} + 2q_0 + 3\right)\right], \tag{10}$$

$$D_0 = \frac{1}{3!}\left[-\frac{H_0'''}{H_0} + 6\frac{H_0''}{H_0}\frac{H_0'}{H_0} - 6\left(\frac{H_0'}{H_0}\right)^3 + \frac{c_0'''}{c_0} - 3\frac{c_0''}{c_0}\frac{H_0'}{H_0} - 3\frac{c_0'}{c_0}\left(\frac{H_0''}{H_0} - 2\left(\frac{H_0'}{H_0}\right)^2\right)\right] \tag{11}$$
$$= \frac{1}{3!}\left[s_0 + 9j_0 + 10q_0j_0 - 15q_0^3 - 27q_0^2 - 18q_0 - 6 + \frac{b}{4}\left(\left(3j_0 - 9q_0^2 - 15q_0 - 11\right) - \frac{3}{4}(q_0+2)b - \frac{1}{16}b^2\right)\right].$$

The transverse comoving distance, denoted as $r_0(\delta z)$, can be derived from Eqs. (7) and (8)

$$r_0(\delta z) \simeq \frac{c_0\delta z}{H_0}\left[1 + \frac{1}{2!}B_0\delta z + \frac{1}{3!}\left(2C_0 + \Omega_k\right)\delta z^2 + \frac{1}{4!}\left(6D_0 + 6B_0\Omega_k\right)\delta z^3 + \cdots\right]. \tag{12}$$

The luminosity distance (d_L) for an object is expressed as $d_L = (L/4\pi F)^{1/2}$, where L denotes the luminosity of the observed object (assumed to be known for high-redshift supernovae), and F represents the energy flux received from the object. This formula describes the luminosity distance in a dynamic, homogeneous, isotropic spacetime. Expressing the luminosity distance as a function of δz involves Eqs. (3) and (12)

$$d_L(\delta z) = (1+\delta z)r_0(\delta z) \simeq \frac{c_0\delta z}{H_0}[1 + \frac{1}{2!}(2+B_0)\delta z + \frac{1}{3!}$$
$$\left(3B_0 + 2C_0 + \Omega_k\right)\delta z^2 + \frac{1}{4!}(8C_0 + 6D_0 + (4 + 6B_0\Omega_k)\delta z^3 + \cdots] \equiv \frac{D_L(\delta z)}{H_0}, \tag{13}$$

introducing the concept of the Hubble-free luminosity distance D_L, which remains independent of H_0. These results remain consistent with those found in references [1,2,12], barring the impact of the time-varying speed of light. Consequently, by eliminating the derivation terms associated with c, the obtained results match those of the reference. Notably, our approach doesn't necessitate relying on lookback time to arrive at these outcomes.

3. Observation

The distance modulus (μ) is the logarithmic measure of the ratio between an astronomical object's intrinsic and observed brightness, used for determining its distance. When dealing with standard candles like Cepheid variables or Type Ia supernovae (SNe Ia), the distance modulus is widely used in astronomy to quantify the distance to celestial objects. It represents the difference between the apparent magnitude (*i.e.*, observed brightness), $m(z)$ (preferably corrected for interstellar absorption effects), and the absolute magnitude (*i.e.*, intrinsic brightness), M, of a standard candle. The luminous distance measured in megaparsecs (Mpc), $d_L(z)$, is connected to the distance modulus through

$$\mu(z) = m(z) - M = 5\log_{10}\left[\frac{d_L(z)}{(\text{Mpc})}\right] + 25. \tag{14}$$

By replacing the luminosity distance in the above equation with the Hubble-free luminosity distance, $D_L(z)$, defined in Eq. (13), we obtain

$$\mu(z) = 5\log_{10}\left[\frac{D_L(z)}{H_0(\text{Mpc})}\right] + 25 = 5\log_{10}\left[\frac{D_L(z)}{(\text{km/s})}\right] - 5\log_{10}\left[\frac{H_0}{(\text{km/s/Mpc})}\right] + 25. \tag{15}$$

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Equations (14) and (15) enable us to express the Hubble parameter in units of (km/s/Mpc) as

$$\log_{10}\left[\frac{H_0}{(\mathrm{km/s/Mpc})}\right] = 0.2M + \log_{10}\left[\frac{D_L(z)}{(\mathrm{km/s})}\right] - 0.2m(z) + 5 \equiv 0.2M + a_B(z) + 5\,, \tag{16}$$

where the Hubble intercept parameter, denoted as a_B and employed in the SH0ES (Supernova H0 for the Equation of State) analysis, defines the x-intercept ($m = 0$) of a Hubble diagram plotting $0.2m$ against a modified $\log_{10} D_L$ term [31]. At low redshifts ($z \ll 1$), one can determine a_B using Eqs. (13) and (16),

$$a_B \approx \log_{10}\Big[\frac{cz}{(\mathrm{km/s})}\Big(1 + \frac{1}{2!}(2+B_0)z + \frac{1}{3!}\left(3B_0 + 2C_0 + \Omega_k\right)z^2 + \frac{1}{4!}\left(8C_0 + 6D_0 + (4+6B_0)\,\Omega_k\right)z^3 + \cdots\Big)\Big] - 0.2m(z)\,, \tag{17}$$

where a_B is measured from a group of SNe Ia with known redshifts and magnitudes $(z, m(0))$.

4. Parameters

Cosmography proves to be a valuable tool in constraining cosmological parameters [32–35]. The Hubble parameter in the meVSL model is derived from the modified Friedmann equation, incorporating the ΛCDM model [36]

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \left[\frac{8\pi G_0}{3}\rho_{\mathrm{m}} + \frac{\Lambda c_0^2}{3} - k\frac{c_0^2}{a^2}\right] a^{b/2}\,, \tag{18}$$

$$1 = \left[\frac{8\pi G_0}{3H^2}\rho_{\mathrm{m}} + \frac{\Lambda c_0^2}{3H^2} - \frac{kc_0^2}{a^2H^2}\right] a^{b/2} \equiv \left[\Omega_{\mathrm{m}} + \Omega_\Lambda + \Omega_k\right] a^{b/2}\,, \tag{19}$$

where one defines the density contrasts of each component. Using these density contrasts, the coefficients in equations (9) to (11) can be expressed as

$$B_0 = -\left(\frac{3}{2}\Omega_{\mathrm{m}0} + \Omega_{k0}\right)\,, \tag{20}$$

$$C_0 = \frac{27}{4}\Omega_{\mathrm{m}0}^2 + 3\left(3\Omega_{k0} - 1\right)\Omega_{\mathrm{m}0} + \left(3\Omega_{k0} - 1\right)\Omega_{k0}\,, \tag{21}$$

$$D_0 = -\frac{405}{8}\Omega_{\mathrm{m}0}^3 - \frac{81}{2}\left(\frac{5}{2}\Omega_{k0} - 1\right)\Omega_{\mathrm{m}0}^2 - \frac{3}{2}\left(45\Omega_{k0}^2 - 27\Omega_{k0} + 2\right)\Omega_{\mathrm{m}0} - 3\Omega_{k0}^2\left(5\Omega_{k0} - 3\right)\,. \tag{22}$$

One can also use density contrasts of the ΛCDM model to express cosmographic parameters as

$$q_0 = \frac{3}{2}\Omega_{\mathrm{m}0} + \Omega_{k0} - 1 - \frac{b}{4}\,, \tag{23}$$

$$j_0 = 1 - \Omega_{k0} + \frac{b}{8}\left[b + 6 - 4\left(3\Omega_{\mathrm{m}0} + 2\Omega_{k0}\right)\right]\,, \tag{24}$$

$$s_0 = 1 + \left(\Omega_{k0} + \frac{3}{2}\Omega_{\mathrm{m}0} - 2\right)\Omega_{k0} - \frac{9}{2}\Omega_{\mathrm{m}0} + \frac{b}{8}\left[12 + 8\left(\Omega_{k0} + 3\Omega_{\mathrm{m}0} - 2\right)\Omega_{k0} + 3\Omega_{\mathrm{m}0}\left(6\Omega_{\mathrm{m}0} - 1\right)\right] + \frac{b^2}{16}\left(11 - 18\Omega_{k0} - 27\Omega_{\mathrm{m}0}\right) + \frac{3}{32}b^3\,. \tag{25}$$

As evident from the above equations, the effects of the meVSL model can manifest in each cosmographic parameter. It can potentially influence the values of the density contrasts in the ΛCDM model. In the subsequent follow-up paper, we will discuss the impact of this varying speed-of-light theoretical model on the values of density contrasts, comparing it with the standard model using several cosmological observational datasets.

5. Discussion

It has been well-established that cosmography provides a model-independent means of constraining cosmological parameters. However, in this paper, we demonstrated that if we apply cosmography to a varying speed of light model, the predictions may differ from those offered by the standard cosmological model, depending on cosmological observables. Specifically, we have presented predictions for the minimally extended varying speed of light model in this paper. According to these predictions, cosmological parameters obtained from Hubble parameters or photometric distances can yield different values in the standard model and the meVSL model. We intend to delve deeper into this theoretical possibility through empirical investigations using specific cosmological observations.

Author Contributions: For research articles with several authors, a short paragraph specifying their individual contributions must be provided. The following statements should be used "Conceptualization, X.X. and Y.Y.; methodology, X.X.; software, X.X.; validation, X.X., Y.Y. and Z.Z.; formal analysis, X.X.; investigation, X.X.; resources, X.X.; data curation, X.X.; writing—original draft preparation, X.X.; writing—review and editing, X.X.; visualization, X.X.; supervision, X.X.; project administration, X.X.; funding acquisition, Y.Y. All authors have read and agreed to the published version of the manuscript.", please turn to the CRediT taxonomy for the term explanation. Authorship must be limited to those who have contributed substantially to the work reported.

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