Preprints.org

Article

Not peer-reviewed version

When Four Cyclic Antipodal Points Are Ordered Counterclockwise

Abraham UNGAR *

Posted Date: 30 August 2023

doi: 10.20944/preprints202308.2018.v1

Keywords: Cyclic antipodal points.

Article

When Four Cyclic Antipodal Points Are Ordered Counterclockwise

Abraham A. Ungar

Department of Mathematics, North Dakota State University, Fargo, ND 58105, USA; abraham.ungar@ndsu.edu

Abstract: A novel theorem is presented about a circle with four antipodal points, arbitrarily ordered counterclockwise (or clockwise), in a Euclidean plane.

Keywords: cyclic antipodal points

MSC: 51N20

1. Four Cyclic Antipodal Points

A cyclic antipodal points of a circle in a Euclidean plane $\mathbb{R}^2$ is a pair (A, A') of points $A, A' \in \mathbb{R}^2$ that are the intersections of the circle with a diameter of the circle. A circle with four cyclic antipodal points is shown in Figure 1. The distance $|AB|$ between two points $A, B \in \mathbb{R}^2$ is given by

$$|AB| = |-A+B|\,. \tag{1}$$

Theorem 1. (A Four Cyclic Antipodal Points Theorem). *Let $\Sigma(O,r)$ be a circle in the Euclidean plane $\mathbb{R}^2$ with radius r, centered at $O \in \mathbb{R}^2$, with four cyclic antipodal points (A, A'), (B, B'), (C, C') and (D, D'), such that the eight points $\{A, B, C, D, A', B', C', D'\} \subset \mathbb{R}^2$ are arbitrarily ordered counterclockwise (or clockwise), as shown in Figure 1.*

Then, the four cyclic antipodal points satisfy the identity

$$\begin{aligned} |AB'||BC'||CD'| - |AB'||BC||CD| - |AB||BC'||CD| \\ - |AB||BC||CD'| = 4r^2|A'D|\,. \end{aligned} \tag{2}$$

Proof. The proof is based on the elegant trigonometric identity

$$\begin{aligned} \sin\frac{\alpha+\pi}{2}\sin\frac{\beta+\pi}{2}\sin\frac{\gamma+\pi}{2} - \sin\frac{\alpha+\pi}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \\ - \sin\frac{\alpha}{2}\sin\frac{\beta+\pi}{2}\sin\frac{\gamma}{2} \\ - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma+\pi}{2} = \sin\frac{\delta}{2} \end{aligned} \tag{3}$$

which holds for all $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}^2$ that satisfy the condition

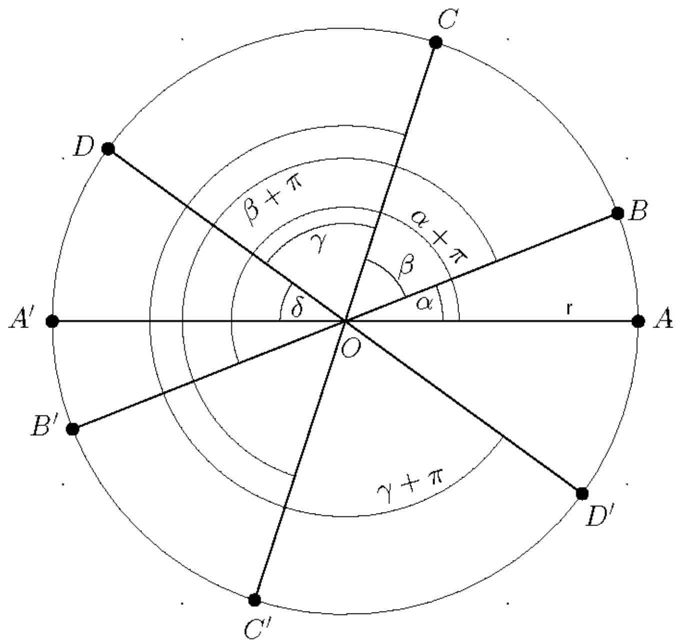
$$\alpha+\beta+\gamma+\delta = \pi\,. \tag{4}$$

The trigonometric identity in (3)–(4) can readily be verified by computer assisted computation.

In order to realize (3)–(4) geometrically by a circle with four cyclic antipodal points, shown in Figure 1, we define the four O-vertex angles in Figure 1 as follows.

$$\begin{aligned} \alpha &= \angle AOB \\ \beta &= \angle BOC \\ \gamma &= \angle COD \\ \delta &= \angle DOA'\,. \end{aligned} \tag{5}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 August 2023 doi:10.20944/preprints202308.2018.v1



$$|AB'||BC'||CD'| - |AB'||BC||CD| - |AB||BC'||CD| - |AB||BC||CD'|$$
$$= 4r^2|A'D|$$

Figure 1. Four cyclic antipodal points, (A, A'), (B, B'), (C, C'), (D, D'), on a circle $\Sigma(O, r)$ centered at O with radius r, and their corresponding O-vertex angles α, β, γ, δ. The points $A, B, C, D, A', B', C', D'$ are arbitrarily ordered counterclockwise, implying $\alpha + \beta + \gamma + \delta = \pi$. The identity of Theorem 1 is depicted, where $|AB| = |-A + B|$, *etc.*

We note that $\alpha + \beta + \gamma + \delta = \pi$, as required by Condition (4), since the points $A, B, C, D, A', B', C', D'$ are ordered counterclockwise.

Then, consequently, the remaining three O-vertex angles in Figure 1 are

$$\begin{aligned} \angle AOB' &= \alpha + \pi \\ \angle BOC' &= \beta + \pi \\ \angle COD' &= \gamma + \pi \,. \end{aligned} \tag{6}$$

Applying the law of cosines to triangle AOB yields

$$\begin{aligned} |AB|^2 &= 2r^2 - 2r^2 \cos\alpha \\ &= 2r^2(1 - \cos\alpha) \\ &= 4r^2 \sin^2 \frac{\alpha}{2} \end{aligned} \tag{7}$$

so that

$$|AB| = 2r \sin \frac{\alpha}{2} \,. \tag{8}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 August 2023 doi:10.20944/preprints202308.2018.v1

Similarly to (8), by means of (5)–(6), we obtain the following seven results:

$$\begin{aligned}\sin\frac{\alpha}{2} &= \frac{1}{2r}|AB|\\ \sin\frac{\beta}{2} &= \frac{1}{2r}|BC|\\ \sin\frac{\gamma}{2} &= \frac{1}{2r}|CD|\\ \sin\frac{\delta}{2} &= \frac{1}{2r}|A'D|\\ \sin\frac{\alpha+\pi}{2} &= \frac{1}{2r}|AB'|\\ \sin\frac{\beta+\pi}{2} &= \frac{1}{2r}|BC'|\\ \sin\frac{\gamma+\pi}{2} &= \frac{1}{2r}|CD'|\,.\end{aligned} \tag{9}$$

Substituting the sines from (9) into the trigonometric identity (3) yields (2), as desired. □

The proof of Theorem 1 is motivated by the proof of Ptolemy's Theorem in [1]. In [1,2] Ptolemy's Theorem is extended to hyperbolic geometry and, similarly, Theorem 1 can be extended to hyperbolic geometry as well.

2. Special Cases

In the special case when $A = B$ and, hence, $A' = B'$, the result (2) of Theorem 1 descends to

$$\begin{aligned}&|BB'|BC'|CD'| - |BB'||BC||CD| - |BB||BC'||CD|\\ &\quad - |BB||BC||CD'| = 4r^2|B'D|\,.\end{aligned} \tag{10}$$

Noting that $|BB'| = 2r$ and $|BB| = 0$, (10) yields

$$|BC'||CD'| - |BC||CD| = 2r|B'D|\,. \tag{11}$$

Formalizing the result in (11) we obtain the result (13) of Corollary 1.

In a second special case, when $D = A'$ and, hence, $D' = A$, the result (2) of Theorem 1 descends to

$$\begin{aligned}&|AB'||BC'||AC| - |AB'||BC||A'C| - |AB||BC'||A'C|\\ &\quad - |AB||BC||AC| = 4r^2|A'A'| = 0\,.\end{aligned} \tag{12}$$

Formalizing the result in (12) we obtain the result (14) of Corollary 1.

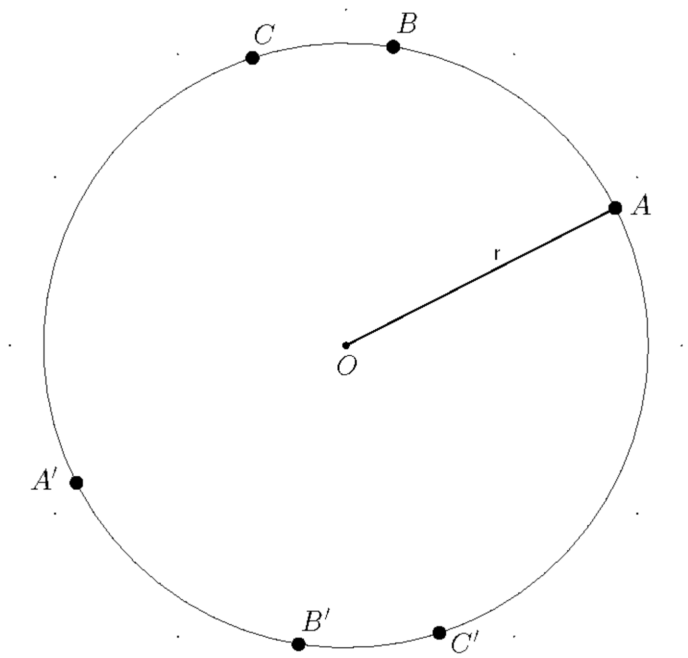
Corollary 1. (A Three Cyclic Antipodal Points Theorem). *Let $\Sigma(O,r)$ be a circle in the Euclidean plane $\mathbb{R}^2$ with radius r, centered at $O \in\mathbb{R}^2$, with three cyclic antipodal points (A,A'), (B,B') and (C,C'). The six points $\{A,B,C,A',B',C'\} \subset\mathbb{R}^2$ are arbitrarily ordered counterclockwise (or clockwise), as shown in Figure 2. Then,*

$$|AB'||BC'| - |AB||BC| = 2r|A'C| \tag{13}$$

and

$$\begin{aligned}&|AB'||BC'||AC| - |AB'||BC||A'C| - |AB||BC'||A'C|\\ &\quad - |AB||BC||AC| = 0\,.\end{aligned} \tag{14}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 August 2023 doi:10.20944/preprints202308.2018.v1



$$|AB'||BC'| - |AB||BC| = 2r|A'C|$$

$$|AB'||BC'||AC| - |AB'||BC||A'C| - |AB||BC'||A'C| - |AB||BC||AC|$$
$$= 0$$

Figure 2. Three cyclic antipodal points, (A, A'), (B, B'), (C, C'), on a circle $\Sigma(O, r)$ centered at O with radius r. The points A, B, C, A', B', C' are ordered counterclockwise. The two identities of Corrolary 1 are shown.

References

1. Abraham A. Ungar. Ptolemy's theorem in the relativistic model of analytic hyperbolic geometry. *Symmetry*, 15(3):649, Mar 2023.
2. Abraham A. Ungar. The hyperbolic ptolemy's theorem in the poincaré ball model of analytic hyperbolic geometry. *Symmetry*, 15(8):1487, Jul 2023.