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Unfolding the Third Approximation of The Time-Dependent Controller's Designed Parameter (TDCDP) of Fokker Planck Kolmogorov (FPK) Probability Density Function(PDF)

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Unfolding the Third Approximation of The Time-Dependent Controller's Designed Parameter (TDCDP) of Fokker Planck Kolmogorov (FPK) Probability Density Function(PDF)

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Abstract: A difficult open problem that has never been fully resolved in the literature to date is the behaviour of the (TDCDP) third approximation. As a giant step towards modern control theory, this cutting-edge paper will advance control theory and other relevant inter-disciplinary subjects. Because control theory is important in industry and engineering, this study will be especially beneficial to anyone working in these disciplines who wish to understand more about recent advances in control theory settings. In contrast, the wealth of applications of Fokker Planck Kolmogorov (FPK) equations numerous disciplines As a result, the study instantly gains more taste and credibility. At the end of the paper, there are several challenging open problems, some concluding remarks, and recommendations for further investigation.

Keywords: control theory; time-dependent controller's designed parameter(TDCDP)

Introduction

It is possible to equationally view the closed-loop system [1–7] by:

$$dx_t = -\varphi x_t dt + \sigma\, dW_t \tag{1}$$

The system's state is represented by x_t, TDCDP is denoted by $\varphi > 0$, the Weiner process is defined by W_t, and $\sigma > 0$.

Fokker Planck Kolmogorov equation is:

$$\frac{\partial p(x,t)}{\partial t} = \varphi \frac{\partial}{\partial x}\left(xp(x,t)\right) + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial t^2} \qquad (2) \tag{2}$$

The PDF, $p(x,t)$ solves (2), with

$$p(x,t) = \sqrt{\left(\frac{\varphi}{\pi\sigma^2(1-e^{-2\varphi t})}\right)}e^{\left(-\frac{\varphi(x-x_0e^{-\varphi t})^2}{\pi\sigma^2(1-e^{-2\varphi t})}\right)} \tag{3}$$

x_0 defines the initial value of x_t

$$\varphi(t) = \frac{W_0\left(-\frac{t\sigma^2}{r(t)}e^{-\frac{t\sigma^2}{r(t)}}\right)}{2t} + \frac{t\sigma^2}{2r(t)} \tag{4}$$

Provided that W_0 is the Lambert W function [8] and $r(t)$ defines a time-dependent variance function. Figure 1(c.f., [7]) visualizes x_t.

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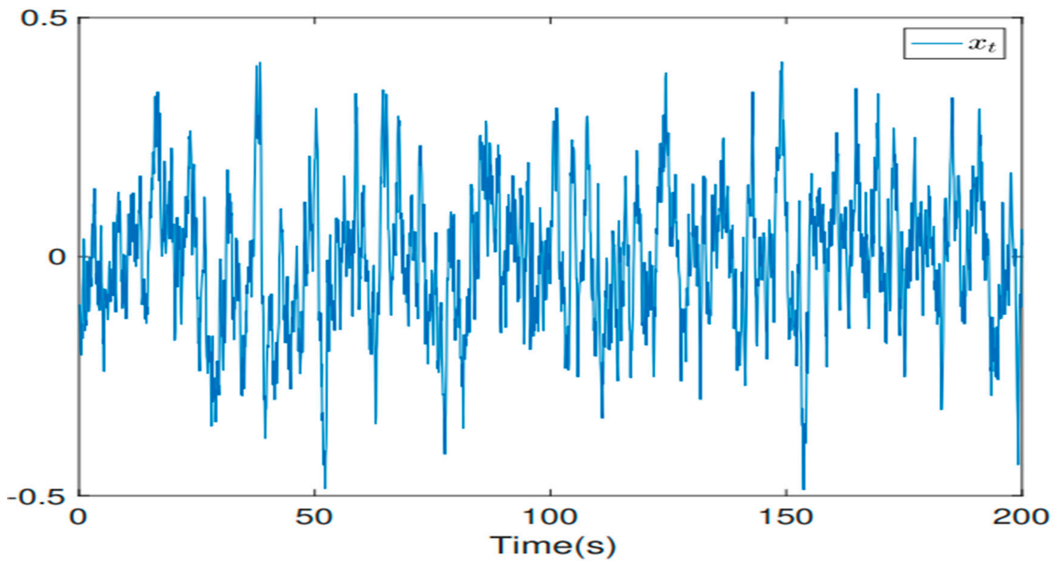

Figure 1.

W_0 is defined as [8]:.

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1} z^n}{n!} \tag{5}$$

Where z is any complex number and it is satisfied when z is real. Also, for real values of z, $W_0(z)$ satisfies:

$$\frac{dW_0(z)}{dz} = \frac{W_0(z)}{z(1+W_0(z))} \tag{6}$$

The following theorem is essential to obtain the main results of section two.

Preliminary Theorem (PT) [9]

Let f be a function that is defined and differentiable on an open interval (c,d).

$$\text{If } f'(x) > (<)0 \ \forall x \in (c,d), \text{ then f increases(decreases) on } (c,d) \tag{7}$$

This paper's contributions are:

- Solving for the first time ever, the TDCDP's third approximation.
- Introducing up-to-date, unsolved issues.

The current work reads: The methodology is highlighted in section two. Section three provides results and discussion. Section four discusses some emerging research questions with future research pathways.

Methodology

Notably, a mathematical approach is undertaken to calculate the threshold based on the preliminary theorem(see Equation (7)). More potentially, calculus and more advanced algebraic forms are utilized to uncover TDCDP's time-dependent lower bound. Looking at the bigger picture, this discovery will lead to a contemporary control theory rather than being limited within the traditional classical frames.

This current paper provides the full answer, by solving the , but yet there are still numerous unexplored applications. This will put the research community into more spacious frontiers of thoughtful innovation.

Communicating (4), and (5), we arrive at the n^{th} approximation of TDCDP, namely, $\varphi_n(t)$:

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$$\varphi_n(t,r(t)) = \frac{\sum_{n=1}^{\infty}\frac{(-n)^{n-1}z^n}{n!}}{2t} + \frac{t\sigma^2}{2r(t)}, \quad z = -\frac{t\sigma^2}{r(t)}e^{-\frac{t\sigma^2}{r(t)}} \tag{8}$$

Results and Discussion

Define

$$\varphi_3(t,r(t)) = \frac{\left(z-z^2+\frac{3}{2}z^3\right)}{2t} + \frac{t\sigma^2}{2r(t)}, \quad \text{z (c.f., (8))} \tag{9}$$

Theorem 1 For φ_3 of (9) satisfies:

i)

$$\varphi_3(t,r(t)) > -\left(\frac{\sigma^2}{r(t)} + \frac{t\sigma^4}{(r(t))^2} + \frac{3t^2\sigma^6}{2(r(t))^3}\right) + \frac{t\sigma^2}{2r(t)} \tag{10}$$

ii) $\varphi_3(t,r(t))$ is temporally forever increasing(decreasing), whenever

$$\frac{\sigma^2}{(r(t))^2}(r - tr^{\cdot}) - \left(e^{-\frac{t\sigma^2}{r(t)}}\left(\frac{\sigma^4}{(r(t))^2} + \frac{t\sigma^4 r^{\cdot}}{(r(t))^3} - \frac{\sigma^2 r^{\cdot}}{(r(t))^2}\right) + e^{-\frac{2t\sigma^2}{r(t)}}\left(\frac{\sigma^4}{(r(t))^2} - \frac{2t\sigma^4 r^{\cdot}}{(r(t))^3} - \frac{2t\sigma^6}{(r(t))^3} + \frac{2t^2\sigma^6 r^{\cdot}}{(r(t))^4}\right) + e^{-\frac{3t\sigma^2}{r(t)}}\left(\frac{3t\sigma^6}{(r(t))^3} - \frac{9t^2\sigma^6 r^{\cdot}}{2(r(t))^4} - \frac{9t^2\sigma^8}{2(r(t))^4} + \frac{9t^3\sigma^8 r^{\cdot}}{2(r(t))^5}\right)\right) > (<\ 0), . = \frac{d}{dt} \tag{11}$$

Proof

(i) Clearly, $\varphi_3(t,r(t))$ (c.f., (9)) read:

$$\varphi_3(t,r(t)) = -\frac{1}{2}\left(\frac{\sigma^2}{r(t)}e^{-\gamma} + \frac{t\sigma^4}{(r(t))^2}e^{-2\gamma} + \frac{3t^2\sigma^6}{2(r(t))^3}e^{-3\gamma}\right) + \frac{t\sigma^2}{2r(t)}, \quad \gamma = \frac{t\sigma^2}{r(t)} \tag{12}$$

Thus,

$$\varphi_3(t,r(t)) = -\frac{1}{2}\left(\frac{\sigma^2}{r(t)}e^{-\gamma} + \frac{t\sigma^4}{(r(t))^2}e^{-2\gamma} + \frac{3t^2\sigma^6}{2(r(t))^3}e^{-3\gamma}\right) + \frac{t\sigma^2}{2r(t)}$$

$$> -\frac{1}{2}\left(\frac{\sigma^2}{r(t)} + \frac{t\sigma^4}{(r(t))^2} + \frac{3t^2\sigma^6}{2(r(t))^3}\right) + \frac{t\sigma^2}{2r(t)} \ \text{(Since, } e^{-\gamma} < 1\text{)}$$

$$> -\left(\frac{\sigma^2}{r(t)} + \frac{t\sigma^4}{(r(t))^2} + \frac{3t^2\sigma^6}{2(r(t))^3}\right) + \frac{t\sigma^2}{2r(t)}$$

(ii) Communicating the preliminary theorem,

$$\varphi_3(t,r(t)) \text{ is forever increasing(decreasing) in } t \text{ if and only if } \frac{d\varphi_3(t,r(t))}{dt} > (<0) \text{ respectively} \tag{13}$$

Following some lengthy mathematical steps, we arrive at the desired result in (11).
Define

$$r(t) = \sigma^2 \text{(c.f., (8))} \tag{14}$$

This established the following theorem.

Theorem 2 For $\varphi_n(t,r(t))$ (c.f., (14)), it holds that:

i)

$$\varphi_3(t,\sigma^2) = -\frac{1}{2}(e^{-3t})\left(e^{2t} + te^t + \frac{3}{2}t^2\right) + \frac{t}{2} \tag{15}$$

ii)

$$\varphi_3(t,\sigma^2) = 0 \quad \text{has} \quad \text{a real root within the unit interval} \tag{16}$$

iii) $\varphi_3(t,\sigma^2)$ is forever increasing in t

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Proof

We have, $\varphi_3(t, r(t))$ (c.f., (9))

It is implied that:

$$r(t) = \sigma^2 \Rightarrow \text{z} = -te^{-t} \tag{17}$$

Thus,

$$\varphi_3(t, \sigma^2) = \frac{\left(-te^{-t} - t^2e^{-2t} - \frac{3}{2}t^3e^{-3t}\right)}{2t} + \frac{t}{2} \quad = -\frac{1}{2}(e^{-3t})\left(e^{2t} + te^t + \frac{3}{2}t^2\right) + \frac{t}{2} \tag{18}$$

Looking at (14), (i) is immediate.

ii) Let

$$\varphi_3(t, \sigma^2) = 0$$

Therefore,

This can be visualized by checking that

$$\varphi_3(0, \sigma^2) = -0.5, \qquad \varphi_3(1, \sigma^2) = 0.2110523365 \tag{19}$$

Looking at Figure 2, it is evident that the root of $\varphi_3(t, \sigma^2) = 0$, will be floating between $0.7, \mathit{and}\ 0.8 \in [0,1]$.

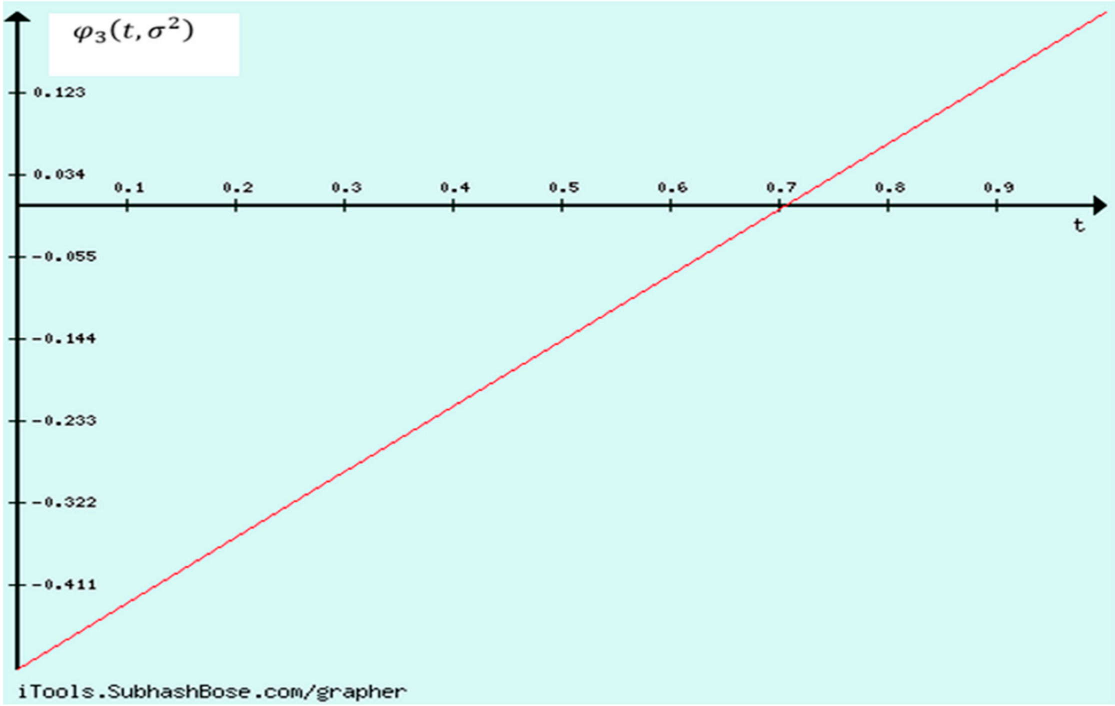

Figure 2. Roots of $\varphi_3(t, \sigma^2) = 0$.

Communicating mathematical analysis, (ii) follows.

(iii)

$$\frac{d\varphi_3(t, \sigma^2)}{dt} = \left(\frac{1}{2} + \frac{1}{2}e^{-3t}\left(e^{2t} + 2te^t + \frac{9}{2}t^2 + 3t - e^t\right)\right) \tag{10}$$

Therefore,

$$\frac{d\varphi_3(t, \sigma^2)}{dt} > 0 \Leftrightarrow \left(\frac{1}{2} + \frac{1}{2}e^{-3t}\left(e^{2t} + 2te^t + \frac{9}{2}t^2 + 3t - e^t\right)\right) > 0 \tag{11}$$

Or

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$$\left(\frac{1}{2}+\frac{1}{2}e^{-3t}\left(e^{2t}+2te^{t}+\frac{9}{2}t^{2}+3t-e^{t}\right)\right)>0 \quad (12)$$

Equivalently,

$$e^{3t}+e^{2t}+2te^{t}+\frac{9}{2}t^{2}>3t+e^{t} \quad (13)$$

Mathematically speaking, since $t>0$

$$e^{3t}+e^{2t}+2te^{t}+\frac{9}{2}t^{2}>e^{t}+t+t+t=3t+e^{t}$$

Hence, (iii) is immediate by engaging the preliminary theorem.
On a different note,

$$lim_{t\to\infty}\varphi_3(t,\sigma^2)=lim_{t\to\infty}\left(\frac{\left(-te^{-t}-t^2e^{-2t}-\frac{3}{2}t^3e^{-3t}\right)}{2t}+\frac{t}{2}\right)=\infty \quad (14)$$

Which consolidates the forever increasability of $\varphi_3(t,\sigma^2)$ in t.

Having a close look at another case, $\varphi_3(t,t^2\sigma^2)$, it can be easily shown by Figure 3, that $\varphi_3(t,t^2\sigma^2)$ decreases drastically as time increases. Moreover,

$$lim_{t\to\infty}\varphi_3(t,t^2\sigma^2)=0 \quad (15)$$

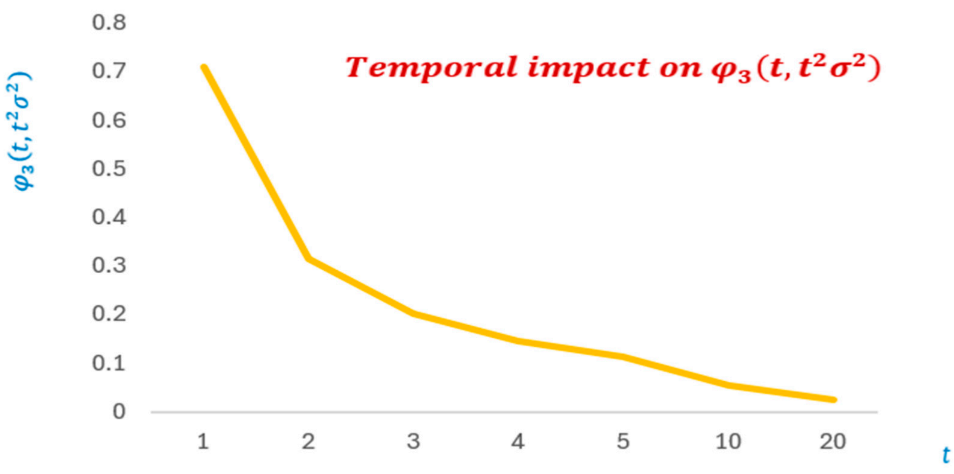

Figure 3.

Conclusion and Future Research

The third approximation of the TDCDP, $\varphi_3(t,r(t))$, has been examined in this explanation. More opportunely, this research has brought attention to a few suggested open issues:

Open Problem 1

Can we solve the ever- challenging open problem of finding the upper bound of $\varphi_3(t,r(t))$ (c.f.(10)). It is expected that this upper bound, if existed, will be time-dependent?

Open Problem 2

Is it possible mathematically wise to unlock the most challenging open problem ever in uncovering the TDCDP's fourth, fifth, sixth… approximations. The proposed open challenges will be solved in the next phase of research, which will also look at further extensions of FPK theory to additional multidisciplinary areas of human understanding.

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