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Article

The Invariant-Mass-Based Equation for Bound States of the Hydrogen and the Helium Atoms

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Abstract: The invariant mass of free particles is used to derive a bound-state equation for several particle atomic systems at rest. This relativistic-kinematic bound-state equation is applied to the hydrogen and helium atoms. The derived equation has the well-known solutions for the single-electron bound states of the hydrogen atom, and the two-electron states of the helium atom. For the hydrogen atom, existence of the two-particle bound states, for which the electron and the proton kinetic energies are of the same order of magnitude, is predicted. The three-particle bound states with the same feature of the kinetic energies can exist in the helium atom. Radiative operators for processes involving the hydrogen two-particle bound states, are obtained. It is discussed that these new two- and three-particle bound states should be optically inactive.

Keywords: bound-state equation; one-electron atom; two-electron atom; radiative transitions

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1. Introduction

In non-relativistic quantum mechanics, bound few-body systems are studied using the Schrödinger equation [1]. The Hamiltonian consist of the sum of the kinetic energies of the particles and the sum of the pairwise interaction between them. Then, the two-body problem is reduced to the single-particle one [1–3]. This statement is also true for the hydrogen atom, for which the solution of the Schrödinger equation is found using the method of variable separation. In the center-of-mass frame, the problem is reduced to the motion of a particle with the electron charge and the reduced mass in the external Coulomb field. The eigenfunctions of the transformed equation are the single-particle ones, and depend only on the relative radius-vector between the electron and the proton, $\psi(\mathbf{r})$, where $\mathbf{r} = \mathbf{r}_e - \mathbf{r}_p$. For the three-body problem, the Faddeev equations are the most often used as non-perturbative formulations of the quantum-mechanical system [1]. Usually, solutions of the equations are carried out using an iteration method.

Note that the hydrogen atom is one of the most studied quantum objects. At present, the optical frequencies of the atom have been measured with very high accuracy [4]. It gives a unique opportunity to use the spectroscopy methods for the experimental verification of atomic level theories.

In quantum electrodynamics, bound several-particle systems are studied using the Bethe-Salpeter equation [5,6]. For two-particle systems and, in particular, for hydrogen atom, the exact solution of the equation is difficult to find even in the simplest ladder approximation for the interaction function [7]. Therefore, the electron motion in the external field of the fixed proton is investigated [5]. In the frame of reference associated with a fixed proton ($\mathbf{r}_p = 0$), the bispinor wave function of the electron depends only on its radius-vector, $\psi(\mathbf{r}_e)$. As a result, the two-particle systems is described by the single-particle bound states for the Dirac equation with the external field [5–9]. For systems of many fermions, the motion of each fermion is described by the Dirac equation in an external field [10]. But now the external potential field requires its self-consistent definition for each particle.

It is usually accepted that in the Hamiltonian $H = H_0 + V$, H_0 is the operator of the energy of free, non-interacting particles, and V is the interaction between them. In the works [11–15] a system of two particles with the same mass was studied. The operator H_0 was represented as the sum of

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the relativistic energies of the free particles whose momenta are equal in absolute value. As a result, the bound-state spectrum of the Hamiltonian $2\sqrt{p^2+m^2}+V$ was studied. That is, the two-particle problem is again reduced to the one-particle problem.

In the present paper, a relativistic-kinematic method to study the bound states of several-particle systems is proposed. The method is based on the fact that in relativistic mechanics the total energy and the total momentum of free particles systems are additive quantities, but their total mass is not additive [16]. However, the mass of the systems is the invariant which does not change when moving from one frame of reference to another. This circumstance makes it possible to use the invariant mass to construct the relativistic-kinematic bound-state equation for composite particle, composed of a number of pairwise interacting particles. Note that this equation is applicable only for the composite at rest.

The invariant-mass-based equations for bound states of the hydrogen and the helium atoms are derived. For the hydrogen atom, existence of the two-particle bound states, for which the electron and the proton kinetic are of the same order of magnitude, is predicted. The three-particle bound states with the same feature of the kinetic energies can exist in the helium atom.

The relativistic-kinematic method allows us to find all possible radiative processes for composites. New one-photon and two-photon radiative operators are obtained for the hydrogen atom. It is discussed that these two-particle bound states of the hydrogen atom and three-particle states for the helium atom are seemly optically inactive, since the optical transitions involving these states, will be suppressed by the large nucleus mass.

Natural units ($\hbar = c = 1$) will be used throughout.

2. The relativistic-kinematic bound-state equation

In relativistic mechanics, the total energy and the total momentum of a the free particles system are additive quantities [16]. For the system with N particles we have:

$$E_N = \sum_{i=1}^{N} \sqrt{m_i^2 + \mathbf{p}_i^2} \tag{1}$$

and

$$\mathbf{P}_N = \sum_{i}^{N} \mathbf{p}_i, \tag{2}$$

where m_i and $\hat{\mathbf{p}}_i$ are the mass and the momentum of the $i-$ particle.

The mass of the system is not additive, and is defined as:

$$m_N = \sqrt{E_N^2 - \mathbf{P}_N^2}. \tag{3}$$

It is important to note that in the relativistic-kinematic theory, the expression for the mass of the system of the free particles (3), taking into account (1) and (2), does not depend on the particle spin. Therefore, this expression (3) is valid for both bosons and fermions. The differences between them can show up in the expressions for the interaction function between particles.

The mass is invariant in all frames of reference, and determines the energy of the system at rest, $H_{0N} = m_N$. Using (3) and replacing the particle momenta $\mathbf{p}_{i=1,\cdots,N}$ with their operators $\hat{\mathbf{p}}_i = -i\nabla_{\mathbf{r}_i}$ where $\mathbf{r}_i$ is the radius-vector of the i-particle, we obtain the differential operator $\hat{H}_{0N}$ which corresponds to the Hamiltonian of the free particle system.

$$\hat{H}_{0N} = \sqrt{\left(\sum_{i=1}^{N} \sqrt{m_i^2 + \hat{\mathbf{p}}_i^2}\right)^2 - \left(\sum_{i}^{N} \hat{\mathbf{p}}_i\right)^2}. \tag{4}$$

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Considering the pair interaction between the particles and using (4), we obtain the Schrödinger-like equation which determines the dynamics of the quantum mechanical system,:

$$i\frac{\partial}{\partial t}\psi(\{\mathbf{r}_i\},t) = H_N\psi(\{\mathbf{r}_i\},t), \tag{5}$$

where H_N is the differential operator,

$$H_N = H_{0N} + \sum_{i<j} V_{ij}(\mathbf{r}_i - \mathbf{r}_i), \tag{6}$$

where V_{ij} is the interaction between i- and j-particles.

For stationary states, Eq. (5) with (6) is reduced to the form:

$$H_N\psi(\{\mathbf{r}_i\}) = E\psi(\{\mathbf{r}_i\}), \tag{7}$$

where E is the energy of the N-particle system.

Note that Eq. (7) with (6) was derived for the N-particle system at rest, and can be used for finding the bound states. The binding energy can be defined as $\mathcal{E} = E - \sum_i^N m_i$. For usual bound states $\mathcal{E} < 0$. For the bound states in the continuum $\mathcal{E} > 0$ [17].

The equation can be applied to study systems with two or more particles. Below we apply Eq. (7) to the two- and three particle systems.

3. The two-particle bound-state equation

For the hydrogen atom, the differential operator $H_{0,N=2}$ is given by:

$$H_{0,N=2} = \sqrt{\left(\sqrt{m^2+p^2} + \sqrt{M^2+q^2}\right)^2 - (\mathbf{p}+\mathbf{q})^2}, \tag{8}$$

where m and $\hat{\mathbf{p}}$ are the mass and the momentum operator of the electron, M and $\hat{\mathbf{q}}$ are the mass and the momentum operator of the proton.

With account for (8), the Schrödinger-like equation takes the form:

$$\left[\sqrt{\left(\sqrt{m^2+p^2} + \sqrt{M^2+q^2}\right)^2 - (\mathbf{p}+\mathbf{q})^2} - \frac{\alpha(1-\hat{\mathbf{v}}_e\hat{\mathbf{v}}_p)}{|\mathbf{r}_e - \mathbf{r}_p|}\right]\psi(\mathbf{r}_e,\mathbf{r}_p) = E\psi. \tag{9}$$

Here $\mathbf{r}_e$ is the electron radius-vector, $\mathbf{r}_p$ is the proton radius-vector of the second particle. The interaction through the vector potential is taken into account, $\hat{\mathbf{v}}_e$ and $\hat{\mathbf{v}}_p$ are the particle's velocity operators.

Note that in quantum electrodynamics the expression $(1 - \hat{\mathbf{v}}_e\hat{\mathbf{v}}_p)$ is replaced with $(1 - \hat{\boldsymbol{\alpha}}_e\hat{\boldsymbol{\alpha}}_p)$, where $\boldsymbol{\alpha}$ matrix should be taken in standard representation [6]. With this replacement, the potential energy on the right side (9) corresponds to the electron-proton interaction function in the ladder approximation with neglecting the interaction retardation. For the hydrogen atom, this retardation contributed to the hyperfine structure of the levels, can be neglected due to the smallness of the fine structure constant.

For the hydrogen atom $m << M$ and the energy $E = \mathcal{E} + m + M$ with $\mathcal{E} \propto -\alpha^2 m$. The average values $| < \hat{\mathbf{p}} > | << m$ and $| < \hat{\mathbf{q}} > | << m$. Then we can expand the left-hand side of Eq. (9) in the power of $\hat{\mathbf{p}}$ and $\hat{\mathbf{q}}$, and restrict ourselves to the term of order of p^4/m^3. The interaction through the vector potential can be omitted since $v_p << v_e \propto \alpha c$, where c is the speed of light in vacuum. As a result, we obtain:

$$\left[\frac{1}{2(m+M)}\left(\sqrt{\frac{M}{m}}\hat{\mathbf{p}} - \sqrt{\frac{m}{M}}\hat{\mathbf{q}}\right)^2 - \frac{\alpha}{|\mathbf{r}_e - \mathbf{r}_p|} - \frac{M^2(M+3m)\hat{\mathbf{p}}^4}{8m^3(M+m)^3}\right]\psi(\mathbf{r}_e,\mathbf{r}_p) = \mathcal{E}\psi. \tag{10}$$

3.1. The center of mass frame

The term proportional to p^4 on the left-hand side of Eq. (9) determines the fine splitting of the energy levels. Given $m << M$, the term is reduced to $p^4/8m^3$ that leads to the well-known expression for the fine structure. Now we omit this term. Then Eq. (10) is reduced to:

$$\left[\frac{1}{2(m+M)}\left(\sqrt{\frac{M}{m}}\hat{\mathbf{p}} - \sqrt{\frac{m}{M}}\hat{\mathbf{q}}\right)^2 - \frac{\alpha}{|\mathbf{r}_e - \mathbf{r}_p|}\right]\psi(\mathbf{r}_e, \mathbf{r}_p) = \mathcal{E}\psi(\mathbf{r}_e, \mathbf{r}_p). \tag{11}$$

One can transform Eq. (11) in terms of new independent variables:

$$\mathbf{R} = \frac{m\mathbf{r}_e + M\mathbf{r}_p}{m+M}, \qquad \mathbf{r} = \mathbf{r}_e - \mathbf{r}_p, \tag{12}$$

Then, it is easy to obtain that the transformed equation is the Schrödinger equation:

$$\left(-\frac{\hbar^2}{2\mu}\Delta_{\mathbf{r}} - \frac{\alpha}{r}\right)\psi(\mathbf{r}) = \mathcal{E}\psi(\mathbf{r}).$$

Here $\mu = \frac{mM}{m+M}$ is the reduced mass. In this equation, the term related to the center-of-mass motion, is absent. This is due to the fact that Eq. (11) were obtained in the reference frame in which the atom is at rest. Then the total momentum of the system is zero. In this case, it makes no sense to use the replacement (12). Instead, solutions of the original equation (11) which depend on the independent variables $\mathbf{r}_e$ and $\mathbf{r}_p$, should be sought.

3.2. Integral presentations

The differential equation (11) corresponds to the following integral equation:

$$\psi(\mathbf{r}_e, \mathbf{r}_p) = -\alpha \int d\mathbf{r}'_e \int d\mathbf{r}'_p \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\mathbf{q}}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{r}_e - \mathbf{r}'_e) + i\mathbf{q}(\mathbf{r}_p - \mathbf{r}'_p)}$$

$$G(\mathbf{p}, \mathbf{q}; \mathcal{E})\frac{1}{|\mathbf{r}'_e - \mathbf{r}'_p|}\psi(\mathbf{r}'_e, \mathbf{r}'_p), \tag{13}$$

where, by analogy with the well-known structure of bound-state integral equations, the function G should be considered as the two-particle propagator in the momentum space:

$$G(\mathbf{p}, \mathbf{q}; \mathcal{E}) = \frac{1}{\mathcal{E} - \frac{1}{2(m+M)}\left(\sqrt{\frac{M}{m}}\mathbf{p} - \sqrt{\frac{m}{M}}\mathbf{q}\right)^2}. \tag{14}$$

In the momentum space Eq. (13) takes the form:

$$\psi(\mathbf{p}, \mathbf{q}) = -\frac{\alpha}{2\pi^2}G(\mathbf{p}, \mathbf{q}; \mathcal{E}) \int \frac{d\mathbf{k}}{(\mathbf{p} - \mathbf{k})^2}\psi(\mathbf{k}, \mathbf{p} + \mathbf{q} - \mathbf{k}) \tag{15}$$

In Eq. (15) the binding energy of the two-particle system is given by:

$$\mathcal{E} = T + U, \tag{16}$$

where the average kinetic energy is:

$$T =< \psi(\mathbf{p}, \mathbf{q})|\frac{1}{2(m+M)}\left(\sqrt{\frac{M}{m}}\mathbf{p} - \sqrt{\frac{m}{M}}\mathbf{q}\right)^2|\psi(\mathbf{p}, \mathbf{q}) > \tag{17}$$

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and the average potential energy is

$$U = - < \psi(\mathbf{p}, \mathbf{q})|\frac{\alpha}{2\pi^2} \int \frac{d\mathbf{k}}{(\mathbf{p}-\mathbf{k})^2} |\psi(\mathbf{k}, \mathbf{p}+\mathbf{q}-\mathbf{k}) > . \tag{18}$$

The wave function is normalized:

$$< \psi(\mathbf{p}, \mathbf{q})|\psi(\mathbf{p}, \mathbf{q}) >= 1 \tag{19}$$

Note that Eq. (15) with definition (14) and Eq. (16) with (17) and (18) are a closed system of two integral equations. This system determines the bound state energy $\mathcal{E}$ and the wave function $\psi(\mathbf{p}, \mathbf{q})$, which must satisfy the normalization condition (19). This system can be solved by an iterative procedure.

Although the hydrogen atom consists of two particles, its well-known states are single-particle ones. Indeed, the known wave functions ψ depend only on the relative radius of the vector between particles. For example, for the ground state $\psi_0(\mathbf{r}) \propto \exp\left(-\frac{|\mathbf{r}_e - \mathbf{r}_p|}{a_B}\right)$, where a_B is the Bohr radius.

According to Eq. (15), two-particle bound-state wave functions, which in the momentum space could depend on the independent momenta of the electron and the proton, $\psi(\mathbf{p}, \mathbf{q})$, could be sought. In this state, the kinetic energies of the proton and the electron are comparable in order of magnitude. The electron kinetic energy must be of the order of $\alpha^2 m$ and, respectively, the average momentum of the electron is $< p > \propto \alpha m$. Then, the proton average momentum should be of the order of $< q > \propto \alpha\sqrt{mM}$. It means that in the supposed state, $< q >>>< p >$. In the stationary steady state, the average momenta of the particles must be zero, $< \mathbf{p} >= 0$ and $< \mathbf{q} >= 0$. In the coordinate space, the electron and the proton could have different scales of intraatomic motions. Characteristic size of the electron orbits are $a_{el} = \lambda\!\!\!^{-}_e/\alpha$, where$\lambda\!\!\!^{-}_e$ is the electron Compton wavelength. Respectively, similar size of the proton motion is approximately 43 times smaller then that for the electron.

Attention is drawn to the symmetry with respect to particle masses in the propagator (14). But this symmetry leads to a surprising result: the proton kinetic energy can be of the order of the electron kinetic energy, and, nevertheless, makes almost no contribution to the electron-proton bound-state energy. Indeed, for $p \simeq q\sqrt{m/M}$ we have $p >> q\frac{m}{M}$. Despite the proton large momentum, the absence of the proton energy helps to the confinement of the particles in the unusual two-particle bound state.

3.3. Radiative transition operators

With the aim to find radiative operators, in Eq. (8) we introduce the canonical momenta:

$$\hat{\mathbf{p}} \to \hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}}_e \qquad \hat{\mathbf{q}} \to \hat{\mathbf{q}} + \frac{e}{c}\hat{\mathbf{A}}_p. \tag{20}$$

Here $\hat{\mathbf{A}}_e$ and $\hat{\mathbf{A}}_p$ are the operators of the vector potentials generated by the electron and proton, respectively. As a result, the differential operator $H_{0,N=2}$ takes the form:

$$H_{0,N=2} = \sqrt{\left(\sqrt{m^2 + (\hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}}_e)^2} + \sqrt{M^2 + (\hat{\mathbf{q}} + \frac{e}{c}\hat{\mathbf{A}}_p)^2}\right)^2 - \left(\hat{\mathbf{p}} - \frac{e}{c}\hat{\mathbf{A}}_e + \hat{\mathbf{q}} + \frac{e}{c}\hat{\mathbf{A}}_p\right)^2}. \tag{21}$$

Then, in the electromagnetic field, Eq. (5) is rewritten as:

$$i\frac{\partial}{\partial t}\psi(\mathbf{r}_e, \mathbf{r}_p, t) = \left[H_2 + \hat{Q} + \hat{H}_{ph}\right]\psi(\mathbf{r}_e, \mathbf{r}_p, t), \tag{22}$$

Here $\hat{H}_2$ is the unperturbed operator given by the left-hand side of (11), $\hat{H}_{ph}$ is the electromagnetic-field Hamiltonian, $\hat{Q}$ is the atom-photon interaction operator, deduced for the single- and two- photon transitions. It consists of three terms:

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1) the electron operators,

$$\hat{Q}_e = \frac{e}{c}\frac{m_p}{m_e(m_e+m_p)}\left[-\hat{\mathbf{A}}_e\hat{\mathbf{p}}_e + \frac{e}{2c}\hat{\mathbf{A}}_e\hat{\mathbf{A}}_e\right], \tag{23}$$

2) the proton operators,

$$\hat{Q}_p = \frac{e}{c}\frac{m_e}{m_p(m_e+m_p)}\left[\hat{\mathbf{A}}_p\hat{\mathbf{p}}_p + \frac{e}{2c}\hat{\mathbf{A}}_p\hat{\mathbf{A}}_p\right], \tag{24}$$

3) the electron-proton operators,

$$\hat{Q}_{ep} = \frac{e}{(m_e+m_p)c}\left[\hat{\mathbf{A}}_e\hat{\mathbf{p}}_p - \hat{\mathbf{A}}_p\hat{\mathbf{p}}_e + \frac{e}{c}\hat{\mathbf{A}}_e\hat{\mathbf{A}}_p\right] \tag{25}$$

The operators (23) present the single-photon and two-photon electron transitions. For these operators, the reduced mass μ is replaced by $\frac{m(m+M)}{M}$. The operators (24) describe single-photon and two-photon proton transitions in hydrogen atom. The effective mass for these processes, $\frac{M(m+M)}{m}$, turns out to be extremely large. Therefore, one should expect very low probabilities for these processes.

The radiative operators (25) are new operators that may be of interest. Here the first term represents the process in which the electron emits the photon, the electromagnetic field of which changes the proton state. A similar process is represented by the second term of the group (25). In this case, the proton emits the photon whose vector potential changes the electron state. The unusual two-photon process corresponds to the third operator in (25). In this process, two photons are simultaneously emitted or absorbed, one of which is by the electron, and the second photon - by the proton. Moreover, the radiative transition energy is now distributed between these two photons.

Of course, in order to calculate the transition probabilities for the group (25) and the two-photon energy distributions, one needs to know the two-particle wave functions $\psi(\mathbf{r}_e, \mathbf{r}_p)$ which satisfy equation (11). In the momentum space, these wave functions are given by Eq. (15).

4. The three-particle bound-state equation

According to Eq. (4), the energy of the three free particles $H_{0,N=3}$ is given by:

$$H_{0,N=3} = \sqrt{\left(\sqrt{m_1^2+\mathbf{p}_1^2} + \sqrt{m_2^2+\mathbf{p}_2^2} + \sqrt{m_3^2+\mathbf{p}_3^2}\right)^2 - (\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)^2}, \tag{26}$$

where m_i and $\mathbf{p}_i$ are the mass and the momentum of i-particle.

Apparently, two three-particle systems may be of interest. One of them is the system of the electrons and the positrons. Also, the helium atom can be considered as the three-particle system [7]. For the helium atom, Eq. (26) is rewritten as:

$$H_{0,N=3} = \sqrt{\left(\sqrt{m^2+\mathbf{p}_1^2} + \sqrt{m^2+\mathbf{p}_2^2} + \sqrt{M^2+\mathbf{p}_3^2}\right)^2 - (\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)^2} \tag{27}$$

where M and $\mathbf{p}_3$ are the mass and the momentum of the helium nucleus, $\mathbf{p}_{1,2}$ are the momenta of the electrons.

Given the interaction between particles, we have the differential operator H_3 for the helium atom:

$$H_3 = H_{0,N=3} + \sum_{i<j} V_{ij}(\mathbf{r}_i - \mathbf{r}_i), \tag{28}$$

where V_{ij} is the interaction between i- and j-particles.

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With account for (28), the Schrödinger-like equation takes the form: three-particle bound-state equation:

$$H_3\psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = (M + 2m + \mathcal{E})\psi, \tag{29}$$

where $\mathcal{E}$ is the binding energy.

4.1. The expansion of the free energy

In the helium atom, $p_{1,2} << m$ and $q << M$. Then we can expand the right-hand side of Eq. (26) in the power of $\hat{\mathbf{p}}_{1,2}$ and $\hat{\mathbf{q}}$, and restrict ourselves to the term of order of $p^4_{1,2}/m^3$. We obtain:

$$H_{0,N=3} = \frac{1}{2(M+2m)}\Big[\Big(\sqrt{\frac{M}{m}}\hat{\mathbf{p}}_1 - \sqrt{\frac{m}{M}}\hat{\mathbf{p}}_3\Big)^2 + \Big(\sqrt{\frac{M}{m}}\hat{\mathbf{p}}_2 - \sqrt{\frac{m}{M}}\hat{\mathbf{p}}_3\Big)^2\Big] + \hat{W}. \tag{30}$$

In Eq. (30), we omitted the term $M + 2m$, which is convenient to introduce in the definition of the binding energy, $\hat{W}$ is the operator determined the fine splitting of the atomic levels:

$$\hat{W} = \frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{2(2m+M)} - \frac{p_1^4 + p_2^4}{8m^3} \tag{31}$$

Note that correlations in the electron motions are taken in the operator (31).

Omitting the $\hat{W}$ term, the differential equation (29) is reduced to:

$$\Big[\frac{1}{2(M+2m)}\Big(\Big(\sqrt{\frac{M}{m}}\hat{\mathbf{p}}_1 - \sqrt{\frac{m}{M}}\hat{\mathbf{p}}_3\Big)^2 + \Big(\sqrt{\frac{M}{m}}\hat{\mathbf{p}}_2 - \sqrt{\frac{m}{M}}\hat{\mathbf{p}}_3\Big)^2\Big)$$

$$+ \frac{\alpha}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{\alpha}{|\mathbf{r}_1 - \mathbf{r}_3|} - \frac{\alpha}{|\mathbf{r}_2 - \mathbf{r}_3|}\Big]\psi = \mathcal{E}\psi. \tag{32}$$

4.2. Integral presentations

The differential equation (32) corresponds to the following integral equation:

$$\psi(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = -\alpha \int\int\int \prod_{i=1}^{3} d\mathbf{r}'_i \int\int\int \prod_{i=1}^{3} \frac{d\mathbf{p}_i}{(2\pi)^3} \exp(i\sum_1^3 \mathbf{p}_i(\mathbf{r}_i - \mathbf{r}'_i))$$

$$G(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathcal{E})\Big(-\frac{1}{|\mathbf{r}'_1 - \mathbf{r}'_2|} + \frac{1}{|\mathbf{r}'_1 - \mathbf{r}'_3|} + \frac{1}{|\mathbf{r}'_2 - \mathbf{r}'_3|}\Big)\psi(\mathbf{r}'_1,\mathbf{r}'_2,\mathbf{r}'_3), \tag{33}$$

where, by analogy with the well-known structure of bound-state integral equations, the function G should be considered as the three-particle propagator in the momentum space:

$$G(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathcal{E}) = \frac{1}{\mathcal{E} - \frac{1}{2(M+2m)}\Big[\Big(\sqrt{\frac{M}{m}}\hat{\mathbf{p}}_1 - \sqrt{\frac{m}{M}}\hat{\mathbf{p}}_3\Big)^2 + \Big(\sqrt{\frac{M}{m}}\hat{\mathbf{p}}_2 - \sqrt{\frac{m}{M}}\hat{\mathbf{p}}_3\Big)^2\Big]}. \tag{34}$$

In the momentum space Eq. (33) takes the form:

$$\psi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) = -\frac{\alpha}{2\pi^2}G(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3;\mathcal{E})\int\frac{d\mathbf{k}}{k^2}$$

$$\Big[\psi(\mathbf{p}_1+\mathbf{k},\mathbf{p}_2-\mathbf{k},\mathbf{p}_3) + \psi(\mathbf{p}_1,\mathbf{p}_2+\mathbf{k},\mathbf{p}_3-\mathbf{k}) + \psi(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2,\mathbf{p}_3+\mathbf{k})\Big] \tag{35}$$

Eqs. (32), (33) and (35) are the different presentations of the three-particle bound-state equation. These equations are greatly simplified if we neglect the motion of the nucleus, $p_3 = 0$. Then, Eq. (32) is rewritten as:

$$\Big[\frac{\hat{\mathbf{p}}_1^2+\hat{\mathbf{p}}_2^2}{2\mu}+\frac{\alpha}{|\mathbf{r}_1-\mathbf{r}_2|}-\frac{\alpha}{r_1}-\frac{\alpha}{r_2}\Big]\psi=\mathcal{E}\psi, \tag{36}$$

Eq. (36) has the usual form of the Schrödinger equation for the helium atom with the electron effective mass $\mu = m(1+\frac{2m}{M})$. For the equation, the eigenfunctions of the three particle system are the two-electron bound states $\psi(\mathbf{r}_1, \mathbf{r}_2)$, where the radius-vectors of the electrons are read off from the nucleus position.

Despite the weak dependence of the operator (34) on the nucleus momentum, the variable $\mathbf{p}_3$ cannot be omitted in the function $\psi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$. For the three-particle bound states, which are of interest to us, the average momentum of the electrons are $< p_{1,2} >\propto \alpha m$, whereas the average nuclear momentum is expected to be $< p_3 >\propto \alpha\sqrt{mM}$ which is much more than that for the electrons. So, the wave function $\psi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ should have the strong dependence on the nuclear momentum. The kinetic energy of the nucleus is $T_{nuc} \propto \alpha^2 m$, which corresponds in order of magnitude to the electron kinetic energies. Characteristic size of the electron orbits are $a_{el} = \lambda\!\!\!^{-}_e/\alpha$ ($\lambda\!\!\!^{-}_e$ is the electron Compton wavelength), and the similar size for the nucleus $a_{nuc} = a_{el} \times \sqrt{\frac{m}{M}} = 0.016 a_{el}$.

It is easy to see from Eq. (30) that the nucleus motion practically does not contribute to the operator $H_{0,N=3}$ representing kinetic energy in the bound state. However, the nucleus state can effect on the radiative transitions in the atom. The three-particle bound state is given by the states of each of the three particles. Changes in the state of one of the particles, for example, due to the one electron optical transition, must lead to a change in the states of the other two particles. This should effect the transition probabilities. Thus, all the three particles must participate even in a single-photon transition. Introducing the canonical momenta of the particles in Eq. (27), it is easy to obtain all radiative operators representing the interaction of the helium atom with an electromagnetic field.

5. Conclusion

In the present paper, the relativistic-kinematic method to study the bound states of several-particle atomic systems is proposed. The method is based on the fact that in relativistic mechanics the total energy and the total momentum of free particles systems are additive quantities, but their total mass is not additive. However, the mass of the systems is the invariant which does not change when moving from one frame of reference to another. This circumstance makes it possible to use the invariant mass to construct the relativistic-kinematic bound-state equation for composite particles at rest.

This relativistic-kinematic bound-state equation is applied to the hydrogen and helium atoms. It is shown that the derived equation has the well-known single-electron bound states for the hydrogen atom, and the two-electron states for the helium atom. For the hydrogen atom, existence of the two-particle bound states, for which the electron and the proton kinetic are of the same order of magnitude, is predicted. The three-particle bound states with the same feature of the kinetic energies can exist in the helium atom.

In the relativistic-kinematic theory, the expression for the total mass of the free particle systems does not depend on the particle spin. Therefore, the derived bound-state equation is valid for both bosons and fermions. The differences between them can show up in the relativistic expressions for the interaction function between particles.

In conclusion, we would like to discuss the following question: if in the hydrogen atom, the two-particle states predicted above, exist, then why they did not detected earlier in spectroscopic studies?

Regardless of whether a two-particle bound state is the final or initial state, any optical process must be accompanied by changes in the internal motions of both the electron and the proton. These particle states are not independent, but form the certain two-particle bound state. Hence, during any radiative transition, both the electron and the proton must change simultaneous their states. The radiative process involving the proton transition, should have a very lower probability, since the process will be suppressed by the large proton mass. Therefore, we expect that these two-particle states can be optically inactive.

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