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Article

Oscillation of Second-Order Hybrid Neutral Delay Difference Equations Using Binomial Form

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Abstract: This paper explores the oscillatory behavior of a class of second-order hybrid-type neutral delay difference equations. A novel approach involves transforming these complex trinomial equations into a simpler binomial form, using solutions from an associated linear difference equation. By employing comparison techniques and summation averaging methods, we establish new oscillation criteria that ensure all solutions exhibit oscillatory behavior. Our findings extend current oscillation theory and are applicable even to non-neutral second-order equations. Couple of examples are provided to demonstrate the significance and originality of these results.

Keywords: trinomial difference equation; neutral; second-order; binomial form; oscillation

MSC: 39A10

1. Introduction

This paper concerned with the oscillation properties of the second-order neutral delay difference equation of the form

$$\Delta(a(n)\Delta z(n)) - p(n)x(n + 1) + q(n)x(\sigma(n)) = 0, n \geq n_0 > 0, \tag{E}$$

where $z(n) = x(n) + b(n)x(\tau(n))$. Hereafter it is assumed that:

- (H1) $\{a(n)\}$, $\{b(n)\}$, $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive real numbers with $\sum_{n=n_0}^{\infty} \frac{1}{a(n)} = \infty$;
(H2) $\{b(n)\}$ is a nonnegative real sequence with $0 \leq b(n) \leq d < 1$;
(H3) $\{\sigma(n)\}$, $\{\tau(n)\}$, are sequences of integers such that $\tau(n) \leq n - 1$ and $\sigma(n) \leq n - 1$ with $\lim_{n \rightarrow \infty} \sigma(n) = \lim_{n \rightarrow \infty} \tau(n) = \infty$.

Under a solution of equation (E) we mean a real sequence $\{x(n)\}$ defined and satisfying (E) for all $n \geq n_0$. We consider only such solutions of (E) that satisfy $\sup\{|x(n)| : n \geq N\} > 0$ for all $n \geq n_0$. We tacitly assume that (E) possesses such solutions.

Definition 1. A solution of (E) is said to be oscillatory if it is either eventually negative nor eventually positive and nonoscillatory otherwise. Equation (E) itself called oscillatory if all its solutions are oscillatory.

Note that if $p(n) = 0$, then (E) is reduced to a second-order neutral delay difference equation of the form

$$\Delta(a(n)\Delta z(n)) + q(n)x(\sigma(n)) = 0, n \geq n_0 > 0 \tag{E1}$$

and if $q(n) = 0$, then (E) reduced to unstable type neutral difference equation of the form

$$\Delta(a(n)\Delta z(n)) - p(n)x(n + 1) = 0, n \geq n_0 > 0. \tag{E2}$$

Therefore, we may refer to equation (E) as a "hybrid type" difference equation.

The theory of oscillation of difference equations and its practical applications has attracted substantial interest in recent decades; see, for example, [1] and the references cited therein. In particular, the analysis of the oscillatory behavior of second-order functional difference equations has gained significant attention due to their fundamental importance in addressing applied problems across various fields, such as population dynamics, networks, economics, and dynamic systems. For further applications, see [2].

Numerous results have been reported in the literature concerning the oscillation and asymptotic behavior of second-order difference equations with linear, sub-linear and super-linear neutral terms; see the monographs [1,3] and the research articles [4–17], as well as the references therein.

Oscillation criteria for second-order difference equations-particularly those involving both positive and negative coefficients-remain a topic of ongoing research. The presence of both positive and negative terms complicates the analysis, as the structure of the set of nonoscillatory solutions of the hybrid equation (E) is not well understood. Nevertheless, a limited number of criteria have been developed to investigate the oscillatory behavior of such equations; see [3,18–25] and the references therein.

In this paper, we adopt a novel approach that overcomes the challenges posed by the coexistence of positive and negative terms in equation (E). By employing comparison techniques and summation averaging methods, we derive sufficient conditions under which every solution of equation (E) is oscillatory. The oscillation criteria established in this work are new and extend the existing results in the literature for both neutral and nonneutral difference equations.

2. Preliminary Results

In this section, we present some preliminary results which are used to prove our main theorems. The main theorems presented in Section 3 relate the properties of solutions of second- order neutral delay difference equations of the form (E) to those of solutions of an auxiliary second-order linear difference equation of the form

$$\Delta(a(n)\Delta u(n)) - p(n)u(n + 1) = 0, n \geq n_0. \tag{1}$$

The first result is based on an equivalent representation for the linear difference operator

$$L(z(n)) = \Delta(a(n)\Delta z(n)) - p(n)z(n + 1) \tag{2}$$

in terms of a positive solution $\{u(n)\}$ of (1).

It is well-known that if $\{a(n)\}$ and $\{p(n)\}$ are positive real sequences, then the equation (1) is nonoscillatory. Note that equation (E) can be written in an equivalent form as

$$\Delta(a(n)\Delta z(n)) - p(n)z(n + 1) + p(n)b(n)x(\tau(n + 1)) + q(n)x(\sigma(n)) = 0. \tag{3}$$

Lemma 1. Assume that (1) has a positive solution $\{u(n)\}$. Then the operator (2) can be represented as

$$L(z(n)) = \frac{1}{u(n + 1)}\Delta\left(a(n)u(n)u(n + 1)\Delta\left(\frac{z(n)}{u(n)}\right)\right).$$

Proof. Using a difference calculus, we find that

$$\begin{aligned} L(z(n)) &= \frac{1}{u(n+1)} \Delta(a(n)u(n)\Delta z(n) - a(n)z(n)\Delta u(n)) \\ &= \Delta(a(n)\Delta z(n)) - \Delta(a(n)\Delta u(n)) \frac{z(n+1)}{u(n+1)} \\ &= \Delta(a(n)\Delta z(n)) - p(n)z(n+1), \end{aligned}$$

where we have used the positive solution of (1). The proof of the lemma is complete.

□

Lemma 2. Let $\{u(n)\}$ be a positive solution of (1). Then equation (E) can be written in the form

$$\Delta(\eta(n)\Delta\mu(n)) + u(n+1)p(n)b(n)x(r(n+1)) + u(n+1)q(n)x^\alpha(r(n)) = 0 \quad (4)$$

where

$$\eta(n) = a(n)u(n)u(n+1), \quad \mu(n) = \frac{z(n)}{u(n)}.$$

Proof. The proof is obvious from Lemma 2.1. □

For our investigation, it is very helpful to have (4) in the canonical form, that is, we assume in the sequel that

$$\sum_{n=n_0}^{\infty} \frac{1}{\eta(n)} = \sum_{n=n_0}^{\infty} \frac{1}{a(n)u(n)u(n+1)} = \infty. \quad (5)$$

Next, we present the structure of nonoscillatory solutions of (1) using discrete Kneser Theorem [1].

Lemma 3. Under the assumption (H1), the equation (1) has a principal (recessive) solution $\{r(n)\}$ satisfying

$$r(n) > 0, \quad \Delta r(n) < 0, \quad \Delta(a(n)\Delta r(n)) \geq 0, \quad n \geq n_0$$

and a nonprincipal (dominant) solution $\{v(n)\}$ satisfying

$$v(n) > 0, \quad \Delta v(n) > 0, \quad \Delta(a(n)\Delta v(n)) \geq 0, n \geq n_0.$$

Next, we study the behavior of solutions of (E) with the help of the equivalent representation (4).

Lemma 4. Let $\{u(n)\}$ be a positive solution of (1) satisfying (5). If $\{x(n)\}$ is a positive solution (E), then the corresponding sequence $\{\mu(n)\}$ satisfies the following condition

$$\mu(n) > 0, \quad \eta(n)\Delta\mu(n) > 0, \quad \Delta(\eta(n)\Delta\mu(n)) \leq 0 \quad (6)$$

for all $n \geq n_0$.

Proof. Let $\{x(n)\}$ be an eventually positive solution of (E), say, $x(n) > 0, x(\tau(n)) > 0$ and $x(\sigma(n)) > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. Then by the definition of $z(n)$, we see that $z(n) > 0$ for all $n \geq n_1$. Since $u(n) > 0$ for all $n \geq n_0$, we find that $\eta(n) = a(n)u(n)u(n+1) > 0$ and $\mu(n) = \frac{z(n)}{u(n)} > 0$ for all $n \geq n_2 \geq n_1$. From (4), we see that $\Delta(\eta(n)\Delta\mu(n)) \leq 0$ for all $n \geq n_2$ and the condition (5) clearly implies that $\eta(n)\Delta\mu(n) > 0$ for all $n \geq n_2$. The proof of the lemma is complete. □

Next, we find the relation between $\{x(n)\}$ and $\{\mu(n)\}$.

Lemma 5. Let $\{u(n)\}$ be a positive solution of (1) satisfies (5). Then

$$x(t) \geq u(n)E(n)\mu(n), n \geq n_2 \geq n_1, \quad (7)$$

where

$$E(n) = \left(1 - b(n) \frac{u(\tau(n))}{u(n)}\right).$$

Proof. From the definition of $\mu(n)$ and its monotonicity behavior, we see that

$$u(n)\mu(n) = z(n) = x(n) + b(n)x(\tau(n)),$$

that is,

$$x(n) \geq u(n)\mu(n) - b(n)u(\tau(n))\mu(\tau(n)).$$

So,

$$x(n) \geq u(n) \left(1 - \frac{b(n)u(\tau(n))}{u(n)}\right) \mu(n),$$

which ends the proof. $\square$

Before presenting our next results, let us define

$$\begin{aligned} Q_1(n) &= u(n+1)b(n)p(n)u(\tau(n))E(\tau(n)), \\ Q_2(n) &= u(n+1)q(n)u(\sigma(n))E(\sigma(n)), \\ \delta(n) &= \min\{\tau(n), \sigma(n)\}, Q_3(n) = Q_1(n) + Q_2(n), \\ \Omega(n) &= \sum_{s=n_0}^{n-1} \frac{1}{\eta(s)}. \end{aligned}$$

Lemma 6. Let $\{u(n)\}$ be a positive solution of (1) satisfying (5). Then equation (E) is oscillatory provided that

$$\Delta(\eta(n)\Delta\mu(n)) + Q_3(n)\mu(\delta(n)) = 0, n \geq n_0, \tag{8}$$

is oscillatory.

Proof. Assume the contrary that $\{x(n)\}$ is an eventually positive solution of (E), say, $x(n) > 0, x(\tau(n)) > 0, x(\sigma(n)) > 0$ for all $n \geq n_1 \geq n_0$ and the corresponding function $z(n) > 0$ for all $n \geq n_1 \geq n_0$. Then by Lemma 4, that $\mu(n) = \frac{z(n)}{u(n)} > 0$ and satisfies condition (6) for all $n \geq n_1 \geq n_0$. Using (7) in (4), we find that $\{\mu(n)\}$ is a positive and increasing solution of

$$\Delta(\eta(n)\Delta\mu(n)) + Q_3(n)\mu(\delta(n)) \leq 0.$$

But by Lemma 1 of [26], the corresponding equation (8) also has a positive solution, and this contradiction ends the proof. $\square$

3. Oscillation Results

In this section, we present oscillation criteria for equation (E) with the help of the equation (8). We begin with the following theorem.

Theorem 1. Let $\{u(n)\}$ be a positive solution (1) such that (5) holds. If

$$\sum_{n=n_0}^{\infty} Q_3(n) = \infty, \tag{9}$$

then equation (E) is oscillatory.

Proof. Assume the contrary that $\{x(n)\}$ is an eventually positive solution of (E), say, $x(n) > 0, x(\sigma(n)) > 0$ and $x(\tau(n)) > 0$ for all $n \geq n_1$ for some $n_1 \geq n_0$. Then the corresponding function

$z(n) > 0, z(\sigma(n)) > 0$ and $z(\tau(n)) > 0$ for all $n \geq n_0$. From Lemma 2, the function $\mu(n) = \frac{z(n)}{u(n)} > 0$ and satisfies condition (6). Since $\{\mu(n)\}$ is increasing, there exists a constant $M > 0$ such that $\mu(n) \geq M > 0$ for all $n \geq n_2 \geq n_1$. Using this in (8), we find that by summing it from n_2 to n ,

$$\sum_{s=n_2}^n M(Q_2(s) + Q_1(s)) \leq \eta(n_2)\Delta\mu(n_2) - \eta(n+1)\Delta\mu(n+1).$$

As $n \rightarrow \infty$, we see that

$$\sum_{n=n_2}^{\infty} M(Q_2(n) + Q_1(n)) \leq \eta(n_2)\Delta\mu(n_2) < \infty,$$

which contradicts (9). The proof of the theorem is complete. $\square$

Remark 1. Note that the above theorem can be applicable to ordinary, delay and advanced type difference equations.

Next, we derive oscillation criteria for equation (E) when the condition (9) fails to hold.

Theorem 2. Let $\{u(n)\}$ be a positive solution of (1) such that (5) holds. If the first order delay difference equation

$$\Delta\omega(n) + Q_3(n)\Omega(\delta(n))\omega(\delta(n)) = 0, \quad (10)$$

is oscillatory, then (E) is oscillatory.

Proof. Assume the contrary that $\{x(n)\}$ is an eventually positive solution of (E), say, $x(n) > 0, x(\tau(n)) > 0$ and $x(\sigma(n)) > 0$ for all $n \geq n_1 \geq n_0$. Then the corresponding function $z(n) > 0, z(\sigma(n)) > 0$ and $z(\tau(n)) > 0$ for all $n \geq n_1$. From Lemma 2, the function $\mu(n) = \frac{z(n)}{u(n)} > 0$ and satisfies condition (6). From the monotonicity $\eta(n)\Delta\mu(n)$, we have,

$$\mu(n) \geq \sum_{s=n_1}^{n-1} \frac{\eta(s)\Delta\mu(s)}{\eta(s)} \geq \Omega(n)\eta(n)\Delta\mu(n) \quad (11)$$

and so

$$\Delta\left(\frac{\mu(n)}{\Omega(n)}\right) = \frac{\Omega(n)\eta(n)\Delta\mu(n) - \mu(n)}{\Omega(n)\Omega(n+1)\eta(n)} \leq 0,$$

which implies $\{\mu(n)/\Omega(n)\}$ is decreasing. Using (11) in (8), we find that $\omega(n) = \eta(n)\Delta\mu(n) > 0$ satisfies the inequality

$$\Delta\omega(n) + Q_3(n)\Omega(\delta(n))\omega(\delta(n)) \leq 0.$$

Then by Lemma 2.7 of [3], we see that the equation (10) also has a positive solution. This contradiction ends the proof. $\square$

Next, we provide explicit criteria for the oscillation of equation (10).

Corollary 1. Let $\{u(n)\}$ be a positive solution (1) such that (5) holds. If

$$\liminf_{n \rightarrow \infty} \sum_{s=\delta(n)}^{n-1} Q_3(s)\Omega(\delta(s)) > \frac{1}{e}, \quad (12)$$

then equation (E) is oscillatory.

Proof. In view of condition (12), Theorem 2.2 of [27] implies that equation (10) is oscillatory and the conclusion follows from Theorem 2. This ends the proof. $\square$

Corollary 2. Let $\{u(n)\}$ be a positive station (1) such that (5) holds. If $\alpha = 1$, $\delta(n) = n - k$, k is a positive integer and

$$\liminf_{n \rightarrow \infty} \sum_{s=n-k}^{n-1} Q_3(s) \Omega(\delta(s)) > \left(\frac{k}{k+1} \right)^{k+1}, \quad (13)$$

then equation (E) is oscillatory.

Proof. In view of condition (13) and Lemma 7.6.1 of [1], it is easy to see that equation (10) is oscillatory and the conclusion follows from Theorem 2. This ends the proof. $\square$

Theorem 3. Let $\{u(n)\}$ be a positive solution (1) such that (5) holds. If $\delta(n) = n - k$, k is a positive integer and

$$\limsup_{n \rightarrow \infty} \left\{ \frac{1}{\Omega(n-k)} \sum_{s=n_1}^{n-k-1} Q_3(s) \Omega(s) \Omega(s-k) + \sum_{s=n-k}^{n-1} Q_3(s) \Omega(s-k) + \Omega(n-k) \sum_{s=n}^{\infty} Q_3(s) \right\} > 1, \quad (14)$$

for some $n_1 \geq n_0 + k$, then (E) is oscillatory.

Proof. Assume that (E) is not oscillatory. Then by Theorem 2, equation (8) is also nonoscillatory and we may assume that it possesses an eventually positive solution $\{\mu(n)\}$ with $\mu(n) > 0$ for $n \geq n_1 \geq n_0 + k$ such that (14) holds. Summing (8) yields

$$\Delta \mu(n) \geq \frac{1}{\eta(n)} \sum_{s=n}^{\infty} Q_3(s) \mu(s-k).$$

Summing once more gives

$$\begin{aligned} \mu(n) &\geq \sum_{s=n_1}^{n-1} \frac{1}{\eta(s)} \sum_{t=s}^{\infty} Q_3(t) \mu(t-k) \\ &= \sum_{s=n_1}^{n-1} \frac{1}{\eta(s)} \sum_{t=s}^{n-1} Q_3(t) \mu(t-k) + \sum_{s=n_1}^{n-1} \frac{1}{\eta(s)} \sum_{t=n}^{\infty} Q_3(t) \mu(t-k). \end{aligned}$$

Employing summation by parts, we have

$$\mu(n) \geq \sum_{s=n_1}^{n-1} Q_3(s) \Omega(s+1) \mu(s-k) + \Omega(n) \sum_{t=n}^{\infty} Q_3(t) \mu(t-k).$$

Hence,

$$\begin{aligned} \mu(n-k) &\geq \sum_{s=n_1}^{n-k-1} Q_3(s) \Omega(s+1) \mu(s-k) + \Omega(n-k) \sum_{t=n-k}^{n-1} Q_3(t) \mu(t-k) \\ &\quad + \Omega(n-k) \sum_{t=n}^{\infty} Q_3(t) \mu(t-k). \end{aligned}$$

Because of the fact that $\mu(n)/\Omega(n)$ is decreasing and $\mu(n)$ is increasing, the previous inequality yields

$$\begin{aligned} \mu(n-k) &\geq \frac{\mu(n-k)}{\Omega(n-k)} \sum_{s=n_1}^{n-k-1} Q_3(s) \Omega(s+1) \Omega(s-k) \\ &\quad + \mu(n-k) \sum_{s=n-k}^{n-1} Q_3(s) \Omega(s-k) + \Omega(n-k) \mu(n-k) \sum_{s=n}^{\infty} Q_3(s). \end{aligned}$$

Dividing the last inequality by $\mu(n - k)$, we have

$$1 \geq \left\{ \frac{1}{\Omega(n - k)} \sum_{s=n_1}^{n-k-1} Q_3(s) \Omega(s + 1) \Omega(s - k) + \sum_{s=n-k}^{n-1} Q_3(s) \Omega(s - k) + \Omega(n - k) \sum_{s=n}^{\infty} Q_3(s) \right\},$$

which is a contradiction. The proof of the theorem is complete. $\square$

Theorem 4. Let $\{u(n)\}$ be a positive solution of (1) such that (5) holds. If

$$\liminf_{n \rightarrow \infty} \Omega(n) \sum_{s=n}^{\infty} Q_3(s) \frac{\Omega(\delta(s))}{\Omega(s)} > \frac{1}{4}, \tag{15}$$

then equation (E) is oscillatory.

Proof. Assume that (E) is not oscillatory. Then by Theorem 2, equation (8) is also nonoscillatory and we assume that it possesses an eventually positive solution $\{\mu(n)\}$ with $\mu(n) > 0$ for $n \geq n_1 \geq n_0$ such that (15) holds. Since $\frac{\mu(n)}{\Omega(n)}$ is decreasing and $\delta(n) \leq n - 1$, we find that

$$\mu(\delta(n)) \geq \frac{\Omega(\delta(n))}{\Omega(n)} \mu(n). \tag{16}$$

Using (16) in (8) gives

$$\Delta(\eta(n) \Delta \mu(n)) + Q_3(n) \frac{\Omega(\delta(n))}{\Omega(n)} \mu(n) \leq 0. \tag{17}$$

Define,

$$w(n) = \frac{\eta(n) \Delta \mu(n)}{\mu(n)}, n \geq n_1.$$

Then $w(n) > 0$ and

$$\begin{aligned} \Delta \omega(n) &= \frac{\Delta(\eta(n) \Delta \mu(n))}{\mu(n)} - \frac{\eta(n + 1) \Delta \mu(n + 1)}{\mu(n) \mu(n + 1)} \Delta \mu(n) \\ &\leq -Q_3(n) \frac{\Omega(\delta(n))}{\Omega(n)} - \frac{w(n + 1) \omega(n)}{\eta(n)}. \end{aligned}$$

Summing the last inequality from n to ∞ , we find that

$$\omega(n) \geq \sum_{s=n}^{\infty} Q_3(s) \frac{\Omega(\delta(s))}{\Omega(s)} + \sum_{s=n}^{\infty} \frac{\omega(s) \omega(s + 1)}{\eta(s)}. \tag{18}$$

Let $\liminf_{n \rightarrow \infty} \Omega(n) \omega(n) = M > 0$. Multiplying (18) by $\Omega(n)$ and using (15), we obtain

$$M > \frac{1}{4} + M^2, \tag{19}$$

since $\Omega(n) \sum_{s=n}^{\infty} \frac{1}{\Omega(s) \Omega(s + 1) \eta(s)} = 1$. The inequality in (19) is not possible and therefore, the proof of the theorem is complete. $\square$

Remark 2. Note that there are many explicit oscillation criteria available in the literature (see, the monograph [1,3] and the references cited therein) for the equation (8), from which we can reduce many oscillation criteria for the studied equation (E).

4. Examples

In this section, we present couple of examples to illustrate the main results.

Example 1. Consider the second-order neutral hybrid type difference equation

$$\Delta(n\Delta z(n)) - \frac{1}{(n+2)}x(n+1) + q_0(n+1)x(n-2) = 0, n \geq 3, \tag{20}$$

where $q_0 > 0$ and $z(n) = x(n) + \frac{1}{2}x(n-1)$.
Here $a(n) = n, b(n) = \frac{1}{2}, p(n) = \frac{1}{n+2}, q(n) = q_0(n+1)$. Now, (1) takes the form

$$\Delta(n\Delta u(n)) - \frac{1}{(n+2)}u(n+1) = 0, n \geq 1. \tag{21}$$

The sequence $\{u(n)\} = \{\frac{1}{n}\}$ is a positive solution of (21) and $\eta(n) = \frac{1}{n+1}$, satisfies condition (5). Further

$$E(n) = \frac{1}{2} \frac{(n-2)}{n-1}, \quad \delta(n) = n-2, Q_1(n) \approx \frac{1}{4n^3}, Q_2(n) \approx \frac{q_0}{2n},$$

and $Q_3(n) \approx \frac{q_0}{2n} + \frac{1}{4n^3}$. The condition (9) becomes

$$\sum_{n=3}^{\infty} \left(\frac{q_0}{2n} + \frac{1}{4n^3} \right) = \infty,$$

that is, condition (9) holds if $q_0 > 0$. Therefore, by Theorem 1 the equation (9) is oscillatory.

Example 2. Consider the second-order neutral type hybrid delay difference equation

$$\Delta(n\Delta z(n)) - \frac{1}{(n+2)}x(n+1) + \frac{q_0(n-2)}{n(n+2)}x(n-2) = 0, n \geq 3, \tag{22}$$

where $z(n) = x(n) + \frac{1}{n}x(n-1)$.
Here $a(n) = n, b(n) = \frac{1}{n}, p(n) = \frac{2}{n(n+2)}, q(n) = \frac{q_0(n-2)}{n(n+2)}, \tau(n) = n-1, \sigma(n) = n-2$ and $\delta(n) = n-2$. Now (1) takes the form

$$\Delta(n\Delta u(n)) - \frac{1}{(n+2)}u(n+1) = 0, n \geq 1. \tag{23}$$

The sequence $\{u(n)\} = \{1/n\}$ is a positive solution of (23) which satisfies (5) since $\eta(n) = \frac{1}{n+1}$. Further calculations show that

$$E(n) = \frac{n-2}{n-1}, \delta(n) = n-2, Q_1(n) \approx \frac{1}{n^3}, Q_2(n) \approx \frac{q_0}{n(n+1)(n+2)}$$

and $\Omega(n) \approx \frac{n^2}{2}$. The condition (15) becomes

$$\liminf_{n \rightarrow \infty} \frac{n^2}{2} \sum_{s=n}^{\infty} \left(\frac{1}{s^3} + \frac{q_0}{s(s+1)(s+2)} \right) = \frac{q_0}{4} > \frac{1}{4}$$

that is, condition (15) holds if $q_0 > 1$. Therefore, by Theorem 4, the equation (22) is oscillatory.

5. Conclusion

In this paper, we investigate the oscillatory behavior of solutions of the hybrid equation (E). We do this by first transforming the studied equation into binomial form. Then, using comparison and summation averaging techniques, we derive new sufficient conditions for all solutions of (E) to

oscillate. The results obtained are novel and complement the existing oscillation theory of difference equations. We also provide couple of examples to demonstrate the importance and novelty of our main findings, as none of the known results are directly applicable to equations (20) and (22).

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