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Article

Parametric Asymptotic Expansions and Confluence for Banach Valued Solutions to Some Singularly Perturbed Nonlinear q-Difference-Differential Cauchy Problem

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Abstract: We investigate a singularly perturbed q-difference differential Cauchy problem with polynomial coefficients in complex time t and space z and with quadratic nonlinearity. We construct local holomorphic solutions on sectors in the complex plane with respect to the perturbation parameter ϵ with values in some Banach space of formal power series in z with analytic coefficients on shrinking domains in t. Two aspects of these solutions are addressed. One feature concerns asymptotic expansions in ϵ for which a Gevrey type structure is unveiled. The other fact deals with confluence properties as $q > 1$ tends to 1. In particular the built up Banach valued solutions are shown to merge in norm to a fully bounded holomorphic map in all the variables t,z and ϵ that solves a nonlinear partial differential Cauchy problem.

Keywords: asymptotic expansion; confluence; formal power series; partial differential equation; $q-$difference equation

MSC: 35R10; 35C10; 35C15; 35C20

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1. Introduction

In this paper we examine a nonlinear singularly perturbed Cauchy problem which couples up two classes of operators applying both on a complex time variable t, formed by compositions of the plain $q-$difference operator $\sigma_{q;t} : t \mapsto qt$ for a prescribed real number $q > 1$ and powers of the basic singularly perturbed differential operator $t \mapsto \epsilon^k t^{k+1}\partial_t$ of irregular type where $k \geq 1$ is a given integer and ϵ stands for a complex parameter.

The problem under study is outlined as follows

$$P(\epsilon^k t^{k+1}\partial_t)\partial_z^S u(t,z,\epsilon) = \mathcal{P}(t,z,\epsilon,\sigma_{q;t},\epsilon^k t^{k+1}\partial_t,\partial_z)u(t,z,\epsilon) + d(z,\epsilon)(u(t,z,\epsilon))^2 \quad (1)$$

for assigned Cauchy data

$$(\partial_z^j u)(t,0,\epsilon) = \varphi_j(t,\epsilon) \ \ 0 \leq j \leq S-1 \quad (2)$$

where

- $S,k \geq 1$ are integers and the element $P(T)$ from the leading term of (1) represents a polynomial belonging to $\mathbb{C}[T]$.
- The linear part $\mathcal{P}(t,z,\epsilon,V_1,V_2,V_3)$ of (1) is a polynomial with complex coefficients in all its arguments except in ϵ whose dependence is bounded analytic on a fixed disc D_{ϵ_0} centered at 0 in $\mathbb{C}$ with given radius $\epsilon_0 > 0$.
- The coefficient $d(z,\epsilon)$ of the nonlinear part of (1) together with the Cauchy data $\varphi_j(t,\epsilon)$, $0 \leq j \leq S-1$, are complex polynomials in their arguments t,z with bounded holomorphic reliance in ϵ on D_{ϵ_0}.

This work is a continuation of the study [8] by A. Lastra and the author. In [8], we consider linear singularly perturbed Cauchy problems whose shape is similar to the linear part of (1),

$$Q(\epsilon^r t^{r+1}\partial_t)\partial_z^\beta y(t,z,\epsilon) = \mathcal{Q}(t,z,\epsilon,\sigma_{q;t},\epsilon^r t^{r+1}\partial_t,\partial_z)y(t,z,\epsilon) \quad (3)$$

for given Cauchy data

$$(\partial_z^j y)(t,0,\epsilon) = \psi_j(t,\epsilon) \ , \ 0 \leq j \leq S-1 \tag{4}$$

where $\beta, r \geq 1$ are integers, $Q(T)$ belongs to $\mathbb{C}[T]$, $\mathcal{Q}(t,z,\epsilon,V_1,V_2,V_3)$ stands for a polynomial in t,z,V_1,V_2,V_3 with bounded holomorphic coefficients in ϵ on a small fixed disc D_{ϵ_0} and where the data ψ_j have the same features as the Cauchy data (2).

Assuming strong restrictions on the profile of $\mathcal{Q}$, which differ from the conditions we impose on $\mathcal{P}$ in the present work (see Section 2.1), we build up a finite set $\{y_p(t,z,\epsilon)\}_{0 \leq p \leq \varsigma-1}$, for some integer $\varsigma \geq 2$, of bounded holomorphic solutions to (3), (4) defined on products $\mathcal{T} \times D_R \times \mathcal{E}_p$, for some radius $R > 0$, where $\mathcal{T}$ is a suitably chosen bounded sector at 0 and where the set of finite sectors $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma-1}$ stands for a good covering in $\mathbb{C}^*$ (see Definition 1 in this paper). These solutions are represented by means of power series in z with Laplace transforms of order r coefficients

$$y_p(t,z,\epsilon) = \sum_{n \geq 0} y_{p,n}(t,\epsilon)\frac{z^n}{n!}$$

where

$$y_{p,n}(t,\epsilon) = k\int_{L_{\gamma p}} \omega_{p,n}(u,\epsilon)\exp\left(-(\frac{u}{\epsilon t})^r\right)\frac{du}{u}$$

along halflines $L_{\gamma_p} = [0,+\infty)e^{\sqrt{-1}\gamma_p}$ in appropriate directions $\gamma_p \in \mathbb{R}$ where the so-called Borel maps $\omega_{p,n}$ are subjected to q-exponential growth relatively to u on L_{γ_p}, namely

$$|\omega_{p,n}(u,\epsilon)| \leq G_n n! |u| \exp\left(k_1 \log^2(|u|+u_0) + \alpha\log(|u|+u_0)\right) \tag{5}$$

for some geometric sequence $(G_n)_{n \geq 0}$ and positive constants $k_1, u_0, \alpha > 0$.

Actually, the main achievement of [8] concerns the construction of asymptotic expansions of these solutions as ϵ tends to 0. Our idea consists in *embedding* the partial maps $(t,z) \mapsto y_p(t,z,\epsilon)$, for all $\epsilon \in \mathcal{E}_p$, in two *distinguished* Banach spaces of formal power series in z with holomorphic coefficients in t on $\mathcal{T}$. Each embedding forsters a different formal Gevrey asymptotic expansion as ϵ tends to 0, one of so-called q-Gevrey/Gevrey mixed type and the second of mere Gevrey type. Namely, let $R_1 > 0$ be a fixed real number.

- We denote $\mathbb{E}_{\mathcal{T};R_1}$ the vector space of formal power series $h(t,z) = \sum_{n \geq 0} h_n(t)z^n/n!$ with coefficients $h_n(t) \in \mathcal{O}_b(\mathcal{T})$, bounded holomorphic on the sector $\mathcal{T}$, equipped with the mixed type sup norm and L_1 norm

  $$||h(t,z)||_{\mathcal{T};R_1} := \sum_{n \geq 0} \sup_{t \in \mathcal{T}} |h_n(t)|\frac{R_1^n}{n!}.$$

  The first main result of [8] states that for each $0 \leq p \leq \varsigma - 1$, the partial map $\epsilon \mapsto y_p(t,z,\epsilon)$ represents a bounded holomorphic map from $\mathcal{E}_p$ into the Banach space $(\mathbb{E}_{\mathcal{T};R_1}, ||.||_{\mathcal{T};R_1})$ provided that $0 < R_1 < R$. Furthermore, there exists a formal power series

  $$\hat{u}_1(t,z,\epsilon) = \sum_{n \geq 0} u_{1,n}(t,z)\epsilon^n \in \mathbb{E}_{\mathcal{T};R_1}[[\epsilon]]$$

  which is the common asymptotic expansion of so-called Gevrey type with mixed order $(1/r;(q,1))$ for all partial maps $\epsilon \mapsto y_p(t,z,\epsilon)$ on $\mathcal{E}_p$, with $0 \leq p \leq \varsigma - 1$. It means that one can find constants $A_p, C_p > 0$ with

  $$||y_p(t,z,\epsilon) - \sum_{n=0}^{N} u_{1,n}(t,z)\epsilon^n||_{\mathcal{T};R_1} \leq C_p(A_p)^N q^{(N+1)^2/2}\Gamma(\frac{N+1}{r})|\epsilon|^{N+1}$$

  for all $N \geq 0$, all $\epsilon \in \mathcal{E}_p$, all $0 \leq p \leq \varsigma - 1$.

- The second Banach space in which the partial map $(t,z) \mapsto y_p(t,z,\epsilon)$ can be embedded for all $\epsilon \in \mathcal{E}_p$ has been at first introduced in [8] and minded in Definition 3 of this paper. It is denoted $\mathcal{O}_{(\mathcal{D}_n)_{n\geq 0};R_1}$, for $0 < R_1 < R$ and represents a modification of the classical space $\mathbb{E}_{\mathcal{T};R_1}$ for which the domain $\mathcal{D}_n$ of each sup norm taken for the coefficient $h_n(t)$ now relies on n and shrinks to the empty set as $n \to +\infty$, meaning in particular that $\cap_{n\geq 0}\mathcal{D}_n = \emptyset$. We show the existence of a formal power series
$$\hat{u}_2(t,z,\epsilon) = \sum_{n\geq 0} u_{2,n}(t,z)\epsilon^n \in \mathcal{O}_{(\mathcal{D}_n)_{n\geq 0};R_1}[[\epsilon]]$$
which is the shared asymptotic expansion of Gevrey type with order $1/r$ for all partial maps $\epsilon \mapsto y_p(t,z,\epsilon)$ on $\mathcal{E}_p$, with $0 \leq p \leq \varsigma - 1$. In other words, constants $D_p, B_p > 0$ can be singled out with
$$||y_p(t,z,\epsilon) - \sum_{n=0}^{N} u_{2,n}(t,z)\epsilon^n||_{(\mathcal{D}_n)_{n\geq 0};R_1} \leq D_p(B_p)^N \Gamma(\frac{N+1}{r})|\epsilon|^{N+1}$$
for all $N \geq 0$, all $\epsilon \in \mathcal{E}_p$, all $0 \leq p \leq \varsigma - 1$.

In the present work, our objectives are similar to the ones of [8]. Namely

- The construction of a set of solutions $u_p(t,z,\epsilon)$, $0 \leq p \leq \varsigma - 1$, for some integer $\varsigma \geq 2$, to (1), (2), well defined and holomorphic with respect to ϵ on bounded sectors $\mathcal{E}_p$.
- The asymptotic analysis of these solutions as ϵ tends to 0.

Furthermore, we address another aspect concerning the so-called *confluence* of this finite set of solutions as q tends to 1. This feature has already been studied in the linear case and in the nonperturbative setting for equations with the shape (3) in [9].

It is noteworthy to stress that the statements of this work are presented for the Cauchy problem (1), (2) which presents a quadratic nonlinearity. Such a restriction has only been favored in order to avoid cumbersome and lenghty computations for the convenience of the reader. The approach we introduce in this work can actually be applied to a wider class of equations with higher degrees polynomial nonlinearities that might also involve actions of the q-difference operator $\sigma_{q;t}$.

In the first main result of this paper (Theorem 1, Section 5), given a set of technical requirements disclosed in Subsection 2.1 imposed on (1), (2), we establish a finite set $\{u_p(t,z,\epsilon)\}_{0\leq p\leq\varsigma-1}$ of solutions to (1), (2), for some integer $\varsigma \geq 2$, that enjoys the next properties. A good covering $\{\mathcal{E}_p\}_{0\leq p\leq\varsigma-1}$ in $\mathbb{C}^*$ and a bounded sector $\mathcal{T}$ centered at 0 can be properly chosen in a way that for each $0 \leq p \leq \varsigma - 1$

- the partial map $\epsilon \mapsto u_p(t,z,\epsilon)$ is bounded holomorphic from $\mathcal{E}_p$ into a Banach space $(\mathcal{O}_{(\mathcal{D}_n)_{n\geq 0};R_1}, ||.||_{(\mathcal{D}_n)_{n\geq 0};R_1})$ for a contracting sequence of domains $\mathcal{D}_n = \mathcal{T} \cap D_{\tilde{R}_0/q^{n\tilde{\Delta}}}$ where the radius $\tilde{R}_0, R_1 > 0$ and $\tilde{\Delta} > 0$ are suitable real numbers.
- the map $u_p(t,z,\epsilon)$ is expressed through a formal power series in z
$$u_p(t,z,\epsilon) = \sum_{n\geq 0} u_{p,n}(t,\epsilon)\frac{z^n}{n!}$$
with Laplace transform of order k coefficients
$$u_{p,n}(t,\epsilon) = k\int_{L_{\gamma_p}} w_{p,n}(u,\epsilon)\exp\left(-(\frac{u}{\epsilon t})^k\right)\frac{du}{u}$$
along halflines L_{γ_p} in fitting directions $\gamma_p \in \mathbb{R}$ where the Borel maps $w_{p,n}$ are *no longer* of q-exponential growth as in [8], see (5), but with exponential growth of order k with respect to u,
$$|w_{p,n}(u,\epsilon)| \leq C_n n!|u|\exp(K_n|u|^k)$$
for some geometric sequence $(C_n)_{n\geq 0}$ and where the type K_n tends to $+\infty$ as $n \to +\infty$ with the shape Cq^{nM_1} for convenient constants $C, M_1 > 0$.

In comparison to our previous work [8], we are not able to construct analytic solutions to (1), (2) in all arguments t, z and ϵ but only analytic in ϵ whose values are located in the second embedding introduced in [8]. However, for some special type of nonlinear q-difference and differential Cauchy problem, analytic solutions both in complex time and space could be exhibited in a recent contribution of the author, see [10]. These problems are expressed as a *coupling* of a nonperturbative version of the linear Cauchy problem (3), (4) and a classical Cauchy-Kowaleski type partial differential equation with quadratic nonlinearity which involves the action of the contractive q-difference operator $t \mapsto q^{-l}t$ for some integers $l \geq 1$.

In general, the construction of genuine holomorphic local solutions to nonlinear equations involving q-difference operators is a difficult endeavour. In the case of nonlinear q-difference equations, a lush literature concerns the so-called q-Painlevé equations which are q-analogs of the celebrated second order nonlinear Painlevé equations. We refer to the book [6] for a comprehensive introduction to the subject. General nonlinear first order q-difference equations have been studied in [11] from the standpoint of mould calculus introduced by J. Écalle. More general nonlinear algebraic q-difference equations have been recently considered in [5] where the authors build up local holomorphic solutions on sectors by means of generalized power series expansions with complex exponents. In the framework of partial q-difference and differential equations, the amount of results is more scarce. We mention however the important result [13] for the construction of convergent power series and logarithmic type solutions to a q-analog of the Briot-Bouquet type partial differential equations extensively studied in the textbook [4].

In the second foremost statement of our work (Theorem 2, Section 6), we prove the existence of a formal power series $\hat{u}(t,z,\epsilon) = \sum_{n\geq 0} h_n(t,z)\epsilon^n$ whose coefficients belong to the Banach space $\mathcal{O}_{(\mathcal{D}_n)_{n\geq 0};R_1}$ which is the common asymptotic expansion of Gevrey type with order $1/k$ for all the partial maps $\epsilon \mapsto u_p(t,z,\epsilon)$ on $\mathcal{E}_p$, for $0 \leq p \leq \varsigma - 1$. This result is in the vein of the one concerning the parametric asymptotic expansion of the second embedding for the solutions $y_p(t,z,\epsilon)$ to (3), (4) obtained in [8]. No Gevrey type expansions with mixed order are reached in the present work. Notice that such double scales expansions were obtained for the holomorphic solutions to the special nonlinear q-difference and differential Cauchy problems investigated in [10].

In the third prominent claim of our study (Theorem 3, Subsection 7.4), we show that for any given sector $\mathcal{E}$ from the good covering $\{\mathcal{E}_p\}_{0\leq p\leq\varsigma-1}$ in $\mathbb{C}^*$, the corresponding solution $\epsilon \mapsto u_{;q}(t,z,\epsilon)$ (where the reliance on the parameter q is flagged by an index $;q$) to (1), (2) merges uniformly on $\mathcal{E}$, as $q \in (1, q_0]$ tends to 1, for some fixed $q_0 > 1$ to a holomorphic function $u_{;1}(t,z,\epsilon)$ in the norm $||.||_{(\hat{\mathcal{D}}_n)_{n\geq 0};\hat{R}_1}$ for domains $\hat{\mathcal{D}}_n = \mathcal{T} \cap D_{\hat{R}_0/q^{n\tilde{\Delta}}}$ and suitable radius $\hat{R}_0, \hat{R}_1 > 0$. The limit map $u_{;1}(t,z,\epsilon)$ is bounded holomorphic on a domain $(\mathcal{T} \cap D_{\check{R}_0}) \times D_{\check{R}_1} \times \mathcal{E}$ for some radius $\check{R}_0, \check{R}_1 > 0$ and solves a nonlinear partial differential Cauchy problem displayed in (109), (110) which is merely reached by setting $q = 1$ in the initial problem (1), (2).

In the context of linear q-difference equations, general statements for the confluence of holomorphic solutions as $q \to 1$ have been established in a recent past for Fuchsian systems in [12] and for so-called equations with irregular singularity involving several slopes in [2]. From the standpoint of nonlinear q-difference equations, confluence still remains a direction of active research to which our present contribution participates. In this trend, we can mention the recent major work [3] on the confluence of some discret solutions for the q-Painlevé VI equations as q tends to 1 to analytic solutions for the famous Painleve VI equations using Hamiltonian systems representations. This last work has been strongly influential for the investigation of confluence properties for the special type of nonlinear q-difference and differential Cauchy problems mentioned earlier in this introduction and undertaken in [10].

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2. Statement of the Main Problem

2.1. The Main Cauchy Problem and the Set of Assumptions Disclosed

Let $k, S \geq 1$ be integers and $q > 1$, $\epsilon_0 > 0$ be real numbers. We consider $P(\tau) \in \mathbb{C}[\tau]$ a polynomial with complex coefficients subjected to the conditions

$$\deg(P) \geq 1 \ , \ P(0) \neq 0. \tag{6}$$

Let us fix a finite subset $\mathcal{A}$ of $\mathbb{N}^4$. We denote D_{ϵ_0} the open disc in $\mathbb{C}$ centered at 0 with radius ϵ_0. The radius $\epsilon_0 > 0$ will be determined later on in the study and will be taken close to 0. The next items represent the coefficients and Cauchy data for the main problem we consider in this work.

- To each element $\underline{l} \in \mathcal{A}$, we attach
  - a polynomial
    $$c_{\underline{l}}(z,\epsilon) = \sum_{h \in I_{\underline{l}}} c_{\underline{l},h}(\epsilon) z^h \tag{7}$$
    in the variable z, where $I_{\underline{l}}$ stands for a finite subset of the natural numbers $\mathbb{N}$, whose coefficients $c_{\underline{l},h}$ are bounded holomorphic functions on the disc D_{ϵ_0}. For later use, we introduce the constants
    $$\sup_{\epsilon \in D_{\epsilon_0}} |c_{\underline{l},h}(\epsilon)| = \mathbf{c}_{\underline{l},h,\epsilon_0} \tag{8}$$
    for $h \in I_{\underline{l}}$.
- We set
  $$d(z,\epsilon) = \sum_{h \in I_d} d_h(\epsilon) z^h \tag{9}$$
  as a polynomial in the variable z, where I_d is a finite subset of $\mathbb{N}$, whose coefficients $d_h(\epsilon)$ are bounded holomorphic maps on the disc D_{ϵ_0}. Furthermore, the constants
  $$\sup_{\epsilon \in D_{\epsilon_0}} |d_h(\epsilon)| = \mathbf{d}_{h,\epsilon_0} \tag{10}$$
  for $h \in I_d$, are assumed close enough to the origin in a manner that will be expounded later on in the paper.
- For all $0 \leq j \leq S-1$, we introduce maps $\varphi_j(t,\epsilon)$ expressed in time rescaled form
  $$\varphi_j(t,\epsilon) = \check{\varphi}_j(\epsilon t, \epsilon) \tag{11}$$
  where $\check{\varphi}_j$ stands for a polynomial in the variable T expanded as
  $$\check{\varphi}_j(T,\epsilon) = \sum_{h \in J_j} p_{j,h}(\epsilon) \Gamma(h/k) T^h \tag{12}$$
  where J_j represents a finite subset of $\mathbb{N} \setminus \{0\}$, the coefficients $p_{j,h}$ stand for bounded holomorphic funtions on the disc D_{ϵ_0} and $\Gamma(x)$ symbolizes the classical Gamma function.

For grounds that will be justified later on in the work, the next list of conditions is required on the finite set $\mathcal{A}$.
1) For all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$, the strict inequalities

$$\Delta_{\underline{l}} > l_0 \ , \ S > l_2 \tag{13}$$

hold.

2) There exists a positive real number $M_1 > 0$ for which

$$SM_1 \geq l_2 M_1 + l_3 k \tag{14}$$

for all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$.

3) For all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$, the lower bounds

$$k\mathrm{deg}(P) \geq l_0 + kl_1 \tag{15}$$

hold.

We consider the next singularly perturbed nonlinear Cauchy problem with polynomial coefficients in complex time t and space z,

$$P(\epsilon^k t^{k+1}\partial_t)\partial_z^S u(t,z,\epsilon) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}} \epsilon^{\Delta_{\underline{l}}} c_{\underline{l}}(z,\epsilon) t^{l_0} \left((\epsilon^k t^{k+1}\partial_t)^{l_1} \partial_z^{l_2} u \right)(q^{l_3}t,z,\epsilon) + d(z,\epsilon)(u(t,z,\epsilon))^2 \tag{16}$$

for prescribed Cauchy data

$$(\partial_z^j u)(t,0,\epsilon) = \varphi_j(t,\epsilon) \ , \ 0 \leq j \leq S-1. \tag{17}$$

We discuss now our main approach that will lead later on to the construction of fitting sets of solutions to our problem. We search for solutions in time rescaled form

$$u(t,z,\epsilon) = U(\epsilon t,z,\epsilon) \tag{18}$$

for some expressions $U(T,z,\epsilon)$. We first observe that $u(t,z,\epsilon)$ formally solves the problem (16), (17) **if** the expression $U(T,z,\epsilon)$ solves the next Cauchy problem

$$P(T^{k+1}\partial_T)\partial_z^S U(T,z,\epsilon) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}} \epsilon^{\Delta_{\underline{l}}-l_0} c_{\underline{l}}(z,\epsilon) T^{l_0} \left((T^{k+1}\partial_T)^{l_1} \partial_z^{l_2} U \right)(q^{l_3}T,z,\epsilon) + d(z,\epsilon)(U(T,z,\epsilon))^2 \tag{19}$$

for given Cauchy data

$$(\partial_z^j U)(T,0,\epsilon) = \check{\varphi}_j(T,\epsilon) \ , \ 0 \leq j \leq S-1. \tag{20}$$

We seek for solutions to (19), (20) presented as formal series in the variable z,

$$U(T,z,\epsilon) = \sum_{n\geq 0} U_n(T,\epsilon)\frac{z^n}{n!} \tag{21}$$

whose coefficients $U_n(T,\epsilon)$ stand for Laplace transforms of order k,

$$U_n(T,\epsilon) = k\int_{L_\gamma} w_n(u,\epsilon)\exp(-(\frac{u}{T})^k)du/u \tag{22}$$

along a prescribed halfline $L_\gamma = [0,+\infty)e^{\sqrt{-1}\gamma}$ for some well chosen direction $\gamma \in \mathbb{R}$. For all $n \geq 0$, the so-called Borel maps $w_n(u,\epsilon)$ are assumed to be holomorphic on a common product $\mathcal{U} \times D_{\epsilon_0}$, where $\mathcal{U}$ is an open unbounded sector edged at 0 containing the halfline $L_\gamma \setminus \{0\}$. For each Laplace transform

(22) to be well defined, we make the assumption that for each $n \geq 0$, there exist constants $C_n, K_n > 0$ with the bounds

$$\sup_{\epsilon \in D_{\epsilon_0}} |w_n(u,\epsilon)| \leq C_n |u| \exp(K_n |u|^k) \tag{23}$$

provided that $u \in \mathcal{U}$. The precise shape of the sequences $(C_n)_{n \geq 0}$ and $(K_n)_{n \geq 0}$ will be given in Section 4.

Once it is assumed that such formal solutions exists, we plan to derive some recursion relations that the sequence of Borel maps $w_n(u,\epsilon)$, $n \geq 0$, are asked to fulfill. Such relations will be described in the next subsection. In later sections of the work (see Sections 3,4 and 5) we will rigorously show the existence of such recursion relations in different kind of function spaces, that will lead to the construction of some Banach valued solutions of the form (21) to (19), (20). Furthermore, asymptotic expansions as ϵ tends to 0 will be extracted in Section 6.

2.2. Some $q-$difference-differential and $q-$difference-convolution relations

In this subsection, we formulate some integro-$q-$difference recurrence relations satisfied by the sequence $(w_n(u,\epsilon))_{n \geq 0}$.

We first need to derive some $q-$difference-differential relations that the sequence of functions $(U_n(T,\epsilon))_{n \geq 0}$ is asked to fulfill. The next lemma is straightforward.

Lemma 1. *The formal series (21) solves the problem (19), (20)* **if** *the sequence $(U_n(T,\epsilon))_{n \geq 0}$ fulfills the next $q-$difference-differential relation*

$$P(T^{k+1}\partial_T)\frac{U_{n+S}(T,\epsilon)}{n!} = \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}} \epsilon^{\Delta_{\underline{l}} - l_0} \sum_{n_1+n_2=n} c_{\underline{l},n_1}(\epsilon) \frac{T^{l_0}((T^{k+1}\partial_T)^{l_1} U_{n_2+l_2})(q^{l_3}T,\epsilon)}{n_2!}$$
$$+ \sum_{n_1+n_2+n_3=n} d_{n_1}(\epsilon) \frac{U_{n_2}(T,\epsilon)}{n_2!} \frac{U_{n_3}(T,\epsilon)}{n_3!} \tag{24}$$

for all $n \geq 0$, together with

$$U_j(T,\epsilon) = \check{\varphi}_j(T,\epsilon) \ , \ 0 \leq j \leq S-1, \tag{25}$$

where, by convention, we set $c_{\underline{l},h}(\epsilon) \equiv 0$ for $h \notin I_{\underline{l}}$ and $d_h(\epsilon) \equiv 0$ whenever $h \notin I_d$.

The next proposition rephrases Proposition 1 from [10] for $\mathbb{C}-$valued holomorphic maps.

Proposition 1. *Let $k \geq 1$ be an integer and let $w : S_{d,\delta} \rightarrow \mathbb{C}$ be a holomorphic function on the open unbounded sector $S_{d,\delta} = \{u \in \mathbb{C}^* : |d - \arg(u)| < \delta\}$, continuous on $S_{d,\delta} \cup \{0\}$. We take for granted the existence of two constants $C > 0$ and $K > 0$ such that*

$$|w(u)| \leq C|u|e^{K|u|^k} \tag{26}$$

for all $u \in S_{d,\delta}$. Then, the Laplace transform of order k of w in the direction d is defined by the integral representation

$$\mathcal{L}_k^d(w(u))(t) = k \int_{L_\gamma} w(u) e^{-(u/t)^k} \frac{du}{u},$$

along a half-line $L_\gamma = \mathbb{R}_+ e^{i\gamma} \subset S_{d,\delta} \cup \{0\}$, where γ depends on t and is chosen in such a way that $\cos(k(\gamma - \arg(t))) \geq \delta_1 > 0$, for some fixed δ_1. The function $\mathcal{L}_k^d(w(u))(t)$ is well defined, holomorphic and bounded in any sector

$$S_{d,\theta,R^{1/k}} = \{t \in \mathbb{C}^* : |t| < R^{1/k} \ , \ |d - \arg(t)| < \theta/2\}, \tag{27}$$

where $\frac{\pi}{k} < \theta < \frac{\pi}{k} + 2\delta$ and $0 < R < \delta_1/K$.
A) The action of the Laplace transform on entire functions is described as follows: If w is an entire function on

$\mathbb{C}$, with growth estimates (26) and with Taylor expansion $w(u) = \sum_{n\geq 1} b_n u^n$, then $\mathcal{L}_k^d(w(u))(t)$ defines an analytic function near the origin w.r.t t with convergent Taylor expansion $\sum_{n\geq 1} \Gamma(\frac{n}{k}) b_n t^n$.
B) The actions of the irregular operator $t^{k+1}\partial_t$ and the monomial t^m on the Laplace transform are expressed through the next formulas

$$\mathcal{L}_k^d(ku^k w(u))(t) = t^{k+1}\partial_t\Big(\mathcal{L}_k^d(w(u))(t)\Big) \ , \ t^m \mathcal{L}_k^d(w(u))(t) = \mathcal{L}_k^d\Big(u \mapsto (u^m \star_k w(u))\Big)(t), \tag{28}$$

for every nonnegative integer m, and for all $t \in S_{d,\theta,R^{1/k}}$ with $0 < R < \delta_1/K$. Here, $u^m \star_k w(u)$ stands for the convolution product

$$\frac{u^k}{\Gamma(\frac{m}{k})} \int_0^{u^k} (u^k - s)^{\frac{m}{k}-1} w(s^{1/k}) \frac{ds}{s}.$$

C) Let $w_1, w_2 : S_{d,\delta} \to \mathbb{C}$ be holomorphic maps with the same feature (26) as w above. Then, the next multiplicative formula

$$\mathcal{L}_k^d(w_1(u))(t) \times \mathcal{L}_k^d(w_2(u))(t) = \mathcal{L}_k^d(w_1(u) \star_k w_2(u))(t) \tag{29}$$

holds for all $t \in S_{d,\theta,R^{1/k}}$ with $0 < R < \delta_1/K$, where $w_1(u) \star_k w_2(u)$ represents the convolution product

$$w_1(u) \star_k w_2(u) := u^k \int_0^{u^k} w_1((u^k - s)^{1/k}) w_2(s^{1/k}) \frac{1}{(u^k - s)s} ds$$

D) The action of the dilation $t \mapsto q^\delta t$ commutes with the Laplace transform, for any integer $\delta \geq 1$, namely

$$\mathcal{L}_k^d(w(u))(q^\delta t) = \mathcal{L}_k^d(w(q^\delta u))(t) \tag{30}$$

holds for all $t \in S_{d,\theta,R_1^{1/k}}$ for $0 < R_1 < \delta_1/(Kq^{k\delta})$.

From the point A) of the above proposition, we first notice that the Cauchy data (20) given as polynomials through (12) can be expressed as Laplace transforms of order k,

$$\check{\varphi}_j(T,\epsilon) = k \int_{L_\gamma} P_j(u,\epsilon) \exp(-(u/T)^k) du/u \tag{31}$$

of polynomials given by

$$P_j(u,\epsilon) = \sum_{h \in J_j} p_{j,h}(\epsilon) u^h \ , \ 0 \leq j \leq S-1. \tag{32}$$

According to the above identities (28), (29), (30) and (31), the next lemma is deduced.

Lemma 2. *The sequence $(U_n(T,\epsilon))_{n\geq 0}$ where U_n are given by (22) conforms to the $q-$difference-differential relations (24) with prescribed S first terms (25)* **if** *the sequence of Borel maps $(w_n(u,\epsilon))_{n\geq 0}$ obeys the next $q-$difference-convolution recurrence relation*

$$
\begin{aligned}
\frac{w_{n+S}(u,\epsilon)}{n!} = & \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \epsilon^{\Delta_{\underline{l}}} \frac{(k(q^{l_3}u)^k)^{l_1}}{P(ku^k)} \Big(\sum_{n_1+n_2=n} c_{\underline{l},n_1}(\epsilon) \frac{w_{n_2+l_2}(q^{l_3}u,\epsilon)}{n_2!} \Big) \\
& + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon^{\Delta_{\underline{l}}-l_0} \frac{u^k}{P(ku^k)\Gamma(l_0/k)} \int_0^{u^k} (u^k-s)^{\frac{l_0}{k}-1} (k(q^{l_3}s^{1/k})^k)^{l_1} \\
& \times \Big(\sum_{n_1+n_2=n} c_{\underline{l},n_1}(\epsilon) \frac{w_{n_2+l_2}(q^{l_3}s^{1/k},\epsilon)}{n_2!} \Big) \frac{ds}{s} \\
& + \frac{u^k}{P(ku^k)} \int_0^{u^k} \Big(\sum_{n_1+n_2+n_3=n} d_{n_1}(\epsilon) \frac{w_{n_2}((u^k-s)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3}(s^{1/k},\epsilon)}{n_3!} \Big) \frac{1}{(u^k-s)s} ds
\end{aligned} \tag{33}
$$

for given

$$w_j(u,\epsilon) = P_j(u,\epsilon) \ , \ 0 \leq j \leq S-1. \tag{34}$$

3. Some $q-$Difference-Convolution Recursion on a Sequence of Discs

In this section, the recursion relation (33), (34) is solved on a well selected sequence of discs D_{R_n}, whose radius R_n tends to 0 as n tends to infinity. Besides, we display sharp bounds control for the sequence of functions $(w_n)_{n\geq 0}$ solving this recursion relation.

Let ζ_j, $1\leq j\leq k\mathrm{deg}(P)$, the complex roots of the polynomial $u\mapsto P(ku^k)$, where $\mathrm{deg}(P)$ stands for the degree of the polynomial P introduced at the onset of Section 2. The second condition of (6) imposed on P grants the existence of a disc D_{R_0} for which

$$\zeta_j \notin D_{R_0} \ , \ 1\leq j\leq k\mathrm{deg}(P). \tag{35}$$

We introduce the constant

$$\Delta = M_1/k \tag{36}$$

where $M_1>0$ is the real number appearing in (14). We introduce the sequence of radius

$$R_n = \frac{R_0}{q^{\Delta n}} \ , \ n\geq 0. \tag{37}$$

Our main objective is the discussion of the next proposition.

Proposition 2. *Assuming the radius $\epsilon_0>0$ and the constants $\mathbf{d}_{h,\epsilon_0}>0$ introduced in (10) small enough, one can single out a unique sequence of functions $(w_n(u,\epsilon))_{n\geq 0}$, where each map $(u,\epsilon)\mapsto w_n(u,\epsilon)$ is bounded holomorphic w.r.t u on the disc D_{R_n} and w.r.t ϵ on D_{ϵ_0}, that fulfills the recursion (33) with S first terms (34). Furthermore, one can choose two constants $C_1,C_2>0$ such that the next bounds hold*

$$\sup_{\epsilon\in D_{\epsilon_0}} |w_n(u,\epsilon)| \leq C_1(C_2)^n n!|u| \tag{38}$$

for all $n\geq 0$, all $u\in D_{R_n}$.

Proof. We will proceed by induction. We name $\mathbb{D}_n$ the property (38) for a fixed given $n\geq 0$. We first check that the property $\mathbb{D}_n$ holds in a straight manner for $0\leq n\leq S-1$ for well chosen $C_1,C_2>0$ since in that case it is imposed that $w_n(u,\epsilon)=P_n(u,\epsilon)$ are polynomials with bounded holomorphic coefficients w.r.t ϵ on the disc D_{ϵ_0} such that $P_n(0,\epsilon)=0$.

Let $n \geq 0$, we assume that $\mathbb{D}_p$ holds for all $p < n + S$ for some given $C_1, C_2 > 0$. Our goal throughout the rest of the proof is to show that $\mathbb{D}_{n+S}$ holds. The induction principle will then imply that the property $\mathbb{D}_p$ holds for all $p \geq 0$.

In the next lemma, we provide upper bounds for some terms w_p for $p < n + S$ which are involved in the identity (33).

Lemma 3. *1) For all $\underline{l} = (0, l_1, l_2, l_3) \in \mathcal{A}$, the next bounds*

$$|w_{n_2+l_2}(q^{l_3}u, \epsilon)| \leq C_1(C_2)^{n_2+l_2}(n_2+l_2)!|q^{l_3}u| \tag{39}$$

hold provided that $u \in D_{R_{n+S}}$, $\epsilon \in D_{\epsilon_0}$, for all $n_2 \leq n$.
2) For all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$, the next inequality

$$|w_{n_2+l_2}(q^{l_3}s^{1/k}, \epsilon)| \leq C_1(C_2)^{n_2+l_2}(n_2+l_2)!|q^{l_3}s^{1/k}| \tag{40}$$

is valid whenever $s \in [0, u^k]$, for all $u \in D_{R_{n+S}}$, $\epsilon \in D_{\epsilon_0}$, for all $n_2 \leq n$.
3) For all $n_2, n_3 \leq n$, the next inequalities

$$|w_{n_2}((u^k - s)^{1/k}, \epsilon)| \leq C_1(C_2)^{n_2}n_2!|(u^k - s)^{1/k}| \tag{41}$$

and

$$|w_{n_3}(s^{1/k}, \epsilon)| \leq C_1(C_2)^{n_3}n_3!|s^{1/k}| \tag{42}$$

hold for all $u \in D_{R_{n+S}}$, $s \in [0, u^k]$, provided that $\epsilon \in D_{\epsilon_0}$.

Proof. We first treat the points 1) and 2). Provided that $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$, we observe that

- If $u \in D_{R_{n+S}}$ then $q^{l_3}u \in D_{R_{n_2+l_2}}$ for $n_2 \leq n$.
- If $s \in [0, u^k]$ for $u \in D_{R_{n+S}}$ then $q^{l_3}s^{1/k} \in D_{R_{n_2+l_2}}$ for $n_2 \leq n$.

Indeed, $u \in D_{R_{n+S}}$ means that

$$|u| \leq R_0/q^{\Delta(n+S)}. \tag{43}$$

According to the inequality

$$S\Delta \geq l_2\Delta + l_3$$

which is deduced from the condition (14) for all $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ and the definition (36), we get

$$q^{l_3}|u| \leq \frac{R_0}{q^{\Delta(n+S)-l_3}} \leq \frac{R_0}{q^{\Delta(n_2+l_2)}} \tag{44}$$

which means that $q^{l_3}u \in D_{R_{n_2+l_2}}$. Besides, if $s \in [0, u^k]$, we notice that $|s^{1/k}| \leq |u|$ and hence that $q^{l_3}|s^{1/k}| \leq q^{l_3}|u|$. From (44), we deduce that $q^{l_3}s^{1/k} \in D_{R_{n_2+l_2}}$.

As a consequence of the above first and second items, we deduce that both inequalities (39) and (40) follow directly from the recusion hypothesis $\mathbb{D}_{n_2+l_2}$ keeping in mind the assumption (13).

We focus on the third point 3). We check that

- If $u \in D_{R_{n+S}}$, then $(u^k - s)^{1/k} \in D_{R_{n_2}}$ provided that $s \in [0, u^k]$ and $n_2 \leq n$.
- If $u \in D_{R_{n+S}}$, then $s^{1/k} \in D_{R_{n_3}}$ as long as $s \in [0, u^k]$ and $n_3 \leq n$.

Indeed, we can parametrize $s \in [0, u^k]$ by $s = u^k s_1$ for $0 \leq s_1 \leq 1$ and obtain

$$|(u^k - s)^{1/k}| = |u|(1 - s_1)^{1/k} \ , \ |s^{1/k}| = |u|s_1^{1/k}.$$

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Hence, for $u \in D_{R_{n+S}}$, we get that

$$|(u^k - s)^{1/k}| \leq \frac{R_0}{q^{\Delta(n+S)}}(1-s_1)^{1/k} \leq \frac{R_0}{q^{\Delta n_2}}$$

since $(1-s_1)^{1/k} \leq 1$ and $n_2 \leq n$, which means that $(u^k - s)^{1/k} \in D_{R_{n_2}}$ and

$$|s^{1/k}| \leq \frac{R_0}{q^{\Delta(n+S)}} s_1^{1/k} \leq \frac{R_0}{q^{\Delta n_3}}$$

owing to $s_1^{1/k} \leq 1$ and $n_3 \leq n$, which says that $s^{1/k} \in D_{R_{n_3}}$.

As a direct outcome of the latter two items, we conclude that the upper bound (41) is a straight effect of the recursion hypothesis $\mathbb{D}_{n_2}$ and that (42) follows from the recursion assumption $\mathbb{D}_{n_3}$. □

According to the recursion (33) and the bounds reached in Lemma 3, we get the next estimates for the term $|w_{n+S}(u,\epsilon)|$,

$$\frac{|w_{n+S}(u,\epsilon)|}{n!} \leq \mathcal{L}_1(u,n,\epsilon) + \mathcal{L}_2(u,n,\epsilon) + \mathcal{L}_3(u,n,\epsilon) \tag{45}$$

where

$$\begin{aligned} &\mathcal{L}_1(u,n,\epsilon) \\ &\leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} |\epsilon|^{\Delta_{\underline{l}}} \frac{(k(q^{l_3}|u|)^k)^{l_1}}{|P(ku^k)|} \Big(\sum_{n_1+n_2=n} |c_{\underline{l},n_1}(\epsilon)| C_1 C_2^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3}|u| \Big) \end{aligned} \tag{46}$$

and

$$\begin{aligned} \mathcal{L}_2(u,n,\epsilon) \leq &\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} |\epsilon|^{\Delta_{\underline{l}}-l_0} \frac{|u|^k}{|P(ku^k)|\Gamma(l_0/k)} \int_0^{|u|^k} (|u|^k - s)^{\frac{l_0}{k}-1} (k(q^{l_3}s^{1/k})^k)^{l_1} \\ &\times \Big(\sum_{n_1+n_2=n} |c_{\underline{l},n_1}(\epsilon)| C_1 (C_2)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} s^{1/k} \Big) \frac{ds}{s} \end{aligned} \tag{47}$$

in a row with

$$\begin{aligned} &\mathcal{L}_3(u,n,\epsilon) \\ &\leq \frac{|u|^k}{|P(ku^k)|} \int_0^{|u|^k} \Big(\sum_{n_1+n_2+n_3=n} |d_{n_1}(\epsilon)| C_1 (C_2)^{n_2} (|u|^k - s)^{1/k} C_1 (C_2)^{n_3} s^{1/k} \frac{1}{(|u|^k - s)s} \Big) ds \end{aligned} \tag{48}$$

provided that $u \in D_{R_{n+S}}$ and $\epsilon \in D_{\epsilon_0}$.

In order to provide upper bounds for the above quantities, the next lemma is needed.

Lemma 4. *The next inequality*

$$n!\frac{(n_2+l_2)!}{n_2!} \leq (n+S)! \tag{49}$$

holds for all integers $n \geq 0$, $n_2 \leq n$ and $l_2 < S$.

Proof. The above inequality results from the next observation

$$\frac{n!}{(n+S)!}\frac{(n_2+l_2)!}{n_2!} = \frac{\Pi_{k=1}^{l_2}(n_2+k)}{\Pi_{k=1}^{S}(n+k)} \leq 1$$

provided that $n_2 \leq n, l_2 < S$. $\square$

We provide upper estimates for the first piece $\mathcal{L}_1(u,n,\epsilon)$. The use of the definition (8), the assumption (13), the requirement (35), the inclusion $D_{R_{n+S}} \subset D_{R_0}$ along with the above Lemma 4 yield the next suitable upper bounds

$$
\begin{aligned}
& n!\mathcal{L}_1(u,n,\epsilon) \\
& \quad \leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \epsilon_0^{\Delta_{\underline{l}}} \frac{(k(q^{l_3}R_0)^k)^{l_1}}{\min_{u\in D_{R_0}}|P(ku^k)|} \Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} |c_{\underline{l},n_1}(\epsilon)| C_1 C_2^{n-n_1+S}(n+S)! q^{l_3}|u| \Big) \\
& \quad \leq C_1(C_2)^{n+S}(n+S)!|u| \times \Big[\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \epsilon_0^{\Delta_{\underline{l}}} \frac{(k(q^{l_3}R_0)^k)^{l_1}q^{l_3}}{\min_{u\in D_{R_0}}|P(ku^k)|} \Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_2^{-n_1} \Big) \Big] \qquad (50)
\end{aligned}
$$

for all $u \in D_{R_{n+S}}$ and $\epsilon \in D_{\epsilon_0}$.

We aim attention to the second part $\mathcal{L}_2(u,n,\epsilon)$. The assumption (13), the requirement (35), the inclusion $D_{R_{n+S}} \subset D_{R_0}$ along with the above Lemma 4 give rise to

$$
\begin{aligned}
n!\mathcal{L}_2(u,n,\epsilon) \leq & \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon_0^{\Delta_{\underline{l}}-l_0} \frac{1}{\min_{u\in D_{R_0}}|P(ku^k)||\Gamma(l_0/k)|}|u| \\
& \times |u|^{k-1}\int_0^{|u|^k} (|u|^k - s)^{\frac{l_0}{k}-1}(k(q^{l_3}s^{1/k})^k)^{l_1} \\
& \times \Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} |c_{\underline{l},n_1}(\epsilon)| C_1(C_2)^{n-n_1+S}(n+S)! q^{l_3}s^{1/k} \Big) \frac{ds}{s} \qquad (51)
\end{aligned}
$$

for all $u \in D_{R_{n+S}}$ and $\epsilon \in D_{\epsilon_0}$. Besides, the change of variable $s = |u|^k s_1$, allows to reshape the next integral as a product

$$
|u|^{k-1}\int_0^{|u|^k} (|u|^k - s)^{\frac{l_0}{k}-1} s^{l_1+\frac{1}{k}} \frac{ds}{s} = |u|^{l_0+kl_1} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1+\frac{1}{k}} \frac{ds_1}{s_1} \qquad (52)
$$

for all $u \in \mathbb{C}$. The combination of (51) and (52) with the definition (8) and the inclusion $D_{R_{n+S}} \subset D_{R_0}$ beget the next fitting upper bounds

$$
\begin{aligned}
n!\mathcal{L}_2(u,n,\epsilon) \leq & \; C_1(C_2)^{n+S}(n+S)!|u| \\
& \times \Big[\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon_0^{\Delta_{\underline{l}}-l_0} \frac{1}{\min_{u\in D_{R_0}}|P(ku^k)||\Gamma(l_0/k)|} (k(q^{l_3})^k)^{l_1} q^{l_3} R_0^{l_0+kl_1} \\
& \times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1+\frac{1}{k}} \frac{ds_1}{s_1} \times \Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_2^{-n_1} \Big) \Big] \qquad (53)
\end{aligned}
$$

whenever $u \in D_{R_{n+S}}$ and $\epsilon \in D_{\epsilon_0}$.

At last, we address the tail piece $\mathcal{L}_3(u,n,\epsilon)$. Namely, from the inclusion $D_{R_{n+S}} \subset D_{R_0}$ and the requirement (35), we first observe that

$$
\begin{aligned}
n!\mathcal{L}_3(u,n,\epsilon) &\leq \frac{1}{\min_{u \in D_{R_0}} |P(ku^k)|}|u| \\
&\times |u|^{k-1} \int_0^{|u|^k} \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_d}} |d_{n_1}(\epsilon)|(C_1)^2 \times \big(\sum_{n_2+n_3=n-n_1} (C_2)^{n_2+n_3} n! \big) \times (|u|^k - s)^{1/k} s^{1/k} \frac{1}{(|u|^k-s)s} \Big) ds \\
&\leq \frac{1}{\min_{u \in D_{R_0}} |P(ku^k)|}|u| \\
&\times |u|^{k-1} \int_0^{|u|^k} \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_d}} |d_{n_1}(\epsilon)|(C_1)^2 n!(n-n_1+1) C_2^{n-n_1} (|u|^k - s)^{1/k} s^{1/k} \frac{1}{(|u|^k-s)s} \Big) ds
\end{aligned} \tag{54}
$$

provided that $u \in D_{R_{n+S}}$ and $\epsilon \in D_{\epsilon_0}$. In addition, the change of variable $s = |u|^k s_1$ enables to write the next integral in factorized form

$$
|u|^{k-1} \int_0^{|u|^k} (|u|^k - s)^{1/k} s^{1/k} \frac{1}{(|u|^k - s)s} ds = |u| \int_0^1 (1-s_1)^{1/k} s_1^{1/k} \frac{1}{1-s_1} \frac{1}{s_1} ds_1 \tag{55}
$$

for all $u \in \mathbb{C}$. The coupling of (54) and (55) together with the inclusion $D_{R_{n+S}} \subset D_{R_0}$ and the definition (10) triggers the following appropriate bounds

$$
\begin{aligned}
n!\mathcal{L}_3(u,n,\epsilon) &\leq C_1 (C_2)^{n+S} (n+S)! |u| \\
&\times \Big[\frac{1}{\min_{u \in D_{R_0}} |P(ku^k)|} R_0 \times \Big(\int_0^1 (1-s_1)^{1/k} s_1^{1/k} \frac{1}{1-s_1} \frac{1}{s_1} ds_1 \Big) \times C_1 \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_d}} \mathbf{d}_{n_1,\epsilon_0} C_2^{-n_1} \Big) \Big]
\end{aligned} \tag{56}
$$

for all $u \in D_{R_{n+S}}$ and $\epsilon \in D_{\epsilon_0}$.

We make the assumption that the radius $\epsilon_0 > 0$ and the constants $\mathbf{d}_{h,\epsilon_0} > 0$ introduced in (10) are selected nearby the origin in a way that the next condition

$$
\begin{aligned}
&\sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0} \epsilon_0^{\Delta_{\underline{l}}} \frac{(k(q^{l_3}R_0)^k)^{l_1} q^{l_3}}{\min_{u \in D_{R_0}} |P(ku^k)|} \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_2^{-n_1} \Big) \\
&+ \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \geq 1} \epsilon_0^{\Delta_{\underline{l}} - l_0} \frac{1}{\min_{u \in D_{R_0}} |P(ku^k)| \Gamma(l_0/k)} (k(q^{l_3})^k)^{l_1} q^{l_3} R_0^{l_0 + kl_1} \\
&\times \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1 + \frac{1}{k}} \frac{ds_1}{s_1} \times \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_2^{-n_1} \Big) \\
&+ \frac{1}{\min_{u \in D_{R_0}} |P(ku^k)|} R_0 \times \Big(\int_0^1 (1-s_1)^{1/k} s_1^{1/k} \frac{1}{1-s_1} \frac{1}{s_1} ds_1 \Big) \times C_1 \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_d}} \mathbf{d}_{n_1,\epsilon_0} C_2^{-n_1} \Big) \leq 1
\end{aligned} \tag{57}
$$

holds. Eventually, we combine the gathering of the above auxiliary bounds (50), (53) and (56) with the initial estimates (45) constrained by (57) from which follow the next bounds

$$
|w_{n+S}(u,\epsilon)| \leq C_1 (C_2)^{n+S} (n+S)! |u| \tag{58}
$$

provided that $u \in D_{R_{n+S}}$ and $\epsilon \in D_{\epsilon_0}$. Therefore, the property $\mathbb{D}_{n+S}$ is valid. Proposition 2 follows. □

4. Some $q-$Difference-Convolution Recursion on Unbounded Sectors

In this section, we solve the recursion relation (33), (34) on suitable unbounded sectors $\mathcal{U}$ together with sharp bounds control for the sequence of holomorphic maps $(w_n)_{n\geq 0}$ solving it. Indeed, we fix an unbounded sector

$$\mathcal{U} = \{u \in \mathbb{C}^*/\alpha < \arg(u) < \beta\} \tag{59}$$

for some given angles $\alpha < \beta$. We assume that

$$\zeta_j \notin \mathcal{U} \ \ , \ \ 1 \leq j \leq k\deg(P) \tag{60}$$

where ζ_j, $1 \leq j \leq k\deg(P)$ represent the roots of the polynomial $u \mapsto P(ku^k)$.

The purpose of this section is to explain the next proposition.

Proposition 3. *Provided that the radius $\epsilon_0 > 0$ and the constants $\mathbf{d}_{h,\epsilon_0} > 0$ introduced in (10) are taken close enough to 0, one can find out a unique sequence of functions $(w_n(u,\epsilon))_{n\geq 0}$, where each map $(u,\epsilon) \mapsto w_n(u,\epsilon)$ is holomorphic on the sector $\mathcal{U}$, continuous on $\mathcal{U} \cup \{0\}$ with respect to u and bounded holomorphic w.r.t ϵ on D_{ϵ_0}, that fulfills the recursion (33) with S first terms (34). In addition, one can find constants $C_3, C_4 > 0$ and $C_5 > 0$ for which the next estimates*

$$\sup_{\epsilon \in D_{\epsilon_0}} |w_n(u,\epsilon)| \leq C_3(C_4)^n n! |u| \exp\left(C_5 q^{nM_1}|u|^k\right) \tag{61}$$

hold for all $n \geq 0$, all $u \in \mathcal{U} \cup \{0\}$, where $M_1 > 0$ appears in the assumption (14).

Proof. The induction principle is applied. We denote $\mathbb{U}_n$ the property (61) for a given integer $n \geq 0$. We observe that the property $\mathbb{U}_n$ is valid whenever $0 \leq n \leq S-1$ for well chosen constants $C_3, C_4, C_5 > 0$ owing to the fact that $w_n(u,\epsilon) = P_n(u,\epsilon)$ represent mere polynomials that vanish at $u = 0$ and possess bounded holomorphic coefficients w.r.t ϵ on the disc D_{ϵ_0}.

Let $n \geq 0$, we take for granted that $\mathbb{U}_p$ is true for all $p < n+S$ for some given $C_3, C_4, C_5 > 0$. Our aim is to prove that $\mathbb{U}_{n+S}$ holds. The induction principle then implies that the bounds $\mathbb{U}_p$ are valid for any integer $p \geq 0$.

According to the recursion (33) and that $\mathbb{U}_p$ is taken for granted for all $p < n+S$, we get the next bounds for the term $|w_{n+S}(u,\epsilon)|$,

$$\frac{|w_{n+S}(u,\epsilon)|}{n!} \leq \mathcal{P}_1(u,n,\epsilon) + \mathcal{P}_2(u,n,\epsilon) + \mathcal{P}_3(u,n,\epsilon) \tag{62}$$

where

$$\mathcal{P}_1(u,n,\epsilon) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} |\epsilon|^{\Delta_{\underline{l}}} \frac{(k(q^{l_3}|u|)^k)^{l_1}}{|P(ku^k)|} \Big(\sum_{n_1+n_2=n} |c_{\underline{l},n_1}(\epsilon)| C_3 (C_4)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} \\ \times |q^{l_3}u| \exp\left(C_5 q^{(n_2+l_2)M_1} |q^{l_3}u|^k\right)\Big) \tag{63}$$

and

$$\mathcal{P}_2(u,n,\epsilon) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} |\epsilon|^{\Delta_{\underline{l}}-l_0} \frac{|u|^k}{|P(ku^k)|\Gamma(l_0/k)} \int_0^{|u|^k} (|u|^k - s)^{\frac{l_0}{k}-1} (k(q^{l_3}s^{1/k})^k)^{l_1} \\ \times \Big(\sum_{n_1+n_2=n} |c_{\underline{l},n_1}(\epsilon)| C_3 (C_4)^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} q^{l_3} s^{1/k} \exp\left(C_5 q^{(n_2+l_2)M_1} (q^{l_3}s^{1/k})^k\right)\Big) \frac{ds}{s} \tag{64}$$

along with

$$\begin{aligned}\mathcal{P}_3(u,n,\epsilon)&\\ =\frac{|u|^k}{|P(ku^k)|}\int_0^{|u|^k}&\Big(\sum_{n_1+n_2+n_3=n}|d_{n_1}(\epsilon)|C_3(C_4)^{n_2}(|u|^k-s)^{1/k}\exp\left(C_5q^{n_2M_1}((|u|^k-s)^{1/k})^k\right)\\ &\times C_3(C_4)^{n_3}s^{1/k}\exp\left(C_5q^{n_3M_1}(s^{1/k})^k\right)\Big)\frac{1}{(|u|^k-s)s}ds \quad (65)\end{aligned}$$

provided that $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

Owing to the condition (14), since $q > 1$, we get that

$$q^{(n_2+l_2)M_1}q^{l_3k} \leq q^{(n+S)M_1} \quad (66)$$

provided that $n_2 \leq n$ and $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$. Besides, taking heed of the conditions (6), (15) and the requirement (60), one can find a constant $\mathbb{M}_{P,k,l_1} > 0$ with

$$\frac{|u|^{kl_1}}{|P(ku^k)|} \leq \mathbb{M}_{P,k,l_1} \quad (67)$$

for all $u \in \mathcal{U}$, provided that $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ with $l_0 = 0$.

With the use of (8) and the hypothesis (13) together with the help of (49), (66) and (67), we get appropriate upper bounds for the first piece $\mathcal{P}_1(u, n, \epsilon)$. Namely,

$$\begin{aligned}n!\mathcal{P}_1(u,n,\epsilon)\leq&\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0}\epsilon_0^{\Delta_{\underline{l}}}(kq^{l_3k})^{l_1}\mathbb{M}_{P,k,l_1}\\ &\times\Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}}|c_{\underline{l},n_1}(\epsilon)|C_3(C_4)^{n-n_1+S}(n+S)!q^{l_3}|u|\exp\left(C_5q^{(n+S)M_1}|u|^k\right)\Big)\\ &\leq C_3(C_4)^{n+S}(n+S)!|u|\exp\left(C_5q^{(n+S)M_1}|u|^k\right)\\ &\times\Big[\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0}\epsilon_0^{\Delta_{\underline{l}}}(kq^{l_3k})^{l_1}\mathbb{M}_{P,k,l_1}q^{l_3}\times\Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}}\mathbf{c}_{\underline{l},n_1,\epsilon_0}C_4^{-n_1}\Big)\Big] \quad (68)\end{aligned}$$

provided that $u \in \mathcal{U}$ and $\epsilon \in D_{\epsilon_0}$.

We now focus on the second piece $\mathcal{P}_2(u, n, \epsilon)$. Bearing in mind (13), (49) and (66), we first observe that

$$\begin{aligned}n!\mathcal{P}_2(u,n,\epsilon)\leq&\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq1}\epsilon_0^{\Delta_{\underline{l}}-l_0}\frac{|u|^k}{|P(ku^k)|\Gamma(l_0/k)}\int_0^{|u|^k}(|u|^k-s)^{\frac{l_0}{k}-1}(kq^{l_3k})^{l_1}s^{l_1}\\ &\times\Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}}|c_{\underline{l},n_1}(\epsilon)|C_3(C_4)^{n-n_1+S}(n+S)!q^{l_3}s^{1/k}\Big)\frac{ds}{s}\times\exp\left(C_5q^{(n+S)M_1}|u|^k\right)\\ &\leq C_3(C_4)^{n+S}(n+S)!|u|\exp\left(C_5q^{(n+S)M_1}|u|^k\right)\\ &\times\Big[\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq1}\epsilon_0^{\Delta_{\underline{l}}-l_0}\frac{|u|^{k-1}}{|P(ku^k)|\Gamma(l_0/k)}\int_0^{|u|^k}(|u|^k-s)^{\frac{l_0}{k}-1}(kq^{l_3k})^{l_1}s^{l_1}\\ &\times\Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}}|c_{\underline{l},n_1}(\epsilon)|C_4^{-n_1}q^{l_3}s^{1/k}\Big)\frac{ds}{s}\Big] \quad (69)\end{aligned}$$

for all $u \in \mathcal{U}$ and $\epsilon \in D_{\epsilon_0}$. In the next step, we need to upper bound the integral expression

$$A(u) = \frac{|u|^{k-1}}{|P(ku^k)|} \int_0^{|u|^k} (|u|^k - s)^{\frac{l_0}{k}-1} s^{l_1} s^{1/k} \frac{ds}{s} \tag{70}$$

for $u \in \mathcal{U}$. Indeed, using the change of variable $s = |u|^k s_1$ for $0 \le s_1 \le 1$ in $A(u)$, we get from (6), (15) and the constraint (60), a constant $\mathbb{M}_{P,k,l_0,l_1} > 0$ with

$$A(u) = \frac{|u|^{l_0+kl_1}}{|P(ku^k)|} \times \Big(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1+\frac{1}{k}} \frac{ds_1}{s_1} \Big) \le \mathbb{M}_{P,k,l_0,l_1} \tag{71}$$

for all $u \in \mathcal{U}$. At last, paying regard to the definition (8), the combination of the above bounds (69) and (71), yields the next suitable estimates for the quantity $\mathcal{P}_2(u,n,\epsilon)$. Namely,

$$\begin{aligned} n!\mathcal{P}_2(u,n,\epsilon) \le{}& C_3 (C_4)^{n+S} (n+S)! |u| \exp\big(C_5 q^{(n+S)M_1} |u|^k\big) \\ &\times \Big[\sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \ge 1} \epsilon_0^{\Delta_{\underline{l}} - l_0} \frac{1}{\Gamma(l_0/k)} (kq^{l_3 k})^{l_1} q^{l_3} \mathbb{M}_{P,k,l_0,l_1} \times \Big(\sum_{\substack{0 \le n_1 \le n \\ n_1 \in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_4^{-n_1} \Big) \Big] \end{aligned} \tag{72}$$

as long as $u \in \mathcal{U}$ and $\epsilon \in D_{\epsilon_0}$.

We turn our attention to the third piece $\mathcal{P}_3(u,n,\epsilon)$. From the straight bounds

$$q^{n_2 M_1} (|u|^k - s) + q^{n_3 M_1} s \le q^{(n+S)M_1} |u|^k$$

for all $u \in \mathbb{C}$, $0 \le s \le |u|^k$ and $n_2, n_3 \le n$, we observe that

$$\begin{aligned} n!\mathcal{P}_3(u,n,\epsilon) \le{}& \frac{|u|^k}{|P(ku^k)|} \int_0^{|u|^k} \Big(\sum_{\substack{0 \le n_1 \le n \\ n_1 \in I_d}} |d_{n_1}(\epsilon)| (C_3)^2 \times \Big(\sum_{n_2+n_3=n-n_1} (C_4)^{n_2+n_3} n! \Big) \Big) \\ &\times \exp\big(C_5 q^{(n+S)M_1} |u|^k\big) (|u|^k - s)^{1/k} s^{1/k} \frac{1}{(|u|^k - s)s} ds \\ \le{}& \frac{|u|^k}{|P(ku^k)|} \int_0^{|u|^k} \Big(\sum_{\substack{0 \le n_1 \le n \\ n_1 \in I_d}} |d_{n_1}(\epsilon)| (C_3)^2 C_4^{n-n_1} (n-n_1+1) n! \\ &\times \exp\big(C_5 q^{(n+S)M_1} |u|^k\big) (|u|^k - s)^{1/k} s^{1/k} \frac{1}{(|u|^k - s)s} ds \end{aligned} \tag{73}$$

Again, we are reduced to upper bound an integral term

$$B(u) = \frac{|u|^{k-1}}{|P(ku^k)|} \times \Big(\int_0^{|u|^k} (|u|^k - s)^{1/k} s^{1/k} \frac{1}{(|u|^k - s)s} ds \Big) \tag{74}$$

for $u \in \mathcal{U}$. Indeed, owing to a change of variable $s = |u|^k s_1$ for $0 \le s_1 \le 1$ in $B(u)$, we get from (6) and the constraint (60), a constant $\mathbb{M}_{P,k} > 0$ can be singled out with

$$B(u) = \frac{|u|}{|P(ku^k)|} \times \Big(\int_0^1 (1-s_1)^{1/k} s_1^{1/k} \frac{1}{(1-s_1)s_1} ds_1 \Big) \le \mathbb{M}_{P,k} \tag{75}$$

for all $u \in \mathcal{U}$. Eventually, paying heed to (10) and teaming up (73) with (75), we reach the next fitting estimates for the term $\mathcal{P}_3(u,n,\epsilon)$. Namely,

$$n!\mathcal{P}_3(u,n,\epsilon) \le C_3 (C_4)^{n+S} (n+S)! |u| \exp\big(C_5 q^{(n+S)M_1} |u|^k\big) \times \Big[\mathbb{M}_{P,k} C_3 \Big(\sum_{\substack{0 \le n_1 \le n \\ n_1 \in I_d}} \mathbf{d}_{n_1,\epsilon_0} C_4^{-n_1} \Big) \Big] \tag{76}$$

whenever $u \in \mathcal{U}$ and $\epsilon \in D_{\epsilon_0}$.

Now, we make the assumption that the radius $\epsilon_0 > 0$ and the constants $\mathbf{d}_{h,\epsilon_0} > 0$ introduced in (10) are chosen in the vicinity of the origin in a way that the next constraint

$$\begin{aligned}\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \epsilon_0^{\Delta_{\underline{l}}}(kq^{l_3k})^{l_1}\mathbb{M}_{P,k,l_1}q^{l_3} \times \Big(\sum_{\substack{0\le n_1\le n\\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0}C_4^{-n_1}\Big) \\ + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\ge 1} \epsilon_0^{\Delta_{\underline{l}}-l_0}\frac{1}{\Gamma(l_0/k)}(kq^{l_3k})^{l_1}q^{l_3}\mathbb{M}_{P,k,l_0,l_1} \times \Big(\sum_{\substack{0\le n_1\le n\\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0}C_4^{-n_1}\Big) \\ + \mathbb{M}_{P,k}C_3\Big(\sum_{\substack{0\le n_1\le n\\ n_1\in I_d}} \mathbf{d}_{n_1,\epsilon_0}C_4^{-n_1}\Big) \le 1 \quad (77)\end{aligned}$$

holds. At last, the collection of all the technical bounds (68), (72) and (76) when applied to the inequality (62) under the constraint (77) gives rise to the estimates

$$|w_{n+S}(u,\epsilon)| \le C_3(C_4)^{n+S}(n+S)!|u|\exp\left(C_5q^{(n+S)M_1}|u|^k\right) \quad (78)$$

for all $u \in \mathcal{U}$, all $\epsilon \in D_{\epsilon_0}$. This means that the property $\mathbb{U}_{n+S}$ holds. This yields Proposition 2. □

5. Layout of Analytic Banach Valued Solutions to the Initial Cauchy Problem (16), (17).

We recall the definition of a good covering in $\mathbb{C}^*$ as given in [7], Chapter XI.

Definition 1. *Let $\varsigma \ge 2$ be an integer. We consider a family $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0\le p\le\varsigma-1}$ of open sectors $\mathcal{E}_p$ centered at 0 (but not containing the origin) with given radius ϵ_0 that conforms to the next three features:*
i) *The intersection of any two consecutive sectors $\mathcal{E}_p \cap \mathcal{E}_{p+1}$ of the family $\underline{\mathcal{E}}$ is not empty for all $0 \le p \le \varsigma - 1$, with the convention that $\mathcal{E}_\varsigma = \mathcal{E}_0$.*
ii) *The intersection of any three elements $\mathcal{E}_{p_1} \cap \mathcal{E}_{p_2} \cap \mathcal{E}_{p_3}$ for $p_1 \ne p_2 \ne p_3$ in $\underline{\mathcal{E}}$ is empty.*
iii) *The union of the sectors $\mathcal{E}_p$ covers some punctured neighborhood $\dot{\mathcal{U}}$ of the origin in $\mathbb{C}^*$, namely*

$$\cup_{p=0}^{\varsigma-1}\mathcal{E}_p = \dot{\mathcal{U}} = \mathcal{U} \setminus \{0\}.$$

A family $\underline{\mathcal{E}}$ endowed with the above three properties is called a good covering in $\mathbb{C}^*$.

A notion of admissible set of sectors is put forward in the next

Definition 2. *Let $\varsigma \ge 2$ be an integer and let $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0\le p\le\varsigma-1}$ be a good covering in $\mathbb{C}^*$ and let $\mathcal{T}$ be a bounded sector edged at 0. We consider a set $\underline{\mathcal{U}} = \{\mathcal{U}_p\}_{0\le p\le\varsigma-1}$ of unbounded sectors $\mathcal{U}_p$ edged at 0, that are subjected to the next properties:*
1) *Each sector $\mathcal{U}_p$ does not contain any of the roots of the polynomial $u \mapsto P(ku^k)$, for $0 \le p \le \varsigma - 1$.*
2) *For all $0 \le p \le \varsigma - 1$, there exists a constant $\Delta_p > 0$ such that for all $\epsilon \in \mathcal{E}_p$ and all $t \in \mathcal{T}$, one can find out a direction $\gamma_p \in \mathbb{R}$ (that may depend on ϵ and t) such that both conditions*

$$L_{\gamma_p} = [0,+\infty)\exp(\sqrt{-1}\gamma_p) \subset \mathcal{U}_p \cup \{0\} \quad (79)$$

and

$$\cos(k(\gamma_p - \arg(\epsilon t))) > \Delta_p \quad (80)$$

hold.

The set of sectors $\underline{\mathcal{D}} = \{\underline{\mathcal{E}}, \mathcal{T}, \underline{\mathcal{U}}\}$ favored with the above two features is called an admissible set of sectors.

In the next definition, we remind the reader the construction of a Banach space of formal power series introduced in the former work [8] by A. Lastra and the author.

Definition 3. *Let $\mathcal{T}$ be a fixed bounded sector edged at 0. Let $\tilde{R}_0, \tilde{\Delta} > 0$ be constants. We mind the sequence $(\tilde{R}_n)_{n \geq 0}$ where*

$$\tilde{R}_n = \tilde{R}_0/q^{n\tilde{\Delta}} \ , \ n \geq 0. \tag{81}$$

For each $n \geq 0$, we consider the domain

$$\mathcal{D}_n = \mathcal{T} \cap D_{\tilde{R}_n}, \tag{82}$$

and we fix some positive radius $R_1 > 0$. We denote $\mathcal{O}_{(\mathcal{D}_n)_{n \geq 0};R_1}$ the vector space of formal power series

$$h(t,z) = \sum_{n \geq 0} h_n(t) \frac{z^n}{n!} \tag{83}$$

where each coefficient h_n belongs to $\mathcal{O}_b(\mathcal{D}_n)$, the vector space of bounded $\mathbb{C}$-valued holomorphic functions on $\mathcal{D}_n$, such that the norm

$$||h(t,z)||_{(\mathcal{D}_n)_{n \geq 0};R_1} := \sum_{n \geq 0} \sup_{t \in \mathcal{D}_n} |h_n(t)| \frac{R_1^n}{n!} \tag{84}$$

is finite.

The next proposition has been discussed in [8].

Proposition 4. *The vector space $\mathcal{O}_{(\mathcal{D}_n)_{n \geq 0};R_1}$ of formal series in the variable z equipped with the mixed type sup-norm and L_1-norm $||.||_{(\mathcal{D}_n)_{n \geq 0};R_1}$ turns out to be a complex Banach space.*

At this stage of the paper, the necessary prefatory material has been prepared in order to show the first main result of our work.

Theorem 1. *We take for granted that the conditions described in Section 2.1 are imposed on the main Cauchy problem (16), (17). Consider a good covering $\underline{\mathcal{E}} = \{\mathcal{E}_p\}_{0 \leq p \leq \varsigma - 1}$ in $\mathbb{C}^*$, a set $\underline{\mathcal{U}} = \{\mathcal{U}_p\}_{0 \leq p \leq \varsigma - 1}$ of unbounded sectors edged at 0 and a bounded sector $\mathcal{T}$ edged at 0 chosen in a way that the data $\underline{\mathcal{D}} = \{\underline{\mathcal{E}}, \mathcal{T}, \underline{\mathcal{U}}\}$ forms an admissible set of sectors.*

Provided that the radius $\epsilon_0 > 0$ and the constants $\mathbf{d}_{h,\epsilon_0} > 0$ introduced in (10) are taken close enough to the origin, for each $0 \leq p \leq \varsigma - 1$, one can construct a solution $u_p(t,z,\epsilon)$ to our main Cauchy problem (16), (17) endowed with the next hallmarks

1. *The partial map $\epsilon \mapsto u_p(t,z,\epsilon)$ represents a bounded holomorphic map from the sector $\mathcal{E}_p$ into the Banach space $(\mathcal{O}_{(\mathcal{D}_n)_{n \geq 0};R_1}, ||.||_{(\mathcal{D}_n)_{n \geq 0};R_1})$, for well chosen constants $\tilde{R}_0 > 0$, $R_1 > 0$ with $\tilde{\Delta} = \Delta$ introduced in (36).*
2. *The map $u_p(t,z,\epsilon)$ is expressed as a formal power in z*

$$u_p(t,z,\epsilon) = \sum_{n \geq 0} u_{p,n}(t,\epsilon) \frac{z^n}{n!} \tag{85}$$

where each coefficient $u_{p,n}(t,\epsilon)$, $n \geq 0$, stands for a Laplace transform of order k,

$$u_{p,n}(t,\epsilon) = k \int_{L_{\gamma_p}} w_{p,n}(u,\epsilon) \exp\left(-\left(\frac{u}{\epsilon t}\right)^k\right) \frac{du}{u} \tag{86}$$

along a halfline $L_{\gamma_p} = [0,+\infty)e^{\sqrt{-1}\gamma_p}$ for the direction γ_p given in Definition 2 2), where the so-called Borel map $w_{p,n}(u,\epsilon)$ fulfills the next two features

   - *the map* $(u,\epsilon) \mapsto w_{p,n}(u,\epsilon)$ *is bounded holomorphic on the product* $D_{R_n} \times D_{\epsilon_0}$ *for* R_n *given by* (37). *Furthermore, two constants* $C_1, C_2 > 0$ *can be pinpointed with the bounds*

     $$|w_{p,n}(u,\epsilon)| \leq C_1 (C_2)^n n! |u| \tag{87}$$

     for all $n \geq 0$*, all* $u \in D_{R_n}$*, all* $\epsilon \in D_{\epsilon_0}$*.*
   - *the map* $(u,\epsilon) \mapsto w_{p,n}(u,\epsilon)$ *is holomorphic on the product* $\mathcal{U}_p \times D_{\epsilon_0}$ *where* $\mathcal{U}_p$ *is the unbounded sector of the set* $\underline{\mathcal{U}}$ *subjected to the inclusion* (79) *from Definition 2 2). Moreover, three constants* $C_3, C_4 > 0$ *and* $C_5 > 0$ *can be chosen with the bounds*

     $$|w_{p,n}(u,\epsilon)| \leq C_3 (C_4)^n n! |u| \exp\big(C_5 q^{nM_1} |u|^k \big) \tag{88}$$

     for all $n \geq 0$*, all* $u \in \mathcal{U}_p \cup \{0\}$*, where* $M_1 > 0$ *stems from the assumption* (14).

3. *The difference of consecutive solutions* $u_{p+1} - u_p$ *can be estimated as follows. For each* $0 \leq p \leq \varsigma - 1$*, one can find two constants* $K_{p,1}, K_{p,2} > 0$ *such that*

   $$||u_{p+1}(t,z,\epsilon) - u_p(t,z,\epsilon)||_{(\mathcal{D}_n)_{n\geq 0};R_1} \leq K_{p,1} \exp\big(-\frac{K_{p,2}}{|\epsilon|^k} \big) \tag{89}$$

   for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$*, with the convention that* $u_\varsigma = u_0$ *and* $\mathcal{E}_\varsigma = \mathcal{E}_0$*.*

Proof. Let $\underline{\mathcal{D}} = \{\underline{\mathcal{E}}, \mathcal{T}, \underline{\mathcal{U}}\}$ be an admissible set of sectors. Provided that $\epsilon_0 > 0$ and the constants $\mathbf{d}_{h,\epsilon_0} > 0$ introduced in (10) are taken close enough to the origin, for a given unbounded sector $\mathcal{U}_p$ from $\underline{\mathcal{U}}$, we consider the unique sequence of functions $(w_{n,p}(u,\epsilon))_{n\geq 0}$ built up in Proposition 3 for the sector $\mathcal{U} = \mathcal{U}_p$ that fulfills the recursion (33) with S first terms (34). By construction, these maps are submitted to the bounds (88). According to Proposition 2 and by a unicity argument, it follows that each map $w_{n,p}(u,\epsilon)$ is also bounded holomorphic on a product $D_{R_n} \times D_{\epsilon_0}$ for R_n given by (37) and obeys the bounds (87).

In the next step, we provide bounds estimates for the Laplace transform $u_{p,n}(t,\epsilon)$ displayed in (86). Namely, according to the bounds (88) and the lower estimates (80), for $u = re^{\sqrt{-1}\gamma_p} \in L_{\gamma_p}$ with $r \geq 0$, we get the next upper estimates

$$\begin{aligned}
&|w_{p,n}(u,\epsilon) \exp\big(-(\frac{u}{\epsilon t})^k \big) \frac{1}{u}| \\
&\leq C_3 (C_4)^n n! \exp\big(C_5 q^{nM_1} r^k \big) \exp\Big(-(\frac{r}{|\epsilon t|})^k \cos(k(\gamma_p - \arg(\epsilon t))) \Big) \\
&\leq C_3 (C_4)^n n! \exp\Big(r^k \big[C_5 q^{nM_1} - \frac{\Delta_p}{|\epsilon t|^k} \big] \Big) \\
&\leq C_3 (C_4)^n n! \exp\Big(r^k \big[C_5 q^{nM_1} - \frac{\Delta_p}{(\epsilon_0 |t|)^k} \big] \Big)
\end{aligned} \tag{90}$$

for all $\epsilon \in \mathcal{E}_p$ and $t \in \mathcal{T}$. In the next lemma, we provide some technical bounds

Lemma 5. *There exists* $\tilde{R}_0 > 0$ *and a constant* $B_6 > 0$ *such that*

$$C_5 q^{nM_1} - \frac{\Delta_p}{(\epsilon_0 |t|)^k} \leq -B_6 \tag{91}$$

provided that $t \in \mathbb{C}$ *with* $|t| \leq \tilde{R}_0 / q^{n\Delta}$*.*

Proof. I straight computation shows that the inequality (91) is equivalent to

$$|t| \leq \frac{\Delta_p^{1/k}/\epsilon_0}{(C_5 q^{nM_1} + B_6)^{1/k}}. \tag{92}$$

For some given $B_6 > 0$, we introduce the constant $\tilde{R}_0 > 0$ defined by

$$\tilde{R}_0 = \min_{n \geq 0} \frac{(\Delta_p^{1/k}/\epsilon_0) q^{n\Delta}}{(C_5 q^{nM_1} + B_6)^{1/k}} = \min_{n \geq 0} \frac{\Delta_p^{1/k}/\epsilon_0}{(C_5 + B_6 q^{-nM_1})^{1/k}} = \frac{\Delta_p^{1/k}/\epsilon_0}{(C_5 + B_6)^{1/k}} \tag{93}$$

by keeping in mind the definition (36). By construction, if $|t| \leq \tilde{R}_0/q^{n\Delta}$, then the inequality (92) holds. The Lemma follows. □

The combination of the bounds (90) and (91) yields the estimates

$$|w_{p,n}(u,\epsilon) \exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{1}{u}| \leq C_3 (C_4)^n n! \exp(-B_6 r^k) \tag{94}$$

provided that $u = re^{\sqrt{-1}\gamma_p} \in L_{\gamma_p}$ with $r \geq 0$, for all $\epsilon \in \mathcal{E}_p$ and $t \in \mathcal{T} \cap D_{\tilde{R}_0/q^{n\Delta}} = \mathcal{D}_n$, for all $n \geq 0$, all $0 \leq p \leq \varsigma - 1$. It follows that each Laplace transform $u_{p,n}(t,\epsilon)$ given by (86) defines a bounded holomorphic function on the product $\mathcal{D}_n \times \mathcal{E}_p$ and is subjected to the bounds

$$|u_{p,n}(t,\epsilon)| \leq k C_3 (C_4)^n n! \int_0^{+\infty} \exp(-B_6 r^k) dr \tag{95}$$

for all $\epsilon \in \mathcal{E}_p$ and $t \in \mathcal{D}_n$. If one sets the formal power series $u_p(t,z,\epsilon)$ by the formal expansion in z given by (85), one checks that for all $\epsilon \in \mathcal{E}_p$, $u_p(t,z,\epsilon)$ belongs to the Banach space $(\mathcal{O}_{(\mathcal{D}_n)_{n\geq 0};R_1}, ||.||_{(\mathcal{D}_n)_{n \geq 0};R_1})$, provided that $0 < R_1 < 1/C_4$. Indeed, from (95)

$$||u_p(t,z,\epsilon)||_{(\mathcal{D}_n)_{n\geq 0};R_1} \leq k C_3 \int_0^{+\infty} \exp(-B_6 r^k) dr \times \sum_{n \geq 0} (C_4 R_1)^n = \frac{k C_3 \int_0^{+\infty} \exp(-B_6 r^k) dr}{1 - C_4 R_1}. \tag{96}$$

Besides, according to Lemma 2 and Lemma 1 from Subsection 2.2 and bearing in mind that the sequence $(w_{n,p}(u,\epsilon))_{n \geq 0}$ obeys the recursion (33) with S first terms (34), it follows that the formal series $u_p(t,z,\epsilon)$ solves the problem (16), (17). As a result, the first 1. and second 2. point of Theorem 1 hold true.

In the last part of the proof, we discuss the third point 3. Let $0 \leq p \leq \varsigma - 1$ be a given integer. Our main objective is to provide sharp bounds for the difference of Laplace transforms $u_{p+1,n}(t,\epsilon) - u_{p,n}(t,\epsilon)$ for each given $n \geq 0$. To that end, we employ a path deformation argument. Namely, according to the construction of the sequence $(w_{n,p}(u,\epsilon))_{n\geq 0}$ at the beginning of the proof, we observe that for each $n \geq 0$, for all given $\epsilon \in D_{\epsilon_0}$, the partial maps $u \mapsto w_{p+1,n}(u,\epsilon)$ and $u \mapsto w_{p,n}(u,\epsilon)$ share a common analytic continuation, that we name $u \mapsto w_n(u,\epsilon)$, on the disc D_{R_n} for $R_n = R_0/q^{n\Delta}$ where $R_0 > 0$ is some radius submitted to (35) and $\Delta > 0$ is given by (36). The classical Cauchy's formula enables the deformation of the oriented path $L_{\gamma_{p+1}} - L_{\gamma_p}$ into the union of

- the two halflines

$$L_{\gamma_{p+1},R_{n+1}} = [\frac{R_0}{q^{(n+1)\Delta}}, +\infty)\exp(\sqrt{-1}\gamma_{p+1}),$$
$$-L_{\gamma_p,R_{n+1}} = -[\frac{R_0}{q^{(n+1)\Delta}}, +\infty)\exp(\sqrt{-1}\gamma_p), \tag{97}$$

- the circle
$$C_{\gamma_p,\gamma_{p+1},R_{n+1}} = \{ \frac{R_0}{q^{(n+1)\Delta}} \exp(\sqrt{-1}\theta)/\theta \in (\gamma_p,\gamma_{p+1})\} \tag{98}$$
with radius R_{n+1} joining the above two halflines,

keeping the value of the difference $u_{p+1,n}(t,\epsilon) - u_{p,n}(t,\epsilon)$ unchanged. As a result, we can decompose the difference $u_{p+1,n} - u_{p,n}$ as a sum of three contributions,

$$\begin{aligned} u_{p+1,n}(t,\epsilon) - u_{p,n}(t,\epsilon) &= k\int_{L_{\gamma_{p+1},R_{n+1}}} w_{p+1,n}(u,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{du}{u} \\ &- k\int_{L_{\gamma_p,R_{n+1}}} w_{p,n}(u,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{du}{u} + k\int_{C_{\gamma_p,\gamma_{p+1},R_{n+1}}} w_n(u,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{du}{u}. \end{aligned} \tag{99}$$

provided that $t \in \mathcal{D}_n$, $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$. In the next step, we find upper estimates for each piece of the above splitting. Let

$$I_1(t,\epsilon) = \Big| k\int_{L_{\gamma_{p+1},R_{n+1}}} w_{p+1,n}(u,\epsilon)\exp\big(-(\frac{u}{\epsilon t})^k\big)\frac{du}{u}\Big|.$$

Based on the estimates (90), we obtain

$$I_1(t,\epsilon) \leq k\int_{R_{n+1}}^{+\infty} n!C_3(C_4)^n \exp\big(r^k[C_5 q^{nM_1} - \frac{\Delta_{p+1}}{|\epsilon t|^k}]\big)dr \tag{100}$$

for all $t \in \mathcal{D}_n$, $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$ and therefore

$$\sup_{t\in\mathcal{D}_n} I_1(t,\epsilon) \leq k\int_{R_{n+1}}^{+\infty} n!C_3(C_4)^n \exp\big(r^k[C_5 q^{nM_1} - \frac{\Delta_{p+1}}{|\epsilon|^k}\frac{q^{nM_1}}{\tilde{R}_0^k}]\big)dr \tag{101}$$

whenever $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$. Now, we choose $\check{\Delta}_{p+1} > 0$, close enough to 0, in a way that

$$C_5 - \frac{\Delta_{p+1}}{|\epsilon|^k}\frac{1}{\tilde{R}_0^k} \leq -\frac{\check{\Delta}_{p+1}}{|\epsilon|^k} \tag{102}$$

for all $\epsilon \in D_{\epsilon_0}$. A direct computation shows that (102) holds provided that $\epsilon_0 > 0$ is constrained to the inequality

$$\epsilon_0 \leq \frac{\big(\frac{\Delta_{p+1}}{\tilde{R}_0^k} - \check{\Delta}_{p+1}\big)^{1/k}}{C_5^{1/k}}$$

which is warranted if we take ϵ_0 small enough. As a result of (101) and (102) we reach the next bounds

$$\begin{aligned} \sup_{t\in\mathcal{D}_n} I_1(t,\epsilon) &\leq k\int_{R_{n+1}}^{+\infty} n!C_3(C_4)^n \exp\big(-r^k q^{nM_1}\frac{\check{\Delta}_{p+1}}{|\epsilon|^k}\big)dr \\ &\leq k\int_{R_{n+1}}^{+\infty} n!C_3(C_4)^n \frac{1}{kr^{k-1}} \times kr^{k-1}\exp\big(-r^k q^{nM_1}\frac{\check{\Delta}_{p+1}}{|\epsilon|^k}\big)dr \\ &\leq kC_3 n!(C_4)^n \frac{1}{k(R_{n+1})^{k-1}}\Big[-\frac{1}{q^{nM_1}\check{\Delta}_{p+1}/|\epsilon|^k}\exp\big(-r^k q^{nM_1}\frac{\check{\Delta}_{p+1}}{|\epsilon|^k}\big)\Big]_{R_{n+1}}^{+\infty} \\ &\leq kC_3(C_4)^n n!\frac{1}{kR_0^{k-1}} q^{-\Delta n+\Delta(k-1)}\frac{\epsilon_0^k}{\check{\Delta}_{p+1}}\exp\big(-\frac{R_0^k}{q^{\Delta k}}\frac{\check{\Delta}_{p+1}}{|\epsilon|^k}\big) \end{aligned} \tag{103}$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$.

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In a similar manner, we get bounds for the second piece of the decomposition (99). Let

$$I_2(t,\epsilon) = \Big| k \int_{L_{\gamma_p,R_{n+1}}} w_{p,n}(u,\epsilon) \exp\big(-(\frac{u}{\epsilon t})^k\big) \frac{du}{u} \Big|.$$

Then, one can single out a tiny constant $\check{\Delta}_p > 0$, in a way that

$$\sup_{t \in \mathcal{D}_n} I_2(t,\epsilon) \leq kC_3(C_4)^n n! \frac{1}{kR_0^{k-1}} q^{-\Delta n + \Delta(k-1)} \frac{\epsilon_0^k}{\check{\Delta}_p} \exp\big(-\frac{R_0^k}{q^{\Delta k}} \frac{\check{\Delta}_p}{|\epsilon|^k}\big) \tag{104}$$

provided that $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$.

Eventually, we control the tail piece of (99). We set

$$I_3(t,\epsilon) = \Big| k \int_{C_{\gamma_p,\gamma_{p+1},R_{n+1}}} w_n(u,\epsilon) \exp\big(-(\frac{u}{\epsilon t})^k\big) \frac{du}{u} \Big|.$$

Owing to the fact that the arc of circle $C_{\gamma_p,\gamma_{p+1},R_{n+1}}$ is contained in the disc D_{R_n}, we deduce from the bounds (87) that

$$I_3(t,\epsilon) \leq \Big| k \int_{\gamma_p}^{\gamma_{p+1}} C_1(C_2)^n n! R_{n+1} \exp\big(-(\frac{R_{n+1}}{|\epsilon t|})^k \cos(k(\theta - \arg(\epsilon t)))\big) d\theta \Big| \tag{105}$$

for all $t \in \mathcal{D}_n$, $\epsilon \in \mathcal{E}_p \cap \mathcal{E}_{p+1}$. By construction of the directions γ_p and γ_{p+1} in Definition 2 2), one can find a constant $\Delta_{p+1,p} > 0$ with

$$\cos(k(\theta - \arg(\epsilon t))) > \Delta_{p+1,p} \tag{106}$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, $t \in \mathcal{T}$ and $\theta \in (\gamma_p, \gamma_{p+1})$. As a consequence of (105) and (106), we reach the upper estimates

$$\sup_{t \in \mathcal{D}_n} I_3(t,\epsilon) \leq k|\gamma_{p+1} - \gamma_p| C_1(C_2)^n n! \frac{R_0}{q^{\Delta(n+1)}} \exp\big(-(\frac{R_0}{\tilde{R}_0 q^{\Delta}})^k \Delta_{p+1,p} \frac{1}{|\epsilon|^k}\big) \tag{107}$$

whenever $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$.

At last, we apply the above bounds (103), (104) and (107) to the decomposition (99) and summing up with respect to n yields the next estimates

$$\begin{aligned} &||u_{p+1}(t,z,\epsilon) - u_p(t,z,\epsilon)||_{(\mathcal{D}_n)_{n \geq 0};R_1} \\ &\leq \exp\big(-\frac{R_0^k}{q^{\Delta k}} \frac{\check{\Delta}_{p+1}}{|\epsilon|^k}\big) \times \sum_{n \geq 0} kC_3(C_4)^n \frac{1}{kR_0^{k-1}} q^{-\Delta n + \Delta(k-1)} \frac{\epsilon_0^k}{\check{\Delta}_{p+1}} R_1^n \\ &+ \exp\big(-\frac{R_0^k}{q^{\Delta k}} \frac{\check{\Delta}_p}{|\epsilon|^k}\big) \times \sum_{n \geq 0} kC_3(C_4)^n \frac{1}{kR_0^{k-1}} q^{-\Delta n + \Delta(k-1)} \frac{\epsilon_0^k}{\check{\Delta}_p} R_1^n \\ &+ \exp\big(-(\frac{R_0}{\tilde{R}_0 q^{\Delta}})^k \Delta_{p+1,p} \frac{1}{|\epsilon|^k}\big) \times \sum_{n \geq 0} k|\gamma_{p+1} - \gamma_p| C_1(C_2)^n \frac{R_0}{q^{\Delta(n+1)}} R_1^n \end{aligned} \tag{108}$$

for all $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$, which hold provided that $R_1 > 0$ is taken small enough and fulfills $0 < R_1 < 1/C_4$ and $0 < R_1 < 1/C_2$. The awaited estimates (89) follow. □

6. Parametric Gevrey Asymptotic Expansions for the Banach Valued Solutions to the Cauchy Problem (16), (17).

The next definition is classical and can be found in the references [1], [7].

Definition 4. *Let* $(\mathbb{E}, ||.||_{\mathbb{E}})$ *be a complex Banach space. Let* $k > 0$ *be a real number and let* $\mathcal{E}$ *be an open bounded sector in* $\mathbb{C}^*$ *edged at 0. Let* $f : \mathcal{E} \to \mathbb{E}$ *be a holomorphic function. The map* f *is said to possess the formal series*

$$\hat{f}(\epsilon) = \sum_{n \geq 0} a_n \epsilon^n \in \mathbb{E}[[\epsilon]]$$

as Gevrey asymptotic expansion of order $1/k$ *on* $\mathcal{E}$ *if for each closed proper subsector* $\mathcal{W}$ *of* $\mathcal{E}$ *edged at 0, one can select two constants* $C, M > 0$ *with*

$$||f(\epsilon) - \sum_{n=0}^{N} a_n \epsilon^n||_{\mathbb{F}} \leq CM^{N+1}\Gamma(\frac{N+1}{k})|\epsilon|^{N+1}$$

for all integers $N \geq 0$*, provided that* $\epsilon \in \mathcal{W}$.

We can state the second prominent result of our work.

Theorem 2. *There exists a formal power series* $\hat{u}(t,z,\epsilon) = \sum_{n \geq 0} h_n(t,z)\epsilon^n$ *in* ϵ *with the next two features*

1. *The coefficients* h_n, $n \geq 0$*, belong to the Banach space* $(\mathcal{O}_{(\mathcal{D}_n)_{n \geq 0};R_1}, ||.||_{(\mathcal{D}_n)_{n \geq 0};R_1})$ *described in Definition 3, where the sector* $\mathcal{T}$*, the sequence* $(\tilde{R}_n)_{n \geq 0}$ *and radius* $R_1 > 0$ *are chosen in Theorem 1.*
2. *For each* $0 \leq p \leq \varsigma - 1$*, the holomorphic map* $\epsilon \mapsto u_p(t,z,\epsilon)$ *from* $\mathcal{E}_p$ *into* $(\mathcal{O}_{(\mathcal{D}_n)_{n \geq 0};R_1}, ||.||_{(\mathcal{D}_n)_{n \geq 0};R_1})$ *built up in Theorem 1, possesses* $\hat{u}(t,z,\epsilon)$ *as Gevrey asymptotic expansion of order* $1/k$ *on* $\mathcal{E}_p$*, for the integer* $k \geq 1$ *introduced in Section 2.1.*

Proof. The proof relies on the following Banach valued version of the classical Ramis-Sibuya theorem stated in Lemma XI-2-6 from [7].

Theorem (Ramis-Sibuya) *Let* $(\mathbb{E}, ||.||_{\mathbb{E}})$ *be a Banach space over* $\mathbb{C}$ *and* $\{\mathcal{E}_p\}_{0 \leq p \leq \varsigma - 1}$ *be a good covering in* $\mathbb{C}^*$ *as described in Definition 1. For each* $0 \leq p \leq \varsigma - 1$*, we consider a holomorphic function* $G_p(\epsilon)$ *from* $\mathcal{E}_p$ *into the Banach space* $(\mathbb{E}, ||.||_{\mathbb{E}})$*. We take for granted that for each* $0 \leq p \leq \varsigma - 1$

1. *the function* $G_p(\epsilon)$ *is bounded on the sector* $\mathcal{E}_p$*,*
2. *the difference* $\Theta_p(\epsilon) = G_{p+1}(\epsilon) - G_p(\epsilon)$ *is exponentially flat of some order* $k > 0$ *on the intersection* $\mathcal{E}_{p+1} \cap \mathcal{E}_p$*, meaning that some constants* $A_p, B_p > 0$ *can be sorted with*

$$||\Theta_p(\epsilon)||_{\mathbb{E}} \leq A_p e^{-B_p/|\epsilon|^k}$$

whenever $\epsilon \in \mathcal{E}_{p+1} \cap \mathcal{E}_p$*, with the convention that* $\mathcal{E}_\varsigma = \mathcal{E}_0$ *and* $G_\varsigma = G_0$.

Then, there exists a formal power series

$$\hat{G}(\epsilon) = \sum_{n \geq 0} G_n \epsilon^n \in \mathbb{E}[[\epsilon]]$$

which is the common Gevrey asymptotic expansion of order $1/k$ *for each map* $G_p(\epsilon)$ *on* $\mathcal{E}_p$, $0 \leq p \leq \varsigma - 1$.

We set $\mathbb{E} = \mathcal{O}_{(\mathcal{D}_n)_{n \geq 0};R_1}$ equipped with the norm $||.||_{\mathbb{E}} = ||.||_{(\mathcal{D}_n)_{n \geq 0};R_1}$. According to Proposition 4, $(\mathbb{E}, ||.||_{\mathbb{E}})$ is a Banach space over $\mathbb{C}$. We apply the above theorem to the maps $G_p : \mathcal{E}_p \to \mathbb{E}$ defined by

$$G_p(\epsilon) := (t,z) \mapsto u_p(t,z,\epsilon).$$

Based on the points 1. and 3. of the statement of Theorem 1, we observe that the set of maps $\{G_p\}_{0 \leq p \leq \varsigma - 1}$ obeys the two requirements 1. and 2. of the above Ramis-Sibuya theorem. As a result, the existence of a formal power series $\hat{G}(\epsilon)$ that we denote $\hat{u}(t,z,\epsilon)$ in the statement of Theorem 2 which stands for the common Gevrey asymptotic expansion of order $1/k$ for the maps $G_p(\epsilon)$ on $\mathcal{E}_p$ for all $0 \leq p \leq \varsigma - 1$ is warranted. □

7. Confluence of the Banach Valued Solutions to (16), (17) as $q > 1$ Tends to 1.

Throughout this section, the notations introduced in the earlier sections of the work are slightly changed. In order to keep track of the dependence of the set of solutions $\{u_p(t,z,\epsilon)\}_{0 \leq p \leq \varsigma - 1}$ to the problem (16), (17) with respect to the parameter $q > 1$, constructed in Theorem 1, we denote $u_{p;q}(t,z,\epsilon)$ the map $u_p(t,z,\epsilon)$. A third index q is added to the Borel maps $w_{p,n}(u,\epsilon)$, by setting $w_{p,n}(u,\epsilon) = w_{p,n;q}(u,\epsilon)$ inside the integral representation (86). The real parameter q is chosen within the range $(1,q_0]$ for some fixed real number $q_0 > 1$.

7.1. The limit nonlinear partial differential Cauchy problem.

In this subsection, a new Cauchy problem is introduced that is called the *limit problem as* $q > 1$ tends to 1. It is displayed as follows

$$P(\epsilon^k t^{k+1}\partial_t)\partial_z^S u_{;1}(t,z,\epsilon) = \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A}} \epsilon^{\Delta_{\underline{l}}} c_{\underline{l}}(z,\epsilon) t^{l_0} (\epsilon^k t^{k+1}\partial_t)^{l_1} \partial_z^{l_2} u_{;1}(t,z,\epsilon) + d(z,\epsilon)(u_{;1}(t,z,\epsilon))^2 \quad (109)$$

for given Cauchy data

$$(\partial_z^j u_{;1})(t,0,\epsilon) = \varphi_j(t,\epsilon) \ , \ 0 \leq j \leq S-1. \quad (110)$$

where all the data $k, S, P, \mathcal{A}$, the constants $\Delta_{\underline{l}}$ and the coefficients $c_{\underline{l}}(z,\epsilon)$ with $\underline{l} \in \mathcal{A}$, $d(z,\epsilon)$ along with the initial data $\varphi_j(t,\epsilon)$ for $0 \leq j \leq S-1$ are the ones already set up in Section 2.1.

We search for solutions to (109), (110) that can be written as a formal power series in the variable z

$$u_{;1}(t,z,\epsilon) = \sum_{n\geq 0} u_{n;1}(t,\epsilon)\frac{z^n}{n!} \quad (111)$$

whose coefficients $u_{n;1}(t,\epsilon)$, $n \geq 0$, are expressed through Laplace transforms of order k,

$$u_{n;1}(t,\epsilon) = k\int_{L_\gamma} w_{n;1}(u,\epsilon)\exp\left(-(\frac{u}{\epsilon t})^k\right)\frac{du}{u} \quad (112)$$

along a prescribed halfline $L_\gamma = [0,+\infty)e^{\sqrt{-1}\gamma}$ for suitable direction $\gamma \in \mathbb{R}$. For all $n \geq 0$, the Borel maps $w_{n;1}(u,\epsilon)$ are supposed to be holomorphic on a common product $\mathcal{U} \times D_{\epsilon_0}$, where $\mathcal{U}$ stands for an open unbounded sector edged at 0 containing the halfline $L_\gamma \setminus \{0\}$. For each Laplace transform (112) to be well defined, the assumption is made that a constant $K > 0$ and a positive sequence $(H_n)_{n\geq 0}$ can be found with bounds of the form

$$\sup_{\epsilon\in D_{\epsilon_0}} |w_{n;1}(u,\epsilon)| \leq H_n|u|\exp(K|u|^k) \quad (113)$$

for all $n \geq 0$, provided that $u \in \mathcal{U}$. The value of the constant $K > 0$ and the exact shape of the sequence $(H_n)_{n\geq 0}$ will be discussed in the forthcoming Proposition 5..

Identical computations as the ones performed in Lemma 1 and Lemma 2 in Section 2.2 give rise to the next lemma.

Lemma 6. *The formal series (111) solves the problem (109), (110)* **if** *the sequence of Borel maps* $(w_{n;1}(u,\epsilon))_{n\geq 0}$ *conforms to the next convolution recurrence relation*

$$
\begin{aligned}
\frac{w_{n+S;1}(u,\epsilon)}{n!} = & \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \epsilon^{\Delta_{\underline{l}}} \frac{(ku^k)^{l_1}}{P(ku^k)} \Big(\sum_{n_1+n_2=n} c_{\underline{l},n_1}(\epsilon) \frac{w_{n_2+l_2;1}(u,\epsilon)}{n_2!} \Big) \\
& + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon^{\Delta_{\underline{l}}-l_0} \frac{u^k}{P(ku^k)\Gamma(l_0/k)} \int_0^{u^k} (u^k-s)^{\frac{l_0}{k}-1}(ks)^{l_1} \\
& \times \Big(\sum_{n_1+n_2=n} c_{\underline{l},n_1}(\epsilon) \frac{w_{n_2+l_2;1}(s^{1/k},\epsilon)}{n_2!} \Big) \frac{ds}{s} \\
& + \frac{u^k}{P(ku^k)} \int_0^{u^k} \Big(\sum_{n_1+n_2+n_3=n} d_{n_1}(\epsilon) \frac{w_{n_2;1}((u^k-s)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;1}(s^{1/k},\epsilon)}{n_3!} \Big) \frac{1}{(u^k-s)s} ds
\end{aligned}
\tag{114}
$$

for given

$$w_{j;1}(u,\epsilon) = P_j(u,\epsilon) \ , \ 0 \leq j \leq S-1. \tag{115}$$

The next proposition can be shown in following exactly the same steps of the proof of Proposition 3 and by merely setting $q=1$ in each inequality involved.

Proposition 5. *Let* $\underline{\mathcal{D}} = \{\underline{\mathcal{E}},\mathcal{T},\underline{\mathcal{U}}\}$ *be an admissible set of sectors as chosen in Definition 2 of Section 5. Let* $\mathcal{U}$ *be one sector belonging to the family of unbounded sectors* $\underline{\mathcal{U}}$*. Provided that the radius* $\epsilon_0 > 0$ *and the constants* $\mathbf{d}_{h,\epsilon_0} > 0$ *introduced in (10) are taken close enough to 0, one can find a unique sequence of functions* $(w_{n;1}(u,\epsilon))_{n\geq 0}$*, where each map* $(u,\epsilon) \mapsto w_{n;1}(u,\epsilon)$

- *is holomorphic with respect to u on* $\mathcal{U}$ *and continuous on* $\mathcal{U}\cup\{0\}$
- *is holomorphic relatively to* ϵ *on* D_{ϵ_0}
- *fulfills the recursion (114) for given S first terms (115)*
- *suffers the next upper bounds : one can single out constants* $C_6, C_7, C_8 > 0$ *(independent of n) with*

$$\sup_{\epsilon\in D_{\epsilon_0}} |w_{n;1}(u,\epsilon)| \leq C_6(C_7)^n n! |u| \exp(C_8|u|^k) \tag{116}$$

for all $u \in \mathcal{U}\cup\{0\}$.

In the next proposition, we build up a genuine holomorphic solution to the Cauchy problem (109), (110).

Proposition 6. *We consider an admissible set of sectors* $\underline{\mathcal{D}} = \{\underline{\mathcal{E}},\mathcal{T},\underline{\mathcal{U}}\}$ *as chosen in Definition 2 of Section 5. Let us pick up one sector* $\mathcal{U}$ *belonging to the family of unbounded sectors* $\underline{\mathcal{U}}$*. We denote* $\mathcal{E}$ *the open sector from the good covering* $\underline{\mathcal{E}}$ *that is related to* $\mathcal{U}$ *under the requirement of Definition 2 2). Granted that the radius* $\epsilon_0 > 0$ *and the constants* $\mathbf{d}_{h,\epsilon_0} > 0$ *introduced in (10) are taken in the vicinity of 0, one can construct a solution* $u_{;1}(t,z,\epsilon)$ *to the limit Cauchy problem (109), (110) favored with the next features*

1. *The map* $u_{;1}(t,z,\epsilon)$ *represents a bounded holomorphic map on the domain* $(\mathcal{T}\cap D_{\check{R}_0}) \times D_{\check{R}_1} \times \mathcal{E}$*, for well chosen constants* $\check{R}_0, \check{R}_1 > 0$.
2. *The map* $u_{;1}(t,z,\epsilon)$ *is expressed as a convergent series in z with the shape (111) with coefficients* $u_{n;1}(t,\epsilon)$*,* $n\geq 1$ *in the form of a Laplace transform of order k given by (112) along a halfline* $L_\gamma = [0,+\infty)e^{\sqrt{-1}\gamma}$ *for the direction* $\gamma\in\mathbb{R}$ *given in Definition 2 2) relatively to the sectors* $\mathcal{E}$ *and* $\mathcal{U}$*. In addition, each Borel map* $w_{n;1}(u,\epsilon)$*,* $n\geq 0$*, is holomorphic on the product* $\mathcal{U}\times D_{\epsilon_0}$ *and is submitted to the bounds (116) for some constants* $C_6, C_7, C_8 > 0$ *and all* $u\in\mathcal{U}\cup\{0\}$.

Proof. We consider the unique sequence of functions $(w_{n;1}(u,\epsilon))_{n\geq 0}$ disclosed in Proposition 5. We need to display upper bounds for the Laplace transform $u_{n;1}(t,\epsilon)$ given by (112). Indeed, according to the upper and lower bounds (116) and (80), for $u = re^{\sqrt{-1}\gamma} \in L_{\gamma}$ with $r \geq 0$, we reach a constant $\Delta_{;1} > 0$ with

$$|w_{n;1}(u,\epsilon) \exp\big(-(\frac{u}{\epsilon t})^{k}\big)\frac{1}{u}| \\ \leq C_6(C_7)^{n} n! \exp\big(C_8 r^{k}\big) \exp\Big(-(\frac{r}{|\epsilon t|})^{k} \cos(k(\gamma - \arg(\epsilon t)))\Big) \\ \leq C_6(C_7)^{n} n! \exp\Big(r^{k}\big[C_8 - \frac{\Delta_{;1}}{(\epsilon_0 |t|)^{k}}\big]\Big) \quad (117)$$

for all $\epsilon \in \mathcal{E}$ and $t \in \mathcal{T}$. As a result, we can find a constant $B_9 > 0$ such that

$$|w_{n;1}(u,\epsilon) \exp\big(-(\frac{u}{\epsilon t})^{k}\big)\frac{1}{u}| \leq C_6(C_7)^{n} n! \exp(-B_9 r^{k}) \quad (118)$$

provided that $u = re^{\sqrt{-1}\gamma} \in L_{\gamma}$, for $r \geq 0$, all $\epsilon \in \mathcal{E}$ and $t \in \mathcal{T}$ constrained to the bounds

$$|t| \leq \frac{(\Delta_{;1})^{1/k}/\epsilon_0}{(C_8 + B_9)^{1/k}} = \check{R}_0.$$

It follows that each Laplace transform $u_{n;1}(t,\epsilon)$ displayed in (112) defines a bounded holomorphic function on the product $(\mathcal{T} \cap D_{\check{R}_0}) \times \mathcal{E}$ and suffers the bounds

$$|u_{n;1}(t,\epsilon)| \leq kC_6(C_7)^{n} n! \int_{0}^{+\infty} \exp(-B_9 r^{k}) dr \quad (119)$$

for all $t \in \mathcal{T} \cap D_{\check{R}_0}$ and $\epsilon \in \mathcal{E}$. If one sets the expression $u_{;1}(t,z,\epsilon)$ by the expansion (111) in z, one checks that $u_{;1}(t,z,\epsilon)$ defines a bounded holomorphic function on the product $(\mathcal{T} \cap D_{\check{R}_0}) \times D_{\check{R}_1} \times \mathcal{E}$ provided that $0 < \check{R}_1 < 1/C_7$. Namely, from (119)

$$|u_{;1}(t,z,\epsilon)| \leq kC_6 \int_{0}^{+\infty} \exp(-B_9 r^{k}) dr \times \sum_{n \geq 0} (C_7 \check{R}_1)^{n} = \frac{kC_6 \int_{0}^{+\infty} \exp(-B_9 r^{k}) dr}{1 - C_7 \check{R}_1} \quad (120)$$

for all $t \in \mathcal{T} \cap D_{\check{R}_0}$, $z \in D_{\check{R}_1}$ and $\epsilon \in \mathcal{E}$. Besides, according to Lemma 6 and the fact discussed in Proposition 5 that the sequence $(w_{n;1}(u,\epsilon))_{n \geq 0}$ obeys the recursion (114) for given S first terms (115), it follows that the series $u_{;1}(t,z,\epsilon)$ solves the limit Cauchy problem (109), (110). Proposition 6 ensues. □

7.2. Bounds for the difference of Borel maps solving the convolution recurrence relation (114) under the action of a q−difference operator

The aim of this subsection is to prove the next technical issue.

Proposition 7. *Let $\beta \in \mathbb{Z}^{*}$ be a non vanishing integer and let $q \in (1, q_0]$. Provided that the radius $\epsilon_0 > 0$ and the constants $\mathbf{d}_{h,\epsilon_0} > 0$ introduced in (10) are sufficiently small, one can choose constants $C_9, C_{10}, C_{11} > 0$ independently of $q \in (1, q_0]$ (where C_{10} can be selected larger than the constant C_7 obtained in Proposition 5) such that*

$$|w_{n;1}(u,\epsilon) - w_{n;1}(q^{\beta}u,\epsilon)| \leq |q^{\beta} - 1| C_9 (C_{10})^{n} n! |u| \exp(C_{11}|u|^{k}) \quad (121)$$

for all integers $n \geq 0$, all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

Proof. We proceed by induction. We call $\mathscr{D}_{n}^{;1;\beta}$ the property (121) for a given natural number $n \geq 0$.

We first discuss the reason for which $\mathscr{D}_{n}^{;1;\beta}$ holds true for $0 \leq n \leq S-1$. By construction, whenever $0 \leq n \leq S-1$, $w_{n;1}(u,\epsilon) = P_n(u,\epsilon)$ is a polynomial in u with bounded holomorphic coefficients in ϵ

on D_{ϵ_0}. Since the partial map $u \mapsto P_n(u,\epsilon)$ admits in particular a derivative $u \mapsto P'_n(u,\epsilon)$ w.r.t u on $\mathbb{C}$, we can rewrite the next difference as an integral

$$P_n(u,\epsilon) - P_n(q^{\beta}u,\epsilon) = \int_{q^{\beta}u}^{u} P'_n(s,\epsilon)ds$$

for all $u \in \mathbb{C}$ and from the parametrization $s = uh + q^{\beta}u(1-h)$ of the segment $[q^{\beta}u, u]$ with $0 \leq h \leq 1$, we obtain the integral representation

$$P_n(u,\epsilon) - P_n(q^{\beta}u,\epsilon) = (1-q^{\beta})u \int_0^1 P'_n(uh + q^{\beta}u(1-h),\epsilon)dh$$

and since $u \mapsto P'_n(uh + q^{\beta}u(1-h),\epsilon)$ is again a polynomial in u, with bounded holomorphic coefficients in ϵ on D_{ϵ_0}, we get that $\mathscr{D}_n^{;1;\beta}$ is valid for $0 \leq n \leq S-1$, for some well chosen constants $C_9, C_{10}, C_{11} > 0$ depending on β, q_0.

Let $n \geq 0$, we take for granted that $\mathscr{D}_p^{;1;\beta}$ holds for any $p < n+S$ for some given constants $C_9, C_{10}, C_{11} > 0$. In the remaining part of the proof, we prove that $\mathscr{D}_{n+S}^{;1;\beta}$ is valid. The induction principle will then imply that the feature $\mathscr{D}_n^{;1;\beta}$ occurs for all $n \geq 0$.

In a first step, we use the recursion (114) with S first terms (115) in order to write the quantity $w_{n+S;1}(q^{\beta}u,\epsilon)$ by means of prior terms $w_{p;1}(u,\epsilon)$ and $w_{p;1}(q^{\beta}u,\epsilon)$ for $p < n+S$. Namely, the next identities

$$\begin{aligned}
\frac{w_{n+S;1}(q^{\beta}u,\epsilon)}{n!} &= \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \epsilon^{\Delta_{\underline{l}}} \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \Big(\sum_{n_1+n_2=n} c_{\underline{l},n_1}(\epsilon) \frac{w_{n_2+l_2;1}(q^{\beta}u,\epsilon)}{n_2!} \Big) \\
&+ \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon^{\Delta_{\underline{l}}-l_0} \frac{(q^{\beta}u)^k}{P(k(q^{\beta}u)^k)\Gamma(l_0/k)} \int_0^{(q^{\beta}u)^k} ((q^{\beta}u)^k - s)^{\frac{l_0}{k}-1}(ks)^{l_1} \\
&\times \Big(\sum_{n_1+n_2=n} c_{\underline{l},n_1}(\epsilon) \frac{w_{n_2+l_2;1}(s^{1/k},\epsilon)}{n_2!} \Big) \frac{ds}{s} \\
&+ \frac{(q^{\beta}u)^k}{P(k(q^{\beta}u)^k)} \\
\times \int_0^{(q^{\beta}u)^k} &\Big(\sum_{n_1+n_2+n_3=n} d_{n_1}(\epsilon) \frac{w_{n_2;1}(((q^{\beta}u)^k - s)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;1}(s^{1/k},\epsilon)}{n_3!} \Big) \frac{1}{((q^{\beta}u)^k - s)s} ds
\end{aligned} \quad (122)$$

hold for prescribed

$$w_{j;1}(q^{\beta}u,\epsilon) = P_j(q^{\beta}u,\epsilon) \;\; , \;\; 0 \leq j \leq S-1. \quad (123)$$

In the next lemma, upper bounds are provided for the quantity

$$\mathcal{A}_{n_2,l_2}(u,\epsilon) := \frac{(ku^k)^{l_1}}{P(ku^k)} \frac{w_{n_2+l_2;1}(u,\epsilon)}{n_2!} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \frac{w_{n_2+l_2;1}(q^{\beta}u,\epsilon)}{n_2!} \quad (124)$$

Lemma 7. *Two constants $K_{j,l_1,P} > 0$, $j = 1,3$, can be singled out for which*

$$|\mathcal{A}_{n_2,l_2}(u,\epsilon)| \leq \left[K_{3,l_1,P} + K_{1,l_1,P}\right] \times |q^{\beta} - 1| C_9 (C_{10})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp(C_{11}|u|^k) \quad (125)$$

holds for all $u \in \mathcal{U} \cup \{0\}$, all $\epsilon \in D_{\epsilon_0}$, all $n_2, l_2 \geq 0$ with $n_2 \leq n$ and $\underline{l} = (0, l_1, l_2, l_3) \in \mathcal{A}$.

Proof. We use the classical identity $ab - cd = (a-c)b + c(b-d)$ which enables to rewrite $\mathcal{A}_{n_2,l_2}(u,\epsilon)$ as a sum

$$\mathcal{A}_{n_2,l_2}(u,\epsilon) = \left[\frac{(ku^k)^{l_1}}{P(ku^k)} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \right] \frac{w_{n_2+l_2;1}(u,\epsilon)}{n_2!} \\ + \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \left[\frac{w_{n_2+l_2;1}(u,\epsilon) - w_{n_2+l_2;1}(q^{\beta}u,\epsilon)}{n_2!} \right] \tag{126}$$

a) According to the assumption (15) and the condition given in Definition 2 1) imposed on the unbounded sector $\mathcal{U}$, we get a constant $K_{1,l_1,P} > 0$ such that

$$\left| \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \right| \leq K_{1,l_1,P} \tag{127}$$

for all $u \in \mathcal{U} \cup \{0\}$, for all $\underline{l} = (0, l_1, l_2, l_3) \in \mathcal{A}$.
b) On the other hand, we can display an integral representation for the next difference

$$\frac{(ku^k)^{l_1}}{P(ku^k)} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} = \int_{q^{\beta}u}^{u} \left(\frac{(ks^k)^{l_1}}{P(ks^k)} \right)' ds \tag{128}$$

where the integrand can be explicitely computed

$$\left(\frac{(ks^k)^{l_1}}{P(ks^k)} \right)' = k^{l_1} \frac{kl_1 s^{kl_1-1} P(ks^k) - \hat{P}_k(s) s^{kl_1}}{(P(ks^k))^2} \tag{129}$$

where $\hat{P}_k(s) = (P(ks^k))'$ is the derivative w.r.t s of the polynomial $P(ks^k)$. Furthermore, since $s \mapsto P(ks^k)$ is a polynomial of degree $k\deg(P)$ together with the condition on $\mathcal{U}$ imposed in Definition 2 1), we notice the next two bounds

$$C_{P,k}(1+|s|)^{k\deg(P)} \leq |P(ks^k)| \leq C_{P,k}^{1}(|s|+1)^{k\deg(P)} \ , \ \ |\hat{P}_k(s)| \leq C_{\hat{P}_k}^{1}(|s|+1)^{k\deg(P)-1} \tag{130}$$

for all $s \in \mathcal{U} \cup \{0\}$, for some constants $C_{P,k}, C_{P,k}^{1}, C_{\hat{P}_k}^{1} > 0$. Consequently, departing from (129), the bounds (130), beget a constant $K_{2,l_1,P} > 0$ for which

$$\left| \left(\frac{(ks^k)^{l_1}}{P(ks^k)} \right)' \right| \leq k^{l_1} \frac{kl_1 |s|^{kl_1-1} C_{P,k}^{1}(|s|+1)^{k\deg(P)} + |s|^{kl_1} C_{\hat{P}_k}^{1}(|s|+1)^{k\deg(P)-1}}{C_{P,k}^{2}(1+|s|)^{2k\deg(P)}} \\ \leq \frac{K_{2,l_1,P}}{(1+|s|)^{k\mathrm{deg(P)}-kl_1+1}} \tag{131}$$

provided that $s \in \mathcal{U} \cup \{0\}$.

From the parametrization $s = uh + q^{\beta}u(1-h)$ of the segment $[q^{\beta}u, u]$ for $0 \leq h \leq 1$ in the integral (128), owing to the previous bounds (131), we deduce

$$\left| \frac{(ku^k)^{l_1}}{P(ku^k)} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \right| \leq |q^{\beta} - 1||u| \int_0^1 \frac{K_{2,l_1,P}}{(1+|uh + q^{\beta}u(1-h)|)^{k\deg(P)-kl_1+1}} dh \tag{132}$$

for all $u \in \mathcal{U} \cup \{0\}$. Since $(1-q^{\beta})h + q^{\beta} \geq d_{q,\beta}$ for all $0 \leq h \leq 1$ where $d_{q,\beta} = 1$ if $\beta > 0$ and $d_{q,\beta} = q^{\beta}$ if $\beta < 0$, we reach a constant $K_{3,l_1,P} > 0$ for which

$$\left| \frac{(ku^k)^{l_1}}{P(ku^k)} - \frac{(k(q^{\beta}u)^k)^{l_1}}{P(k(q^{\beta}u)^k)} \right| \leq |q^{\beta}-1| K_{2,l_1,P} \frac{|u|}{(1+|u|d_{q,\beta})^{k\deg(P)-kl_1+1}} \leq |q^{\beta}-1|K_{3,l_1,P} \tag{133}$$

whenever $u \in \mathcal{U} \cup \{0\}$, by taking heed of the condition (15).
c) In accordance with (116), we already know that

$$\begin{aligned} |w_{n_2+l_2;1}(u,\epsilon)| &\leq C_6 (C_7)^{n_2+l_2}(n_2+l_2)!|u| \exp(C_8|u|^k) \\ &\leq C_9 (C_{10})^{n_2+l_2}(n_2+l_2)!|u| \exp(C_{11}|u|^k) \end{aligned} \tag{134}$$

for all $n_2, l_2 \geq 0$, whenever $u \in \mathcal{U} \cup \{0\}$, $\epsilon \in D_{\epsilon_0}$, provided that we select $C_9 > C_6$ and $C_{10} > C_7$ and $C_{11} > C_8$.
d) Due to (13), we notice that $n_2 + l_2 < n + S$, therefore by the induction hypothesis, the property $\mathscr{D}^{;1;\beta}_{n_2+l_2}$ is valid, which yields the upper bounds

$$|w_{n_2+l_2;1}(u,\epsilon) - w_{n_2+l_2;1}(q^{\beta}u,\epsilon)| \leq |q^{\beta}-1| C_9 (C_{10})^{n_2+l_2}(n_2+l_2)!|u| \exp(C_{11}|u|^k) \tag{135}$$

whenever $u \in \mathcal{U} \cup \{0\}$, $\epsilon \in D_{\epsilon_0}$. Collecting the set of bounds (127), (133) along with (134) and (135), we arrive at the forecast bounds (125). □

The next lemma is devoted to upper bounds for the quantity

$$\begin{aligned} \mathcal{B}_{n_2,l_2}(u,\epsilon) := &\frac{u^k}{P(ku^k)} \int_0^{u^k} (u^k - s)^{\frac{l_0}{k}-1} s^{l_1} \frac{w_{n_2+l_2;1}(s^{1/k},\epsilon)}{n_2!} \frac{ds}{s} \\ &- \frac{(q^{\beta}u)^k}{P(k(q^{\beta}u)^k)} \int_0^{(q^{\beta}u)^k} ((q^{\beta}u)^k - s)^{\frac{l_0}{k}-1} s^{l_1} \frac{w_{n_2+l_2;1}(s^{1/k},\epsilon)}{n_2!} \frac{ds}{s} \end{aligned} \tag{136}$$

We first rephrase the above difference by performing the parametrization $s = u^k s_1$, $0 \leq s_1 \leq 1$ in the first integral part of $\mathcal{B}_{n_2,l_2}(u,\epsilon)$ and $s = (q^{\beta}u)^k s_1$, $0 \leq s_1 \leq 1$ in the second. Indeed,

$$\begin{aligned} \mathcal{B}_{n_2,l_2}(u,\epsilon) := &\frac{u^{l_0+kl_1}}{P(ku^k)} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(us_1^{1/k},\epsilon)}{n_2!} \frac{ds_1}{s_1} - \\ &\frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)} \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(q^{\beta}us_1^{1/k},\epsilon)}{n_2!} \frac{ds_1}{s_1} \end{aligned} \tag{137}$$

Lemma 8. *We can find two constants $L_{j,l_0,l_1,P} > 0$, $j = 1,3$, such that*

$$\begin{aligned} |\mathcal{B}_{n_2,l_2}(u,\epsilon)| \leq &\left[L_{3,l_0,l_1,P} + L_{1,l_0,l_1,p} \right] \\ &\times |q^{\beta}-1| C_9 \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (C_{10})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp(C_{11}|u|^k) \end{aligned} \tag{138}$$

for all $u \in \mathcal{U} \cup \{0\}$, $\epsilon \in D_{\epsilon_0}$, all $n_2, l_2 \geq 0$ with $n_2 \leq n$ and $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$ with $l_0 \geq 1$.

Proof. The proof follows the same guideline as the one of Lemma 7. Namely, the identity $ab - cd = (a-c)b + c(b-d)$ allows us to restate the expression of $\mathcal{B}_{n_2,l_2}(u,\epsilon)$ as a sum of differences

$$\mathcal{B}_{n_2,l_2}(u,\epsilon) = \left[\frac{u^{l_0+kl_1}}{P(ku^k)} - \frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)} \right] \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(us_1^{1/k},\epsilon)}{n_2!} \frac{ds_1}{s_1}$$
$$+ \frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)} \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \left\{ \frac{w_{n_2+l_2;1}(us_1^{1/k},\epsilon)}{n_2!} - \frac{w_{n_2+l_2;1}(q^{\beta}us_1^{1/k},\epsilon)}{n_2!} \right\} \frac{ds_1}{s_1} \right). \quad (139)$$

a) The assumption (15) and the condition given in Definition 2 1) imposed on the unbounded sector $\mathcal{U}$ yields a constant $L_{1,l_0,l_1,P} > 0$ can be found with

$$\left| \frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)} \right| \leq L_{1,l_0,l_1,P} \quad (140)$$

for all $u \in \mathcal{U} \cup \{0\}$.

b) By adapting the arguments outlined in the paragraph b) in the proof of Lemma 7, we can gather two constants $L_{j,l_0,l_1,P} > 0$, $j = 2,3$ such that

$$\left| \frac{u^{l_0+kl_1}}{P(ku^k)} - \frac{(q^{\beta}u)^{l_0+kl_1}}{P(k(q^{\beta}u)^k)} \right| \leq |q^{\beta}-1| L_{2,l_0,l_1,P} \frac{|u|}{(1+|u|d_{q,\beta})^{k\deg(P)-(l_0+kl_1)+1}} \leq |q^{\beta}-1| L_{3,l_0,l_1,P} \quad (141)$$

for all $u \in \mathcal{U} \cup \{0\}$.

c) Bearing in mind the bounds (134) reached in the point c) of Lemma 7, we obtain upper bounds for the next integral piece

$$\left| \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(us_1^{1/k},\epsilon)}{n_2!} \frac{ds_1}{s_1} \right|$$
$$\leq C_9 \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (C_{10})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp(C_{11}|u|^k) \quad (142)$$

for all $u \in \mathcal{U} \cup \{0\}$, $\epsilon \in D_{\epsilon_0}$.

d) According to the bounds (135) obtained from our induction hypothesis, we arrive at

$$\left| \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \left\{ \frac{w_{n_2+l_2;1}(us_1^{1/k},\epsilon)}{n_2!} - \frac{w_{n_2+l_2;1}(q^{\beta}us_1^{1/k},\epsilon)}{n_2!} \right\} \frac{ds_1}{s_1} \right|$$
$$\leq |q^{\beta}-1| C_9 \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (C_{10})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp(C_{11}|u|^k) \quad (143)$$

whenever $u \in \mathcal{U} \cup \{0\}$, $\epsilon \in D_{\epsilon_0}$.

Combining the bounds (140), (141), (142), (143) along with the factorization (139) triggers the foretold estimates (138). □

The forthcoming lemma discusses upper bounds for the difference of nonlinear terms

$$\begin{aligned}\mathcal{C}_{n_2,n_3}(u,\epsilon) \\ = \frac{u^k}{P(ku^k)} \times \int_0^{u^k} \frac{w_{n_2;1}((u^k-s)^{1/k},\epsilon)}{n_2!}\frac{w_{n_3;1}(s^{1/k},\epsilon)}{n_3!}\frac{1}{(u^k-s)s}ds - \frac{(q^{\beta}u)^k}{P(k(q^{\beta}u)^k)} \\ \times \int_0^{(q^{\beta}u)^k} \frac{w_{n_2;1}(((q^{\beta}u)^k-s)^{1/k},\epsilon)}{n_2!}\frac{w_{n_3;1}(s^{1/k},\epsilon)}{n_3!}\frac{1}{((q^{\beta}u)^k-s)s}ds \quad (144)\end{aligned}$$

Lemma 9. *We can find constants $M_{j,P} > 0$, $j = 1,3$, such that*

$$\begin{aligned}|\mathcal{C}_{n_2,n_3}(u,\epsilon)| \\ \leq [M_{1,P}+M_{3,P}]\left(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}}s_1^{1-\frac{1}{k}}}ds_1\right)|q^{\beta}-1|(C_9)^2(C_{10})^{n_2+n_3}|u|\exp(C_{11}|u|^k) \quad (145)\end{aligned}$$

holds for all $u \in \mathcal{U} \cup \{0\}$, $\epsilon \in D_{\epsilon_0}$, provided that the integers $n_2, n_3 \geq 0$ obey $n_2+n_3 \leq n$.

Proof. We first reshape the expression of $\mathcal{C}_{n_2,n_3}(u,\epsilon)$ by using the parametrization $s = u^k s_1$, $0 \leq s_1 \leq 1$, in the first term of (144) and $s = (q^{\beta}u)^k s_1$, $0 \leq s_1 \leq 1$ in the second. Indeed,

$$\begin{aligned}\mathcal{C}_{n_2,n_3}(u,\epsilon) \\ = \frac{1}{P(ku^k)} \times \int_0^1 \frac{w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)}{n_2!}\frac{w_{n_3;1}(us_1^{1/k},\epsilon)}{n_3!}\frac{1}{(1-s_1)s_1}ds_1 - \frac{1}{P(k(q^{\beta}u)^k)} \\ \times \int_0^1 \frac{w_{n_2;1}(q^{\beta}u(1-s_1)^{1/k},\epsilon)}{n_2!}\frac{w_{n_3;1}(q^{\beta}us_1^{1/k},\epsilon)}{n_3!}\frac{1}{(1-s_1)s_1}ds_1.\end{aligned}$$

Furthermore, the identity $ab-cd = (a-c)b+c(b-d)$ enables the above expression to be written as a sum of differences

$$\mathcal{C}_{n_2,n_3}(u,\epsilon) = \mathcal{C}^1_{n_2,n_3}(u,\epsilon) + \mathcal{C}^2_{n_2,n_3}(u,\epsilon) \quad (146)$$

where

$$\begin{aligned}\mathcal{C}^1_{n_2,n_3}(u,\epsilon) := \left[\frac{1}{P(ku^k)} - \frac{1}{P(k(q^{\beta}u)^k)}\right] \\ \times \int_0^1 \frac{w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)}{n_2!}\frac{w_{n_3;1}(us_1^{1/k},\epsilon)}{n_3!}\frac{1}{(1-s_1)s_1}ds_1 \quad (147)\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}^2_{n_2,n_3}(u,\epsilon) := \frac{1}{P(k(q^{\beta}u)^k)} \\ \times \left[\int_0^1 \left\{\frac{w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)}{n_2!}\frac{w_{n_3;1}(us_1^{1/k},\epsilon)}{n_3!} - \frac{w_{n_2;1}(q^{\beta}u(1-s_1)^{1/k},\epsilon)}{n_2!}\frac{w_{n_3;1}(q^{\beta}us_1^{1/k},\epsilon)}{n_3!}\right\}\right. \\ \left.\times \frac{1}{(1-s_1)s_1}ds_1\right]. \quad (148)\end{aligned}$$

We control the quantity $\mathcal{C}^1_{n_2,n_3}(u,\epsilon)$. Namely, comparable computations as the ones performed in the paragraph b) in the proof of Lemma 7 yield a constant $M_{2,P} > 0$ with

$$\left|\frac{1}{P(ku^k)} - \frac{1}{P(k(q^{\beta}u)^k)}\right| \leq |q^{\beta}-1|M_{2,P} \times \frac{|u|}{(1+|u|d_{q,\beta})^{k\deg(P)+1}} \quad (149)$$

for all $u \in \mathcal{U} \cup \{0\}$, for the constant $d_{q,\beta} = 1$ if $\beta > 0$ and $d_{q,\beta} = q^{\beta}$ if $\beta < 0$. Besides, from the bounds (116) in Proposition 5, we observe that

$$|w_{n_2;1}(u,\epsilon)| \leq C_6(C_7)^{n_2} n_2!|u|\exp(C_8|u|^k) \leq C_9(C_{10})^{n_2} n_2!|u|\exp(C_{11}|u|^k) \tag{150}$$

along with

$$|w_{n_3;1}(u,\epsilon)| \leq C_6(C_7)^{n_3} n_3!|u|\exp(C_8|u|^k) \leq C_9(C_{10})^{n_3} n_3!|u|\exp(C_{11}|u|^k) \tag{151}$$

for all integers $n_2, n_3 \geq 0$, provided that $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$, under the condition that we choose $C_9 > C_6$, $C_{10} > C_7$ and $C_{11} > C_8$. We deduce that the integral part of $\mathcal{C}^{1}_{n_2,n_3}(u,\epsilon)$ is bounded as follows.

$$\begin{aligned} &\Big| \int_0^1 \frac{w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;1}(us_1^{1/k},\epsilon)}{n_3!} \frac{1}{(1-s_1)s_1} ds_1 \Big| \\ &\qquad \leq (C_9)^2(C_{10})^{n_2+n_3}|u|^2 \int_0^1 \exp\left(C_{11}|u|^k(1-s_1)\right) \exp(C_{11}|u|^k s_1) \frac{1}{(1-s_1)^{1-\frac{1}{k}} s_1^{1-\frac{1}{k}}} ds_1 \\ &\qquad\qquad \leq (C_9)^2(C_{10})^{n_2+n_3}|u| \times \Big(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}} s_1^{1-\frac{1}{k}}} ds_1 \Big) \times |u|\exp(C_{11}|u|^k) \end{aligned} \tag{152}$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$. As a result of (149) with (152) and by keeping on mind (6), we reach a constant $M_{1,P} > 0$ for which

$$\begin{aligned} &|\mathcal{C}^{1}_{n_2,n_3}(u,\epsilon)| \\ &\qquad \leq |q^{\beta} - 1| M_{1,P} \times \Big(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}} s_1^{1-\frac{1}{k}}} ds_1 \Big) (C_9)^2 (C_{10})^{n_2+n_3}|u|\exp(C_{11}|u|^k) \end{aligned} \tag{153}$$

provided that $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

The term $\mathcal{C}^{2}_{n_2,n_3}(u,\epsilon)$ is now framed. We need to further split this term as a sum of differences using again the identity $ab - cd = (a-c)b + c(b-d)$. Indeed,

$$\mathcal{C}^{2}_{n_2,n_3}(u,\epsilon) = \mathcal{C}^{2.1}_{n_2,n_3}(u,\epsilon) + \mathcal{C}^{2.2}_{n_2,n_3}(u,\epsilon) \tag{154}$$

where

$$\begin{aligned} &\mathcal{C}^{2.1}_{n_2,n_3}(u,\epsilon) := \frac{1}{P(k(q^{\beta}u)^k)} \\ &\quad \times \Big[\int_0^1 \Big\{ \frac{w_{n_2;1}(u(1-s_1)^{1/k},\epsilon) - w_{n_2;1}(q^{\beta}u(1-s_1)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;1}(us_1^{1/k},\epsilon)}{n_3!} \Big\} \frac{1}{(1-s_1)s_1} ds_1 \Big] \end{aligned} \tag{155}$$

and

$$\begin{aligned} &\mathcal{C}^{2.2}_{n_2,n_3}(u,\epsilon) := \frac{1}{P(k(q^{\beta}u)^k)} \\ &\quad \times \Big[\int_0^1 \Big\{ \frac{w_{n_2;1}(q^{\beta}u(1-s_1)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;1}(us_1^{1/k},\epsilon) - w_{n_3;1}(q^{\beta}us_1^{1/k},\epsilon)}{n_3!} \Big\} \frac{1}{(1-s_1)s_1} ds_1 \Big]. \end{aligned} \tag{156}$$

Since the integers $n_2, n_3 \geq 0$ are not larger than n, owing to the induction hypothesis, both properties $\mathcal{D}^{;1;\beta}_{n_2}$ and $\mathcal{D}^{;1;\beta}_{n_3}$ are valid and displayed as

$$|w_{n_2;1}(u,\epsilon) - w_{n_2;1}(q^{\beta}u,\epsilon)| \leq |q^{\beta} - 1| C_9(C_{10})^{n_2} n_2!|u|\exp(C_{11}|u|^k) \tag{157}$$

in a row with

$$|w_{n_3;1}(u,\epsilon) - w_{n_3;1}(q^{\beta}u,\epsilon)| \leq |q^{\beta}-1|C_9(C_{10})^{n_3}n_3!|u|\exp(C_{11}|u|^k) \tag{158}$$

as long as $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$. The bounds (157) together with (151) give rise to the next estimates for the integral piece of (155),

$$\begin{aligned}
&\Big| \int_0^1 \Big\{ \frac{w_{n_2;1}(u(1-s_1)^{1/k},\epsilon) - w_{n_2;1}(q^{\beta}u(1-s_1)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;1}(us_1^{1/k},\epsilon)}{n_3!} \Big\} \frac{1}{(1-s_1)s_1} ds_1 \Big| \\
&\leq \int_0^1 \{ |q^{\beta}-1|C_9(C_{10})^{n_2}|u|(1-s_1)^{1/k}\exp\left(C_{11}|u|^k(1-s_1)\right) \times C_9(C_{10})^{n_3}|u|s_1^{1/k}\exp(C_{11}|u|^k s_1) \} \\
&\quad \times \frac{1}{(1-s_1)s_1}ds_1 \leq |q^{\beta}-1|(C_9)^2(C_{10})^{n_2+n_3}|u|^2\exp(C_{11}|u|^k) \times \left(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}}s_1^{1-\frac{1}{k}}}ds_1 \right)
\end{aligned} \tag{159}$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$. In a similar manner, the bounds (158) along with the first inequality of (150) beget upper estimates for the integral piece of (156). Namely,

$$\begin{aligned}
&\Big| \int_0^1 \Big\{ \frac{w_{n_2;1}(q^{\beta}u(1-s_1)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;1}(us_1^{1/k},\epsilon) - w_{n_3;1}(q^{\beta}us_1^{1/k},\epsilon)}{n_3!} \Big\} \frac{1}{(1-s_1)s_1} ds_1 \Big| \\
&\qquad \leq \int_0^1 C_6(C_7)^{n_2}|q^{\beta}u|(1-s_1)^{1/k}\exp\left(C_8|q^{\beta}u|^k(1-s_1)\right) \\
&\qquad \times |q^{\beta}-1|C_9(C_{10})^{n_3}|u|s_1^{1/k}\exp(C_{11}|u|^k s_1)\frac{1}{(1-s_1)s_1}ds_1 \\
&\qquad \leq q^{\beta}|q^{\beta}-1|(C_9)^2(C_{10})^{n_2+n_3}|u|^2\exp(C_{11}|u|^k) \times \left(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}}s_1^{1-\frac{1}{k}}}ds_1 \right)
\end{aligned} \tag{160}$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$, under the additional assumption that $C_8 q^{\beta k} < C_{11}$, together with $C_6 < C_9$ and $C_7 < C_{10}$.

Besides, the assumption (6) and the condition given in Definition 2 1) imposed on the unbounded sector $\mathcal{U}$ yields a constant $M_{P,1,\beta} > 0$ with

$$\frac{|u|}{|P(k(q^{\beta}u)^k)|} \leq M_{P,1,\beta} \tag{161}$$

for all $u \in \mathcal{U} \cup \{0\}$.

By stacking up the above bounds (159), (160) and (161), we deduce from the decomposition (154) the next appropriate upper bounds for $\mathcal{C}^2_{n_2,n_3}(u,\epsilon)$,

$$\begin{aligned}
|\mathcal{C}^2_{n_2,n_3}(u,\epsilon)| &\leq |q^{\beta}-1|M_{P,1,\beta} \times [1+q^{\beta}] \\
&\times \left(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}}s_1^{1-\frac{1}{k}}}ds_1 \right)(C_9)^2(C_{10})^{n_2+n_3}|u|\exp(C_{11}|u|^k)
\end{aligned} \tag{162}$$

provided that $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

At last, the combination of the two bounds (153) and (162) applied to the splitting (146) triggers the awaited bounds (145). □

With the help of the above lemma 7, 8 and 9, we get from the recursion (114) with S first terms (115) along with the identities (122) and (123) the next bounds

$$\frac{|w_{n+S;1}(u,\epsilon) - w_{n+S;1}(q^{\beta}u,\epsilon)|}{n!} \leq \mathcal{Q}_1(u,n,\epsilon) + \mathcal{Q}_2(u,n,\epsilon) + \mathcal{Q}_3(u,n,\epsilon) \tag{163}$$

for

$$\mathcal{Q}_1(u,n,\epsilon) := \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} |\epsilon|^{\Delta_{\underline{l}}} \sum_{n_1+n_2=n} |c_{\underline{l},n_1}(\epsilon)| \\ \times \left[K_{3,l_1,P} + K_{1,l_1,P}\right] |q^{\beta}-1| C_9 (C_{10})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp(C_{11}|u|^k) \quad (164)$$

and

$$\mathcal{Q}_2(u,n,\epsilon) := \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} |\epsilon|^{\Delta_{\underline{l}}-l_0} \frac{k^{l_1}}{\Gamma(l_0/k)} \sum_{n_1+n_2=n} |c_{\underline{l},n_1}(\epsilon)| \times \left[L_{3,l_0,l_1,P} + L_{1,l_0,l_1,p}\right] \\ \times |q^{\beta}-1| C_9 \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (C_{10})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp(C_{11}|u|^k) \quad (165)$$

with

$$\mathcal{Q}_3(u,n,\epsilon) := \sum_{n_1+n_2+n_3=n} |d_{n_1}(\epsilon)| \left[M_{1,P} + M_{3,P}\right] \left(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}} s_1^{1-\frac{1}{k}}} ds_1 \right) \\ \times |q^{\beta}-1| (C_9)^2 (C_{10})^{n_2+n_3} |u| \exp(C_{11}|u|^k) \quad (166)$$

provided that $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

Bearing in mind the assumptions (8), (13) and the bounds (49) from Lemma 4, we deduce the accurate bounds

$$n!\mathcal{Q}_1(u,n,\epsilon) \leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \epsilon_0^{\Delta_{\underline{l}}} \Big(\sum_{\substack{0\leq n_1\leq n \\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_{10}^{-n_1} \Big) \left[K_{3,l_1,P} + K_{1,l_1,P}\right] \\ \times |q^{\beta}-1| C_9 (C_{10})^{n+S} (n+S)! |u| \exp(C_{11}|u|^k) \quad (167)$$

together with

$$n!\mathcal{Q}_2(u,n,\epsilon) \leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon_0^{\Delta_{\underline{l}}-l_0} \frac{k^{l_1}}{\Gamma(l_0/k)} \Big(\sum_{\substack{0\leq n_1\leq n \\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_{10}^{-n_1} \Big) \\ \times \left[L_{3,l_0,l_1,P} + L_{1,l_0,l_1,p}\right] \\ \times |q^{\beta}-1| C_9 \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) (C_{10})^{n+S} (n+S)! |u| \exp(C_{11}|u|^k) \quad (168)$$

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for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$. Eventually, based on the assumption (10), we reach

$$\begin{aligned} n!\mathcal{Q}_3(u,n,\epsilon) &\leq \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_d}} |d_{n_1}(\epsilon)| \times [M_{1,P} + M_{3,P}] \Big(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}} s_1^{1-\frac{1}{k}}} ds_1 \Big) \\ &\quad \times |q^{\beta} - 1| (C_9)^2 \times \Big(\sum_{n_2+n_3=n-n_1} (C_{10})^{n_2+n_3} n! \Big) \Big) |u| \exp(C_{11}|u|^k) \\ &\leq \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_d}} |d_{n_1}(\epsilon)| [M_{1,P} + M_{3,P}] \Big(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}} s_1^{1-\frac{1}{k}}} ds_1 \Big) \\ &\quad \times |q^{\beta} - 1| (C_9)^2 C_{10}^{n-n_1} (n-n_1+1) n! |u| \exp(C_{11}|u|^k) \\ &\leq \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_d}} \mathbf{d}_{n_1,\epsilon_0} C_{10}^{-n_1} \Big) \times [M_{1,P} + M_{3,P}] \Big(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}} s_1^{1-\frac{1}{k}}} ds_1 \Big) \\ &\quad \times |q^{\beta} - 1| (C_9)^2 (C_{10})^{n+S} (n+S)! |u| \exp(C_{11}|u|^k) \end{aligned} \tag{169}$$

whenever $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

Now, we take for granted that the radius $\epsilon_0 > 0$ and the constants $\mathbf{d}_{h,\epsilon_0} > 0$ given in (10) are suitably chosen near the origin in a manner that the next constraint

$$\begin{aligned} &\sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0=0} \epsilon_0^{\Delta_{\underline{l}}} \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_{10}^{-n_1} \Big) \times [K_{3,l_1,P} + K_{1,l_1,P}] \\ &+ \sum_{\underline{l}=(l_0,l_1,l_2,l_3) \in \mathcal{A}; l_0 \geq 1} \epsilon_0^{\Delta_{\underline{l}} - l_0} \frac{k^{l_1}}{\Gamma(l_0/k)} \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_{10}^{-n_1} \Big) [L_{3,l_0,l_1,P} + L_{1,l_0,l_1,p}] \\ &\quad \times \Big(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \Big) \\ &+ \Big(\sum_{\substack{0 \leq n_1 \leq n \\ n_1 \in I_d}} \mathbf{d}_{n_1,\epsilon_0} C_{10}^{-n_1} \Big) \times [M_{1,P} + M_{3,P}] \times \Big(\int_0^1 \frac{1}{(1-s_1)^{1-\frac{1}{k}} s_1^{1-\frac{1}{k}}} ds_1 \Big) C_9 \leq 1 \end{aligned} \tag{170}$$

holds. As a result, the gathering of the bounds (167), (168) and (169) under the contingency (170) yields the next estimates

$$|w_{n+S;1}(u,\epsilon) - w_{n+S;1}(q^{\beta}u,\epsilon)| \leq |q^{\beta} - 1| C_9 (C_{10})^{n+S} (n+S)! |u| \exp(C_{11}|u|^k) \tag{171}$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$. This means that the property $\mathscr{D}_{n+S}^{;1;\beta}$ is valid. Proposition 7 follows. □

7.3. Bounds for the difference of the sequences of Borel maps solving the recursions (33), (34) and (114), (115)

This subsection is devoted to the explanation of the next proposition.

Proposition 8. *Let $\underline{\mathcal{D}} = \{\underline{\mathcal{E}}, \mathcal{T}, \underline{\mathcal{U}}\}$ be an admissible set of sectors as described in Definition 2 of Section 5. Let $\mathcal{U}$ be one sector belonging to the family of unbounded sectors $\underline{\mathcal{U}}$. Let $q \in (1, q_0]$. Provided that $\epsilon_0 > 0$ and the constants $\mathbf{d}_{h,\epsilon_0} > 0$ are taken small enough, we consider*

- *the unique sequence of functions $(w_{n;q}(u,\epsilon))_{n \geq 0}$ that fulfills the recursion (33) with S first terms (34), built up in Proposition 3 and subjected to the bounds (61) on $\mathcal{U} \cup \{0\}$.*
- *the unique sequence of functions $(w_{n;1}(u,\epsilon))_{n \geq 0}$ which obeys the recursion (114) with S initial terms (115), crafted in Proposition 5 and submitted to the bounds (116) on $\mathcal{U} \cup \{0\}$.*

Then, for $\epsilon_0 > 0$ *and the constants* $\mathbf{d}_{h,\epsilon_0} > 0$ *close enough to 0, there exist constants* $C_{12}, C_{13} > 0$ *and* $C_{14} > 0$ *with*

$$|w_{n;q}(u,\epsilon) - w_{n;1}(u,\epsilon)| \leq (q-1)C_{12}(C_{13})^n n! |u| \exp\left(C_{14} q^{nM_1} |u|^k\right) \tag{172}$$

for all $u \in \mathcal{U} \cup \{0\}$ *and* $\epsilon \in D_{\epsilon_0}$*, for the constant* $M_1 > 0$ *defined in (14).*

Proof. We proceed again by means of the induction principle. We call $\mathcal{D}_n^{;q;1}$ the property (172) for any prescribed integer $n \geq 0$.

We first observe that the feature $\mathcal{D}_n^{;q;1}$ holds in a straight manner for $0 \leq n \leq S-1$, since in that case $w_{n;q}(u,\epsilon) = w_{n;1}(u,\epsilon) = P_n(u,\epsilon)$ for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

We set $n \geq 0$. We take for granted that the hypothesis $\mathcal{D}_h^{;q;1}$ holds true for all integers $h < n+S$ for some constants $C_{12}, C_{13} > 0$ and $C_{14} > 0$. In the remaining part of the proof, we will prove that the property $\mathcal{D}_{n+S}^{;q;1}$ is then effective. It will follow from the induction principle, that the property $\mathcal{D}_n^{;q;1}$ is true for all integers $n \geq 0$.

According to the recursions (33) and (114), we can express the difference $w_{n+S;q}(u,\epsilon) - w_{n+S;1}(u,\epsilon)$ in term of the lower indexed quantities $w_{h;q}(u,\epsilon)$ and $w_{h;1}(u,\epsilon)$ with $h < n+S$. Namely,

$$\begin{aligned}
\frac{w_{n+S;q}(u,\epsilon) - w_{n+S;1}(u,\epsilon)}{n!} &= \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \epsilon^{\Delta_{\underline{l}}} \frac{(ku^k)^{l_1}}{P(ku^k)} \\
&\times \Big(\sum_{n_1+n_2=n} c_{\underline{l},n_1}(\epsilon) [\frac{w_{n_2+l_2;q}(q^{l_3}u,\epsilon)}{n_2!} q^{l_3kl_1} - \frac{w_{n_2+l_2;1}(u,\epsilon)}{n_2!}] \Big) \\
&+ \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon^{\Delta_{\underline{l}}-l_0} \frac{u^k}{P(ku^k)} \frac{k^{l_1}}{\Gamma(l_0/k)} \\
&\times \Big(\sum_{n_1+n_2=n} c_{\underline{l},n_1}(\epsilon) \Big[q^{l_3kl_1} \int_0^{u^k} (u^k-s)^{\frac{l_0}{k}-1} s^{l_1} \frac{w_{n_2+l_2;q}(q^{l_3}s^{1/k},\epsilon)}{n_2!} \frac{ds}{s} \\
&- \int_0^{u^k} (u^k-s)^{\frac{l_0}{k}-1} s^{l_1} \frac{w_{n_2+l_2;1}(s^{1/k},\epsilon)}{n_2!} \frac{ds}{s} \Big] \Big) \\
&+ \frac{u^k}{P(ku^k)} \sum_{n_1+n_2+n_3=n} d_{n_1}(\epsilon) \times \Big[\int_0^{u^k} \frac{w_{n_2;q}((u^k-s)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;q}(s^{1/k},\epsilon)}{n_3!} \frac{1}{(u^k-s)s} ds \\
&- \int_0^{u^k} \frac{w_{n_2;1}((u^k-s)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;1}(s^{1/k},\epsilon)}{n_3!} \frac{1}{(u^k-s)s} ds \Big].
\end{aligned} \tag{173}$$

In the next lemma, bounds are provided for the front piece of (173),

$$\mathcal{A}_{n_2,l_2}^{;q;1}(u,\epsilon) := \frac{(ku^k)^{l_1}}{P(ku^k)} \times [\frac{w_{n_2+l_2;q}(q^{l_3}u,\epsilon)}{n_2!} q^{l_3kl_1} - \frac{w_{n_2+l_2;1}(u,\epsilon)}{n_2!}]. \tag{174}$$

Lemma 10. *The next inequality*

$$\begin{aligned}
|\mathcal{A}_{n_2,l_2}^{;q;1}(u,\epsilon)| &\leq K_{1,l_1,P} \times [q^{l_3kl_1}(q^{l_3} + \frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}] \\
&\times (q-1)C_{12}(C_{13})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \exp\left(C_{14} q^{(n+S)M_1} |u|^k\right)
\end{aligned} \tag{175}$$

holds for all $u \in \mathcal{U} \cup \{0\}$ *and* $\epsilon \in D_{\epsilon_0}$*, for the constant* $K_{1,l_1,P} > 0$ *introduced in (127) of Lemma 7.*

Proof. Based on the identity $ab - cd = (a-c)b + c(b-d)$, we reorganize the expression $\mathcal{A}^{;q;1}_{n_2,l_2}(u,\epsilon)$ as a sum of differences

$$\mathcal{A}^{;q;1}_{n_2,l_2}(u,\epsilon) = \frac{(ku^k)^{l_1}}{P(ku^k)} \times \Big[q^{l_3kl_1}\big(\frac{w_{n_2+l_2;q}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(u,\epsilon)}{n_2!}\big) + \frac{w_{n_2+l_2;1}(u,\epsilon)}{n_2!}(q^{l_3kl_1} - 1)\Big]. \quad (176)$$

1) We deal with the first part of the righthandside of (176). In order to enable the use of the induction hypothesis, an auxiliary term is inserted in the expression, namely

$$\begin{aligned} &|w_{n_2+l_2;q}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(u,\epsilon)| \\ &= |w_{n_2+l_2;q}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(q^{l_3}u,\epsilon) + w_{n_2+l_2;1}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(u,\epsilon)| \\ &\leq |w_{n_2+l_2;q}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(q^{l_3}u,\epsilon)| + |w_{n_2+l_2;1}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(u,\epsilon)|. \quad (177) \end{aligned}$$

a) Since $n_2 + l_2 < n + S$ owing to (13), by induction hypothesis, we acknowledge that the property $\mathcal{D}^{;q;1}_{n_2+l_2}$ holds. As a result, we reach

$$\begin{aligned} &|w_{n_2+l_2;q}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(q^{l_3}u,\epsilon)| \\ &\leq (q-1)C_{12}(C_{13})^{n_2+l_2}(n_2+l_2)!|q^{l_3}u| \exp\left(C_{14}q^{(n_2+l_2)M_1}|q^{l_3}u|^k\right) \end{aligned}$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$. Besides, the condition (14) with the assumption $q > 1$ implies that

$$q^{(n_2+l_2)M_1}q^{l_3k} \leq q^{(n+S)M_1} \quad (178)$$

provided that $n_2 \leq n$ for $\underline{l} = (l_0, l_1, l_2, l_3) \in \mathcal{A}$. As a result of these latter two inequalities, we arrive at

$$\begin{aligned} &|w_{n_2+l_2;q}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(q^{l_3}u,\epsilon)| \\ &\leq (q-1)C_{12}(C_{13})^{n_2+l_2}(n_2+l_2)!q^{l_3}|u| \exp\left(C_{14}q^{(n+S)M_1}|u|^k\right) \quad (179) \end{aligned}$$

whenever $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.
b) Taking heed of the auxiliary bounds (121) for the special case $\beta = l_3$ and $n = n_2 + l_2$, we know that

$$|w_{n_2+l_2;1}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(u,\epsilon)| \leq |q^{l_3} - 1|C_9(C_{10})^{n_2+l_2}(n_2+l_2)!|u| \exp(C_{11}|u|^k) \quad (180)$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

Eventually, under the choice that $C_{12} > C_9$, $C_{13} > C_{10}$ and $C_{14} > C_{11}$, it follows from (177) together with (179) and (180) that

$$\begin{aligned} &|w_{n_2+l_2;q}(q^{l_3}u,\epsilon) - w_{n_2+l_2;1}(u,\epsilon)| \\ &\leq \left[q^{l_3} + \frac{q^{l_3}-1}{q-1}\right](q-1)C_{12}(C_{13})^{n_2+l_2}(n_2+l_2)!|u| \exp\left(C_{14}q^{(n+S)M_1}|u|^k\right) \quad (181) \end{aligned}$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

2) We address the second piece of (176). Bearing in mind the bounds (116) from Proposition 5, we know that

$$|w_{n_2+l_2;1}(u,\epsilon)| \leq C_6(C_7)^{n_2+l_2}(n_2+l_2)!|u| \exp(C_8|u|^k)$$

for all $u \in \mathcal{U} \cup \{0\}$, all $\epsilon \in D_{\epsilon_0}$. Under the choice $C_{12} > C_6$, $C_{13} > C_7$ and $C_{14} > C_8$, we get

$$|w_{n_2+l_2;1}(u,\epsilon)| \leq C_{12}(C_{13})^{n_2+l_2}(n_2+l_2)!|u|\exp(C_{14}q^{(n+S)M_1}|u|^k) \tag{182}$$

provided that $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

At last, gathering the bounds (127) of Lemma 7, (181) and (182) spawns the awaited inequality (175). □

In the forthcoming lemma, bounds are exhibited for the piece

$$\mathcal{B}^{;q;1}_{n_2,l_2}(u,\epsilon) := \frac{u^k}{P(ku^k)} \times \Big[q^{l_3kl_1}\int_0^{u^k}(u^k-s)^{\frac{l_0}{k}-1}s^{l_1}\frac{w_{n_2+l_2;q}(q^{l_3}s^{1/k},\epsilon)}{n_2!}\frac{ds}{s} \\ -1\times\int_0^{u^k}(u^k-s)^{\frac{l_0}{k}-1}s^{l_1}\frac{w_{n_2+l_2;1}(s^{1/k},\epsilon)}{n_2!}\frac{ds}{s}\Big]. \tag{183}$$

Lemma 11. *The following inequality*

$$|\mathcal{B}^{;q;1}_{n_2,l_2}(u,\epsilon)| \leq L_{1,l_0,l_1,P} \times \big[q^{l_3kl_1}(q^{l_3}+\frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}\big] \\ \times (q-1)C_{12}(C_{13})^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!} \times \Big(\int_0^1(1-s_1)^{\frac{l_0}{k}-1}s^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1\Big) \\ \times |u|\exp(C_{14}q^{(n+S)M_1}|u|^k) \tag{184}$$

holds for all $u \in \mathcal{U} \cup \{0\}$, all $\epsilon \in D_{\epsilon_0}$, where $L_{1,l_0,l_1,P} > 0$ is the constant set up in (140) of Lemma 8.

Proof. Our favorite identity $ab - cd = (a-c)b + c(b-d)$ warrants $\mathcal{B}^{;q;1}_{n_2,l_2}(u,\epsilon)$ to be expressed as a sum of differences

$$\mathcal{B}^{;q;1}_{n_2,l_2}(u,\epsilon) = \frac{u^k}{P(ku^k)} \times \Big[q^{l_3kl_1}\int_0^{u^k}(u^k-s)^{\frac{l_0}{k}-1}s^{l_1} \\ \times\big(\frac{w_{n_2+l_2;q}(q^{l_3}s^{1/k},\epsilon)-w_{n_2+l_2;1}(s^{1/k},\epsilon)}{n_2!}\big)\frac{ds}{s} \\ +\Big(\int_0^{u^k}(u^k-s)^{\frac{l_0}{k}-1}s^{l_1}\frac{w_{n_2+l_2;1}(s^{1/k},\epsilon)}{n_2!}\frac{ds}{s}\Big)\times(q^{l_3kl_1}-1)\Big]. \tag{185}$$

We parametrize the first and the second integral appearing in the latter expression of $\mathcal{B}^{;q;1}_{n_2,l_2}(u,\epsilon)$ by means of $s = u^k s_1$ for $0 \leq s_1 \leq 1$. This yields the next expression

$$\mathcal{B}^{;q;1}_{n_2,l_2}(u,\epsilon) = \frac{u^{l_0+kl_1}}{P(ku^k)} \times \Big[q^{l_3kl_1}\int_0^1(1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1} \\ \times\big(\frac{w_{n_2+l_2;q}(q^{l_3}us_1^{1/k},\epsilon)-w_{n_2+l_2;1}(us_1^{1/k},\epsilon)}{n_2!}\big)\frac{ds_1}{s_1} \\ +\Big(\int_0^1(1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{w_{n_2+l_2;1}(us_1^{1/k},\epsilon)}{n_2!}\frac{ds_1}{s_1}\Big)\times(q^{l_3kl_1}-1)\Big]. \tag{186}$$

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1) We handle the first part of the righthandside of (186). As in Lemma 10, we insert an auxiliary term in order to apply the induction hypothesis. Indeed,

$$\begin{aligned}&|w_{n_2+l_2;q}(q^{l_3}us_1^{1/k},\epsilon) - w_{n_2+l_2;1}(us_1^{1/k},\epsilon)|\\ &\quad = |w_{n_2+l_2;q}(q^{l_3}us_1^{1/k},\epsilon) - w_{n_2+l_2;1}(q^{l_3}us_1^{1/k},\epsilon) + w_{n_2+l_2;1}(q^{l_3}us_1^{1/k},\epsilon) - w_{n_2+l_2;1}(us_1^{1/k},\epsilon)|\\ &\quad \leq |w_{n_2+l_2;q}(q^{l_3}us_1^{1/k},\epsilon) - w_{n_2+l_2;1}(q^{l_3}us_1^{1/k},\epsilon)| + |w_{n_2+l_2;1}(q^{l_3}us_1^{1/k},\epsilon) - w_{n_2+l_2;1}(us_1^{1/k},\epsilon)|. \quad (187)\end{aligned}$$

a) Since $n_2 + l_2 < n + S$ owing to (13), the induction hypothesis ensures that the property $\mathcal{D}^{;q;1}_{n_2+l_2}$ holds true. As a consequence, we reach

$$\begin{aligned}&|w_{n_2+l_2;q}(q^{l_3}us_1^{1/k},\epsilon) - w_{n_2+l_2;1}(q^{l_3}us_1^{1/k},\epsilon)|\\ &\qquad \leq (q-1)C_{12}(C_{13})^{n_2+l_2}(n_2+l_2)!|q^{l_3}us_1^{1/k}|\exp(C_{14}q^{(n_2+l_2)M_1}|q^{l_3}u|^k)\end{aligned}$$

for $u \in \mathcal{U} \cup \{0\}$, $0 \leq s_1 \leq 1$, $\epsilon \in D_{\epsilon_0}$ and from the inequality (178) provided that $n_2 \leq n$ and that $\underline{l} \in \mathcal{A}$, we deduce

$$\begin{aligned}&|w_{n_2+l_2;q}(q^{l_3}us_1^{1/k},\epsilon) - w_{n_2+l_2;1}(q^{l_3}us_1^{1/k},\epsilon)|\\ &\qquad \leq (q-1)C_{12}(C_{13})^{n_2+l_2}(n_2+l_2)!|q^{l_3}us_1^{1/k}|\exp(C_{14}q^{(n+S)M_1}|u|^k) \quad (188)\end{aligned}$$

whenever $u \in \mathcal{U} \cup \{0\}, 0 \leq s_1 \leq 1, \epsilon \in D_{\epsilon_0}$.

b) Taking into account the auxiliary bounds (121) specialized for $\beta = l_3$ and $n = n_2 + l_2$, we know that

$$\begin{aligned}&|w_{n_2+l_2;1}(q^{l_3}us_1^{1/k},\epsilon) - w_{n_2+l_2;1}(us_1^{1/k},\epsilon)|\\ &\qquad \leq |q^{l_3}-1|C_9(C_{10})^{n_2+l_2}(n_2+l_2)!|us_1^{1/k}|\exp(C_{11}|u|^k) \quad (189)\end{aligned}$$

for all $u \in \mathcal{U} \cup \{0\}, 0 \leq s_1 \leq 1$ and $\epsilon \in D_{\epsilon_0}$.

According to the choice $C_{12} > C_9$, $C_{13} > C_{10}$ and $C_{14} > C_{11}$ made above, the combination of (187), (188) and (189) gives rise to

$$\begin{aligned}&\Big|\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1} \times \Big(\frac{w_{n_2+l_2;q}(q^{l_3}us_1^{1/k},\epsilon) - w_{n_2+l_2;1}(us_1^{1/k},\epsilon)}{n_2!}\Big)\frac{ds_1}{s_1}\Big|\\ &\qquad \leq \Big[q^{l_3} + \frac{q^{l_3}-1}{q-1}\Big](q-1)C_{12}(C_{13})^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!}\\ &\qquad\quad \times |u|\Big(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s_1^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1\Big)\exp(C_{14}q^{(n+S)M_1}|u|^k) \quad (190)\end{aligned}$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

2) We focus on the second part of the righthandside of (186). Keeping in mind the bounds (116) from Proposition 5, we notice that

$$|w_{n_2+l_2;1}(us_1^{1/k},\epsilon)| \leq C_6(C_7)^{n_2+l_2}(n_2+l_2)!|us_1^{1/k}|\exp(C_8|u|^k) \quad (191)$$

for all $u \in \mathcal{U} \cup \{0\}$, $0 \leq s_1 \leq 1$ and $\epsilon \in D_{\epsilon_0}$. Under the choice made overhead, $C_{12} > C_6$, $C_{13} > C_7$ and $C_{14} > C_8$, we obtain

$$\Big| \int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{w_{n_2+l_2;1}(us_1^{1/k},\epsilon)}{n_2!} \frac{ds_1}{s_1} \Big| \leq C_{12}(C_{13})^{n_2+l_2} \frac{(n_2+l_2)!}{n_2!} |u| \\ \times \Big(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \Big) \exp(C_{14} q^{(n+S)M_1} |u|^k) \quad (192)$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

Eventually, the collection of the bounds (140) of Lemma 8, (190) and (192) begets the expected inequality (184). □

In the next lemma, bounds are displayed for the nonlinear term of (173),

$$\mathcal{C}_{n_2,n_3}^{;q;1}(u,\epsilon) := \frac{u^k}{P(ku^k)} \times \Big[\int_0^{u^k} \frac{w_{n_2;q}((u^k-s)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;q}(s^{1/k},\epsilon)}{n_3!} \frac{1}{(u^k-s)s} ds \\ - \int_0^{u^k} \frac{w_{n_2;1}((u^k-s)^{1/k},\epsilon)}{n_2!} \frac{w_{n_3;1}(s^{1/k},\epsilon)}{n_3!} \frac{1}{(u^k-s)s} ds \Big]. \quad (193)$$

Lemma 12. *The next inequality*

$$|\mathcal{C}_{n_2,n_3}^{;q;1}(u,\epsilon)| \leq \mathbb{M}_{P,k} [C_3 + C_6] (q-1) C_{12} (C_{13})^{n_2+n_3} |u| \exp(C_{14} q^{(n+S)M_1} |u|^k) \quad (194)$$

holds for all $u \in \mathcal{U} \cup \{0\}$, $\epsilon \in D_{\epsilon_0}$, where the constant $\mathbb{M}_{P,k} > 0$ appears in (75), $C_3 > 0$ stems from (61) and $C_6 > 0$ springs from (116).

Proof. The identity $ab - cd = (a-c)b + c(b-d)$ allows to express (193) as a sum of differences

$$\mathcal{C}_{n_2,n_3}^{;q;1}(u,\epsilon) = \frac{u^k}{P(ku^k)} \times \Big[\int_0^{u^k} \Big(\frac{w_{n_2;q}((u^k-s)^{1/k},\epsilon) - w_{n_2;1}((u^k-s)^{1/k},\epsilon)}{n_2!} \Big) \\ \times \frac{w_{n_3;q}(s^{1/k},\epsilon)}{n_3!} \frac{1}{(u^k-s)s} ds + \int_0^{u^k} \frac{w_{n_2;1}((u^k-s)^{1/k},\epsilon)}{n_2!} \times \Big(\frac{w_{n_3;q}(s^{1/k},\epsilon) - w_{n_3;1}(s^{1/k},\epsilon)}{n_3!} \Big) \\ \times \frac{1}{(u^k-s)s} ds \Big]. \quad (195)$$

The first and second integrals entailed in the expression (195) are parametrized through $s = u^k s_1$ for $0 \leq s_1 \leq 1$, leading to the next equality

$$\mathcal{C}_{n_2,n_3}^{;q;1}(u,\epsilon) = \frac{1}{P(ku^k)} \times \Big[\int_0^1 \Big(\frac{w_{n_2;q}(u(1-s_1)^{1/k},\epsilon) - w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)}{n_2!} \Big) \\ \times \frac{w_{n_3;q}(us_1^{1/k},\epsilon)}{n_3!} \times \frac{1}{(1-s_1)s_1} ds_1 + \int_0^1 \frac{w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)}{n_2!} \times \Big(\frac{w_{n_3;q}(us_1^{1/k},\epsilon) - w_{n_3;1}(us_1^{1/k},\epsilon)}{n_3!} \Big) \\ \times \frac{1}{(1-s_1)s_1} ds_1 \Big]. \quad (196)$$

1) a) On the ground of the induction hypothesis $\mathcal{D}_{n_2}^{;q;1}$ for $n_2 \leq n < n+S$, we already notice that

$$|w_{n_2;q}(u(1-s_1)^{1/k},\epsilon) - w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)| \\ \leq (q-1) C_{12} (C_{13})^{n_2} n_2! |u| (1-s_1)^{1/k} \exp(C_{14} q^{n_2 M_1} |u|^k (1-s_1)) \quad (197)$$

provided that $u \in \mathcal{U} \cup \{0\}$, $0 \leq s_1 \leq 1$ and $\epsilon \in D_{\epsilon_0}$.

b) In view of the bounds (61), we observe that

$$|w_{n_3;q}(us_1^{1/k},\epsilon)| \leq C_3(C_4)^{n_3}n_3!|u|s_1^{1/k}\exp(C_5q^{n_3M_1}|u|^ks_1)$$
$$\leq C_3(C_{13})^{n_3}n_3!|u|s_1^{1/k}\exp(C_{14}q^{n_3M_1}|u|^ks_1) \quad (198)$$

whenever $u \in \mathcal{U} \cup \{0\}$, $0 \leq s_1 \leq 1$ and $\epsilon \in D_{\epsilon_0}$, assuming that $C_{13} > C_4$ and $C_{14} > C_5$.

2) a) Based on the induction hypothesis $\mathcal{D}_{n_3}^{;q;1}$ for $n_3 \leq n < n+S$, we know that

$$|w_{n_3;q}(us_1^{1/k},\epsilon) - w_{n_3;1}(us_1^{1/k},\epsilon)| \leq (q-1)C_{12}(C_{13})^{n_3}n_3!|u|s_1^{1/k}\exp(C_{14}q^{n_3M_1}|u|^ks_1) \quad (199)$$

as long as $u \in \mathcal{U} \cup \{0\}$, $0 \leq s_1 \leq 1$ and $\epsilon \in D_{\epsilon_0}$.

b) By dint of the bounds (116), we see that

$$|w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)| \leq C_6(C_7)^{n_2}n_2!|u|(1-s_1)^{1/k}\exp\left(C_8|u|^k(1-s_1)\right)$$
$$\leq C_6(C_{13})^{n_2}n_2!|u|(1-s_1)^{1/k}\exp\left(C_{14}q^{n_2M_1}|u|^k(1-s_1)\right) \quad (200)$$

holds for $u \in \mathcal{U} \cup \{0\}$, $0 \leq s_1 \leq 1$ and $\epsilon \in D_{\epsilon_0}$, provided that we choose $C_{13} > C_7$, $C_{14} > C_8$.

Stacking up the latter bounds (197), (198), (199) and (200), we reach

$$\Big| \int_0^1 \Big(\frac{w_{n_2;q}(u(1-s_1)^{1/k},\epsilon) - w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)}{n_2!} \Big)$$
$$\times \frac{w_{n_3;q}(us_1^{1/k},\epsilon)}{n_3!} \times \frac{1}{(1-s_1)s_1} ds_1 \Big|$$
$$\leq C_3(q-1)C_{12}(C_{13})^{n_2+n_3}|u|^2$$
$$\times \Big(\int_0^1 (1-s_1)^{1/k}s_1^{1/k}\frac{1}{(1-s_1)s_1}\exp(C_{14}q^{(n+S)M_1}|u|^k(1-s_1+s_1))ds_1 \Big) \quad (201)$$

together with

$$\Big| \int_0^1 \frac{w_{n_2;1}(u(1-s_1)^{1/k},\epsilon)}{n_2!} \times \Big(\frac{w_{n_3;q}(us_1^{1/k},\epsilon) - w_{n_3;1}(us_1^{1/k},\epsilon)}{n_3!} \Big)$$
$$\times \frac{1}{(1-s_1)s_1} ds_1 \Big|$$
$$\leq C_6(q-1)C_{12}(C_{13})^{n_2+n_3}|u|^2$$
$$\times \Big(\int_0^1 (1-s_1)^{1/k}s_1^{1/k}\frac{1}{(1-s_1)s_1}\exp(C_{14}q^{(n+S)M_1}|u|^k(s_1+1-s_1))ds_1 \Big) \quad (202)$$

for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

At last, gathering (75), (201) and (202) gives rise to (194). □

Based on the above lemma 10, 11 and 12, we obtain from the recursion (173) the next bounds

$$\frac{w_{n+S;q}(u,\epsilon) - w_{n+S;1}(u,\epsilon)}{n!} \leq \mathcal{R}_1(u,n,\epsilon) + \mathcal{R}_2(u,n,\epsilon) + \mathcal{R}_3(u,n,\epsilon) \quad (203)$$

where

$$\begin{aligned}\mathcal{R}_1(u,n,\epsilon) := \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} &|\epsilon|^{\Delta_{\underline{l}}} \times \Big[\sum_{n_1+n_2=n} |c_{\underline{l},n_1}(\epsilon)| K_{1,l_1,P} \\ &\times [q^{l_3kl_1}(q^{l_3} + \frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}] \\ &\times (q-1)C_{12}(C_{13})^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!}|u| \exp\left(C_{14}q^{(n+S)M_1}|u|^k\right)\Big] \end{aligned} \quad (204)$$

along with

$$\begin{aligned}\mathcal{R}_2(u,n,\epsilon) := \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} &|\epsilon|^{\Delta_{\underline{l}}-l_0}\frac{k^{l_1}}{\Gamma(l_0/k)} \times \Big[\sum_{n_1+n_2=n} |c_{\underline{l},n_1}(\epsilon)| L_{1,l_0,l_1,P} \\ &\times [q^{l_3kl_1}(q^{l_3} + \frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}] \\ &\times (q-1)C_{12}(C_{13})^{n_2+l_2}\frac{(n_2+l_2)!}{n_2!} \times \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1}s^{l_1}\frac{1}{s_1^{1-\frac{1}{k}}}ds_1\right) \\ &\times |u|\exp(C_{14}q^{(n+S)M_1}|u|^k)\Big] \end{aligned} \quad (205)$$

and

$$\begin{aligned}\mathcal{R}_3(u,n,\epsilon) := \sum_{n_1+n_2+n_3=n} &|d_{n_1}(\epsilon)|\mathbb{M}_{P,k}[C_3+C_6] \\ &\times (q-1)C_{12}(C_{13})^{n_2+n_3}|u|\exp(C_{14}q^{(n+S)M_1}|u|^k) \end{aligned} \quad (206)$$

provided that $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$.

According to (8), (13) and (49) from Lemma 4, we obtain the next bounds for the quantities $\mathcal{R}_1$ and $\mathcal{R}_2$. Namely,

$$\begin{aligned} n!\mathcal{R}_1(u,n,\epsilon) \leq \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} &\epsilon_0^{\Delta_{\underline{l}}}\Big[\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} |c_{\underline{l},n_1}(\epsilon)|K_{1,l_1,P} \\ &\times [q^{l_3kl_1}(q^{l_3} + \frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}] \\ &\times (q-1)C_{12}(C_{13})^{n-n_1+S}(n+S)!|u|\exp\left(C_{14}q^{(n+S)M_1}|u|^k\right)\Big] \\ &\leq (q-1)C_{12}(C_{13})^{n+S}(n+S)!|u|\exp\left(C_{14}q^{(n+S)M_1}|u|^k\right) \\ \times \Big[\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} &\epsilon_0^{\Delta_{\underline{l}}}\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0}C_{13}^{-n_1}K_{1,l_1,P} \times [q^{l_3kl_1}(q^{l_3} + \frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}]\Big] \end{aligned} \quad (207)$$

together with

$$
\begin{aligned}
n!\mathcal{R}_2(u,n,\epsilon) := & \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon_0^{\Delta_{\underline{l}}-l_0} \frac{k^{l_1}}{\Gamma(l_0/k)} \times \Big[\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} |c_{\underline{l},n_1}(\epsilon)| L_{1,l_0,l_1,P} \\
& \times [q^{l_3kl_1}(q^{l_3}+\frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}] \\
& \times (q-1)C_{12}(C_{13})^{n-n_1+S}(n+S)! \times \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) \times |u| \exp(C_{14}q^{(n+S)M_1}|u|^k) \Big] \\
& \leq (q-1)C_{12}(C_{13})^{n+S}(n+S)!|u| \exp(C_{14}q^{(n+S)M_1}|u|^k) \\
& \times \Big[\sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon_0^{\Delta_{\underline{l}}-l_0} \frac{k^{l_1}}{\Gamma(l_0/k)} \times \sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_{13}^{-n_1} L_{1,l_0,l_1,P} \\
& \times [q^{l_3kl_1}(q^{l_3}+\frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}] \times \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) \Big] \qquad (208)
\end{aligned}
$$

for all $u \in \mathcal{U}\cup\{0\}$ and $\epsilon \in D_{\epsilon_0}$.

Furthermore, owing to (10), we reach the next bounds for $\mathcal{R}_3$. Indeed,

$$
\begin{aligned}
n!\mathcal{R}_3(u,n,\epsilon) \leq & \sum_{\substack{0\leq n_1\leq n\\ n_1\in I_d}} |d_{n_1}(\epsilon)| \mathbb{M}_{P,k}[C_3+C_6](q-1)C_{12} \\
& \times \Big(\sum_{n_2+n_3=n-n_1} C_{13}^{n_2+n_3} n! \Big) |u| \exp(C_{14}q^{(n+S)M_1}|u|^k) \\
& \leq \sum_{\substack{0\leq n_1\leq n\\ n_1\in I_d}} |d_{n_1}(\epsilon)| \mathbb{M}_{P,k}[C_3+C_6](q-1)C_{12}C_{13}^{n-n_1}(n-n_1+1)n! \times |u| \exp(C_{14}q^{(n+S)M_1}|u|^k) \\
& \leq (q-1)C_{12}(C_{13})^{n+S}(n+S)!|u| \exp(C_{14}q^{(n+S)M_1}|u|^k) \times \Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_d}} \mathbf{d}_{n_1,\epsilon_0} C_{13}^{-n_1} \Big) \mathbb{M}_{P,k}[C_3+C_6] \qquad (209)
\end{aligned}
$$

as long as $u \in \mathcal{U}\cup\{0\}$ and $\epsilon \in D_{\epsilon_0}$.

At last, we take for granted that the radius $\epsilon_0 > 0$ and the constants $\mathbf{d}_{h,\epsilon_0} > 0$ given in (10) are close enough to 0 in a way that the next condition

$$
\begin{aligned}
& \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0=0} \epsilon_0^{\Delta_{\underline{l}}} \sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_{13}^{-n_1} K_{1,l_1,P} \times [q^{l_3kl_1}(q^{l_3}+\frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}] \\
& + \sum_{\underline{l}=(l_0,l_1,l_2,l_3)\in\mathcal{A};l_0\geq 1} \epsilon_0^{\Delta_{\underline{l}}-l_0} \frac{k^{l_1}}{\Gamma(l_0/k)} \times \sum_{\substack{0\leq n_1\leq n\\ n_1\in I_{\underline{l}}}} \mathbf{c}_{\underline{l},n_1,\epsilon_0} C_{13}^{-n_1} L_{1,l_0,l_1,P} \\
& \times [q^{l_3kl_1}(q^{l_3}+\frac{q^{l_3}-1}{q-1}) + \frac{q^{l_3kl_1}-1}{q-1}] \times \left(\int_0^1 (1-s_1)^{\frac{l_0}{k}-1} s_1^{l_1} \frac{1}{s_1^{1-\frac{1}{k}}} ds_1 \right) \\
& + \Big(\sum_{\substack{0\leq n_1\leq n\\ n_1\in I_d}} \mathbf{d}_{n_1,\epsilon_0} C_{13}^{-n_1} \Big) \mathbb{M}_{P,k}[C_3+C_6] \leq 1 \qquad (210)
\end{aligned}
$$

holds.

As a consequence of the bounds (203), (207), (208) and (209) under the assumption (210), the next estimates

$$|w_{n+S;q}(u,\epsilon) - w_{n+S;1}(u,\epsilon)| \leq (q-1)C_{12}(C_{13})^{n+S}(n+S)!|u| \exp(C_{14}q^{(n+S)M_1}|u|^k) \tag{211}$$

holds for all $u \in \mathcal{U} \cup \{0\}$ and $\epsilon \in D_{\epsilon_0}$. Hence, the property $\mathcal{D}_{n+S}^{;q;1}$ is granted. Proposition 8 ensues. □

7.4. Confluence for the Banach valued holomorphic solutions of the Cauchy problem (16), (17) as $q \rightarrow 1$.

We establish the third and last main result of the work.

Theorem 3. *Let* $\underline{\mathcal{D}} = \{\underline{\mathcal{E}}, \mathcal{T}, \underline{\mathcal{U}}\}$ *be an admissible set of sectors as chosen in Definition 2. Let* $\mathcal{U}$ *be one sector belonging to the family of unbounded sectors* $\underline{\mathcal{U}}$*. Let* $\mathcal{E}$ *be the one bounded sector from the set* $\underline{\mathcal{E}}$ *that is related to* $\mathcal{U}$ *under the requirement of Definition 2 2). For* $\epsilon_0 > 0$ *and the constants* $\mathbf{d}_{h,\epsilon_0} > 0$ *introduced in (10) taken small enough, let* $q \in (1, q_0]$.

- *We denote* $u_{;q}(t,z,\epsilon)$ *the bounded holomorphic map* $\epsilon \mapsto u_{;q}(t,z,\epsilon)$ *from* $\mathcal{E}$ *into* $\mathcal{O}_{(\mathcal{D}_n)_{n\geq 0};R_1}$ *built up in Theorem 1 which obeys (16), (17), for* $\mathcal{D}_n = \mathcal{T} \cap D_{\tilde{R}_0/q^{n\Delta}}$ *where* $\tilde{R}_0 > 0$ *and* $R_1 > 0$ *are well chosen radius and* $\Delta > 0$ *is given in (36).*
- *We consider the bounded holomorphic solution* $u_{;1}(t,z,\epsilon)$ *to the limit Cauchy problem (109), (110) on the domain* $(\mathcal{T} \cap D_{\check{R}_0}) \times D_{\check{R}_1} \times \mathcal{E}$ *for well chosen radius* $\check{R}_0, \check{R}_1 > 0$*, constructed in Proposition 6.*

Then, one can find a constant $C > 0$ *(independent of* $q \in (1, q_0]$*) such that*

$$\sup_{\epsilon \in \mathcal{E}} ||u_{;q}(t,z,\epsilon) - u_{;1}(t,z,\epsilon)||_{(\hat{\mathcal{D}}_n)_{n\geq 0};\hat{R}_1} \leq C(q-1) \tag{212}$$

for all $q \in (1, q_0]$*, where* $\hat{\mathcal{D}}_n = \mathcal{T} \cap D_{\hat{R}_0/q^{n\Delta}}$*, for some radius* $\hat{R}_0 > 0$, $\hat{R}_1 > 0$ *that are properly chosen and independent of* q.

In particular, we observe that the solution $u_{;q}(t,z,\epsilon)$ *to (16), (17) merges uniformly in* ϵ *on* $\mathcal{E}$ *for the norm* $||.||_{(\hat{\mathcal{D}}_n)_{n\geq 0};\hat{R}_1}$ *(that rely on* q*) to the solution* $u_{;1}(t,z,\epsilon)$ *of (109), (110) as* q *tends to 1.*

Proof. According to the expansions (85) and (111), we can express both expressions $u_{;q}(t,z,\epsilon)$ and $u_{;1}(t,z,\epsilon)$ as formal power series in z with Laplace transforms as coefficients. Namely,

$$u_{;q}(t,z,\epsilon) = \sum_{n\geq 0} u_{n;q}(t,\epsilon)\frac{z^n}{n!} \quad , \quad u_{;1}(t,z,\epsilon) = \sum_{n\geq 0} u_{n;1}(t,\epsilon)\frac{z^n}{n!} \tag{213}$$

where

$$u_{n;q}(t,\epsilon) = k\int_{L_\gamma} w_{n;q}(u,\epsilon)\exp\left(-(\frac{u}{\epsilon t})^k\right)\frac{du}{u},$$
$$u_{n;1}(t,\epsilon) = k\int_{L_\gamma} w_{n;1}(u,\epsilon)\exp\left(-(\frac{u}{\epsilon t})^k\right)\frac{du}{u} \tag{214}$$

along a halfline $L_\gamma = [0,+\infty)e^{\sqrt{-1}\gamma} \subset \mathcal{U} \cup \{0\}$, for a direction $\gamma \in \mathbb{R}$ (that might rely on ϵ and t) submitted to the condition

$$\cos\left(k(\gamma - \arg(\epsilon t))\right) > \delta \tag{215}$$

for some $\delta > 0$, provided that $\epsilon \in \mathcal{E}$ and $t \in \mathcal{T}$.

Owing to the bounds (172) reached in Proposition 8, for each $n \geq 0$ and fixed $u = re^{\sqrt{-1}\gamma} \in L_{\gamma}$, $t \in \mathcal{T}$ and $\epsilon \in \mathcal{E}$, we observe that

$$\begin{aligned} |w_{n;q}(u,\epsilon) - w_{n;1}(u,\epsilon)||\exp\big(-(\frac{u}{\epsilon t})^{k}\big)||\frac{1}{u}| &\leq (q-1)C_{12}(C_{13})^{n}n!\exp\big(C_{14}q^{nM_1}r^{k}\big)\\ &\times \exp\big(-(\frac{r}{|\epsilon t|})^{k}\cos\big(k(\gamma - \arg(\epsilon t))\big)\big)\\ &\leq (q-1)C_{12}(C_{13})^{n}n!\exp\big(r^{k}[C_{14}q^{nM_1} - \frac{\delta}{(\epsilon_0|t|)^{k}}]\big). \end{aligned} \tag{216}$$

In the next lemma some technical bounds are exhibited. Its statement and proof are similar to those of Lemma 5.

Lemma 13. *There exist two constants $\hat{R}_0 > 0$ and $B_{14} > 0$ (both independent of q) such that*

$$C_{14}q^{nM_1} - \frac{\delta}{(\epsilon_0|t|)^{k}} \leq -B_{14} \tag{217}$$

provided that $t \in \mathbb{C}$ subjected to $|t| \leq \hat{R}_0/q^{n\Delta}$.

Proof. A direct computation reveals that the inequality (217) is tantamount to

$$|t| \leq \frac{\delta^{1/k}/\epsilon_0}{(C_{14}q^{nM_1} + B_{14})^{1/k}}. \tag{218}$$

Bearing in mind the definition of Δ given in (36), for prescribed $B_{14} > 0$ (chosen independently of q), we introduce the constant $\hat{R}_0 > 0$ defined by

$$\hat{R}_0 = \min_{n \geq 0} \frac{(\delta^{1/k}/\epsilon_0)q^{n\Delta}}{(C_{14}q^{nM_1} + B_{14})^{1/k}} = \min_{n \geq 0} \frac{\delta^{1/k}/\epsilon_0}{(C_{14} + B_{14}q^{-nM_1})^{1/k}} = \frac{\delta^{1/k}/\epsilon_0}{(C_{14} + B_{14})^{1/k}} \tag{219}$$

which turns out not to rely on q. By construction, we observe that if $|t| \leq \hat{R}_0/q^{n\Delta}$, then the inequality (218) holds. □

As a result of (216) and (217), we obtain

$$|w_{n;q}(u,\epsilon) - w_{n;1}(u,\epsilon)||\exp\big(-(\frac{u}{\epsilon t})^{k}\big)||\frac{1}{u}| \leq (q-1)C_{12}(C_{13})^{n}n!\exp(-B_{14}r^{k}) \tag{220}$$

provided that $u = re^{\sqrt{-1}\gamma} \in L_{\gamma}$, $\epsilon \in \mathcal{E}$ and $t \in \hat{\mathcal{D}}_n = \mathcal{T} \cap D_{\hat{R}_0/q^{n\Delta}}$.

In conclusion, as a consequence of (220), we arrive at

$$\begin{aligned} ||u_{;q}(t,z,\epsilon) - u_{;1}(t,z,\epsilon)||_{(\hat{\mathcal{D}}_n)_{n\geq 0};\hat{R}_1} &:= \sum_{n \geq 0} \sup_{t \in \hat{\mathcal{D}}_n} |u_{n;q}(t,\epsilon) - u_{n;1}(t,\epsilon)|\frac{\hat{R}_1^{n}}{n!}\\ &\leq \sum_{n \geq 0}(q-1)C_{12}(C_{13})^{n}k\Big(\int_0^{+\infty}\exp(-B_{14}r^{k})dr\Big)\hat{R}_1^{n}\\ &\leq (q-1)C_{12}k\Big(\int_0^{+\infty}\exp(-B_{14}r^{k})dr\Big)\frac{1}{1 - C_{13}\hat{R}_1} \end{aligned} \tag{221}$$

for all $\epsilon \in \mathcal{E}$, provided that $\hat{R}_1 > 0$ is chosen such that $C_{13}\hat{R}_1 < 1$ and unrelated to q. This achieves the expected bounds (212) for a constant $C > 0$ independent of q. □

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