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Article

# Green Measures for a Class of non-Markov Processes

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**Abstract:** In this paper, we investigate the Green measure for a class of non-Gaussian processes in  $\mathbb{R}^d$ . These measures are associated with the family of generalized grey Brownian motions  $B_{\beta,\alpha}$ ,  $0 < \beta \leq 1$ ,  $0 < \alpha \leq 2$ . This family includes both fractional Brownian motion, Brownian motion, and other non-Gaussian processes. We show that the perpetual integral exists with probability 1 for  $d\alpha > 2$  and  $1 < \alpha \leq 2$ . The Green measure then generalizes those measures of all these classes.

**Keywords:** Fractional Brownian motion; generalized grey Brownian motion; Green measure; subordination

## 1. Introduction

The goal of this paper (see Theorem 1 and Corollary 1 below) is to prove the existence of the Green measure for a class of non-Gaussian processes in  $\mathbb{R}^d$ , called generalized grey Brownian motion (ggBm for short). We denote this family of processes by  $B_{\beta,\alpha}$  with parameters  $0 < \beta \leq 1$  and  $0 < \alpha \leq 2$ . More precisely, for a Borel function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , the potential of  $f$  (see [1,2] for details) is defined as

$$V_{\beta,\alpha}(f, x) = \int_0^\infty \mathbb{E}[f(x + B_{\beta,\alpha}(t))] dt, \quad x \in \mathbb{R}^d. \quad (1)$$

We would like to investigate the class of functions  $f$  for which the potential of  $f$  has the representation

$$V_{\beta,\alpha}(f, x) = \int_{\mathbb{R}^d} f(y) \mathcal{G}_{\beta,\alpha}(x, dy), \quad (2)$$

where  $\mathcal{G}(x, \cdot) := \mathcal{G}_{\beta,\alpha}(x, \cdot)$  is a Radon measure on  $\mathbb{R}^d$  called Green measure corresponding to the ggBm  $B_{\beta,\alpha}$ , see Definition 2 below. First, we establish the existence of the perpetual integral (cf. Theorem 1)

$$\int_0^\infty f(x + B_{\beta,\alpha}(t)) dt$$

with probability 1. This leads to an explicit representation of the Green measure for ggBm, namely (cf. Corollary 1)

$$\mathcal{G}_{\beta,\alpha}(x, dy) = \frac{D}{|x - y|^{d-2/\alpha}} dy, \quad d\alpha > 2, \quad 1 < \alpha \leq 2,$$

where  $D$  is a constant that depends on  $\beta, \alpha$ , and the dimension  $d$ ; see Theorem 1 for the explicit expression. Note that as  $d\alpha > 2$  and  $1 < \alpha \leq 2$ , the Green measure  $\mathcal{G}_{\beta,\alpha}(x, \cdot)$  exists for  $d \geq 2$ , since  $d > 2/\alpha \in [1, 2)$ . The Brownian case ( $\alpha = 1$ ) is covered only for  $d \geq 3$ .

We emphasize that the existence of the Green measure for a given process  $X$  is not always guaranteed. As an example, the  $d$ -dimensional Brownian motion (Bm) starting at  $x \in \mathbb{R}^d$  has a density given by  $p_t(x, y) = (2\pi t)^{-d/2} \exp(-|x - y|^2/(2t))$ ,  $y \in \mathbb{R}^d$ . It is not difficult to see that  $\int_0^\infty p_t(x, y) dt$  does not exist for  $d = 1, 2$ . This implies the non-existence of the Green measure of Bm for  $d = 1, 2$ . On the other hand, for  $d \geq 3$ , the Green measure of Bm on  $\mathbb{R}^d$  exists and is given by  $\mathcal{G}(x, dy) = C(d)|x - y|^{2-d} dy$ , where  $C(d)$  is a constant depending on the dimension  $d$ ; see [3] and the references therein for more details. In a two-dimensional space, the Green measure of ggBm is determined by the parameter  $\alpha$ . The Green measure of ggBm for  $d = 1$  requires further analysis (for Bm see [4], Ch. 4) which we will postpone for a future paper.

The paper is organized as follows. In Section 2 we recall the definition and main properties of ggBm that will be needed later. In Section 3 we show the existence of the perpetual integral with probability 1, which leads to the explicit formula for the Green measure for ggBm. In Section 4, we discuss the results obtained, connect them with other topics, and draw conclusions.

## 2. Generalized Grey Brownian Motion

We recall the class of non-Gaussian processes, called the generalized grey Brownian motion, which we study below. This class of processes was first introduced by Schneider [5,6], and was generalized by Mura et al. (see [7,8]) as a stochastic model for slow/fast anomalous diffusion described by the time fractional diffusion equation.

### 2.1. Definition and Properties

For  $0 < \beta \leq 1$  the Mittag-Leffler (entire) function  $E_\beta$  is defined by the Taylor series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}, \quad (3)$$

where

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}, \operatorname{Re}(z) \geq 0$$

is the Euler gamma function.

The  $M$ -Wright function is a special case of the class of Wright functions  $W_{\lambda,\mu}$ ,  $\lambda > -1$ ,  $\mu \in \mathbb{C}$  via

$$M_\beta(z) := W_{-\beta, 1-\beta}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\beta n + 1 - \beta)}.$$

The special choice  $\beta = 1/2$  yields the Gaussian density on  $[0, \infty)$

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right). \quad (4)$$

The Mittag-Leffler function  $E_\beta$  is the Laplace transform of the  $M$ -Wright function, that is,

$$E_\beta(-s) = \int_0^{\infty} e^{-s\tau} M_\beta(\tau) d\tau. \quad (5)$$

The generalized moments of the density  $M_\beta$  of order  $\delta > -1$  are finite and are given (see [7]) by

$$\int_0^{\infty} \tau^\delta M_\beta(\tau) d\tau = \frac{\Gamma(\delta + 1)}{\Gamma(\beta\delta + 1)}. \quad (6)$$

**Definition 1.** Let  $0 < \beta \leq 1$  and  $0 < \alpha \leq 2$  be given. A  $d$ -dimensional continuous stochastic process  $B_{\beta,\alpha} = \{B_{\beta,\alpha}(t), t \geq 0\}$  starting at  $0 \in \mathbb{R}^d$  and defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , is a ggBm in  $\mathbb{R}^d$  (see [8] for  $d = 1$ ) if:

1.  $\mathbb{P}(B_{\beta,\alpha}(0) = 0) = 1$ , that is,  $B_{\beta,\alpha}$  starts at zero  $\mathbb{P}$ -almost surely ( $\mathbb{P}$ -a.s.).
2. Any collection  $\{B_{\beta,\alpha}(t_1), \dots, B_{\beta,\alpha}(t_n)\}$  with  $0 \leq t_1 < t_2 < \dots < t_n < \infty$  has a characteristic function given, for any  $\theta = (\theta_1, \dots, \theta_n) \in (\mathbb{R}^d)^n$  with  $\theta_k = (\theta_{k,1}, \dots, \theta_{k,d})$ ,  $k = 1, \dots, n$ , by

$$\mathbb{E} \left[ \exp \left( i \sum_{k=1}^n (\theta_k, B_{\beta,\alpha}(t_k))_{\mathbb{R}^d} \right) \right] = E_\beta \left[ -\frac{1}{2} \sum_{j=1}^d (\theta_{\cdot,j}, \gamma_\alpha \theta_{\cdot,j})_{\mathbb{R}^n} \right], \quad (7)$$

where  $\mathbb{E}$  denotes the expectation w.r.t.  $\mathbb{P}$  and

$$\gamma_\alpha := \gamma_{\alpha,n} := (t_k^\alpha + t_j^\alpha - |t_k - t_j|^\alpha)_{k,j=1}^n.$$

3. The joint probability density function of  $(B_{\beta,\alpha}(t_1), \dots, B_{\beta,\alpha}(t_n))$  is equal to

$$\rho_\beta(\theta, \gamma_\alpha) = \frac{(2\pi)^{-\frac{nd}{2}}}{(\det \gamma_\alpha)^{d/2}} \int_0^\infty \tau^{-\frac{nd}{2}} e^{-\frac{1}{2\tau} \sum_{j=1}^d (\theta_{\cdot,j} \gamma_\alpha^{-1} \theta_{\cdot,j})} M_\beta(\tau) d\tau. \quad (8)$$

The following are the most important key properties of ggBm:

(P1). For each  $t \geq 0$ , the moments of any order of  $B_{\beta,\alpha}(t)$  are given by

$$\begin{cases} \mathbb{E}[|B_{\beta,\alpha}(t)|^{2n+1}] &= 0, \\ \mathbb{E}[|B_{\beta,\alpha}(t)|^{2n}] &= \frac{(2n)!}{2^n \Gamma(\beta n + 1)} t^{\alpha n}. \end{cases}$$

(P2). The covariance function has the form

$$\mathbb{E}[B_{\beta,\alpha}(t), B_{\beta,\alpha}(s)] = \frac{d}{2\Gamma(\beta + 1)} (t^\alpha + s^\alpha - |t - s|^\alpha), \quad t, s \geq 0. \quad (9)$$

(P3). For each  $t, s \geq 0$ , the characteristic function of the increments is

$$\mathbb{E}[e^{i(k, B_{\beta,\alpha}(t) - B_{\beta,\alpha}(s))}] = E_\beta \left( -\frac{|k|^2}{2} |t - s|^\alpha \right), \quad k \in \mathbb{R}^d. \quad (10)$$

(P4). The process  $B_{\beta,\alpha}$  is non-Gaussian,  $\alpha/2$ -self-similar with stationary increments.

(P5). The ggBm is not a semimartingale. Furthermore,  $B_{\alpha,\beta}$  cannot be of finite variation in  $[0, 1]$  and, by scaling and stationarity of the increment, on any interval in  $\mathbb{R}^+$ .

(P5). For  $n = 1$ , the density  $\rho_\beta(x, t)$ ,  $x \in \mathbb{R}^d$ ,  $t > 0$ , is the fundamental solution of the following fractional differential equation (see [9])

$$\mathbb{D}_t^{2\beta} \rho_\beta(x, t) = \Delta_x \rho_\beta(x, t),$$

where  $\Delta_x$  is the  $d$ -dimensional Laplacian in  $x$  and  $\mathbb{D}_t^{2\beta}$  is the Caputo-Dzherbashian fractional derivative; see [10] for the definition and properties.

## 2.2. Representations of Generalized Grey Brownian Motion

The ggBm admits different representations in terms of well-known processes. It follows from (7) that ggBm has an elliptical distribution, see Section 2 in [11] or Section 3 in [12]. On the other hand, ggBm is also given as a product (see [7] for  $d = 1$ ) of two processes as follows

$$\{B_{\beta,\alpha}(t), t \geq 0\} \stackrel{\mathcal{L}}{=} \{\sqrt{Y_\beta} B^{\alpha/2}(t), t \geq 0\}. \quad (11)$$

Here,  $\stackrel{\mathcal{L}}{=}$  means equality in law, the nonnegative random variable  $Y_\beta$  has density  $M_\beta$  and  $B^{\alpha/2}$  is a  $d$ -dimensional fBm with Hurst parameter  $\alpha/2$  and independent of  $Y_\beta$ .

We give another representation of ggBm  $B_{\beta,\alpha}$  as a subordination of fBm (see Prop. 2.14 in [13] for  $d = 1$ ) which is used below. For completeness, we give the short proof.

**Proposition 1.** *The ggBm has the following representation*

$$\{B_{\beta,\alpha}(t), t \geq 0\} \stackrel{\mathcal{L}}{=} \{B^{\alpha/2}(t Y_\beta^{1/\alpha}), t \geq 0\}. \quad (12)$$

**Proof.** We must show that both representations (11) and (12) have the same finite-dimensional distribution. For every  $\theta = (\theta_1, \dots, \theta_n) \in (\mathbb{R}^d)^n$ , we have

$$\begin{aligned} \mathbb{E} \left[ \exp \left( i \sum_{k=1}^n (\theta_k, B^{\alpha/2}(t_k Y_{\beta}^{1/\alpha})) \right) \right] &= \int_0^{\infty} \mathbb{E} \left[ \exp \left( i \sum_{k=1}^n (\theta_k, B^{\alpha/2}(t_k y^{1/\alpha})) \right) \right] M_{\beta}(y) dy \\ &= \int_0^{\infty} \mathbb{E} \left[ \exp \left( i \sum_{k=1}^n (\theta_k, y^{1/2} B^{\alpha/2}(t_k)) \right) \right] M_{\beta}(y) dy \\ &= \mathbb{E} \left[ \exp \left( i \sum_{k=1}^n (\theta_k, Y_{\beta}^{1/2} B^{\alpha/2}(t_k)) \right) \right]. \end{aligned}$$

In the second equality, we used the  $\alpha/2$ -self-similarity of fBm. This completes the proof.  $\square$

### 3. The Green Measure for Generalized Grey Brownian Motion

In this section we show the existence of the Green measure for the ggBm, see (1) and (2). Let us begin by discussing the existence of the Green measure for a general stochastic process  $X$ .

Let  $X = \{X(t), t \geq 0\}$  be a stochastic process in  $\mathbb{R}^d$  starting from  $x \in \mathbb{R}^d$ . If  $X(t), t \geq 0$ , has a probability distribution  $\rho_{X(t)}(x, \cdot)$ , then Eq. (1) becomes

$$V_X(x, f) = \int_0^{\infty} \int_{\mathbb{R}^d} f(y) \rho_{X(t)}(x, dy) dt. \quad (13)$$

Then, applying the Fubini theorem, the Green measure  $\mathcal{G}_X(x, \cdot)$  of  $X$  is given by

$$\mathcal{G}_X(x, dy) = \int_0^{\infty} \rho_{X(t)}(x, dy) dt,$$

assuming the existence of  $\mathcal{G}_X(x, \cdot)$  as a Radon measure on  $\mathbb{R}^d$ . That is, for every bounded Borel set  $B \in \mathcal{B}_b(\mathbb{R}^d)$  we have

$$\mathcal{G}_X(x, B) = \int_0^{\infty} \rho_{X(t)}(x, B) dt < \infty.$$

If the probability distribution  $\rho_{X(t)}(x, \cdot)$  is also absolutely continuous with respect to the Lebesgue measure, say  $\rho_{X(t)}(x, dy) = \rho_t(x, y) dy$ , then the function

$$g_X(x, y) := \int_0^{\infty} \rho_t(x, y) dt, \quad \forall y \in \mathbb{R}^d, \quad (14)$$

is called the Green function of the stochastic process  $X$ . Moreover, the Green measure in this case is given by  $\mathcal{G}_X(x, dy) = g_X(x, y) dy$ .

This leads us to the following definition of the Green measure of a stochastic process  $X$ .

**Definition 2.** Let  $X = \{X(t), t \geq 0\}$  be a stochastic process on  $\mathbb{R}^d$  starting from  $x \in \mathbb{R}^d$  and  $\rho_{X(t)}(x, \cdot)$  be the probability distribution of  $X(t), t \geq 0$ . The Green measure of  $X$  is defined as a Radon measure on  $\mathbb{R}^d$  by

$$\mathcal{G}_X(x, B) := \int_0^{\infty} \rho_{X(t)}(x, B) dt, \quad B \in \mathcal{B}_b(\mathbb{R}^d),$$

or

$$\int_{\mathbb{R}^d} f(y) \mathcal{G}_X(x, dy) = \int_{\mathbb{R}^d} f(y) \int_0^{\infty} \rho_{X(t)}(x, dy) dt, \quad f \in C_0(\mathbb{R}^d)$$

whenever these integrals exist.

In other words,  $\mathcal{G}_X(x, B)$  is the expected length of time the process remains in  $B$ .

In order to state the main theorem which establishes the existence of the Green measure for ggBm, first, we introduce a proper Banach space of functions  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  such that

$$\int_0^\infty f(x + B_{\beta,\alpha}(t)) dt$$

is finite  $\mathbb{P}$ -a.s. Without loss of generality, we may assume that  $f \geq 0$  above. We define the space  $CL(\mathbb{R}^d)$  of continuous real-valued on  $\mathbb{R}^d$  by

$$CL(\mathbb{R}^d) := \{f : \mathbb{R}^d \rightarrow \mathbb{R} \mid f \text{ is continuous, bounded and } f \in L^1(\mathbb{R}^d)\}.$$

The space  $CL(\mathbb{R}^d)$  becomes a Banach space with the norm

$$\|f\|_{CL} := \|f\|_\infty + \|f\|_1, \quad \forall f \in CL(\mathbb{R}^d),$$

where  $\|\cdot\|_\infty$  denotes the sup-norm and  $\|\cdot\|_1$  is the norm in  $L^1(\mathbb{R}^d)$ . The choice of  $CL(\mathbb{R}^d)$  allows us to show that the family of random variables (also known as perpetual integral functionals)

$$\int_0^\infty f(x + B_{\beta,\alpha}(t)) dt, \quad f \in CL(\mathbb{R}^d)$$

have finite expectations  $\mathbb{P}$ -a.s.

**Theorem 1.** Let  $f \in CL(\mathbb{R}^d)$  and  $x \in \mathbb{R}^d$  be given and consider ggBm  $B_{\beta,\alpha}$  with  $d\alpha > 2$  and  $1 < \alpha \leq 2$ . Then, the perpetual integral functional  $\int_0^\infty f(x + B(t)) dt$  is finite  $\mathbb{P}$ -a.s. and its expectation equals

$$\mathbb{E} \left[ \int_0^\infty f(x + B_{\beta,\alpha}(t)) dt \right] = D \int_{\mathbb{R}^d} \frac{f(x+y)}{|y|^{d-2/\alpha}} dy, \quad (15)$$

where  $D = D(\beta, \alpha, d) = \frac{1}{\alpha} 2^{-1/\alpha} \pi^{-\frac{d}{2}} \Gamma\left(\frac{d}{2} - \frac{1}{\alpha}\right) \frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1-\frac{\beta}{\alpha})}$ .

**Proof.** Given  $x \in \mathbb{R}^d$  and  $f \in CL(\mathbb{R}^d)$  non-negative, let  $\rho_\beta(\cdot, t^\alpha)$  denote the density of  $B_{\beta,\alpha}(t)$ ,  $t \geq 0$ , given by (see (8) with  $n = 1$ )

$$\rho_\beta(y, t^\alpha) = \frac{1}{(2\pi t^\alpha)^{d/2}} \int_0^\infty \tau^{-d/2} e^{-\frac{|y|^2}{2t^\alpha \tau}} M_\beta(\tau) d\tau, \quad y \in \mathbb{R}^d.$$

First, we show the equality (15). It follows from the above considerations that

$$\begin{aligned} \mathbb{E} \left[ \int_0^\infty f(x + B_{\beta,\alpha}(t)) dt \right] &= \int_0^\infty \int_{\mathbb{R}^d} f(x+y) \rho_t^{\beta,\alpha}(y) dy dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} f(x+y) \frac{1}{(2\pi t^\alpha)^{d/2}} \int_0^\infty \tau^{-d/2} M_\beta(\tau) e^{-\frac{|y|^2}{2t^\alpha \tau}} d\tau dy dt. \end{aligned}$$

Using Fubini's Theorem, we first compute the  $t$ -integral and use the assumption  $d\alpha > 2$ . We obtain

$$\int_0^\infty \frac{1}{(2\pi t^\alpha \tau)^{d/2}} e^{-\frac{|y|^2}{2t^\alpha \tau}} dt = C(\alpha, d) \frac{\tau^{-\frac{1}{\alpha}}}{|y|^{d-2/\alpha}},$$

where

$$C(\alpha, d) := \frac{1}{\alpha} 2^{-1/\alpha} \pi^{-\frac{d}{2}} \Gamma\left(\frac{d}{2} - \frac{1}{\alpha}\right).$$

Next we compute the  $\tau$ -integral using (6) so that

$$\int_0^\infty \tau^{-1/\alpha} M_\beta(\tau) d\tau = \frac{\Gamma(1 - \frac{1}{\alpha})}{\Gamma(1 - \frac{\beta}{\alpha})}, \quad \alpha > 1.$$

Combining gives

$$\mathbb{E} \left[ \int_0^\infty f(x + B_{\beta,\alpha}(t)) dt \right] = D \int_{\mathbb{R}^d} \frac{f(x+y)}{|y|^{d-2/\alpha}} dy,$$

where  $D = D(\beta, \alpha, d) = C(\alpha, d) \frac{\Gamma(1 - \frac{1}{\alpha})}{\Gamma(1 - \frac{\beta}{\alpha})}$ . Therefore, the equality (15) is shown.

Now we show that the right-hand side of (15) is finite for every non-negative  $f \in CL(\mathbb{R}^d)$ . To see this, we may use the local integrability of  $|y|^{d-2/\alpha}$  in  $y$  and obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{f(x+y)}{|y|^{d-2/\alpha}} dy &= \int_{\{|y| \leq 1\}} \frac{f(x+y)}{|y|^{d-2/\alpha}} dy + \int_{\{|y| > 1\}} \frac{f(x+y)}{|y|^{d-2/\alpha}} dy \\ &\leq C_1 \|f\|_\infty + C_2 \|f\|_1 \leq C \|f\|_{CL}. \end{aligned}$$

Therefore, the integral in (15) is, in fact, well defined. In other words, the integral  $\int_0^\infty f(x + B_{\beta,\alpha}(t)) dt$  exists with probability 1. This completes the proof.  $\square$

As a consequence of the above theorem, we immediately obtain the Green measure of ggBm  $B_{\beta,\alpha}$ , that is, comparing (2) and (15).

**Corollary 1.** *The Green measure of ggBm  $B_{\beta,\alpha}$  for  $d\alpha > 2$  is given by*

$$\mathcal{G}_{\beta,\alpha}(x, dy) = \frac{D}{|x-y|^{d-2/\alpha}} dy.$$

**Remark 1.** 1. *It is possible to show that given  $f \neq 0$ , the perpetual integral  $\int_0^\infty f(x + B_{\beta,\alpha}(t)) dt$  is a non-constant random variable. As a consequence, for  $f \geq 0$  the variance of  $\int_0^\infty f(x + B_{\beta,\alpha}(t)) dt$  is strictly positive. The proof uses the notion of conditional full support of ggBm. We will not provide a detailed explanation of this result that closely follows the ideas of Theorem 2.2 in [3] to which we address the interested readers.*

2. *Note also that the functional in (1)*

$$V_{\beta,\alpha}(\cdot, x) : CL(\mathbb{R}^d) \longrightarrow \mathbb{R}$$

*is continuous. In fact, from the proof of Theorem 1 for any  $f \in CL(\mathbb{R}^d)$  yields*

$$|V_{\beta,\alpha}(f, x)| \leq K \|f\|_{CL},$$

*where  $K$  is a constant depending on the parameters  $\beta, \alpha$ , and  $d$ .*

#### 4. Discussion and Conclusions

We have derived the Green measure for the class of stochastic processes called the generalized grey Brownian motion in Euclidean space  $\mathbb{R}^d$  for  $d \geq 2$ . This class includes, in particular, fractional Brownian motion and other non-Gaussian processes. To address the case where  $d = 1$ , a renormalization process is needed. However, this will be postponed to future work. For  $\beta = \alpha = 1$  ggBm  $B_{1,1}$  is nothing but a Brownian motion. In this case, the Green measure exists for  $d \geq 3$ .



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