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Fractional Neutral Integro-Differential Equations with Neumann-Type Boundary Conditions

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Abstract: We primarily investigate the existence of solutions for fractional neutral integro-differential equations subjected to Neumann-type boundary conditions, which is crucial for understanding natural phenomena. Taking into account factors such as neutral type, fractional-order integrals, and fractional-order derivatives, we employ probability density functions, Laplace transforms, and resolvent operators to formulate a well-defined concept of a mild solution for the specified equation. Following this, by integrating fixed point theorems, we establish the existence of mild solutions under more relaxed conditions.

Keywords: fractional neutral integro-differential equations; resolvent family; probability density function; mild solutions

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1. Introduction

Fractional derivatives, with their capacity to describe nonlocal and long-range dependencies, offer a more precise representation of the dynamic behaviors observed in complex systems compared to traditional integer derivatives. Consequently, fractional differential equations find extensive practical applications across various fields [1–13].

Current research in the field primarily addresses various types of fractional differential equations, including fractional ordinary differential equations, neutral differential equations, functional differential equations, and impulsive differential equations. Additionally, studies explore equations within Banach spaces, covering a range of partial differential equations like Laplace, diffusion, wave, Schrödinger, Navier-Stokes, Heisenberg, Langevin, Fokker-Planck, and others. Uncertain differential equations, encompassing stochastic and fuzzy differential equations, as well as dynamic systems such as chaotic, Hamiltonian, Lorenz, financial systems, and neural networks, are also significant areas of focus.

It is noteworthy that fractional diffusion equations effectively capture the characteristics of anomalous diffusion phenomena, including those with long-tail distributions. Mu et al. [14] have conducted a comprehensive study on the existence and regularity of solutions to fractional diffusion equations

$$\begin{cases} \partial_t^\alpha u(x,t) = Au(x,t) + f(x,t),\ (x,t) \in \Omega \times (0,b), \\ \sum\limits_{i,j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j}u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,b), \\ u(x,0) = u_0(x),\ x \in \Omega. \end{cases} \tag{1}$$

Here, ∂_t^α is the α order partial Caputo derivative with respect to t, $\alpha \in (0,1)$, $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, $T > 0$, f is weighted Hölder continuous;

$$Au(x,t) = \sum_{i,j=1}^{n} \partial_{x_i}\left[a_{ij}(x)\partial_{x_j}u(x,t)\right] - p(x)u(x,t), \tag{2}$$

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a_{ij} are real-valued functions satisfy that

$$a_{ij} \in L^{\infty}(\Omega),\ 1 \leq i,\ j \leq n,$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\vartheta_i\vartheta_j \geq \varsigma|\vartheta|^2,\ \vartheta \in \mathbb{R}^n, \text{a.e. } x \in \Omega, \tag{3}$$

with some $\varsigma > 0$, $p(x)$ is also a real valued function satisfying

$$p \in L^{\infty}(\Omega),\ p(x) \geq p_0 > 0, \text{a.e. } x \in \Omega,$$

By incorporating the integral term, the fractional diffusion equation extends its capacity to model a broader spectrum of complex diffusion phenomena in practical settings [15].

Moreover, with the rise of researches on the dynamic behavior of delay systems, fractional neutral equations have also attracted wide attention [16–18]. Bedi et al. [19] considered a neutral fractional differential equations system with the Atangana-Baleanu-Caputo derivatives, some controllability results be established. Zhou et al. [20] consider the neutral evolution equation

$$\begin{cases} {}^cD^q[x(t) - h(t, x_t)] + Bx(t) = f(t, x_t),\ t \in (0, T], \\ x_0(\varsigma) + (g(x_{t_1}, \ldots, x_{t_n}))(\varsigma) = \varphi(\varsigma),\ \varsigma \in [-r, 0], \end{cases}$$

where ${}^cD_t^{\alpha}$ is the α order Caputo derivative, $\alpha \in (0,1)$, t_i is a monotonically increasing sequence on $(0, T]$ for $i = 1, \ldots, n$, $-B$ generates an analytic semigroup on a Banach space, g, h, φ are given, and the delay term $x_t(\varsigma) = x(t + \varsigma)$ for $\varsigma \in [-r, 0]$.

Based on the above study, this paper will investigate the existence of mild solutions to the fractional neutral integro-differential diffusion equations with Neumann-type boundary conditions:

$$\begin{cases} {}^c\partial_t^{\alpha}[u(x,t) + H(x,t,u(x,t))] = Au(x,t) + I_t^{\beta} f(x,t,u(x,t)),\ (x,t) \in \Omega \times J', \\ \sum\limits_{i,j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j}u(x,t) = 0,\ (x,t) \in \partial\Omega \times J', \\ u(x,0) + h(u) = u_1(x),\ x \in \Omega, \end{cases} \tag{4}$$

where $J' = (0, T]$, $T > 0$, I_t^{β} is the β order partial Riemann-Liouville integral with respect to t, $\beta > 0$, h, u_1, H and f are given functions satisfying some assumptions. Moreover, $Au(x,t)$ satisfies Eq. (2), a_{ij} are real-valued functions satisfy Eq. (3) and

$$a_{ij} \in C^1(\overline{\Omega}),\ 1 \leq i,\ j \leq n,$$

$p(x)$ is also a real valued function satisfying

$$p \in C(\overline{\Omega}),\ p(x) \geq p_0 > 0, x \in \overline{\Omega}.$$

Due to the presence of neutral terms H, integral terms $I_t^{\beta} f$, and non-local terms h, the representation of mild solutions to Eq. (4) will become highly challenging. We will utilize Laplace transforms, resolvent family, and probability density functions to find the solution to the equivalent integral equation of Eq. (4) and investigate the boundedness, strong continuity, and compactness related to the resolvent family. Finally, we will discuss the existence of admissible solutions under weaker conditions.

The resolvent family serves as a potent instrument for examining certain fractional differential equations, as documented in references [2,8,9,21,22]. Reference [8] offers adequate integral estimations necessary for constructing a class of α-times resolvent families. However, due to the simultaneous inclusion of derivative and integral terms in this paper, the single-parameter resolvent family is

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deemed unsuitable. Mu et al. [23] demonstrated the existence of mild solutions for fractional diffusion equations with Dirichlet boundary conditions using the (α,β)-resolvent family, which is also relevant to Eq. (4).

This article is structured as follows: Section 2 furnishes the essential background required for the ensuing discussions, encompassing topics such as boundedness, strong continuity, compactness, and the definition of mild solutions in the context of the resolvent family. Section 3 presents a sequence of research findings concerning the existence of mild solutions under less stringent conditions.

2. Preliminaries

In this paper, let $X = L^2(\Omega)$ with the norm $\|\cdot\|$, $J = [0,T]$, let $Z \subset \mathbb{R}$, $1 \le p \le \infty$, for measurable functions $F : Z \to \mathbb{R}$. We define the norm

$$\|F\|_{L^p Z} = \begin{cases} \left(\int_Z |F(t)|^p dt\right)^{\frac{1}{p}}, \ 1 \le p < \infty, \\ \inf\limits_{\mu(\bar{Z})=0} \{ \sup\limits_{t\in Z-\bar{Z}} |F(t)|\}, \ p = \infty, \end{cases}$$

where $\mu(\bar{Z})$ is the Lebesgue measure on $\bar{Z}$, $L^p(Z,\mathbb{R})$ denotes the Banach space consists of all measurable functions F with $\|F\|_{L^p Z} < \infty$.

We introduce several definitions and notations that are consistently utilized throughout this paper. For detailed discussions on fractional integrals and derivatives, the reader is referred to references [1,3].

Definition 1. *For $\beta > 0$ and a function $f \in L^1[0,\infty)$, the β-order Riemann-Liouville fractional integral of f with respect to $t > 0$ is defined as follows*

$$I_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds. \tag{5}$$

Definition 2. *Given α within the interval $(0,1)$ and an absolutely continuous function f on $[0,\infty)$, we define the α-order Riemann-Liouville fractional derivative of f with respect to $t > 0$as*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t f(s)(t-s)^{-\alpha} ds. \tag{6}$$

Definition 3. *For $\alpha \in (0,1)$ and an absolutely continuous function f defined on $[0,\infty)$, its α order Caputo derivative with respect to $t > 0$ can be written as*

$${}^C D_t^\alpha f(t) = D_t^\alpha (f(t) - f(0)). \tag{7}$$

Remark 4. *We can find the following two properties in [1]*

(i) For $\alpha,\beta \ge 0$, $I_t^\alpha I_t^\beta = I_t^{\alpha+\beta}$;

(ii) For $m > 0$ and $f \in L^1(0,m)$, $D_t^\alpha I_t^\alpha f = f$.

In addition, if f is an abstract function taking values in Banach spaces, the integrals and derivatives presented in Eq. (5) to Eq. (7) should be interpreted in the sense of Bochner.

Definition 5. *([24]) Define the function $h(s)$ on the measure space $(S,\mathcal{J},m)$ with values in X. It is termed Bochner m-integrable, if there exists a sequence $\{h_n(s)\}$ approximating $h(s)$, such that*

$$\lim_{n\to\infty} \int_S \|h(s) - h_n(s)\| m(ds) = 0.$$

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For any set $B \in \mathcal{J}$, the Bochner m-integral of $h(s)$ on B is

$$\int_B h(s)m(ds) = s - \lim_{n\to\infty}\int_S C_B(s)h_n(s)m(ds),$$

C_B is the characteristic function of set B.

Definition 6. *([25]) Consider W as a metric space, and let U be a bounded subset of W. The Kuratowski measure of noncompactness is then defined as follows*

$$\nu(U) = \inf\{\delta > 0 | U = \bigcup_{i=1}^{m} U_i, diamU_i \le \delta\}.$$

Lemma 7. *([26]) If R be a Banach space, $Z_i \subset R$ for $i = 1, 2$. Then $\nu(Z_i)$ satisfies*
(i) $\nu(Z_i) = 0$ if and only if Z_i is relatively compact;
(ii) $\nu(Z_i) = \nu(\overline{Z_i})$;
(iii) if $Z_1 \subset Z_2$, then $\nu(Z_1) \le \nu(Z_2)$;
(iv) $\nu(Z_1 + Z_2) \le \nu(Z_1) + \nu(Z_2)$;
(v) $\nu(cZ_i) = |c|\nu(Z_i)$ for any $c \in \mathbb{R}$;
(vi) $\nu(Z_i) \ge 0$.

Let $C(J, X)$ represent the Banach space of all continuous functions mapping from J to X, which is endowed with a norm

$$\|\cdot\|_\infty = \sup_{t\in J}\{\|\cdot(t)\|\}.$$

Let $B_k = \{u \in C(J, X) | \|u\| \le k\}$. It is evident that B_k is a bounded, closed, and convex subset in $C(J, X)$.

Consider the operator $A : D(A) \subset X \to X$, where $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. For details on these spaces, see [27]. We define $(Au)(t)x = Au(x, t)$, establishing that A generates an analytic semigroup $\{T(t)\}_{t\ge 0}$ on X. For simplicity, and without loss of generality, we assume $0 \in \rho(A)$ and the semigroup $\{T(t)\}_{t\ge 0}$ is uniformly bounded. Additionally, we define the fractional power A^μ as a closed linear operator on its domain $D(A^\mu)$ for $\mu \in (0, 1]$, and it satisfies $N \ge 1$ for some $\|T(t)\| \le N$.

Set $u(t)(x) = u(x, t)$, $H(t)(x) = H(x, t)$ and $f(t, u(t))(x) = f(x, t, u(x, t))$. Consequently, Eq. (4) can be recast as an abstract problem incorporating nonlocal initial conditions

$$\begin{cases} {}^cD_t^\alpha(u(t) + H(t, u(t))) = Au(t) + I_t^\beta f(t, u(t)),\ t \in J', \\ u(0) + h(u) = u_1, \end{cases} \tag{8}$$

where ${}^cD_t^\alpha$ represents the α order Caputo derivative and I_t^β denotes the β order Riemann-Liouville integral. $u_1 \in X$, $h : C(J, X) \to X$, and $H, f : J \times X \to X$.

Remark 8. *If $\alpha \in (0, 1)$, Bajlekova [28, Chapter 2] considered the special case where $\beta = 0$ and $H(t, u(t)) = f(t, u(t)) = 0$ in Eq. (8)*

$$\begin{cases} {}^cD_t^\alpha u(t) = Au(t),\ t > 0, \\ u(0) = x_0 \in X_1, \end{cases} \tag{9}$$

where X_1 is a Banach space. An α-times resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ was applied to obtain the result of uniquely solvable for Eq. (9) with the solution: $u(t) = S_\alpha(t)x_0$. Furthermore, a necessary and sufficient condition is given for Eq. (9) to be well-posed.

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Lemma 9. *If $u \in C(J,X)$, ${}^cD_t^\alpha u \in C(J',X)$, $u(t) \in D(A)$ for $t \in J'$ and u satisfies Eq. (8), then we have*

$$\begin{aligned} u(t) = &S_{\alpha,1-\alpha}(t)\big[u_1 - h(u) + H(0,u(0))\big] - H(t,u(t)) \\ &+ \int_0^t (t-s)^{\alpha-1} A\Psi(t-s)H(s,u(s))ds + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds, \end{aligned} \tag{10}$$

where

$$\Psi(t) = \int_0^\infty \alpha\theta\zeta_\alpha(\theta)T(t^\alpha\theta)d\theta. \tag{11}$$

The family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$ is the resolvent family generated by the operator A and

$$\lambda^{-\beta}(\lambda^\alpha I - A)^{-1}x = \int_0^\infty e^{-\lambda t}S_{\alpha,\beta}(t)xdt \tag{12}$$

for all $x \in X$ [23].

Proof. By Definition 1, Definition 3 and Remark 4, Eq. (8) can be rewritten as

$$\begin{aligned} u(t) = &u_1 - h(u) - H(t,u(t)) + H(0,u(0)) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Au(s)ds \\ &+ \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s,u(s))ds. \end{aligned} \tag{13}$$

Apply the Laplace transform to Eq. (13), let

$$\varphi(\lambda) = \int_0^\infty e^{-\lambda t}u(t)dt, \;\; \varphi_1(\lambda) = \int_0^\infty e^{-\lambda t}f(t,u(t))dt, \;\; \varphi_2(\lambda) = \int_0^\infty e^{-\lambda t}H(t,u(t))dt.$$

Then

$$\begin{aligned} \varphi(\lambda) &= \frac{1}{\lambda}(u_1 - h(u) + H(0,u(0))) - \varphi_2(\lambda) + \frac{1}{\lambda^\alpha}A\varphi(\lambda) + \frac{1}{\lambda^{\alpha+\beta}}\varphi_1(\lambda) \\ &= \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}(u_1 - h(u) + H(0,u(0))) - \lambda^\alpha(\lambda^\alpha I - A)^{-1}\varphi_2(\lambda) \\ &\quad + \lambda^{-\beta}(\lambda^\alpha I - A)^{-1}\varphi_1(\lambda) \\ &=: I_1 - I_2 + I_3. \end{aligned} \tag{14}$$

For I_2, refer to [4], we have

$$I_2 = \int_0^\infty e^{-\lambda t}\left[H(t,u(t)) - \int_0^t (t-s)^{\alpha-1}A\Psi(t-s)H(s,u(s))ds\right]dt,$$

where $\zeta_\alpha(\theta)$ is the probability density function defined on $(0,\infty)$, and

$$\begin{aligned} &\zeta_\alpha(\theta) \geq 0, \\ &\int_0^\infty \theta^\nu\zeta_\alpha(\theta)d\theta = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \;\; \nu > -1. \end{aligned} \tag{15}$$

Then take inverse Laplace transform on both sides of Eq. (14), we obtain that

$$\begin{aligned} u(t) = &S_{\alpha,1-\alpha}(t)\big[u_1 - h(u) + H(0,u(0))\big] - H(t,u(t)) \\ &+ \int_0^t (t-s)^{\alpha-1}A\Psi(t-s)H(s,u(s))ds + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds, \end{aligned}$$

the proof is completed. □

Based on the previous work, we define the mild solution of Eq. (8) as follows.

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Definition 10. *The function u, belonging to $C(J,X)$, is termed a mild solution to Eq.(8) if it fulfills the conditions outlined in u satisfies Eq. (10).*

At the end of this section, we list some lemmas that need to be used in this article, they can be found in [14,20,23,29–31], where $R_{\alpha,\beta}(t) = t^{1-\alpha-\beta} S_{\alpha,\beta}(t)$.

Lemma 11. *$\Psi(t)$ and $S_{\alpha,\beta}(t)$ are bounded, that is,*

$$\|\Psi(t)\| \le \frac{M_\omega}{\Gamma(\alpha)},$$

and

$$\|S_{\alpha,\beta}(t)\| \le M t^{\alpha+\beta-1}, \ \ t > 0, \tag{16}$$

where $M, C > 0$ are constants.

Lemma 12. *The families $\{R_{\alpha,\beta}(t)\}_{t\ge 0}$ and $\{\Psi(t)\}_{t\ge 0}$ exhibit strong continuity.*

Lemma 13. *Should $T(t)$ be compact for any $t > 0$, then $R_{\alpha,\beta}(t)$ and $\Psi(t)$ also exhibit compactness for $t > 0$.*

Lemma 14. *For $x \in X$ and $t \in J$,*

$$A\Psi(t)x = A^{1-\rho}\Psi(t)A^{\rho}x, \tag{17}$$

and for $0 < t < \infty$,

$$\|A^{\eta}\Psi(t)\| \le C t^{-\alpha\eta}, \tag{18}$$

where $\eta \in (0,2)$ and $\rho \in (0,1)$, $C > 0$ is a constant.

Lemma 15. *Assuming $G : J \to X$ is measurable and $\|G\|$ satisfies Lebesgue integrability conditions, then G qualifies as Bochner integrable.*

Lemma 16. *Let D be a bounded, convex, and closed subset of a Banach space, with $0 \in D$. Consider a continuous mapping $N : D \to D$. If, for any subset $V \subset D$, the conditions $V = \overline{conv}N(V)$ or $V = N(V) \cup \{0\}$ imply $\nu(V) = 0$, then it follows that N possesses a fixed point.*

Lemma 17. *Given B, a closed bounded and convex subset of a Banach space, and assuming F, a completely continuous mapping B into itself, it follows that F possesses a fixed point within B.*

3. Main Results

To demonstrate the existence of a mild solution to the Eq. (8), it is essential to outline the requisite assumptions.

(H_1) For any value of $t > 0$, $T(t)$ possesses compactness;

(H_2) For almost every $t \in J$, the functions $f(t,z)$ and $H(t,z)$ are continuous with respect to z in X, and for every $z \in X$, they are strongly measurable with respect to t over J;

(H_2') For each element $z \in X$ and for the majority of elements $t \in J$, the functions $f(t,z)$ and $H(t,z)$ are strongly measurable;

(H_3) The function $\varpi \in C(J,\mathbb{R}^+)$ exists such that for every $z \in X$ and the majority of $t \in J$, the inequality $\|f(t,z)\| \le \varpi(t)\|z\|$ holds;

(H_4) The mapping $h : C(J,X) \to X$ demonstrates complete continuity, and a constant $L > 0$ can be identified such that for all $u \in C(J,X)$, $\|h(u)\| \le L\|u\|_\infty$;

(H_4') In the case of the function $h : C(J,X) \to X$, a constant $L > 0$ is established, ensuring that $\|h(u)\| \le L\|u\|_\infty$ for any u;

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(H_5) The function $H : J \times X \to X$ exhibits continuity. Furthermore, specific constants $\rho \in (0,1)$ and $\xi_1, \xi_2 > 0$ are identified, ensuring that $H \in D(A^\rho)$. Additionally, for any pair of elements $z_1, z_2 \in X$ and any $t \in J$, the function $A^\rho H(t,z)$ demonstrates strong measurability with respect to t within J. It is also established that

$$\|A^\rho H(t,z_1) - A^\rho H(t,z_2)\| \le \xi_1 \|z_1 - z_2\|, \tag{19}$$

$$\|A^\rho H(t,z)\| \le \xi_2 \|z\|; \tag{20}$$

(H_6) For every bounded subset X_1 of X and each $t \in J$, there are constant $a > 0$ and a function $\varpi \in C(J, \mathbb{R}^+)$ such that the measures satisfy $\nu(h(X_1)) \le a\nu(X_1)$, $\nu(A^\rho H(t,X_1)) \le \varpi(t)\nu(X_1)$, and $\nu(f(t,X_1)) \le \varpi(t)\nu(X_1)$ for $0 < \rho < 1$;

(H_7) There exists a constant $L^{'} > 0$ such that for any $y_1, y_2 \in B_{k_1}$, the inequalities

$$\|h(y_1) - h(y_2)\| \le L^{'} \|y_1 - y_2\|_\infty$$

and

$$\|f(t, y_1(t)) - f(t, y_2(t))\| \le \varpi(t)\|y_1 - y_2\|_\infty,$$

hold, where

$$k_1 = \frac{\alpha\rho M(\alpha+\beta)(\|u_1\| + \|h(0)\|)}{\alpha\rho(\alpha+\beta)(1 - ML' - (M+1)\xi_2\|A^{-\rho}\|) - C(\alpha+\beta)\xi_2 T^{\alpha\rho} - \alpha\rho M T^{\alpha+\beta}\|\varpi\|_\infty}.$$

Theorem 18. *Under the conditions specified by assumptions (H_1) to (H_5), and assuming that $H(t,u(t))$ is compact for $t > 0$, the following inequality holds:*

$$ML + M\xi_2\|A^{-\rho}\| + \xi_2\|A^{-\rho}\| + \frac{C\xi_2}{\alpha\rho}T^{\alpha\rho} + \frac{MT^{\alpha+\beta}}{\alpha+\beta}\|\varpi\|_\infty < 1, \tag{21}$$

where $C > 0$ denotes a constant. Then Eq. (8) has a mild solution.

Proof. We choose k_0 such that

$$k_0 = M\left(\|u_1\| + Lk_0 + \xi_2 k_0\|A^{-\rho}\| + \frac{k_0 T^{\alpha+\beta}}{\alpha+\beta}\|\varpi\|_\infty\right) + \xi_2 k_0\|A^{-\rho}\| + \frac{C\xi_2 k_0}{\alpha\rho}T^{\alpha\rho}. \tag{22}$$

For any $u \in B_{k_0}$, by Lemma 11, (H_4) and (H_5),

$$\left\|S_{\alpha,1-\alpha}(t)\left[u_1 - h(u) + H(0,u(0))\right]\right\| \le M\left(\|u_1\| + Lk_0 + \xi_2 k_0\|A^{-\rho}\|\right), \tag{23}$$

and

$$\|H(t,u(t))\| \le \xi_2 k_0 \|A^{-\rho}\|. \tag{24}$$

Furthermore, given (H_2), the function $f(t,u(t))$ is measurable over the interval J. Additionally, considering the stipulations of Lemma 11 and assumption (H_3),

$$\int_0^t \left\|S_{\alpha,\beta}(t-s)f(s,u(s))\right\|ds \le \frac{Mk_0 T^{\alpha+\beta}}{\alpha+\beta}\|\varpi\|_\infty, \tag{25}$$

so $\|S_{\alpha,\beta}(t-s)f(s,u(s))\|$ is Lebesgue integrable with respect to $s \in J$ and $t \in J$. Then, according to Lemma 15, $S_{\alpha,\beta}(t-s)f(s,u(s))$ achieves Bochner integrability with respect to $s \in J$ and for each $t \in J$. According to (H_5), $A^\rho H(t,u(t))$ is strongly measurable. Given that $\{T(t)\}_{t\ge 0}$ is analytic, it follows that for any $t \in J^{'}$ and $\theta > 0$, $s \to (t-s)^{\alpha-1}AT((t-s)^\alpha\theta)$ is continuous in the uniform operator topology over the interval $[0,t)$. Consequently, $(t-s)^{\alpha-1}A\Psi(t-s)H(s,u(s))$ also maintains continuity in $[0,t)$.

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By Eq. (17), Eq. (18) and Eq. (20), for $t \in J$ and $u \in B_{k_0}$,

$$\int_0^t \left\|(t-s)^{\alpha-1} A\Psi(t-s)H(s,u(s))\right\|ds \le \frac{C\xi_2 k_0}{\alpha\rho} T^{\alpha\rho}, \tag{26}$$

similarly, $(t-s)^{\alpha-1}A\Psi(t-s)H(s,u(s))$ is Bochner integrable with respect to both $s \in J$ and $t \in J$. We are now in a position to define an operator G on B_{k_0} as follows:

$$\begin{aligned}(Gu)(t) &= S_{\alpha,1-\alpha}(t)\left[u_1 - h(u) + H(0,u(0))\right] - H(t,u(t)) \\ &\quad + \int_0^t (t-s)^{\alpha-1} A\Psi(t-s)H(s,u(s))ds + \int_0^t S_{\alpha,\beta}(t-s)f(s,u(s))ds,\end{aligned}$$

where $u \in B_{k_0}$. We will demonstrate that the operator G possesses a fixed point within B_{k_0}.

Firstly, we establish that the operator G is completely continuous. Assume that$u_n, u \in B_{k_0}$ and $u_n \to u$ as $n \to \infty$, then

$$\begin{aligned}&\left\|(Gu_n)(t) - (Gu)(t)\right\| \\ &\le \left\|S_{\alpha,1-\alpha}(t)\left[h(u_n) - h(u)\right]\right\| + \left\|H(t,u_n(t)) - H(t,u(t))\right\| \\ &\quad + \int_0^t (t-s)^{\alpha-1}\left\|A\Psi(t-s)\right\|\left\|H(s,u_n(s)) - H(s,u(s))\right\|ds \\ &\quad + \int_0^t \left\|S_{\alpha,\beta}(t-s)\right\|\left\|f(s,u_n(s)) - f(s,u(s))\right\|ds \\ &=: I_4 + I_5 + I_6 + I_7.\end{aligned}$$

Clearly, in accordance with Eq. (16), (H_2) and (H_4), the terms $I_4, I_5 \to 0$. Referencing Lemma 14 and Eq. (19), we have

$$\begin{aligned}I_6 &\le \int_0^t (t-s)^{\alpha-1}\left\|A^{1-\rho}\Psi(t-s)\right\|\left\|A^\rho H(s,u_n(s)) - A^\rho H(s,u(s))\right\|ds \\ &\le C\xi_1\|u_n - u\|_\infty \int_0^t (t-s)^{\alpha\rho-1}ds \\ &\le \frac{C\xi_1 k_0}{\alpha\rho} T^{\alpha\rho},\end{aligned}$$

Then, applying the Lebesgue Dominated Convergence Theorem [32] and assumption (H_2), $I_6, I_7 \to 0$ as $n \to \infty$ [23]. Consequently, it establishes the continuity of the operator G on B_{k_0}.

Next, we will demonstrate that the set $\{Gu|u \in B_{k_0}\}$ is relatively compact. To establish this, it is necessary to show that the set is uniformly bounded and equicontinuous, and that for any $t \in J$, $\{(Gu)(t)|u \in B_{k_0}\}$ possesses the relative compactness in X. We have established that $\|(Gu)(t)\| \le k_0$, as derived from Eq. (23) to Eq. (26), it confirms that the set $\{Gu|u \in B_{k_0}\}$ is uniformly bounded. Considering $u \in B_{k_0}$ and any interval where $0 \le t_1 < t_2 \le T$, then

$$\|(Gu)(t_2) - (Gu)(t_1)\| \le T_1 + T_2 + T_3 + T_4 + T_5 + T_6.$$

where

$$\begin{aligned}T_1 &= \left\|\left[S_{\alpha,1-\alpha}(t_2) - S_{\alpha,1-\alpha}(t_1)\right]\left[u_1 - h(u) + H(0,u(0))\right]\right\|, \\ T_2 &= \left\|H(t_2,u(t_2)) - H(t_1,u(t_1))\right\|, \\ T_3 &= \int_0^{t_1} \left\|S_{\alpha,\beta}(t_2-s) - S_{\alpha,\beta}(t_1-s)\right\|\left\|f(s,u(s))\right\|ds, \\ T_4 &= \int_{t_1}^{t_2} \left\|S_{\alpha,\beta}(t_2-s)f(s,u(s))\right\|ds, \\ T_5 &= \int_0^{t_1} \left\|(t_2-s)^{\alpha-1}A\Psi(t_2-s) - (t_1-s)^{\alpha-1}A\Psi(t_1-s)\right\|\left\|H(s,u(s))\right\|ds, \\ T_6 &= \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\left\|A\Psi(t_2-s)\right\|\left\|H(s,u(s))\right\|ds.\end{aligned}$$

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We can prove that $T_1, T_3 \to 0$ as $t_1 \to t_2$ [23]. Given that $u \in B_{k_0}$ and in light of assumption (H_2), $T_2 \to 0$ as $t_1 \to t_2$. Applying Eq. (16) together with (H_3) to T_4, we find that

$$T_4 \le \int_{t_1}^{t_2} M(t_2-s)^{\alpha+\beta-1} k_0 \|\varpi\|_\infty ds = \frac{Mk_0\|\varpi\|_\infty}{\alpha+\beta}(t_2-t_1)^{\alpha+\beta},$$

therefore, $T_4 \to 0$ as $t_1 \to t_2$. Since

$$\begin{aligned} T_5 \le & \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] \|A\Psi(t_2-s)H(s,u(s))\| ds \\ & + \int_0^{t_1} (t_1-s)^{\alpha-1} \|A\Psi(t_2-s) - A\Psi(t_1-s)\| \|H(s,u(s))\| ds \\ & =: T_{51} + T_{52}, \end{aligned}$$

by Eq. (18) and (H_5),

$$\begin{aligned} T_{51} &\le \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] \|A^{1-\rho}\Psi(t_2-s)\| \|A^\rho H(s,u(s))\| ds \\ &\le C\xi_2 k_0 \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] ds \int_0^{t_1} (t_1-s)^{\alpha(\rho-1)} ds \\ &= \frac{t_2^\alpha - t_1^\alpha - (t_2-t_1)^\alpha}{\alpha} \cdot \frac{C\xi_2 k_0}{\alpha(\rho-1)+1} t_1^{\alpha(\rho-1)+1} \\ &\le C t_1^{\alpha(\rho-1)+1} \xi_2 k_0 \left[t_2^\alpha - t_1^\alpha - (t_2-t_1)^\alpha\right], \end{aligned}$$

where $C > 0$ is a constant, then $T_{51} \to 0$ as $t_1 \to t_2$. If $t_1 = 0$ and t_2 is within J', it is clear that $T_{52} \to 0$ as $t_1 \to t_2$. Additionally, for $t_1 > 0$ and a sufficiently small $\epsilon > 0$,

$$\begin{aligned} T_{52} \le & \int_0^{t_1-\epsilon} (t_1-s)^{\alpha-1} \|\Psi(t_2-s) - \Psi(t_1-s)\| \|AH(s,u(s))\| ds \\ & + \int_{t_1-\epsilon}^{t_1} (t_1-s)^{\alpha-1} \|\Psi(t_2-s) - \Psi(t_1-s)\| \|AH(s,u(s))\| ds \\ & \le \xi_2 k_0 \|A^{1-\rho}\| \int_0^{t_1-\epsilon} (t_1-s)^{\alpha-1} ds \sup_{s\in[0,t_1-\epsilon]} \|\Psi(t_2-s) - \Psi(t_1-s)\| \\ & + C\xi_2 k_0 \|A^{1-\rho}\| \int_{t_1-\epsilon}^{t_1} (t_1-s)^{\alpha-1} ds \\ & = \frac{\xi_2 k_0}{\alpha} (t_1^\alpha - \epsilon^\alpha) \|A^{1-\rho}\| \sup_{s\in[0,t_1-\epsilon]} \|\Psi(t_2-s) - \Psi(t_1-s)\| + C\xi_2 k_0 \epsilon^\alpha \|A^{1-\rho}\|, \end{aligned}$$

since $\Psi(t)$ is continuous in the uniform operator topology for $t > 0$ [33], then $T_{52} \to 0$ independently with $u \in B_{k_0}$ as $t_1 \to t_2$ and $\epsilon \to 0$, hence, $T_5 \to 0$ as $t_1 \to t_2$. Furthermore, according to Eq. (18) and (H_5), $T_6 \le \frac{C\xi_2 k_0}{\alpha\rho}(t_2-t_1)^{\alpha\rho}$, $C > 0$ is a constant, so $T_6 \to 0$ as $t_1 \to t_2$, ensuring the equicontinuity of $\{Gu | u \in B_{k_0}\}$.

Finally, to establish the relative compactness of the set $\{(Gu)(t) | u \in B_{k_0}\}$ in X, according to [23, Theorem 2.11] and due to the compactness of $H(t,u(t))$, it is necessary to demonstrate that the set $\{(\mathcal{J}u)(t) | u \in B_{k_0}\}$ is also relatively compact in X, where

$$(\mathcal{J}u)(t) = \int_0^t (t-s)^{\alpha-1} A\Psi(t-s) H(s,u(s)) ds.$$

Obviously, $(\mathcal{J}u)(0)$ possesses the relative compactness in X. Let $0 < t \le T$ be fixed, for arbitrary values $0 < p_1 < t$ and $p_2 > 0$, we define an operator $\mathcal{V}$ on B_{k_0} as

$$(\mathcal{V}u)(t) = \int_0^{t-p_1} \int_{p_2}^\infty \alpha\theta(t-s)^{\alpha-1} A\zeta_\alpha(\theta) T((t-s)^\alpha\theta) H(s,u(s)) d\theta ds,$$

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then

$$(\mathcal{V}u)(t) = T(p_1^\alpha p_2) \int_0^{t-p_1} \int_{p_2}^{\infty} \alpha\theta(t-s)^{\alpha-1} A\zeta_\alpha(\theta) T((t-s)^\alpha\theta - p_1^\alpha p_2) H(s, u(s)) d\theta ds.$$

By (H_1), $T(p_1^\alpha p_2)(p_1^\alpha p_2 > 0)$ is compact, then for arbitrary $0 < p_1 < t$ and $p_2 > 0$, $\{(\mathcal{V}u)(t)|u \in B_{k_0}\}$ is relatively compact in X. For any $u \in B_{k_0}$, as indicated by $\|T(t)\| \le N$, Eq. (15) and (H_5),

$$\begin{aligned}
&\|(\mathcal{J}u)(t) - (\mathcal{V}u)(t)\| \\
&\le \alpha \int_0^t \int_0^{p_2} \|\theta(t-s)^{\alpha-1} A\zeta_\alpha(\theta) T((t-s)^\alpha\theta) H(s, u(s))\| d\theta ds \\
&\quad + \alpha \int_{t-p_1}^t \int_{p_2}^\infty \|\theta(t-s)^{\alpha-1} A\zeta_\alpha(\theta) T((t-s)^\alpha\theta) H(s, u(s))\| d\theta ds \\
&\le \alpha N \xi_2 k_0 \|A^{1-\rho}\| \int_0^t (t-s)^{\alpha-1} ds \int_0^{p_2} \theta\zeta_\alpha(\theta) d\theta \\
&\quad + \alpha N \xi_2 k_0 \|A^{1-\rho}\| \int_{t-p_1}^t (t-s)^{\alpha-1} ds \int_0^\infty \theta\zeta_\alpha(\theta) d\theta \\
&\le N T^\alpha \xi_2 k_0 \|A^{1-\rho}\| \int_0^{p_2} \theta\zeta_\alpha(\theta) d\theta + \frac{N p_1^\alpha}{\Gamma(1+\alpha)} \xi_2 k_0 \|A^{1-\rho}\|.
\end{aligned}$$

thus, it is possible to identify relatively compact sets arbitrarily close to $\{(\mathcal{J}u)(t)|u \in B_{k_0}\}$, where $t > 0$. Consequently, $\{(\mathcal{J}u)(t)|u \in B_{k_0}\}$ itself is relatively compact in X. Following this, $\{(Gu)(t)|u \in B_{k_0}\}$ also achieves relative compactness in X. According to the Arzela-Ascoli theorem [24], the set $\{Gu|u \in B_{k_0}\}$ is relatively compact. The compactness, combined with the continuity of G, establishes that G is completely continuous. Applying Lemma 17, we find that G has a fixed point on B_{k_0}, implying that Eq. (8) admits a mild solution. The proof is thereby complete. □

Theorem 19. *Under the conditions set forth by assumptions (H_2)-(H_3), (H_4') and (H_5)-(H_6), if the inequality holds:*

$$Ma + \|A^{-\rho}\|\|\varpi\|_\infty + CT^{\alpha\rho}\|\varpi\|_\infty + \frac{MT^{\alpha+\beta}}{\alpha+\beta}\|\varpi\|_\infty < 1,$$

and if Eq. (21) is satisfied, then Eq. (8) admits a mild solution.

Proof. Based on Theorem 18, we establish a proof that demonstrates the continuity of $G : B_{k_0} \to B_{k_0}$. Additionally, the set $\{Gu|u \in B_{k_0}\}$ is uniformly bounded and equicontinuous, where k_0 satisfies Eq. (22). Consider a subset V of B_{k_0} such that $V \subset \overline{G(V)} \cup \{0\}$. Due to the boundedness and equicontinuity of V, it follows that $v(t) = \nu(V(t))$ is continuous for any $t \in J$. By employing Lemma 7, Eq. (16), Lemma 14, (H_5) and (H_6), we have

$$\begin{aligned}
\|v\|_\infty &\le \sup_{t\in J} \|\nu((GV)(t) \cup \{0\})\| \\
&\le \sup_{t\in J} \|\nu((GV)(t))\| \\
&\le \nu(h(V)) \sup_{t\in J} \|S_{\alpha,1-\alpha}(t)\| + \|A^{-\rho}\|\varpi(t)\nu(V(t)) \\
&+ \sup_{t\in J} \int_0^t (t-s)^{\alpha-1} \|A^{1-\rho}\Psi(t-s)\| \varpi(s)\nu(V(s)) ds \\
&+ \sup_{t\in J} \int_0^t \|S_{\alpha,\beta}(t-s)\| \varpi(s)\nu(V(s)) ds \\
&\le Ma\|v\|_\infty + \|A^{-\rho}\|\|\varpi\|_\infty\|v\|_\infty + \sup_{t\in J} \int_0^t C(t-s)^{\alpha\rho-1}\varpi(s)v(s) ds \\
&+ M \sup_{t\in J} \int_0^t (t-s)^{\alpha+\beta-1}\varpi(s)v(s) ds \\
&\le \left(Ma + \|A^{-\rho}\|\|\varpi\|_\infty + CT^{\alpha\rho}\|\varpi\|_\infty + \frac{MT^{\alpha+\beta}}{\alpha+\beta}\|\varpi\|_\infty \right) \|v\|_\infty,
\end{aligned}$$

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since

$$Ma + \|A^{-\rho}\|\|\varpi\|_\infty + CT^{\alpha\rho}\|\varpi\|_\infty + \frac{MT^{\alpha+\beta}}{\alpha+\beta}\|\varpi\|_\infty < 1,$$

then we have $\|v\|_\infty = 0$, it indicates that $v(t) = \nu(V(t)) = 0$, ensuring that $V(t)$ is relatively compact in X. By applying the Arzela-Ascoli theorem [24], $\{Gu|u \in B_{k_0}\}$ possesses the relative compactness. It confirms that $\nu(V) = 0$. Lemma 16 ensures that G has a fixed point on B_{k_0}. In other words, Eq. (8) has a mild solution. Hence, the proof is complete. □

Theorem 20. *Suppose that* $(H_2^{'})$, (H_3), (H_5) *and* (H_7) *hold, then Eq.* (8) *is guaranteed to has a unique mild solution provided that*

$$ML^{'} + (1+M)\xi_2\|A^{-\rho}\| + \frac{C\xi_2}{\alpha\rho}T^{\alpha\rho} + \frac{MT^{\alpha+\beta}}{\alpha+\beta}\|\varpi\|_\infty < 1$$

and

$$ML^{'} + \xi_1\|A^{-\rho}\| + \frac{C\xi_1}{\alpha\rho}T^{\alpha\rho} + \frac{MT^{\alpha+\beta}\|\varpi\|_\infty}{\alpha+\beta} < 1. \tag{27}$$

Proof. Given that $u \in B_{k_1}$, and drawing on the approach of Theorem 18, we can demonstrate that $S_{\alpha,1-\alpha}(t)(u_1 - h(u) + H(0,u(0)))$ exist, $S_{\alpha,\beta}(t-s)f(s,u(s))$ and $(t-s)^{\alpha-1}A\Psi(t-s)H(s,u(s))$ are Bochner integrable with respect to $s \in J$ and $t \in J$, and G maps B_{k_1} into itself. Now we only need to prove that G has a unique fixed point on B_{k_1}.

For arbitrary $u, v \in B_{k_1}$ and $t \in J$, by Eq. (16), Lemma 14, $(H_5)$and (H_7), we obtain

$$\begin{aligned}
&\|(Gu)(t) - (Gv)(t)\| \\
&\leq \|S_{\alpha,1-\alpha}(t)[h(v) - h(u)]\| + \|H(t,v(t)) - H(t,u(t))\| \\
&\quad + \int_0^t (t-s)^{\alpha-1}\|A^{1-\rho}\Psi(t-s)\|\|A^\rho H(s,u(s)) - A^\rho H(s,v(s))\|ds \\
&\quad + \int_0^t \|S_{\alpha,\beta}(t-s)\|\|f(s,u(s)) - f(s,v(s))\|ds \\
&\leq ML^{'}\|u-v\|_\infty + \xi_1\|A^{-\rho}\|\|u-v\|_\infty + C\xi_1\|u-v\|_\infty \int_0^t (t-s)^{\alpha\rho-1}ds \\
&\quad + M\|u-v\|_\infty \int_0^t (t-s)^{\alpha+\beta-1}\varpi(s)ds \\
&\leq \left(ML^{'} + \xi_1\|A^{-\rho}\| + \frac{C\xi_1 T^{\alpha\rho}}{\alpha\rho} + \frac{MT^{\alpha+\beta}\|\varpi\|_\infty}{\alpha+\beta}\right)\|u-v\|_\infty.
\end{aligned}$$

It follows from Eq. (27) that G is a contraction mapping. Therefore, by applying the Banach fixed point theorem, we conclude that Eq. (8) has a unique mild solution, the proof is complete. □

4. Discussion

This paper defines the mild solutions of fractional neutral equations with Neumann boundary conditions through the Laplace transform, a resolvent family $\{S_{\alpha,\beta}(t)\}_{t\geq 0}$, and the function $\Psi(t)$. It also establishes several sufficient conditions for the existence of mild solutions to the equations. Importantly, it demonstrates that deriving (16) from

$$S_{\alpha,\beta}(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}s^{\alpha-1}\Psi(s)ds$$

in [23]. Furthermore, the study achieves the same results as those reported in [23] without the use of path integration.

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On the other hand, the probability density function $\zeta_\alpha(\theta)$ plays a significant role in studying solutions of fractional differential equations. Building on the findings presented in this article, further investigation into the solution of differential equations of the form

$$^{c}D_t^{\alpha}u(x,t) + \sum_{i=1}^{m} b_i^c D_t^{\alpha_i} u(x,t) = Au(x,t) + f(x,t), \tag{28}$$

which involve multiple fractional derivatives, is warranted, where $t > 0$, $x \in \mathbb{R}^n$, $0 < \alpha_m < \cdots < \alpha_1 < \alpha \leq 1$, $b_i > 0$, $i = 1, \ldots, m$, A generates an analytic semigroup with boundness. Bazhlekova established the fundamental properties, primarily complete monotonicity, of the Prabhakar-type generalizations for multinomial Mittag-Leffler functions. These properties were investigated through the use of Laplace transform and Bernstein functions in studying Eq. (28), resulting in several derived estimates.

As the results in [33,34], $\mathcal{L}[\zeta_\alpha(\theta)] = E_\alpha(-z)$ and the multinomial Mittag-Leffler function

$$E_{(a_1,a_2,\ldots,a_n),b}(z_1,z_2,\ldots,z_n) = \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_n=k \\ l_1\geq 0,\ldots l_n\geq 0}} \frac{k!}{\prod_{i=1}^{n} l_i!} \frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma(b + \sum_{i=1}^{n} a_i l_i)}. \tag{29}$$

Since the Laplace transform in $\mathbb{R}^n$ is defined as

$$\mathcal{L}[\varphi(\mathbf{t})] = \hat{\varphi}(\mathbf{s}) = \int_{\mathbb{R}^n_{+\cdots+}} e^{-\mathbf{st}} \varphi(\mathbf{t}) d\mathbf{t}, \tag{30}$$

where $\mathbf{s} \in \mathbb{C}^n$, $\mathbb{R}^n_{+\cdots+} = \{\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{R}^n, t_j > 0, j = 1, \ldots, n\}$. Then, we will consider obtaining the multinomial form of $\zeta_\alpha(\theta)$ by Eq. (29) and Eq. (30) to investigate solutions of Eq. (28).

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