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[Marino Badiale](#) and [Isabella Cravero](#) *

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Article

A Nonlinear ODE Model for a Consumeristic Society

Marino Badiale and Isabella Cravero * 

Dipartimento di Matematica, Università di Torino; marino.badiale@unito.it

* Correspondence: isabella.cravero@unito.it.

Abstract: In this paper we introduce an ODE system to model the interaction between nature resources and human exploitation in a rich consumeristic society. In this model the rate of change of population depends on wealth per capita, and the rate of consumption has a quadratic growth with respect to population and wealth. As in our previous work [3], we distinguish between renewable and non renewable resources, and we introduce a replenishment term for non renewable resources. We first get some information on the asymptotic behaviour of wealth and population, then we compute all the equilibria of the system and we study their stability when the parameter of exploitation of resources is small. Some numerical simulations confirm the theoretical results, and give suggestions for future research.

Keywords: human and nature dynamics; ODE systems for society dynamics; Ecological- Economic Modeling

1. Introduction

In this paper we study a model of interactions between society and nature, inspired by the HANDY model introduced in [7] (see also [8]). The original HANDY model is a set of four differential equations whose variables are natural resources, wealth produced by human work, and the population, splitted in the two classes of Commoner and Elite. Using computer simulations, the paper [7] shows different possibilities for the evolution of society, including collapse. It is a simple model that seems to allow a rich panel of behaviour, so it attracted attention, and several papers have been devoted to its exploration, see [1,2,4,9–11]. In [2] various developments of the original model are introduced, among others the division of natural resources between renewable and non renewable. This idea has been developed in the paper [10], where some general results are obtained and again numerical simulations show different possibilities for the evolution of society. In [10] the dynamics of nonrenewable resources is given by the following equation

$$\frac{dy_n}{dt} = -\delta_n x_c y_n,$$

where y_n are the nonrenewable resources, x_c is the Commoner population and $\delta_n > 0$ a parameter. Clearly, this equation translates the idea of a given amount and an irreversible depletion of y_n . In our previous work [3] we have pursued this idea but we have introduced a replenishment term for nonrenewable resources. This is due to the optimistic argument, that is often used in public debate on these subjects, that human ingenuity and scientific progress can substitute depleted resources with new ones. The term of replenishment that we have added is given by

$$k \frac{xw}{xw + 1} z$$

depending on xw , where x is population and w is wealth, while z is the level of non renewables and k a constant. Hence, the dynamics of non renewable resources z in [3] is given by

$$z' = k \frac{xw}{xw + 1} z - \delta z x,$$

The replenishment term is obtained from a function $\frac{t}{t+1}$ which, in population dynamics, is a Holling II type function (see [6], page 25 and passim), used to model saturation effects. The equation above states that the replenishment is possible but cannot be above a level k , and we can call this an hypothesis

of “moderate optimism”. Another characteristic of [3] is that we dropped the distinction between Commoner and Elite, so we obtained a four variable model, which is as follows

$$\begin{cases} x'(t) = (\beta - \alpha_m)x - \frac{1}{\rho}(\alpha_M - \alpha_m)(\rho x - w)^+ \\ y'(t) = \gamma y(\lambda - y) - \delta_1 xy \\ z'(t) = -\delta_2 xz + k_1 \frac{\delta_3 xw}{\delta_3 xw + k_2} z \\ w'(t) = \delta_1 xy + \delta_2 xz - \delta_3 xw - s \left[x - \frac{1}{\rho}(\rho x - w)^+ \right]. \end{cases} \quad (1)$$

Here, as we said above, x is the population, y, z indicate respectively renewable and nonrenewable resources, w is the wealth. An interesting result in [3] is that it cannot happen that x, w go to infinity as t goes to infinity. This means that, even assuming “moderate optimism”, there cannot be an indefinite growth in population and wealth. In [3] we also obtained all the critical points of the system, and we studied their stability. In the final section of [3] numerical simulations support the theoretical results and also suggest the possibility, not yet treated in a rigorous way, of periodic orbits and chaotic dynamics.

The present paper originated from the attempt to adapt the HANDY model to contemporary, rich, consumeristic societies (like contemporary western countries). Indeed, the HANDY model seems to be a very nice and simple way to treat the interaction of nature and society for a large class of historical societies, but, in our opinion, it has some serious flaws when dealing with contemporary, rich, consumeristic societies. The main one is probably the fact that in the original model the birth rate is constant and the death rate decays when wealth increases, leading to the conclusion that a wealthy society should show sustained population growth. This is definitely not true in contemporary western societies, and the reason is due to the fact that birth rate is not constant, and indeed, in most of contemporary rich societies, it is falling. Hence, in the present paper we introduce an equation for population dynamics which is different from that of HANDY model, and it is as follows

$$x' = -cx + d \frac{\mu x w}{w^2 + \mu^2 x^2} x.$$

Here we have a negative term $-cx$ which is contrasted by the term

$$d \frac{\mu x w}{w^2 + \mu^2 x^2} x = d \frac{\frac{w}{\mu x}}{\left(\frac{w}{\mu x}\right)^2 + 1} x$$

depending on the wealth per capita w/x via a constant μ . When $w/\mu x$ is very small or very large, the positive term is small and the population falls. In an intermediate region, the population grows. This seems fitting better with the recent history of rich countries.

Another difference compared to the HANDY model (and to [3]) is that there the depletion of natural resources depend on the population, while it seems reasonable that in a consumeristic society it should depend more on wealth than on population. Hence, a first idea should be to substitute populations with wealth (x with w) in the depletion terms, giving rise to quadratic terms w^2 . We will pursue this idea in a forthcoming paper. In the present paper, however, we are interested to preserve some influence of population on consumption of natural resources, so we introduce the function $m = m(x, w) = \max\{x, w\}$ and we substitute x with m in the depletion terms. We notice also that the consumption term in the equation for w' in [3], which is given by $s \left[x - \frac{1}{\rho}(\rho x - w)^+ \right]$, has a linear growth, while here, to model a consumeristic society, we use a quadratic term $m(x, w)w$, where the consumption is led mostly by wealth. In this way, we can put together the negative terms in the equation for w' in (1). Summing up all this ideas, we introduce the model that we are going to study. We apply some standard rescaling setting $k_2 = 1$, and we assume $\delta_1 = \delta_2$, that is, we assume that

the rate of depletion is the same for renewable and nonrenewable resources. This is a simplifying assumption, that we hope to drop in future research. In our previous paper [3], some simulations seemed to suggest that the dynamics do not change too much taking different values for δ_1 and δ_2 . In section 7 of the present paper, some simulations show a different dependence of the dynamics on δ_1 and δ_2 , and this seems an interesting topic for future research. In conclusion our model is as follows.

$$\begin{cases} x' = -cx + d \frac{\mu x w}{w^2 + \mu^2 x^2} x, \\ y' = \gamma y(\lambda - y) - \delta y m(x, w), \\ z' = k \frac{m(x, w) w}{m(x, w) w + 1} z - \delta z m(x, w), \\ w' = \delta y m(x, w) + \delta z m(x, w) - \sigma m(x, w) w. \end{cases} \quad (2)$$

where, as we said above, $m(x, w) = \max\{x, w\} = \frac{1}{2}(x + w + |x - w|)$. We assume $d > 2c$, which is necessary to have a positive growth of x for some range of $\frac{w}{\mu x}$. We also assume $\mu \geq 1$, which means that the negative effect on the population growth of the increasing wealth becomes relevant only for large values of wealth per capita. We also assume $\mu \neq \frac{d + \sqrt{d^2 - 4c^2}}{2c}$, which is just a technical assumption that we hope to drop in future work.

The paper is organized as follows: after the introduction, in section 2 we obtain general results of existence and positivity of the solutions, and also some results on their asymptotic behavior. In particular, we prove that it can't happen $x \rightarrow +\infty$ or $w \rightarrow +\infty$ or $z \rightarrow +\infty$ as $t \rightarrow +\infty$.

In sections 3, 4, 5, 6 we compute all the equilibrium points of system (2), and we study their stability properties. We distinguish the cases $x < w$, $x > w$, $x = w = 0$, while the case $x = w > 0$ is excluded by our hypotheses. There is a rich variety of such equilibria. We are able to give a complete description of the stability properties for some of them, but it seems very difficult to do the same for all of them. So we decided to study the stability in the limit of $\delta \rightarrow 0^+$, which is reasonable, in our opinion, because δ is a parameter depending on human choices. Using standard linearization techniques and asymptotic developments, we are able to describe the stability properties of all the equilibria when $\delta \rightarrow 0^+$ (and the other parameters are fixed).

In sections 7 we give some numerical simulations, both for corroborate the theoretical results of the preceding sections and to try to get some hint on the cases for which we don't have such results. It is interesting to see that in some simulations the conditions stated by our results for small δ 's (stability or instability) are preserved for larger δ 's, while they are not in other cases: so this seems to be an interesting subject of future research.

We end this introduction by indicating, for the reader's sake, some of the results obtained in the present paper.

- In section 2 we obtain that it cannot be $x(t) \rightarrow +\infty$ or $w(t) \rightarrow +\infty$ or $z(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. We got a similar result in [3] and, as in that case, the result does not necessarily mean that x, w, z are bounded, but that, if they grow above a critical level, then they will start to oscillate.
- The critical points with $x = w = 0$ are all unstable.
- Most of the equilibria are unstable as $\delta \rightarrow 0^+$, but for some ranges of the parameters, there are asymptotically stable equilibrium points, and also some with positive values for populations and wealth: see below cases ii8) and iii3). See also Figure 13 in section 7.

2. General Results on the Solutions

We set $X = (x, y, z, w) \in \mathbb{R}^4$ and $F(X) = (f_1(X), f_2(X), f_3(X), f_4(X))$

where

$$\begin{cases} f_1(X) = -cx + d \frac{\mu x w}{w^2 + \mu^2 x^2} x \\ f_2(X) = \gamma y(\lambda - y) - \delta y m(x, w) \\ f_3(X) = k \frac{m(x, w) w}{m(x, w) w + 1} z - \delta z m(x, w) \\ f_4(X) = \delta y m(x, w) + \delta z m(x, w) - \sigma m(x, w) w \end{cases}$$

We are interested in non negative solutions, so we work the cone

$$\mathcal{C} = \{(x, y, z, w) \in \mathbb{R}^4 \mid x > 0, y > 0, z > 0, w > 0\}$$

or possibly in its closure

$$\bar{\mathcal{C}} = \{(x, y, z, w) \in \mathbb{R}^4 \mid x \geq 0, y \geq 0, z \geq 0, w \geq 0\}.$$

It is obvious that F is locally Lipschitz at any point of $\bar{\mathcal{C}}$, so the standard theory of ODE gives the following result:

Proposition 1. For any $(t_0, X_0) \in \mathbb{R} \times \bar{\mathcal{C}}$, there exists a unique maximal solution $X(t)$ to the Cauchy problem

$$\begin{cases} X' = F(X) \\ X(t_0) = X_0 \end{cases} \quad (3)$$

defined in $J = (a, b)$, with $a < t_0 < b$.

In the following, we will set $t_0 = 0$ and

$$X_0 = (x_0, y_0, z_0, w_0) \text{ with } x_0, y_0, z_0 > 0 \text{ and } w_0 \geq 0.$$

As first thing we prove that, if X_0 is as above, the solution stays in \mathcal{C} for any $t > 0$.

Proposition 2. Let

$$X(t) = (x(t), y(t), z(t), w(t))$$

be a solution to (3) with $t_0 = 0$ and $x_0, y_0, z_0 > 0$, $w_0 \geq 0$. Then $X(t) \in \mathcal{C}$ for every $t \in (0, b)$.

Proof. It's straightforward to derive that $x(t), y(t), z(t) > 0$ for any $t \in J = (a, b)$, because the zero constant is a solution in the three equations for $x'(t), y'(t), z'(t)$, so it can't be crossed, and $x_0, y_0, z_0 > 0$. As for w , we first observe that there exists a $\varepsilon > 0$ such that $w(t) > 0, \forall t \in (0, \varepsilon)$: this is obvious if $w_0 > 0$, due to continuity, whereas if $w_0 = 0$ we have $w'(0) = \delta x_0 y_0 + \delta x_0 z_0 > 0$, and the thesis follows. Now let us then define

$$t_1 = \sup\{t > 0 \mid w(s) > 0 \text{ for every } s \in (0, t)\}.$$

If $t_1 = b$, we easily conclude that $w(t) > 0$ in $(0, b)$. If $t_1 < b$, it's easy to see that $w(t_1) = 0$ and $w(t) > 0$ for any $t \in (0, t_1)$, so $w'(t_1) \leq 0$. But from the equation we get

$$w'(t_1) = \delta x(t_1) y(t_1) + \delta x(t_1) z(t_1) > 0,$$

and the contradiction proves that $t_1 = b$ and thus $w(t) > 0, \forall t \in (0, b)$. \square

Proposition 3. Under the same assumptions as in Proposition 2 it holds $b = +\infty$.

Proof. We argue by contradiction, so we assume $b < +\infty$. From the first equation we easily get $x' \leq c_1 x$ for some $c_1 > 0$, hence

$$x(t) \leq x_0 e^{c_1 t} \leq x_0 e^{c_1 b} := c_2.$$

As for $y(t)$, using the same argument as in [3], we get that y is bounded in $[0, b)$ and, more precisely, we have $0 \leq y(t) \leq \max\{\lambda, y_0\}$, $\forall t \in [0, b)$. Notice that this holds also when $b = +\infty$. From the equation for $z(t)$ we get $z' \leq kz$, hence

$$z(t) \leq z_0 e^{kt} \leq z_0 e^{kb} := c_3,$$

and thus,

$$\begin{aligned} w' &\leq m(x, w)(\delta y + \delta z) \leq c_4(w + c_2) \\ &= c_4 w + c_5. \end{aligned}$$

From this it follows that $w(t) \leq c_6$ if $b < +\infty$. Therefore, if $b < +\infty$ we get that $X(t)$ is bounded in $[0, b)$, which is impossible due to known theorems. Hence, $b = +\infty$.

□

We now prove some asymptotic results on x and w .

Proposition 4. *It cannot be $\lim_{t \rightarrow +\infty} x(t) = +\infty$.*

Proof. We argue by contradiction, so we assume $\lim_{t \rightarrow +\infty} x(t) = +\infty$. For the sake of simplicity, let us set $m(t) = m(x(t), w(t))$. As $x(t) \rightarrow +\infty$, of course we have also $m(t) \rightarrow +\infty$. To get the result, we have to study the asymptotic behavior of y, z, w , in the hypothesis $x(t) \rightarrow +\infty$.

Claim 1: $\lim_{t \rightarrow +\infty} y(t) = 0$. We have

$$y' \leq y(\gamma\lambda - \gamma y - m).$$

Given that $y(t)$ is bounded and $m(t) \rightarrow +\infty$, $y(t)$ is decreasing as $t \rightarrow +\infty$. Hence, there exists a limit $l_+ = \lim_{t \rightarrow +\infty} y(t) \geq 0$. If $l_+ > 0$, then $y'(t) \rightarrow -\infty$, which is a contradiction with known theorems. Therefore $l_+ = 0$.

Claim 2: $\lim_{t \rightarrow +\infty} z(t) = 0$. We have

$$z' \leq kz - \delta z m = z(k - \delta m) < 0$$

as $t \rightarrow +\infty$. Thus, there exists a limit

$$l_+ = \lim_{t \rightarrow +\infty} z(t) \geq 0.$$

If $l_+ > 0$, then $\lim_{t \rightarrow +\infty} z'(t) = -\infty$, which is, as above, a contradiction.

Claim 3: $w(t)$ is bounded on $[0, +\infty)$.

Let $T > 0$ be such that for all $t \geq T$, $\frac{\delta y(t) + \delta z(t)}{\sigma} < 1$. We distinguish two cases:

1. Suppose $w(t) \geq \frac{\delta y(t) + \delta z(t)}{\sigma}$ for all $t \geq T$. In this case, we have

$$w' = m(x, w)(\delta y + \delta z - \sigma w) \leq 0,$$

which means that $w(t)$ is decreasing on $(T, +\infty)$ and thus it is bounded in $[0, +\infty)$.

2. Suppose now that there exists $t_1 \geq T$ such that $w(t_1) < \frac{\delta y(t_1) + \delta z(t_1)}{\sigma}$. Let $t > t_1$. If $w(t) \leq \frac{\delta y(t) + \delta z(t)}{\sigma}$, then $w(t) < 1$. Otherwise, if $w(t) > \frac{\delta y(t) + \delta z(t)}{\sigma}$, define

$$t_2 = \inf \left\{ \tau \in (t_1, t) \mid w(s) > \frac{\delta y(s) + \delta z(s)}{\sigma}, \forall s \in (\tau, t) \right\}.$$

We have that $t_2 > t_1$. Because of continuity we also have $w(s) > \frac{\delta y(s) + \delta z(s)}{\sigma} \forall s \in (t_2, t)$ and $w(t_2) = \frac{\delta y(t_2) + \delta z(t_2)}{\sigma} < 1$. Hence we have $w'(s) < 0$ in (t_2, t) implying that $w(s)$ is decreasing in (t_2, t) and thus $w(t) \leq w(t_2) < 1$. In conclusion, $w(t) < 1$ for every $t \geq t_1$, and therefore w is bounded on $[0, +\infty)$.

End of the proof: from the previous claims, we obtain, as w is bounded and $x(t) \rightarrow +\infty$, that $\frac{w}{\mu x} \rightarrow 0$ as $t \rightarrow +\infty$, and thus $\frac{\mu x w}{w^2 + \mu^2 x^2} \rightarrow 0$. This implies $x' < 0$ on a half-line $[t_3, +\infty)$, and therefore we get a contradiction with the hypothesis that $\lim_{t \rightarrow +\infty} x(t) = +\infty$.

□

Proposition 5. *It cannot be $\lim_{t \rightarrow +\infty} w(t) = +\infty$.*

Proof. Since $m = m(x, w) \geq w$ we can repeat the previous arguments and obtain $\lim_{t \rightarrow +\infty} y(t) = \lim_{t \rightarrow +\infty} z(t) = 0$. From this we deduce, again as above, that $w(t)$ is bounded, and this is a contradiction. □

Proposition 6. *It cannot be $\lim_{t \rightarrow +\infty} z(t) = +\infty$.*

Proof. Also in this case we argue by contradiction. We will show that, if $\lim_{t \rightarrow +\infty} z(t) = +\infty$, then it holds also $\lim_{t \rightarrow +\infty} w(t) = +\infty$, and this impossible by Proposition 5. So, let us assume $\lim_{t \rightarrow +\infty} z(t) = +\infty$. As a consequence, for any $\gamma > 0$ we can fix $T_\gamma > 0$ such that for all $t \geq T_\gamma$, $z(t) \geq \gamma(1 + \frac{\sigma}{\delta})$.

Claim: it cannot be $w(t) \leq \frac{\delta}{\sigma}(y + z)(t)$ for all $t \geq T_\gamma$. Indeed, in this case it would be $w'(t) \geq 0$ for all $t \geq T_\gamma$, implying w is increasing on $[T_\gamma, +\infty)$. Thus, there would be a limit $l_+ = \lim_{t \rightarrow +\infty} w(t) > 0$. By Proposition 5 it cannot be $l_+ = +\infty$, so it must be $l_+ < +\infty$, hence $\delta y + \delta z - \sigma w$ would go to $+\infty$, leading to $w' \geq aw$ for some $a > 0$ in a half-line $(\alpha, +\infty)$. This of course implies $\lim_{t \rightarrow +\infty} w(t) = +\infty$ contradicting the assumption $l_+ < +\infty$. The contradiction proves the claim.

Thanks to the previous claim, we can state that there exists $t_\gamma \geq T_\gamma$ such that

$$w(t_\gamma) > \frac{\delta}{\sigma}(y(t_\gamma) + z(t_\gamma)).$$

Let us now prove that $w(t) > \gamma$ for all $t \geq t_\gamma$. Fix $t > t_\gamma$. If $w(t) \geq \frac{\delta}{\sigma}(y(t) + z(t))$, then $w(t) \geq \frac{\delta}{\sigma}z(t) > \gamma$ (since $t > t_\gamma \geq T_\gamma$). Otherwise, if $w(t) < \frac{\delta}{\sigma}(y(t) + z(t))$, define

$$\tau_1 = \inf \left\{ \tau \in (t_\gamma, t) \mid w(s) < \frac{\delta}{\sigma}(y(s) + z(s)), \forall s \in (\tau, t) \right\}.$$

By continuity, $w(s) < \frac{\delta}{\sigma}(y(s) + z(s))$ for all s in (τ_1, t) . Hence w is increasing in (τ_1, t) , implying $w(t) \geq w(\tau_1)$. Furthermore, by standard continuity arguments, $w(\tau_1) = \frac{\delta}{\sigma}(y(\tau_1) + z(\tau_1)) \geq \gamma$. Thus, $w(t) \geq \gamma$ for all $t \geq t_\gamma$. Since this holds for any $\gamma > 0$, we conclude that $\lim_{t \rightarrow +\infty} w(t) = +\infty$. However, Proposition 5 states that this is not possible, and the contradiction proves that $z(t)$ cannot tend to $+\infty$ as t tends to $+\infty$.

□

3. Steady States if $x \leq w$

We are now going to compute all the steady states of the system (2) in \bar{C} , and to study their stability. As we said in the introduction, in most cases we will be able to get an answer about the stability only for $\delta \rightarrow 0^+$. In this section we deal with the case $x \leq w$. In this subset the equilibrium points are the solutions of the following system

$$x \left[-c + d \frac{\mu x w}{w^2 + \mu^2 x^2} \right] = 0 \quad (4a)$$

$$y(\gamma\lambda - \gamma y - \delta w) = 0 \quad (4b)$$

$$z w \left[k \frac{w}{w^2 + 1} - \delta \right] = 0 \quad (4c)$$

$$w[\delta y + \delta z - \sigma w] = 0 \quad (4d)$$

We can consider the following cases:

- i) $w = 0$. In this case we get $x = 0$, and from (4b) we have two possibilities: either $y = 0$ or $y = \lambda$. There are no conditions on z , so we have two families of critical points:

$$\{(0, 0, Z, 0), Z \geq 0\} \quad (5a)$$

$$\{(0, \lambda, Z, 0), Z \geq 0\} \quad (5b)$$

We notice that the points (5a) correspond to what is called "desert state" in [11], while the points

(5b) correspond to what is called "nature state" in the same paper.

- ii) $w \neq 0$. In this situation we have

$$(4a) \Rightarrow x = 0, \quad \text{or} \quad c = d \frac{\mu x w}{w^2 + \mu^2 x^2}$$

$$(4b) \Rightarrow y = 0, \quad \text{or} \quad \gamma\lambda - \gamma y - \delta w = 0$$

$$(4c) \Rightarrow z = 0, \quad \text{or} \quad \delta w^2 - k w + \delta = 0$$

and we can count 8 different cases.

We note that, if $x \neq 0$ from (4a) we have

$$d \frac{\mu x w}{w^2 + \mu^2 x^2} = c \Rightarrow d \frac{\frac{w}{\mu x}}{\left(\frac{w}{\mu x}\right)^2 + 1} = c \Rightarrow d \frac{t}{t^2 + 1} = c \Rightarrow ct^2 - dt + c = 0$$

with $t = \frac{w}{\mu x}$ and so

$$\begin{aligned} t_1 &= \frac{d - \sqrt{d^2 - 4c^2}}{2c} \in (0, 1) \\ t_2 &= \frac{d + \sqrt{d^2 - 4c^2}}{2c} > 1. \end{aligned} \quad (6)$$

Notice that the hypothesis $d > 2c$ rules out the case $t_1 = t_2 = 1$.

We then obtain two possible values for x given by

$$x_1 = \frac{w}{\mu t_1}, \quad x_2 = \frac{w}{\mu t_2}, \quad (7)$$

with t_1, t_2 as in (6).

If $y \neq 0$ then, from (4b), $\gamma\lambda - \gamma y - \delta w = 0$, i.e.

$$y = \lambda - \frac{\delta}{\gamma} w. \quad (8)$$

Finally, if $z \neq 0$ then

$$w_1 = \frac{k - \sqrt{k^2 - 4\delta^2}}{2\delta}, \quad w_2 = \frac{k + \sqrt{k^2 - 4\delta^2}}{2\delta}, \quad \text{with } k \geq 2\delta. \quad (9)$$

We can now write down a complete list of equilibrium points, in the half-space $x \leq w$, when $w > 0$.

- ii.1)** First, we observe that if $x = y = z = 0$, then from (4d) we get $w = 0$, and thus we exclude this possibility, because we have already dealt with it (see above the case $w = 0$).
- ii.2)** If $x = y = 0$ and $z \neq 0$ from (4c) we have $w = w_{1,2}$ and from (4d) $z_{1,2} = \frac{\sigma}{\delta} w_{1,2}$. Therefore, we have the two equilibrium points

$$P_i = \left(0, 0, \frac{\sigma}{\delta} w_i, w_i\right), \quad i = 1, 2, \quad k \geq 2\delta, \quad (10)$$

where w_1, w_2 are given by (9).

- ii.3)** If $x = z = 0$, $y = \lambda - \frac{\delta}{\gamma} w$, then from (4d) we have $w = \frac{\delta}{\sigma} y$ so that

$$w = \frac{\gamma\lambda\delta}{\gamma\sigma + \delta^2}, \quad y = \frac{\gamma\lambda\sigma}{\gamma\sigma + \delta^2}, \quad (11)$$

and we obtain the equilibrium point

$$P = \left(0, \frac{\gamma\lambda\sigma}{\gamma\sigma + \delta^2}, 0, \frac{\gamma\lambda\delta}{\gamma\sigma + \delta^2}\right). \quad (12)$$

- ii.4)** If $x = 0$, $y = \lambda - \frac{\delta}{\gamma} w$, $z \neq 0$, we obtain $w = w_{1,2}$, hence from (4d) we deduce

$$z = -\lambda + \left(\frac{\delta}{\gamma} + \frac{\sigma}{\delta}\right)w.$$

We then get two equilibrium points

$$P_i = \left(0, \lambda - \frac{\delta}{\gamma} w_i, -\lambda + \left(\frac{\delta}{\gamma} + \frac{\sigma}{\delta}\right)w_i, w_i\right), \quad i = 1, 2, \quad k \geq 2\delta. \quad (13)$$

- ii.5)** If $x > 0$ and $y = z = 0$, from (4d) we obtain $w = 0$ and we exclude this case because we are assuming $x \leq w$.
- ii.6)** If $x > 0$, $y = 0$ and $z > 0$, from (4c) we get $w = w_{1,2}$, with $k \geq 2\delta$. For x we get (7) with $d \geq 2c$, and from (4d) $z = \frac{\sigma w}{\delta}$. We have then 4 equilibrium points

$$P_{i,j} = \left(\frac{w_j}{\mu t_i}, 0, \frac{\sigma w_j}{\delta}, w_j\right), \quad i, j = 1, 2, \quad k \geq 2\delta, \quad d \geq 2c. \quad (14)$$

with w_j as in (9) and t_i as in (6).

- ii.7)** If $x > 0$, $y > 0$, $z = 0$, from (4d) we deduce for y and w the same values as in (11). Also, x is as in (7), and we obtain the two equilibrium points

$$P_i = \left(\frac{\delta\gamma\lambda}{\mu t_i(\gamma\sigma + \delta^2)}, \frac{\gamma\lambda\sigma}{\gamma\sigma + \delta^2}, 0, \frac{\delta\gamma\lambda}{\gamma\sigma + \delta^2}\right), \quad i = 1, 2, \quad d \geq 2c. \quad (15)$$

- ii.8)** If $x > 0$, $y > 0$, $z > 0$, then $\lambda - \frac{\delta}{\gamma} w$ and arguing as in [ii.7] we obtain 4 equilibrium points

$$P_{i,j} = \left(\frac{w_j}{\mu t_i}, \lambda - \frac{\delta}{\gamma} w_j, -\lambda + \left(\frac{\delta}{\gamma} + \frac{\sigma}{\delta}\right)w_j, w_j\right), \quad i, j = 1, 2, \quad k \geq 2\delta, \quad d \geq 2c. \quad (16)$$

with w_j as in (9) and t_i as in (6).

In the two next sections we are going to discuss the stability of these equilibrium points. In the case $w = 0$ we will obtain a complete result. For all the other equilibria above, we will study the stability as $\delta \rightarrow 0^+$.

4. Instability of Equilibria in the Case $w = 0$

In this section we study the stability of the two families of equilibria (5a), (5b). We will obtain that they all are unstable.

Proposition 7. *Let $Z \geq 0$ and $P = (0, 0, Z, 0)$. Then P is an unstable equilibrium point.*

Proof. Let $\epsilon = \min\left\{\frac{\gamma\lambda}{3\delta}, \frac{\lambda}{3}\right\}$ and let $B_\epsilon(P)$ be the ball with center P and radius ϵ . Let $Q = (x_q, y_q, z_q, w_q)$ be in $B_\epsilon(P) \cap \mathcal{C}$, where \mathcal{C} is the positive cone defined above. Let $u(t, Q) = (x(t), y(t), z(t), w(t))$ be the trajectory starting from Q , so that $u(0, Q) = Q$. We want to prove that there exists $t > 0$ such that $u(t, Q) \notin B_\epsilon(P)$. By contradiction, assume $u(t, Q) \in B_\epsilon(P)$ for every $t \geq 0$. Then $0 < y(t) < \frac{\lambda}{3}$ and $m(x(t), w(t)) < \frac{\gamma\lambda}{3\delta}$ for every $t \geq 0$, thus $\gamma y < \frac{\gamma\lambda}{3}$ and $\delta m(x, w) < \frac{\gamma\lambda}{3}$, hence

$$\gamma\lambda - \gamma y - \delta m(x, w) > \gamma\lambda - \frac{\gamma\lambda}{3} - \frac{\gamma\lambda}{3} = \frac{\gamma\lambda}{3}$$

in $[0, +\infty)$. From this, we obtain

$$y' = y(\lambda\gamma - \gamma y - \delta m(x, w)) > y \frac{\gamma\lambda}{3}$$

and it easily follows that

$$y(t) \geq y_0 e^{\frac{\gamma\lambda}{3}t} \rightarrow +\infty \quad \text{for } t \rightarrow +\infty.$$

This contradicts the assumption $u(t, Q) \in B_\epsilon(P)$ and the contradiction proves our thesis, which in turn implies the instability of P .

□

Proposition 8. *Let $Z \geq 0$ and $P = (0, \lambda, Z, 0)$. Then P is an unstable equilibrium point.*

Proof. Let $\epsilon > 0$ be such that $\epsilon < \frac{\lambda\delta}{2(\delta+\sigma)}$. This implies

$$\delta(\lambda - \epsilon) - \sigma\epsilon > \frac{\delta\lambda}{2}.$$

Let $B_\epsilon(P)$ be the ball centered at P with radius ϵ and let $Q = (x_q, y_q, z_q, w_q) \in B_\epsilon(P) \cap \mathcal{C}$. As in the previous proposition, we argue by contradiction. Let $u(t, Q)$ be the trajectory starting from Q . If it were $u(t, Q) \in B_\epsilon(P)$ for any $t \geq 0$ we would have $y(t) > \lambda - \epsilon$ and $w(t) < \epsilon$ for every $t \geq 0$, hence

$$\delta y + \delta z - \sigma w \geq \delta(\lambda - \epsilon) - \sigma\epsilon > \frac{\delta\lambda}{2}$$

thus

$$w' = m(x, w)(\delta y + \delta z - \sigma w) \geq m(x, w) \frac{\delta\lambda}{2} \geq w \frac{\delta\lambda}{2}.$$

Therefore, we obtain

$$w(t) \geq w_0 e^{\frac{\delta\lambda}{2}t}$$

for every $t \geq 0$ and thus $w \rightarrow +\infty$, contradicting the hypothesis $u(t, Q) \in B_\epsilon(P)$ for every $t \geq 0$. The contradiction proves that there exists $t > 0$ such that $u(t, Q) \notin B_\epsilon(P)$ and since this holds for every $Q \in B_\epsilon(P) \cap \mathcal{C}$, it follows that P is unstable.

□

5. Stability of Equilibria in the Case $w \neq 0$

In this section we study the stability of equilibrium points with $x \leq w$ and $w \neq 0$. These are the equilibria listed above from ii2) to ii8), recalling that we have ruled out the cases ii1) and ii5). We will limit our study to what happens as $\delta \rightarrow 0^+$, assuming that the other parameters remain fixed. We will analyze the Jacobian matrix at equilibria, and this is not possible if an equilibrium lies on the hyperplane $x = w$, because there the vector field F is not differentiable. So let us prove, as first thing, the following proposition.

Proposition 9. *If $P = (x_0, y_0, z_0, w_0)$ is an equilibrium point and $w_0 > 0$, then $x_0 \neq w_0$.*

Proof. We argue by contradiction. If P is an equilibrium point and $x_0 = w_0 > 0$, then by (4a) we obtain

$$-c + \frac{d\mu}{1 + \mu^2} = 0,$$

that is, $\mu = \frac{d + \sqrt{d^2 - 4c^2}}{2c}$ or $\mu = \frac{d - \sqrt{d^2 - 4c^2}}{2c} \leq 1$. Our hypotheses on μ rule out both these possibilities.

□

To study the stability of the problem, we introduce the Jacobian matrix of the differential model (2). Recall that we are studying the case $x \leq w$, $w > 0$, and by the previous proposition we can assume $x < w$. We obtain

$$J(X) = \begin{pmatrix} -c + 2d\mu \frac{xw^3}{(w^2 + \mu^2 x^2)^2} & 0 & 0 & \frac{d\mu x^2(\mu^2 x^2 - w^2)}{(w^2 + \mu^2 x^2)^2} \\ 0 & \gamma\lambda - 2\gamma y - \delta w & 0 & -\delta y \\ 0 & 0 & \frac{kw^2}{w^2 + 1} - \delta w & \frac{2kwz}{(w^2 + 1)^2} - \delta z \\ 0 & \delta w & \delta w & \delta y + \delta z - 2\sigma w \end{pmatrix}.$$

As we can see, $\rho_1 = -c + 2d\mu \frac{xw^3}{(w^2 + \mu^2 x^2)^2}$ is always an eigenvalue of the Jacobian calculated at the critical points. In cases from ii.2) to ii.4), as $x = 0$, we get $\rho_1 = -c < 0$. In cases ii.6) to ii.8) we note that

$$\frac{\mu x w^3}{(w^2 + \mu^2 x^2)^2} = \left(\frac{w}{\mu x}\right)^3 \frac{1}{\left(1 + \left(\frac{w}{\mu x}\right)^2\right)^2} = \frac{t^3}{(1 + t^2)^2}$$

where $t = w/\mu x$, so

$$\rho_1(t) = -c + 2d\mu \frac{xw^3}{(w^2 + \mu^2 x^2)^2} = -c + 2d \frac{t^3}{(1 + t^2)^2} = \frac{c}{d} (2tc - d)$$

since we are in the case in which $d \frac{t}{1 + t^2} = c$. When $t = t_2 = \frac{d + \sqrt{d^2 - 4c^2}}{2c}$ we have

$$\rho_1(t_2) = \frac{c}{d} \sqrt{d^2 - 4c^2} > 0 \quad (17)$$

and when $t = t_1 = \frac{d - \sqrt{d^2 - 4c^2}}{2c}$ we have

$$\rho_1(t_1) = -\frac{c}{d}\sqrt{d^2 - 4c^2} < 0. \quad (18)$$

Thus, in cases from ii.6) to ii.8), the points corresponding to t_2 are always unstable, for any value of the parameters, while for equilibrium points corresponding to $t = t_1$ a more detailed analysis will be carried on.

Let's now study the various cases previously determined.

ii.2) As we just said an eigenvalue of the Jacobian matrix $J(P_i)$ is given by $\rho_1 = -c < 0$. So we have to study the eigenvalues of the submatrix

$$\begin{pmatrix} \gamma\lambda - \delta w & 0 & 0 \\ 0 & \frac{kw^2}{w^2 + 1} - \delta w & \frac{2kwz}{(w^2 + 1)^2} - \delta z \\ \delta w & \delta w & \delta z - 2\sigma w \end{pmatrix}. \quad (19)$$

We note that we have the eigenvalue $\rho_2 = \gamma\lambda - \delta w$ and, for w_1, w_2 as in (9), with $k \geq 2\delta, \delta \rightarrow 0^+$, we obtain the following expansions

$$w_1 = \frac{\delta}{k} + \mathcal{O}(\delta^3), \quad (20)$$

$$w_2 = \frac{k}{\delta} - \frac{\delta}{k} + \mathcal{O}(\delta^3). \quad (21)$$

We need to distinguish the two cases $P_1 = (0, 0, \frac{\sigma}{\delta}w_1, w_1)$ and $P_2 = (0, 0, \frac{\sigma}{\delta}w_2, w_2)$. Due to the continuous dependence of the eigenvalues on the coefficients of the matrix, we can conclude that the point P_1 is unstable as $\delta \rightarrow 0^+$ because $J(P_1)$ has the eigenvalue $\rho_2 = \gamma\lambda - \delta w_1 > 0$, given that $\delta w_1 \rightarrow 0^+$. We have then proved the following proposition.

Proposition 10. *The point $P_1 = (0, 0, \frac{\sigma}{\delta}w_1, w_1)$, with w_1 as in (9), is an unstable equilibrium point as $\delta \rightarrow 0^+$.*

As to P_2 we notice that $\delta w_2 \rightarrow k$ for $\delta \rightarrow 0^+$, and therefore we must take in account the sign of $\gamma\lambda - k$. If $\gamma\lambda - k > 0$ we have that the matrix $J(P_2)$ has an eigenvalue $\rho_2 = \rho_2(\delta)$ which, by continuous dependance of the eigenvalues of a matrix on the entries, tends to the value $\gamma\lambda - k > 0$ as $\delta \rightarrow 0^+$. We then obtain, in this case, the instability for the point P_2 as $\delta \rightarrow 0^+$. Let us now assume $\gamma\lambda - k < 0$. We consider the submatrix $\hat{J}(P_2)$ defined as follows

$$\hat{J}(P_2) = \begin{pmatrix} \frac{kw_2^2}{w_2^2 + 1} - \delta w_2 & \frac{2kw_2z}{(w_2^2 + 1)^2} - \delta z \\ \delta w_2 & \delta z - 2\sigma w_2 \end{pmatrix} = \begin{pmatrix} \mathcal{O}(\delta^2) & -\frac{k\sigma}{\delta} + \mathcal{O}(\delta) \\ k + \mathcal{O}(\delta^2) & -\frac{k\sigma}{\delta} + \mathcal{O}(\delta) \end{pmatrix}. \quad (22)$$

We easily get that, as $\delta \rightarrow 0^+$, the trace of the matrix $\hat{J}(P_2)$ becomes negative and its determinant positive. This implies that the eigenvalues of $\hat{J}(P_2)$ have strictly negative real parts. Now, the eigenvalues of the matrix $J(P_2)$ are given by $\rho_1 = -c < 0, \rho_2 = \gamma\lambda - k$ and by the two eigenvalues of $\hat{J}(P_2)$. Hence, we have proved the following proposition.

Proposition 11. As $\delta \rightarrow 0^+$, the point $P_2 = \left(0, 0, \frac{\sigma}{\delta}w_2, w_2\right)$, with w_2 as in (9), is an unstable equilibrium point if $\gamma\lambda - k > 0$, and it is an asymptotically stable equilibrium point if $\gamma\lambda - k < 0$.

ii.3) We refer to P as (12). We consider $J(P)$ for $\delta \rightarrow 0^+$ and we obtain

$$J(P) = \begin{pmatrix} -c & 0 & 0 & 0 \\ 0 & -\gamma\lambda + \mathcal{O}(\delta^2) & 0 & -\lambda\delta + \mathcal{O}(\delta^3) \\ 0 & 0 & \frac{k\lambda - \sigma}{\sigma^2}\lambda\delta^2 + \mathcal{O}(\delta^4) & 0 \\ 0 & \frac{\lambda}{\sigma}\delta^2 + \mathcal{O}(\delta^4) & \frac{\lambda}{\sigma}\delta^2 + \mathcal{O}(\delta^4) & -\lambda\delta + \mathcal{O}(\delta^3) \end{pmatrix}$$

With some lengthy but standard computations we obtain the following eigenvalues

$$\begin{aligned} \rho_1 &= -c < 0, & \rho_2 &= -\lambda\delta + \mathcal{O}(\delta^2) < 0, \\ \rho_3 &= -\gamma\lambda + \mathcal{O}(\delta) < 0, & \rho_4 &= \frac{\lambda(k\lambda - \sigma)}{\sigma^2}\delta^2 + \mathcal{O}(\delta^4). \end{aligned}$$

As a consequence, we have the following result.

Proposition 12. The point $P = \left(0, \frac{\gamma\lambda\sigma}{\gamma\sigma + \delta^2}, 0, \frac{\gamma\lambda\delta}{\gamma\sigma + \delta^2}\right)$ is an unstable equilibrium point if $k\lambda - \sigma > 0$ and is an asymptotically stable equilibrium point if $k\lambda - \sigma < 0$ as $\delta \rightarrow 0^+$.

ii.4) The Jacobian matrix calculated at the points P_i , $i = 1, 2$ is as follows

$$J(P_i) = \begin{pmatrix} -c & 0 & 0 & 0 \\ 0 & -\gamma\lambda + \delta w & 0 & -\delta y \\ 0 & 0 & 0 & \delta z \left(\frac{2\delta}{k w} - 1 \right) \\ 0 & \delta w & \delta w & -\sigma w \end{pmatrix}, \quad i = 1, 2.$$

We use the Routh-Hurwitz criterion (see [5]). As $\rho_1 = -c$ is an eigenvalue, we can write the characteristic polynomial of $J(P_i)$, $i = 1, 2$, in the following form

$$p_4(\rho) = (\rho + c)p_3(\rho). \quad (23)$$

with $p_3(\rho) = \rho^3 + a_1\rho^2 + a_2\rho + a_3$. According to the Routh-Hurwitz criterion, the necessary and sufficient conditions for all roots of $p_3(\rho)$ to have a strictly negative real part are the following

$$\begin{aligned} a_1 &> 0 \\ a_1 a_2 &> a_3 > 0. \end{aligned} \quad (24)$$

Let's study the stability of the point P_1 obtained by setting $w = w_1$ as in (9). After some lengthy but standard computations we get

$$\begin{aligned}
 a_1 &= \gamma\lambda + \mathcal{O}(\delta) > 0 \\
 a_2 &= \frac{\gamma\lambda\sigma}{k}\delta + \mathcal{O}(\delta^3) > 0 \\
 a_3 &= \frac{\gamma\lambda(k\lambda - \sigma)}{k^2}\delta^3 + \mathcal{O}(\delta^5) \\
 a_1 a_2 - a_3 &= \frac{\gamma^2\lambda^2\sigma}{k}\delta + \mathcal{O}(\delta^2) > 0.
 \end{aligned}$$

Hence, we get instability when $k\lambda - \sigma < 0$, stability when $k\lambda - \sigma > 0$. However, the asymptotic development of the z component of P_1 gives

$$z = -\lambda + \left(\frac{\delta}{\gamma} + \frac{\sigma}{\delta}\right)w_1 = -\frac{1}{k}(k\lambda - \sigma) + \mathcal{O}(\delta^3),$$

hence, in the case of stability, the equilibrium P_1 is not in $\bar{\mathcal{C}}$, as $\delta \rightarrow 0$. We conclude that the following proposition holds true.

Proposition 13. *The point $P_1 = \left(0, \lambda - \frac{\delta}{\gamma}w_1, -\lambda + \left(\frac{\delta}{\gamma} + \frac{\sigma}{\delta}\right)w_1, w_1\right)$, $k \geq 2\delta$, with w_1 as in (9), is an unstable equilibrium as $\delta \rightarrow 0^+$ if $k\lambda - \sigma < 0$. If $k\lambda - \sigma > 0$ the point P_1 is an asymptotically stable equilibrium of the ODE system (2), but it is not in $\bar{\mathcal{C}}$, as $\delta \rightarrow 0^+$.*

We now study the point P_2 with $w = w_2$ as in (9). After some computations, we obtain the following results for the coefficients of p_3 :

$$\begin{aligned}
 a_1 &= \frac{k\sigma}{\delta} + \mathcal{O}(1) > 0 \\
 a_2 &= \frac{k\gamma\lambda\sigma}{\delta} + \mathcal{O}(\delta) > 0 \\
 a_3 &= \frac{k^2\sigma(\gamma\lambda - k)}{\delta} + \mathcal{O}(\delta) \\
 a_1 a_2 - a_3 &= \frac{\gamma\lambda\sigma^2k^2}{\delta^2} + \mathcal{O}(\delta^{-1}) > 0.
 \end{aligned}$$

Hence, by applying the Routh-Hurwitz criterion again, we get that, as $\delta \rightarrow 0^+$, the critical point P_2 is stable when $\gamma\lambda - k > 0$, unstable when $\gamma\lambda - k < 0$. But in P_2 we have

$$y = \lambda - \frac{\delta}{\gamma}w_2 = \lambda - \frac{k}{\gamma} + \mathcal{O}(\delta^2).$$

Hence, in the instability case $\gamma\lambda - k < 0$, we have $y < 0$ as $\delta \rightarrow 0^+$, and P_2 is not in $\bar{\mathcal{C}}$. We have then proved the following proposition.

Proposition 14. *The point $P_2 = \left(0, \lambda - \frac{\delta}{\gamma}w_2, -\lambda + \left(\frac{\delta}{\gamma} + \frac{\sigma}{\delta}\right)w_2, w_2\right)$, $k \geq 2\delta$, with w_2 as in (9), is an asymptotically stable equilibrium point if $\gamma\lambda - k > 0$ as $\delta \rightarrow 0^+$. When $\gamma\lambda - k < 0$ P_2 is an unstable equilibrium point of the ODE system (2), but it is not in $\bar{\mathcal{C}}$, as $\delta \rightarrow 0^+$.*

ii.6) In this case we have 4 critical point $P_{i,j} = P(t_i, w_j)$, $i, j = 1, 2$ as in (14) with t_i, w_j given in (6), (9) respectively. For sake of simplicity, we report the expression of the Jacobian matrix calculated at these points.

$$J(P_{i,j}) = \begin{pmatrix} -c + 2d\mu \frac{xw^3}{(w^2 + \mu^2 x^2)^2} & 0 & 0 & \frac{d\mu x^2(\mu^2 x^2 - w^2)}{(w^2 + \mu^2 x^2)^2} \\ 0 & \gamma\lambda - \delta w & 0 & 0 \\ 0 & 0 & 0 & \frac{2kwz}{(w^2 + 1)^2} - \delta z \\ 0 & \delta w & \delta w & -\sigma w \end{pmatrix}$$

Based on what is established in (17), the following proposition holds.

Proposition 15. *The points $P_{2,j} = \left(\frac{w_j}{\mu t_2}, 0, \frac{\sigma w_j}{\delta}, w_j\right)$, $j = 1, 2$, $k \geq 2\delta$, $d \geq 2c$, with t_2 as in (6) and w_j as in (9), are unstable equilibrium points.*

Let's now begin the study in the case $t = t_1 = \frac{d - \sqrt{d^2 - 4c^2}}{2c}$ where $\rho_1(t_1) < 0$ as seen in (18). Let's then analyze the eigenvalue $\rho_2 = \gamma\lambda - \delta w$. We have

$$\begin{aligned} \rho_2 &= \gamma\lambda - k + \mathcal{O}(\delta^2) \quad \text{if } w = w_2 \\ \rho_2 &= \gamma\lambda + \mathcal{O}(\delta^2) \quad \text{if } w = w_1. \end{aligned}$$

Therefore, the Jacobian calculated at point $P_{1,1}$ has a positive eigenvalue for $\delta \rightarrow 0^+$ and thus the equilibrium $P_{1,1}$ is unstable. As for point $P_{1,2}$, we must distinguish between two cases. If $\gamma\lambda - k > 0$ then this equilibrium will also be unstable, so let us see what happens if $\gamma\lambda - k < 0$. In this case the eigenvalues of $J(P_{1,2})$ are as follows. A first one is given by $\rho_1(t_1)$ as in (18), a second one is $\rho_2 = \gamma\lambda - k + \mathcal{O}(\delta^2) < 0$. The last two are the eigenvalues of the following matrix $\hat{J}(P_{1,2})$:

$$\hat{J}(P_{1,2}) = \begin{pmatrix} 0 & \frac{2kw_2z}{(w_2^2 + 1)^2} - \delta z \\ \delta w_2 & -\sigma w_2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{k\sigma}{\delta} + \mathcal{O}(\delta) \\ k + \mathcal{O}(\delta^2) & -\frac{k\sigma}{\delta} + \mathcal{O}(\delta) \end{pmatrix}.$$

As $\delta \rightarrow 0^+$ this matrix has strictly negative trace and strictly positive determinant, so its eigenvalues have strictly negative real parts. Thus we have proved that, in this case, the point $P_{1,2}$ is a stable equilibrium. Let's summarize everything in the following proposition.

Proposition 16. *The point $P_{1,1} = \left(\frac{w_1}{\mu t_1}, 0, \frac{\sigma w_1}{\delta}, w_1\right)$, $k \geq 2\delta$, $d \geq 2c$, with t_1 as in (6) and w_1 as in (9), is an unstable equilibrium. The point $P_{1,2} = \left(\frac{w_2}{\mu t_1}, 0, \frac{\sigma w_2}{\delta}, w_2\right)$, $k \geq 2\delta$, $d \geq 2c$, with t_1 as in (6) and w_2 as in (9), is an unstable equilibrium if $\gamma\lambda - k > 0$ and is an asymptotically stable equilibrium if $\gamma\lambda - k < 0$.*

ii.7) Based on what is established in (17), the following proposition holds.

Proposition 17. *The point $P_2 = \left(\frac{\delta\gamma\lambda}{\mu t_2(\gamma\sigma + \delta^2)}, \frac{\gamma\lambda\sigma}{\gamma\sigma + \delta^2}, 0, \frac{\gamma\lambda\delta}{\gamma\sigma + \delta^2}\right)$ is an unstable equilibrium point.*

Let's then analyze the behavior of the Jacobian matrix calculated at point $P_1 = \left(\frac{\delta\gamma\lambda}{\mu t_1(\gamma\sigma + \delta^2)}, \frac{\gamma\lambda\sigma}{\gamma\sigma + \delta^2}, 0, \frac{\delta\gamma\lambda}{\gamma\sigma + \delta^2} \right)$. In this case, the Jacobian matrix is almost the same one computed at point P in case ii.3). Indeed, the two points differ only for their first entry, and the two Jacobian matrices only for the entries $j_{1,1}$ (first row, first column). Hence, the two matrices have different ρ_1 eigenvalues ($\rho_1 = -c$ in case ii.3), $\rho_1 = -\frac{c}{d}\sqrt{d^2 - 4c^2}$ in the present case), but both are negative, while the other three eigenvalues are the same as in the case ii.3). Using the results we have obtained in that case, we get the following proposition

Proposition 18. *The point $P_1 = \left(\frac{\delta\gamma\lambda}{\mu t_1(\gamma\sigma + \delta^2)}, \frac{\gamma\lambda\sigma}{\gamma\sigma + \delta^2}, 0, \frac{\gamma\lambda\delta}{\gamma\sigma + \delta^2} \right)$ is an unstable equilibrium if $k\lambda - \sigma > 0$ and is an asymptotically stable equilibrium if $k\lambda - \sigma < 0$ as $\delta \rightarrow 0^+$.*

ii.8) In this case we have 4 critical point $P_{i,j} = P(t_i, w_j)$, $i, j = 1, 2$ as in (16) with t_i, w_j given in (6), (9) respectively. Based on what is established in (17), the following proposition holds.

Proposition 19. *The points $P_{2,j} = \left(\frac{w_j}{\mu t_2}, \lambda - \frac{\delta}{\gamma}w_j, -\lambda + \left(\frac{\delta}{\gamma} + \frac{\sigma}{\delta} \right)w_j, w_j \right)$, $j = 1, 2$, $k \geq 2\delta$, $d \geq 2c$ with t_2 as in (6) and w_j as in (9), are unstable equilibria.*

Regarding the points $P_{1,j}$ ($j = 1, 2$) we argue as we have just done for the previous case ii.7). Indeed, in this case, the Jacobian matrix is almost the same one computed at point P_2 in case ii.4): the two points differ only for their first entry, and the two Jacobian matrices only for the entries $j_{1,1}$, hence the two matrices have different ρ_1 eigenvalues, which are both negative, while the other three eigenvalues are the same in the two cases. Using the arguments and the results of case ii.4), we get the following proposition

Proposition 20. *The point $P_{1,1} = \left(\frac{w_1}{\mu t_1}, \lambda - \frac{\delta}{\gamma}w_1, -\lambda + \left(\frac{\delta}{\gamma} + \frac{\sigma}{\delta} \right)w_1, w_1 \right)$, with w_1 as in (9), is an unstable equilibrium as $\delta \rightarrow 0^+$ if $k\lambda - \sigma < 0$. If $k\lambda - \sigma > 0$ the point $P_{1,1}$ is an asymptotically stable equilibrium of the ODE system (2), but it is not in \bar{C} , as $\delta \rightarrow 0^+$. The point $P_{1,2} = \left(\frac{w_2}{\mu t_1}, \lambda - \frac{\delta}{\gamma}w_2, -\lambda + \left(\frac{\delta}{\gamma} + \frac{\sigma}{\delta} \right)w_2, w_2 \right)$, $k \geq 2\delta$, with w_2 as in (9) is an asymptotically stable equilibrium if $\gamma\lambda - k > 0$ as $\delta \rightarrow 0^+$. When $\gamma\lambda - k < 0$ $P_{1,2}$ is an unstable equilibrium of ODE system (2), but it is not in \bar{C} , as $\delta \rightarrow 0^+$.*

6. Fixed points and stability if $x > w$

We are now going to look for the equilibria, and their stability, in the case $x > w$. To compute the equilibria we need to consider the system

$$x \left[-c + d \frac{\mu x w}{w^2 + \mu^2 x^2} \right] = 0 \quad (25a)$$

$$y(\gamma\lambda - \gamma y - \delta x) = 0 \quad (25b)$$

$$xz \left[k \frac{w}{xw + 1} - \delta \right] = 0 \quad (25c)$$

$$x[\delta y + \delta z - \sigma w] = 0 \quad (25d)$$

We note that x cannot be zero since $x > w \geq 0$. Therefore, as equation (4a) coincides with (25a), we will always have the relation

$$w = t_i \mu x, \quad i = 1, 2 \quad (26)$$

with t_i given by (6). Moreover, it cannot be $y = z = 0$ because this would lead, from (25d), to $w = 0$ and, as just stated, $x = 0$.

We distinguish the following cases:

iii.1) $y = 0$ and $z \neq 0$. From (25d) we obtain $z = \frac{\sigma}{\delta}w$. From (25c), we obtain

$$\begin{aligned} w_1(t) &= \frac{k\mu t - \sqrt{k^2\mu^2 t^2 - 4\delta^2 t\mu}}{2\delta}, \\ w_2(t) &= \frac{k\mu t + \sqrt{k^2\mu^2 t^2 - 4\delta^2 t\mu}}{2\delta}, \quad k^2\mu t - 4\delta^2 \geq 0. \end{aligned} \quad (27)$$

Let us set $w_{i,j} = w_j(t_i)$, with $i, j = 1, 2$, and t_i as in (6). We have the expression for x from (26). Hence, we obtain the following equilibria

$$P_{i,j} = \left(\frac{w_{i,j}}{\mu t_i}, 0, \frac{\sigma}{\delta} w_{i,j}, w_{i,j} \right) \quad i, j = 1, 2. \quad (28)$$

iii.2) $z = 0$ and $y \neq 0$. From (25d) and (26), we have $y = \frac{\sigma}{\delta}w = \sigma t_i \mu x / \delta$ that substitute in (25b) give us the expression for x

$$x = \frac{\delta \gamma \lambda}{\delta^2 + \gamma \mu \sigma t_i}, \quad i = 1, 2. \quad (29)$$

Then we have two equilibrium points

$$P_i = \left(\frac{\delta \gamma \lambda}{\delta^2 + \gamma \mu \sigma t_i}, \frac{\gamma \lambda \mu \sigma t_i}{\delta^2 + \gamma \mu \sigma t_i}, 0, \frac{\delta \gamma \lambda \mu t_i}{\delta^2 + \gamma \mu \sigma t_i} \right) \quad i = 1, 2. \quad (30)$$

iii.3) $y \neq 0$, $z \neq 0$. From (25c) we obtain the expression (27) for w and from (26) the expression for x . Then (25b) gives us the expression for y and finally the (25d) the value of z . So we have four equilibrium points

$$P_{i,j} = \left(\frac{w_{i,j}}{\mu t_i}, \lambda - \frac{\delta}{\gamma \mu t_i} w_{i,j}, -\lambda + \left(\frac{\delta}{\gamma \mu t_i} + \frac{\sigma}{\delta} \right) w_{i,j}, w_{i,j} \right) \quad i, j = 1, 2. \quad (31)$$

with $k^2\mu t_i - 4\delta^2 \geq 0$.

Let us now study the stability of these equilibria. For this, we compute the Jacobian matrix of the differential model (2) with $x > w$. Recalling that $c = d \frac{t}{t^2 + 1}$ and considering (26) we obtain

$$J(X) = \begin{pmatrix} \frac{dt(t^2 - 1)}{(t^2 + 1)^2} & 0 & 0 & -\frac{d(t^2 - 1)}{\mu(t^2 + 1)^2} \\ -\delta y & \gamma \lambda - 2\gamma y - \delta x & 0 & 0 \\ \frac{k\omega z}{(x\omega + 1)^2} - \delta z & 0 & \frac{kx\omega}{x\omega + 1} - \delta x & \frac{kxz}{(x\omega + 1)^2} \\ \delta y + \delta z - \sigma w & \delta x & \delta x & -\sigma x \end{pmatrix},$$

so that $j_{1,4} = -\frac{j_{1,1}}{\mu t}$. Since $t_1 \in (0, 1)$ and $t_2 > 1$ we have that if $t = t_2$ then $j_{1,1} > 0$, $j_{1,4} < 0$ and vice versa for $t = t_1$.

iii.1) As $y = 0$, in this case the entry $j_{2,2} = \gamma \lambda - \delta x$ is an eigenvalue (with $x = x_{i,j} = w_{i,j} / \mu t_i$). Let us call ρ_1 this eigenvalue. Considering the asymptotic for $\delta \rightarrow 0^+$ we obtain

$$\begin{aligned} w_1(t) &= \frac{1}{k}\delta + \mathcal{O}(\delta^3) \\ w_2(t) &= \frac{k\mu t}{\delta} + \mathcal{O}(\delta) \end{aligned} \quad (32)$$

and we can state

$$\begin{aligned} \rho_1 &= \gamma\lambda + \mathcal{O}(\delta^2) && \text{if } w = w_{i,1}, i = 1,2, \\ \rho_1 &= \gamma\lambda - k + \mathcal{O}(\delta^2) && \text{if } w = w_{i,2}, i = 1,2. \end{aligned} \quad (33)$$

Therefore, the Jacobian matrix calculated at the points $P_{i,1}$ for $\delta \rightarrow 0^+$ has at least one positive eigenvalue, and therefore the equilibria $P_{i,1}$ are unstable. We thus state the following proposition.

Proposition 21. *The critical points $P_{i,1} = \left(\frac{w_{i,1}}{\mu t_i}, 0, \frac{\sigma}{\delta} w_{i,1}, w_{i,1}\right)$, $i = 1, 2$ with $w_{i,1}$ given in (27) and t_i given in (6), are unstable equilibria for $\delta \rightarrow 0^+$.*

The behavior of the points $P_{i,2}$ depends instead on the sign of $\gamma\lambda - k$. If $\gamma\lambda - k > 0$ these equilibria are unstable. We then study the behavior of such points under the assumption $\gamma\lambda - k < 0$. We examine the asymptotic behavior for $\delta \rightarrow 0^+$. The asymptotic of the Jacobian matrix $J(P_{i,2})$ becomes

$$J(P_{i,2}) = \begin{pmatrix} \frac{dt(t^2 - 1)}{(t^2 + 1)^2} & 0 & 0 & -\frac{d(t^2 - 1)}{\mu(t^2 + 1)^2} \\ 0 & \gamma\lambda - k + \mathcal{O}(\delta^2) & 0 & 0 \\ -\frac{k\mu\sigma t}{\delta} + \mathcal{O}(\delta) & 0 & 0 & \frac{\sigma}{k\mu t}\delta + \mathcal{O}(\delta^3) \\ 0 & k + \mathcal{O}(\delta^2) & k + \mathcal{O}(\delta^2) & -\frac{k\sigma}{\delta} + \mathcal{O}(\delta) \end{pmatrix}.$$

We use the Routh-Hurwitz criterion. We introduce the characteristic polynomial of $J(P_{i,2})$

$$p_4(\rho) = (\rho - \gamma\lambda + k + \mathcal{O}(\delta^2))(\rho^3 + a_1\rho^2 + a_2\rho + a_3). \quad (34)$$

Through some lengthy but standard computations we get

$$\begin{aligned} a_1 &= \frac{k\sigma}{\delta} + \mathcal{O}(1) \\ a_2 &= \frac{k\sigma dt(1 - t^2)}{(t^2 + 1)^2\delta} + \mathcal{O}(\delta) \\ a_3 &= \frac{(1 - t^2)dk^2\sigma t}{(t^2 + 1)^2\delta} + \mathcal{O}(\delta). \end{aligned}$$

Let us recall that the necessary and sufficient condition for the roots of the polynomial $p_3(\rho) = \rho^3 + a_1\rho^2 + a_2\rho + a_3$ to have strictly negative real parts is that $a_1, a_3 > 0$ e $a_1a_2 - a_3 > 0$. We recall that $t_1 \in (0, 1)$ and $t_2 > 1$ as given in (6). Therefore, $a_2 < 0$ and $a_3 < 0$ if $t = t_2$ making $P_{2,2}$ unstable. Conversely, $a_2 > 0$ and $a_3 > 0$ if $t = t_1$ with

$$a_1a_2 - a_3 = \frac{k^2\sigma^2 dt(1 - t^2)}{(t^2 + 1)^2\delta^2} + \mathcal{O}(\delta^{-1}) > 0.$$

We can therefore summarize in the following proposition.

Proposition 22. *The equilibrium point $P_{2,2} = \left(\frac{w_{2,2}}{\mu t_2}, 0, \frac{\sigma}{\delta} w_{2,2}, w_{2,2}\right)$, with $w_{2,2}$ given in (27) and t_2 given in (6), is unstable for $\delta \rightarrow 0^+$. The equilibrium point $P_{1,2} = \left(\frac{w_{1,2}}{\mu t_1}, 0, \frac{\sigma}{\delta} w_{1,2}, w_{1,2}\right)$, with*

$w_{1,2}$ given in (27) and t_1 given in (6), is unstable for $\delta \rightarrow 0^+$ if $\gamma\lambda - k > 0$. The equilibrium $P_{1,2}$ is asymptotically stable for $\delta \rightarrow 0^+$ if $\gamma\lambda - k < 0$.

iii.2) In this case, we have two critical points P_i , $i = 1, 2$ as in (30). By doing the asymptotic expansion of the variables, for $\delta \rightarrow 0^+$, we obtain

$$\begin{aligned} x_i &= \frac{\delta\gamma\lambda}{\delta^2 + \gamma\mu\sigma t_i} = \frac{\lambda}{\mu\sigma t_i}\delta + \mathcal{O}(\delta^3) \\ y_i &= \frac{\gamma\lambda\mu\sigma t_i}{\delta^2 + \gamma\mu\sigma t_i} = \lambda + \mathcal{O}(\delta^2) \\ z_i &= 0 \\ w_i &= \frac{\delta\gamma\lambda\mu t_i}{\delta^2 + \gamma\mu\sigma t_i} = \frac{\lambda}{\sigma}\delta + \mathcal{O}(\delta^3) \end{aligned} \quad (35)$$

and the asymptotic Jacobian matrix calculated at these points becomes

$$J(P_i) = \begin{pmatrix} \frac{dt(t^2 - 1)}{(t^2 + 1)^2} & 0 & 0 & -\frac{d(t^2 - 1)}{\mu(t^2 + 1)^2} \\ -\lambda\delta + \mathcal{O}(\delta^3) & -\gamma\lambda + \mathcal{O}(\delta^2) & 0 & 0 \\ 0 & 0 & \frac{\lambda(k\lambda - \sigma)}{\mu\sigma^2 t}\delta^2 + \mathcal{O}(\delta^4) & 0 \\ 0 & \frac{\lambda}{\mu\sigma t}\delta^2 + \mathcal{O}(\delta^4) & \frac{\lambda}{\mu\sigma t}\delta^2 + \mathcal{O}(\delta^4) & -\frac{\lambda}{\mu t}\delta + \mathcal{O}(\delta^3) \end{pmatrix},$$

with $t = t_1, t_2$. For $\delta = 0$, this matrix reduces to a diagonal matrix with an eigenvalue given by

$$\rho_1 = \frac{dt(t^2 - 1)}{(t^2 + 1)^2}.$$

We have repeatedly noted that $\rho_1 > 0$ if $t = t_2$ and $\rho_1 < 0$ if $t = t_1$. Due to the continuity of eigenvalues with respect to the elements of the matrix, in the case $t = t_2$ for $\delta \rightarrow 0^+$ there is an eigenvalue of the Jacobian matrix that tends to a positive value, and therefore the critical point P_2 is unstable. If $t = t_1$, we obtain instead $\rho_1 < 0$, hence we must apply the Routh-Hurwitz criterion to the characteristic polynomial $p_4(\rho)$. The criterion states, in this case, that the necessary and sufficient conditions for all roots of characteristic polynomial $p_4(\rho) = \rho^4 + a_1\rho^3 + a_2\rho^2 + a_3\rho + a_4$ to have a strictly negative real part are the following

$$\begin{cases} a_i > 0 & i = 0, \dots, 4 \\ a_1a_2 - a_3 > 0 \\ a_1a_2a_3 - a_1^2a_4 - a_3^2 > 0. \end{cases} \quad (36)$$

After lengthy computations we obtain the following asymptotic developments of the coefficients of the polynomial $p_4(\rho)$ as $\delta \rightarrow 0^+$

$$\begin{aligned} a_4 &= -\frac{(k\lambda - \sigma)\gamma\lambda^3(1-t^2)d}{\mu^2\sigma^2t(t^2+1)^2}\delta^3 + \mathcal{O}(\delta^5) \\ a_3 &= \frac{\gamma\lambda^2d(1-t^2)}{\mu(t^2+1)^2}\delta + \mathcal{O}(\delta^2) \\ a_2 &= \frac{\gamma\lambda(1-t^2)dt}{(t^2+1)^2} + \mathcal{O}(\delta) \\ a_1 &= \gamma\lambda + \frac{dt(1-t^2)}{(t^2+1)^2} + \mathcal{O}(\delta) \\ a_1a_2 - a_3 &= a_1a_2 + \mathcal{O}(\delta) \\ a_1a_2a_3 - a_1^2a_4 - a_3^2 &= a_1a_2a_3 + \mathcal{O}(\delta^2). \end{aligned} \quad (37)$$

Recalling that $1 - t_1^2 > 0$, we have that $a_1, a_2, a_3 > 0$ for $t = t_1$. Also, we have $a_4 > 0$ if $k\lambda - \sigma < 0$. Analyzing the orders of magnitude of the coefficients, the last two conditions are definitely met when the leading terms of the coefficients of the characteristic polynomial are positive. On the other hand $p_4(\rho)$ obviously has a positive root if $a_4 < 0$. We can therefore conclude with the following proposition

Proposition 23. *The point $P_2 = \left(\frac{\delta\gamma\lambda}{\delta^2 + \gamma\mu\sigma t_2}, \frac{\gamma\lambda\mu\sigma t_2}{\delta^2 + \gamma\mu\sigma t_2}, 0, \frac{\delta\gamma\lambda\mu t_2}{\delta^2 + \gamma\mu\sigma t_2} \right)$, for t_2 defined in (6), is an unstable equilibrium point. The point $P_1 = \left(\frac{\delta\gamma\lambda}{\delta^2 + \gamma\mu\sigma t_1}, \frac{\gamma\lambda\mu\sigma t_1}{\delta^2 + \gamma\mu\sigma t_1}, 0, \frac{\delta\gamma\lambda\mu t_1}{\delta^2 + \gamma\mu\sigma t_1} \right)$, for t_1 defined in (6), is an asymptotically stable equilibrium point if $k\lambda - \sigma < 0$ and is unstable if $k\lambda - \sigma > 0$.*

iii.3) In this case, we have four equilibrium points varying with the values of $w_{i,j}$ see (27) and t_i , see (6). We start with $w_{i,1}$ for $i = 1, 2$. Considering the asymptotic expansions of the variables for $\delta \rightarrow 0^+$, we obtain

$$\begin{aligned} x_{i,1} &= \frac{w_{i,1}}{\mu t_i} = \frac{\delta}{k\mu t_i} + \mathcal{O}(\delta^3) \\ y_{i,1} &= \lambda - \frac{\delta}{\gamma}x_{i,1} = \lambda + \mathcal{O}(\delta^2) \\ z_{i,1} &= -y_{i,1} + \frac{\sigma}{\delta}w_{i,1} = \frac{\sigma}{k} - \lambda + \mathcal{O}(\delta^2) \\ w_{i,1} &= \frac{k\mu t_i - \sqrt{k^2\mu^2 t_i^2 - 4\delta^2 t_i \mu}}{2\delta} = \frac{\delta}{k} + \mathcal{O}(\delta^3). \end{aligned} \quad (38)$$

Notice that, when $\lambda k - \sigma > 0$, we obtain $z_{i,1} < 0$ as $\delta \rightarrow 0^+$, hence in this case the points $P_{i,1}$ are not in the cone \mathcal{C} .

The developments above, when substituted into the Jacobian matrix, lead to

$$J(P_{i,1}) = \begin{pmatrix} \frac{dt(t^2-1)}{(t^2+1)^2} & 0 & 0 & -\frac{d(t^2-1)}{\mu(t^2+1)^2} \\ -\lambda\delta + \mathcal{O}(\delta^3) & -\gamma\lambda + \mathcal{O}(\delta^2) & 0 & 0 \\ \mathcal{O}(\delta^3) & 0 & 0 & \frac{\sigma - k\lambda}{k\mu t} \delta + \mathcal{O}(\delta^3) \\ 0 & \frac{\delta^2}{kt\mu} + \mathcal{O}(\delta^4) & \frac{\delta^2}{kt\mu} + \mathcal{O}(\delta^4) & -\frac{\sigma}{kt\mu} \delta + \mathcal{O}(\delta^3) \end{pmatrix}.$$

We now argue as in the previous case, noticing that, when $\delta = 0$, this matrix reduces to a diagonal matrix with an eigenvalue

$$\rho_1 = \frac{dt(t^2 - 1)}{(t^2 + 1)^2}.$$

As we know, this value satisfies $\rho_1 > 0$ if $t = t_2 > 1$ and $\rho_1 < 0$ if $t = t_1 \in (0, 1)$. Therefore, when $t = t_2$, due to the continuity of eigenvalues with respect to the elements of the matrix, there exists at least one positive eigenvalue as $\delta \rightarrow 0^+$, and thus the equilibrium point $P_{2,1}$ is unstable.

We now proceed by studying the stability of the point $P_{1,1}$. We apply again Routh-Hurwitz criterion, and we study the characteristic polynomial $p_4(\rho)$ of the matrix and its coefficients a_i , $i = 1, \dots, 4$. We want to study what happens for $\delta \rightarrow 0^+$, so we look at the asymptotic behavior of the a_i 's. We start noticing that

$$\begin{aligned} a_4 &= \frac{(k\lambda - \sigma)\gamma\lambda(1 - t^2)d}{k^2\mu^2t(t^2 + 1)^2}\delta^3 + \mathcal{O}(\delta^5) \\ a_3 &= \frac{\gamma\lambda\sigma(1 - t^2)dt}{k\mu t(t^2 + 1)^2}\delta + \mathcal{O}(\delta^3) \\ a_2 &= \frac{\gamma\lambda(1 - t^2)dt}{(t^2 + 1)^2} + \mathcal{O}(\delta) \\ a_1 &= \frac{\gamma\lambda(1 + t^2)^2 + dt(1 - t^2)}{(t^2 + 1)^2} + \mathcal{O}(\delta) \\ a_1a_2 - a_3 &= a_1a_2 + \mathcal{O}(\delta) \\ a_1a_2a_3 - a_1^2a_4 - a_3^2 &= a_1a_2a_3 + \mathcal{O}(\delta^2) \end{aligned} \quad (39)$$

We have that $a_1, a_2, a_3 > 0$ for $t = t_1$ and $a_4 > 0$ for $t = t_1$ and $k\lambda - \sigma > 0$. Under these conditions, the $a_1a_2 - a_3$ and $a_1a_2a_3 - a_1^2a_4 - a_3^2$ are also positive, as $\delta \rightarrow 0^+$.

We recall that the equilibria $P_{i,1}$ are not in \mathcal{C} when $k\lambda - \sigma > 0$. We collect our results in the following proposition:

Proposition 24. $P_{2,1}$ is an unstable equilibrium. If $k\lambda - \sigma > 0$ the equilibrium $P_{1,1}$ is asymptotically stable but is not in $\bar{\mathcal{C}}$, together with $P_{2,1}$, as $\delta \rightarrow 0^+$. If $k\lambda - \sigma < 0$ the two equilibria are in \mathcal{C} and are unstable, as $\delta \rightarrow 0^+$.

We now proceed with the study of the stability of the points $P_{i,2}$. The asymptotic developments in this case are

$$\begin{aligned} x_{i,2} &= \frac{w_{i,2}}{\mu t_i} = \frac{k}{\delta} + \mathcal{O}(\delta) \\ y_{i,2} &= \lambda - \frac{\delta}{\gamma}x_{i,2} = \lambda - \frac{k}{\gamma} + \mathcal{O}(\delta^2) \\ z_{i,2} &= -y_{i,2} + \frac{\sigma}{\delta}w_{i,2} = \frac{\sigma k \mu t_i}{\delta^2} + \mathcal{O}(1) \\ w_{i,2} &= \frac{k\mu t_i + \sqrt{k^2\mu^2t_i^2 - 4\delta^2t_i\mu}}{2\delta} = \frac{k\mu t_i}{\delta} + \mathcal{O}(\delta) \end{aligned} \quad (40)$$

which lead to the Jacobian matrix calculated at $P_{i,2}$

$$J(P_{i,2}) = \begin{pmatrix} \frac{dt(t^2-1)}{(t^2+1)^2} & 0 & 0 & -\frac{d(t^2-1)}{\mu(t^2+1)^2} \\ \left(-\lambda + \frac{k}{\gamma}\right)\delta + \mathcal{O}(\delta^3) & -\gamma\lambda + k + \mathcal{O}(\delta^2) & 0 & 0 \\ -\frac{\sigma k \mu t}{\delta} + \mathcal{O}(\delta) & 0 & \mathcal{O}(\delta^6) & \frac{\sigma \delta}{k \mu t} + \mathcal{O}(\delta^3) \\ 0 & k + \mathcal{O}(\delta^2) & k + \mathcal{O}(\delta^2) & -\frac{\sigma k}{\delta} + \mathcal{O}(\delta) \end{pmatrix}.$$

We now apply the Routh-Hurwitz criterion to the characteristic polynomial $p_4(\rho) = \rho^4 + a_1\rho^3 + a_2\rho^2 + a_3\rho + a_4$ with

$$\begin{aligned} a_4 &= \frac{(\gamma\lambda - k)(1 - t^2)dk^2\sigma t}{(t^2 + 1)^2\delta} + \mathcal{O}(\delta) \\ a_3 &= \frac{\gamma\lambda\sigma k(1 - t^2)dt}{(t^2 + 1)^2\delta} + \mathcal{O}(\delta) \\ a_2 &= k\sigma \left[(\gamma\lambda - k) + \frac{dt(1 - t^2)}{(t^2 + 1)^2} \right] \frac{1}{\delta} + \mathcal{O}(1) \\ a_1 &= \frac{\sigma k}{\delta} + \mathcal{O}(1) \end{aligned} \quad (41)$$

If we assume $\gamma\lambda - k < 0$ and $t = t_1 \in (0, 1)$ we have $a_4 < 0$, while if $t = t_2 > 1$ it holds $a_2, a_3 < 0$, and in both cases we obtain that there is an eigenvalue with strictly positive real part. Hence, in this case, the two equilibrium points we are dealing with are unstable. Notice also that from $\gamma\lambda - k < 0$ we derive $y_{i,2} < 0$ as $\delta \rightarrow 0^+$, so in this case the two points are not in the positive cone.

Let us now assume $\gamma\lambda - k > 0$. If $t = t_2 > 1$ we obtain $a_4 < 0$ and the equilibrium is unstable. If $t = t_1 \in (0, 1)$ all the coefficients a_i are positive. For the stability we have then to verify that, at least for $\delta \rightarrow 0^+$, it holds

$$a_1a_2 > a_3, \quad a_1a_2a_3 > a_1^2a_4 + a_3^2.$$

We have that a_1a_2 is led by a term of the type c/δ^2 , with $c > 0$, while $a_3 = \mathcal{O}(\delta^{-1})$, so the first inequality above is verified for small δ 's. Regarding the last condition, that is $a_1a_2a_3 - a_1^2a_4 - a_3^2 > 0$, we notice that $a_3^2 = \mathcal{O}(\delta^{-2})$ while $a_1a_2a_3 = \mathcal{O}(\delta^{-3})$ and $a_1^2a_4 = \mathcal{O}(\delta^{-3})$. Therefore, the last condition is simplified by requiring $a_2a_3 - a_1a_4 > 0$. With simple calculations, we obtain

$$a_2a_3 - a_1a_4 = \frac{\sigma^2 k^2 dt(1 - t^2)}{(t^2 + 1)^2} \left[(\gamma\lambda - k)^2 + \gamma\lambda \frac{dt(1 - t^2)}{(t^2 + 1)^2} \right] \frac{1}{\delta^2} + \mathcal{O}(\delta^{-1}),$$

so the second condition is verified for $t = t_1$ and $\delta \rightarrow 0^+$. We summarize in the following proposition

Proposition 25. *If $\gamma\lambda - k < 0$ the two points $P_{i,2}$ are both unstable equilibria of the system (2), but are not in $\bar{\mathcal{C}}$, as $\delta \rightarrow 0^+$. If $\gamma\lambda - k > 0$ and $\delta \rightarrow 0^+$, the point $P_{2,2}$ is an unstable equilibrium, while the point $P_{1,2}$ is an asymptotically stable equilibrium (and both are in \mathcal{C}).*

7. Simulation results

We now introduce the results derived from simulations conducted with Matlab's ode45 solver, which applies an explicit adaptive Runge-Kutta method. In all depicted graphs, the evolution of variables is illustrated using distinct colors: the population in magenta, renewable resources in green, non-renewable resources in black, and accumulated wealth in blue. Dashed lines indicate the coordinates of the equilibrium point under consideration, with matching colors for corresponding components (i.e., magenta for the population, and so forth).

In these simulations, we first wanted to test our theoretical analyses, and therefore we have depicted in each chart the dashed line that represents the equilibrium point and the curve obtained by uniformly perturbing the the initial value with respect to the equilibrium point. Alongside these verifications, we analyzed the behavior of the solution curves as the depletion coefficient δ of both renewable and non-renewable resources increases, sometimes obtaining results different from those achieved for a small δ (in the literature small δ 's are usually considered). We also analyzed the behavior of the solutions obtained by considering different depletion factors between renewable and non-renewable resources. Finally, we tested the stability of some equilibrium points supporting positive values for all the variables. On the x-axis, we consider a generic unit for time which does not necessarily translate into days, months, or years. It's a generic unit that doesn't necessarily span too long intervals. Indeed, we suspend our simulation when the behavior appears to be identified. In the captions, we have included the data related to the equilibrium point examined and, in particular, the value assumed and the eigenvalues of the Jacobian matrix calculated at that point. In these experiments, we kept certain parameter values constant, setting them to the values generally found in the literature (see [10,11]). However, we deliberately altered other parameters to study their behavior. In particular, if not stated otherwise in the comment of the individual figure, we consider the natures carrying capacity $\lambda = 100$, the natures regeneration factor $\gamma = 0.01$, the wealth consumption $\sigma = 5$, the factor that regulates the possible increase of non-renewable resources $k = 1$ and finally, the factors that govern the dynamics of populations $c = 1, d = 3, \mu = 2$. Our simulations, on the other hand, were conducted by varying the parameter δ .

7.1. Analysis of theoretical results and δ increase effects

In analyzing the correctness of our theoretical studies, we started from the case $x \leq w$, that is when wealth is greater than the current population. When $w = 0$ the critical points are given by two families of points that we have denoted by Q and R , representing the desert state and the nature state, respectively. These are two families of unstable points that in Figure 1 we have represented in the case of $\delta = 3, Z = 1$. For better readability of the chart, achieved with variable values of the same order of magnitude, we have set $\lambda = 10, \gamma = 0.5$.

We then analyzed the situation near the critical points of case ii.2) both to verify the correctness of our analyses and to see the possible scenario near such points. The behavior was confirmed in both the case of a small δ and the case of a possible large value. We have illustrated in Figure 2 the case with $\delta = 0.49 < k/2$. In case ii.2), we also analyzed the behavior of P_2 with a large delta value, specifically $\delta = 5$, by setting $k = 2\delta + 1$. We obtained results aligned with what was predicted by our analyses for small δ , both in the case of $\gamma\lambda - k < 0$ and in the case of $\gamma\lambda - k > 0$.

We recall that, as for case ii.3), we have a single stable critical point for $k\lambda - \sigma < 0$ and unstable in the opposite case. These results were confirmed by numerous simulations with small δ . It should be noted that, in the case of stability, we had to reduce the perturbation applied to P to achieve convergence to this equilibrium point. With the usual perturbation $\epsilon = 10^{-1}$, convergence occurs towards point P_1 of case ii.7), which differs from P only in the value assumed by the population. In Figure 3, we report the results obtained about the analysis of point P in case ii.3) for large δ . This point is stable regardless of the value assumed by $k\lambda - \sigma$. However, we note that the basin of attraction of this point is extremely small. As in the case mentioned above for small δ , we had to reduce the usual perturbation to a value equal to 10^{-3} to achieve convergence to point P and not to point P_1 of case ii.7). In the case of Figure 3,

however, the point P is close to zero in each of its components, therefore, a perturbation of the order of 10^{-3} seems to us to be quite reasonable; a situation different is, for example, if $k\lambda - \sigma = -2$, obtained with $k = 0.3, \lambda = 10$ and $\delta = 0.1$ for which $P = (0, 8.3333, 0, 0.1667)$ and the perturbation must be equally small.

In Figure 4, we report the analysis of point P_{11} of ii.6), as δ varies, always unstable according to the theory.

In Figures 5, 6, and 7, we have reported some of our experiments regarding the critical points indicated in case ii.7). Specifically, in Figures 5 and 6, we analyzed point P_1 , as δ varies, unstable if $k\lambda - \sigma > 0$, stable otherwise. For the sake of greater readability of the graphs, these figures refer to $\gamma = 0.1$. While stability can occur as δ varies, see Figure 6, the instability of the point is not obtained for large values of δ , differently from what our theoretical results state for $\delta \rightarrow 0^+$, see Figure 5. Finally, in Figure 7, we report the analysis of point P_2 , always unstable as δ varies.

We now report some experiments conducted in the case $x > w$.

Specifically, in Figure 8, we analyze, as δ varies, the behavior of solutions starting nearby the equilibrium point, predicted to be unstable if $\gamma\lambda - k > 0$ and indicated as P_{12} in the situation iii.1). In the case of $\gamma\lambda - k < 0$, for which we have stability for small values of δ , we obtained a situation of instability by pushing δ to the maximum value allowed while keeping $k = 1$ fixed. Finally, in Figures 9 and 10, we have reported some of our studies on point P_1 of case iii.2). Here too, we are in a situation in which we have instability for small δ 's, while the simulations suggest a change to stability for larger δ 's.

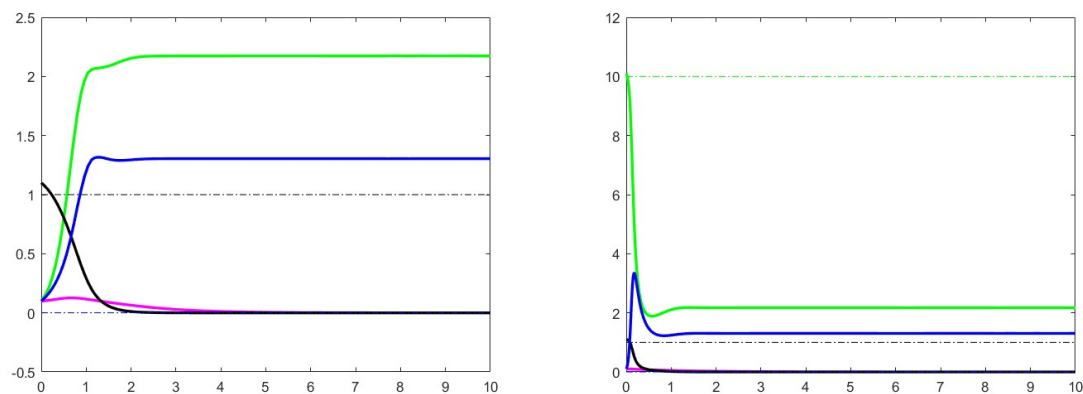


Figure 1. Case i) with $\delta = 3$. Scenario around the unstable critical points. On the left: $Q = (0, 0, 1, 0)$. On the right: $R = (0, 10, 1, 0)$.

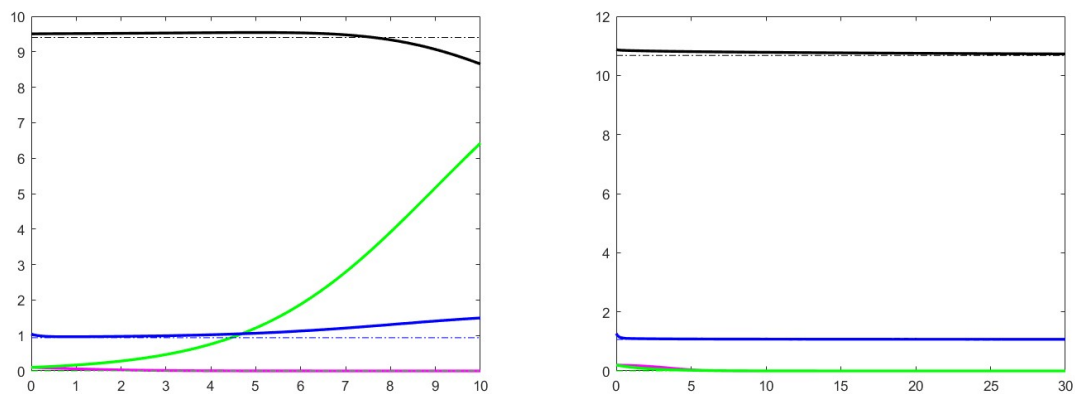


Figure 2. Case ii.2), analysis of the points P_1 and P_2 , with small $\delta = 0.4990 < \frac{k}{2}$. On the left: $P_1 = (0, 0, 9.4054, 0.9387)$, with $\lambda = 100$. The eigenvalues of $J(P_1)$ are $\rho_k = 0.0294, -4.7227, -1.0, 0.5316$, unstable according to the theory. On the right: $P_2 = (0, 0, 10.6748, 1.0653)$ with $\lambda = 10$ so that $\gamma\lambda - k = -0.9$. The eigenvalues of $J(P_2)$ are $\rho_k = -0.0338, -5.2929, -1.0000, -0.4316$, stable according to the theory.

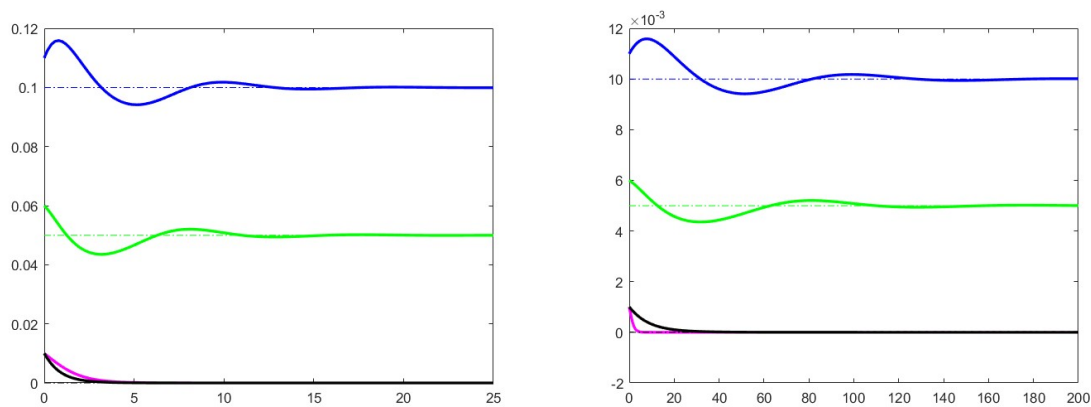


Figure 3. Case ii.3), analysis of the point P with $\delta = 10$. On the left: $P = (0, 0.0500, 0, 0.01000)$ when $k\lambda - \sigma = 95$, stable, contrary the theory for $\delta \rightarrow 0^+$. The eigenvalues of $J(P)$ are $\rho_k = -0.2501 \pm 0.6612i, -1.0000, -0.9896$. With a perturbation $\epsilon \geq 10^{-1}$ the point converges to $P_1 = (0.1308, 0.00500, 0, 0.01000)$ of case ii.7) with the same convergence conditions for $\delta \rightarrow 0$. On the right: $P = (0, 0.00500, 0, 0.01000)$, $k = 0.3, \lambda = 10$, so that $k\lambda - \sigma = -2$. The eigenvalues of $J(P)$ are $-0.0250 \pm 0.0661i, -1.0000, -0.0999$. Stability is observed over very long intervals. With a perturbation $\epsilon \geq 10^{-2}$, we have convergence to $P_1 = (0.0131, 0.00500, 0, 0.01000)$ of case ii.7).

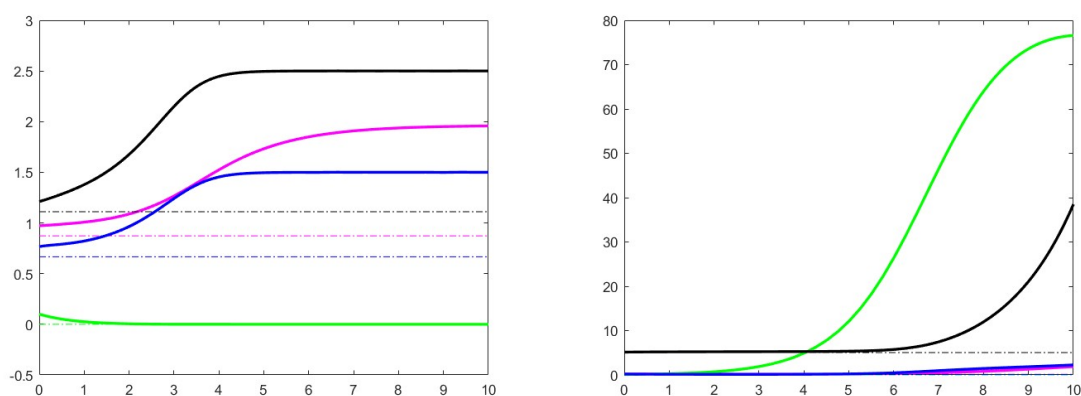


Figure 4. Case ii.6), analysis of the point P_{11} as δ varies, always unstable as $\delta \rightarrow 0$. On the left: $P_{11} = (0.8727, 0, 1.1111, 0.6667)$ with $\delta = 3, k = 2\delta + 0.5$. Eigenvalues of $J(P_{11})$: $\rho_k = -0.7454, -3.9779, 0.6446, -1.0000$. On the right: $P_{11} = (0.1322, 0, 5.0510, 0.1010)$ with $\delta = 0.1, k = 1$. Eigenvalues of $J(P_{11})$: $\rho_k = -0.7454, -0.5148, 0.0097, 0.9899$.

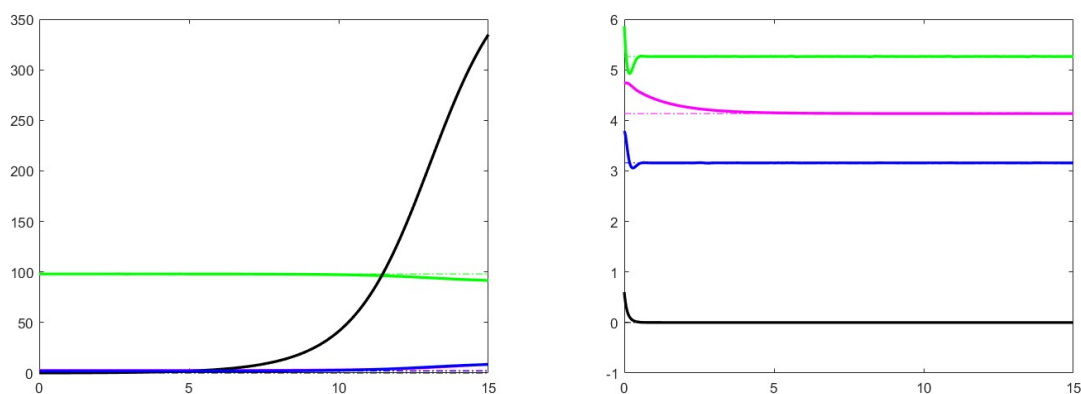


Figure 5. Case ii.7), analysis of the point P_1 as δ varies with $k\lambda - \sigma = 95$, unstable case. On the left: $P_1 = (2.5667, 98.0392, 0, 1.9608)$ with $\delta = 0.1$. Eigenvalues of $J(P_1)$: $\rho_k = -0.7454, -9.8039 \pm 1.3865i, 0.5975$. On the right: $P_1 = (4.1337, 5.2632, 0, 3.1579)$ stable point with $\delta = 3$, differently from the case of small δ . Eigenvalues of $J(P_1)$: $\rho_k = -0.7454, -8.1579 \pm 9.5574i, -8.5648$.

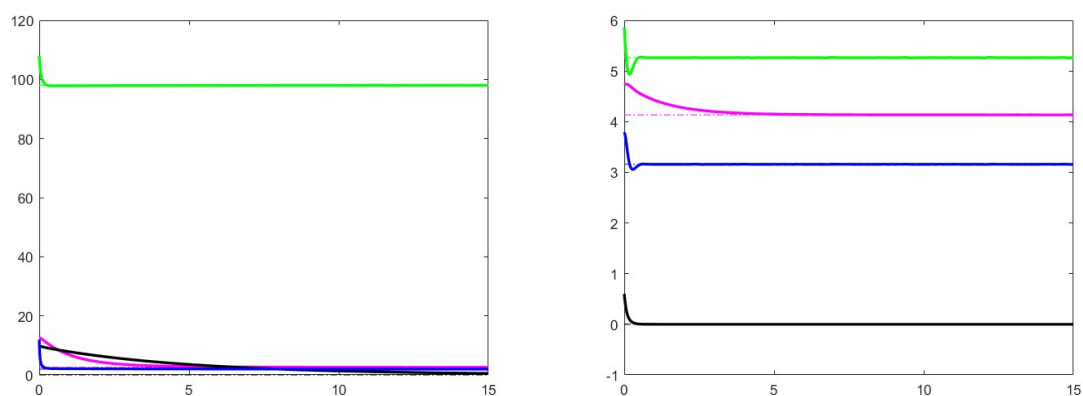


Figure 6. Case ii.7), analysis of the point P_1 with $k\lambda - \sigma = -4$, stable case, obtained by changing $k = 10^{-2}$. The simulations are in accordance with the theory as δ varies. On the left: $P_1 = (2.5667, 98.0392, 0, 1.9608)$ with $\delta = 0.1$. Eigenvalues of $J(P_1)$: $\rho_k = -0.7454, -9.8039 \pm 1.3865i, -0.1881$. On the right: $P_1 = (4.1337, 5.2632, 0, 3.1579)$ with $\delta = 3$. Eigenvalues of $J(P_1)$: $\rho_k = -0.7454, -8.1579 \pm 9.5574i, -9.4646$.

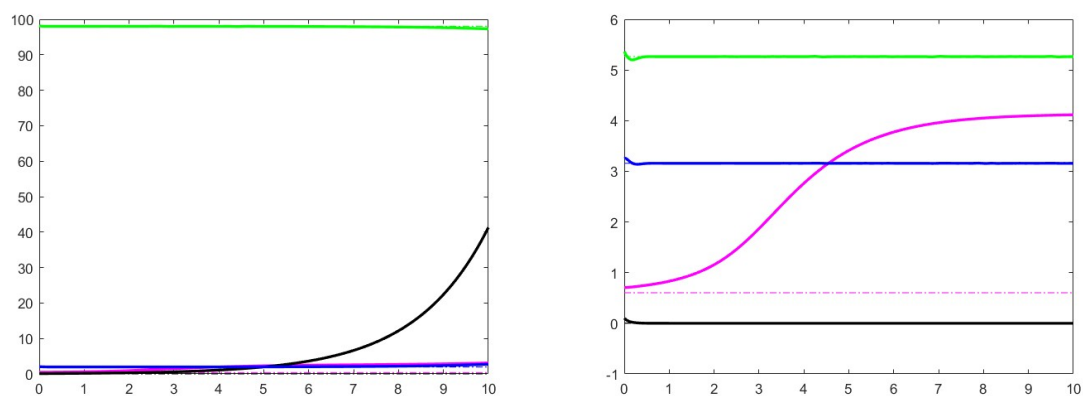


Figure 7. Case ii.7), analysis of the point P_2 always unstable as δ varies. On the left: $P_2 = (0.3745, 98.0392, 0, 1.9608)$ with $\delta = 0.1$. Eigenvalues of $J(P_2)$: $\rho_k = 0.7454, -9.8039 \pm 1.3865i, 0.5975$. On the right: $P_2 = (0.6031, 5.2632, 0, 3.1579)$ with $\delta = 3$. Eigenvalues of $J(P_2)$: $\rho_k = 0.7454, -8.1579 \pm 9.5574i, -8.5648$.

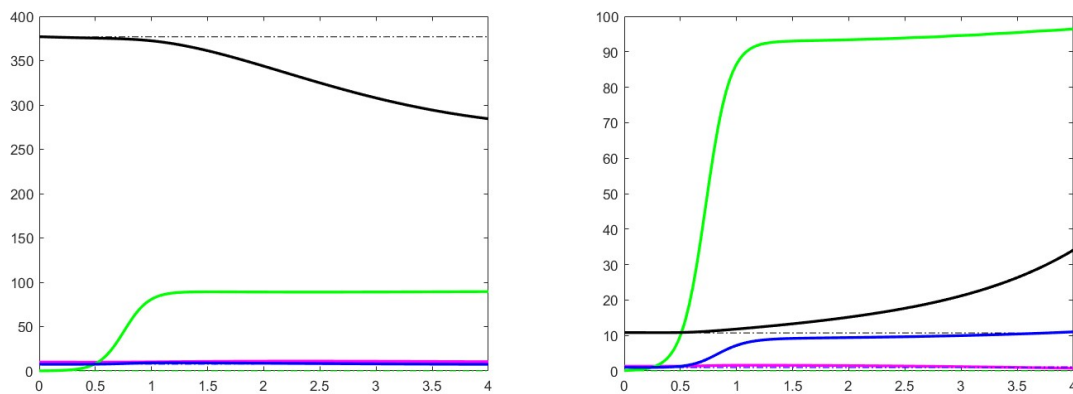


Figure 8. Case iii.1) analysis of the point P_{12} unstable if $\gamma\lambda - k = 9 > 0$, with $\gamma = 0.1$. On the left: case $\delta = 0.1$, $P_{12} = (9.8673, 0, 376.8988, 7.5380)$. Eigenvalues of $J(P_{12})$: $\rho_k = -0.3587 \pm 0.7661i, -49.3647, 9.0133$. On the right: $\delta = k\sqrt{\mu E_1}/2 - 0.001 = 0.4360$, $P_{12} = (1.2243, 0, 10.7251, 0.9353)$. Eigenvalues of $J(P_{12})$: $\rho_k = -0.0631, -0.4081, -6.3956, 9.4662$.

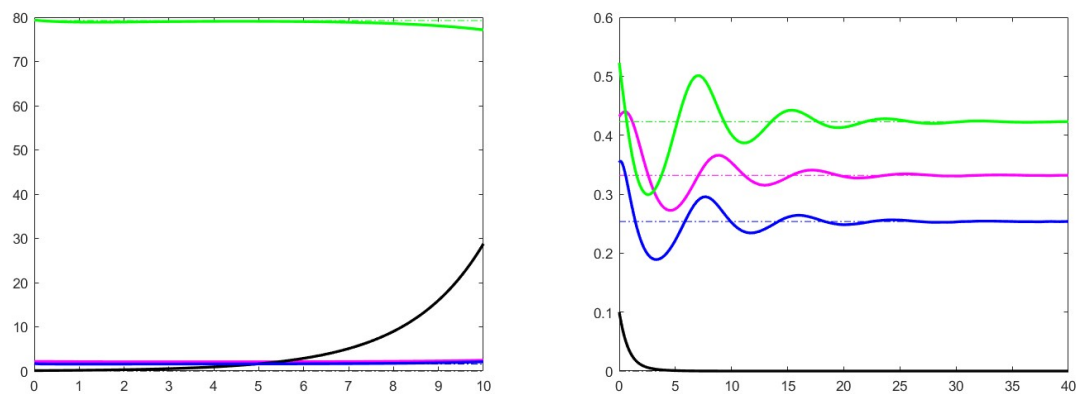


Figure 9. Case iii.2), analysis of the point P_1 , unstable if $k\lambda - \sigma = 95 > 0$ for $\delta \rightarrow 0^+$. On the left: $\delta = 0.1$, $P_1 = (2.0748, 79.2516, 0, 1.5850)$, eigenvalues $\rho_k = -0.7603 \pm 0.4076i, -10.3915, 0.5593$. On the right: $\delta = 3$, $P_1 = (0.3319, 0.4226, 0, 0.2536)$, eigenvalues $\rho_k = -0.1567 \pm 0.7521i, -2.0957, -0.9181$ then a stability situation, differently from the case of small δ .

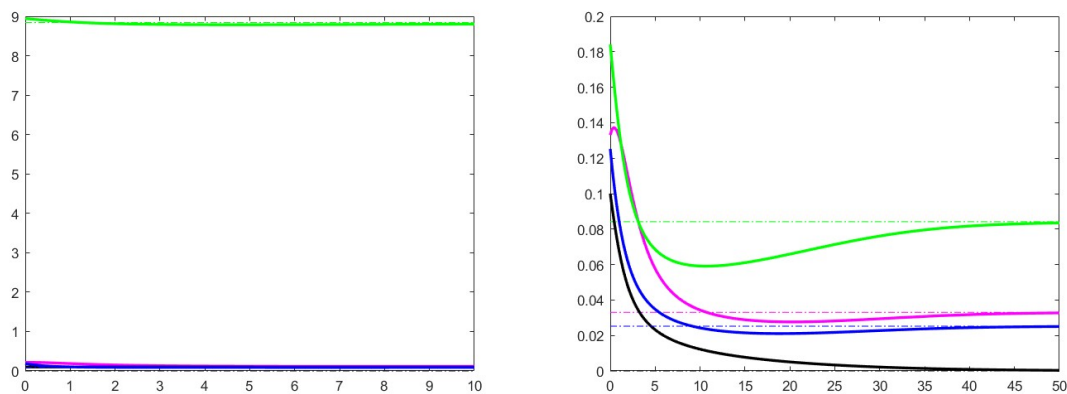


Figure 10. Case iii.2), analysis of the point P_1 when $k\lambda - \sigma = -3 < 0$ obtained with $k = 0.7, \sigma = 10, \lambda = 10$, stability situation for $\delta \rightarrow 0^+$, confirmed also for large δ . On the left: $\delta = 0.1, P_1 = (0.1157, 8.8425, 0, 0.0884)$, eigenvalues $\rho_k = -0.1032, -0.7095, -1.1786, -0.0045$. On the right: $\delta = 3, P_1 = (0.0331, 0.0842, 0, 0.0253)$, eigenvalues $\rho_k = -0.8086, -0.1341 \pm 0.1118i, -0.0986$.

7.2. Analysis of the $\delta_1 \neq \delta_2$ case

We wanted to explore situations similar to those seen in Figures 3 and 9, but considering $\delta_1 \neq \delta_2$. We have presented some results of these analyses in the Figures 11 and 12.

For the case ii.3), we examine the point P for $\delta_1 \neq \delta_2$, in the case $k\lambda - \sigma > 0$ unstable for small δ 's, from theoretical results. The simulations suggest a change to stability for large values of δ . The critical point depends only on δ_1 but it seems that the behavior is dictated by δ_2 as can be seen in Figure 11.

In case iii.2), the analysis of point P_1 with $\delta_1 \neq \delta_2$ when $k\lambda - \sigma > 0$, gives a situation of stability for $\delta \rightarrow 0^+$. Referring to Figure 9, we had seen that instability was verified for small δ , while the simulations suggested stability for large δ . The equilibrium points do not depend on δ_2 , and indeed we obtain the same equilibrium points as in Figure 9 for cases $\delta_1 = \delta$. Even in this case, it would seem that δ_2 controls the behavior of the equilibrium point, whose coordinates depend only on δ_1 .

We would have liked to further analyze the scenario presented in ii.4), because the critical point depends on both δ_1 and δ_2 , but studies for significantly different values of the two parameters are not possible because, as they increase, the point exits the cone.

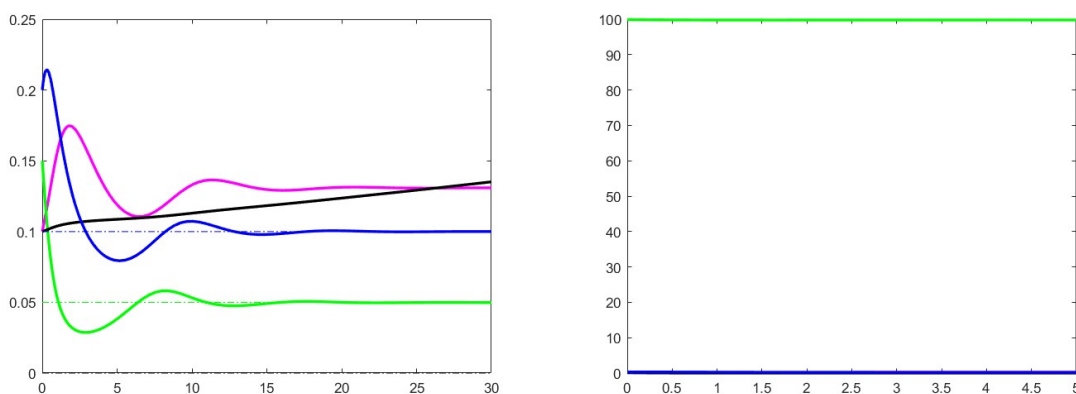


Figure 11. Case ii.3). Analysis of point P for $\delta_1 \neq \delta_2$, in the case $k\lambda - \sigma = 95$ unstable from theoretical results. The critical point depends only on δ_1 . On the left: $P = (0, 0.0500, 0, 0.1000)$ with $\delta_1 = 10, \delta_2 = 0.01$ unstable, according to the theory. Eigenvalues: $\rho_k = -0.2501 \pm 0.6612i, -1.0000, 0.0089$. On the right: $P = (0, 99.8004, 0, 0.1996)$ and $\delta_1 = 0.01, \delta_2 = 10$. Eigenvalues: $-0.9980 \pm 0.0446i, -1.0000, -1.9577$. The point is stable according to the behavior observed for large δ .

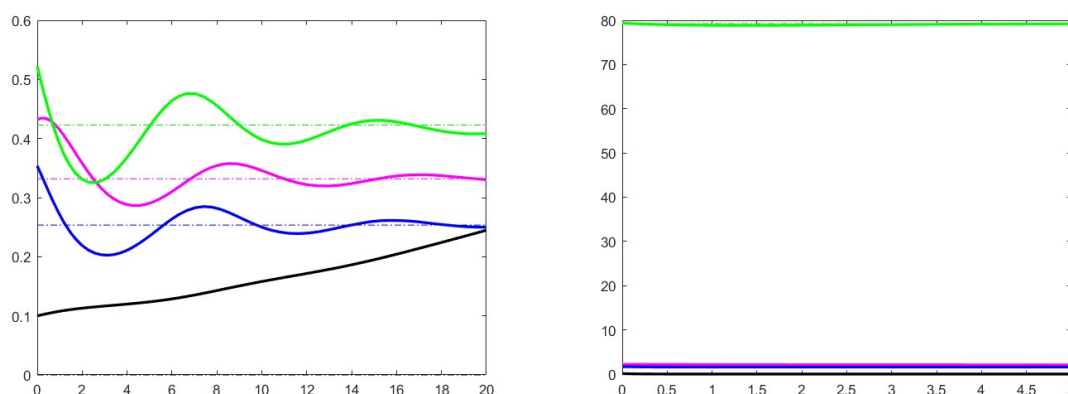


Figure 12. Case iii.2), analysis of point P_1 with $\delta_1 \neq \delta_2$ in the case $k\lambda - \sigma = 95$, instability situation as $\delta \rightarrow 0$. On the left: $P_1 = (0.3319, 0.4226, 0, 0.2536)$ with $\delta_1 = 3, \delta_2 = 0.1$, eigenvalues $\rho_k = -0.1567 \pm 0.7521i, -2.0957, 0.0444$, unstable. On the right: $P_1 = (2.0748, 79.2516, 0, 1.5850)$ with $\delta_1 = 0.1, \delta_2 = 3$, eigenvalues $\rho_k = -0.7603 \pm 0.4076i, -10.3915, -5.4577$, stable.

7.3. Stability for $x > 0$

A situation of stability with a positive population is represented in Figure 13. We analyzed the scenario indicated in cases ii.8) and iii.3) in the predicted cases of stability as $\delta \rightarrow 0^+$.

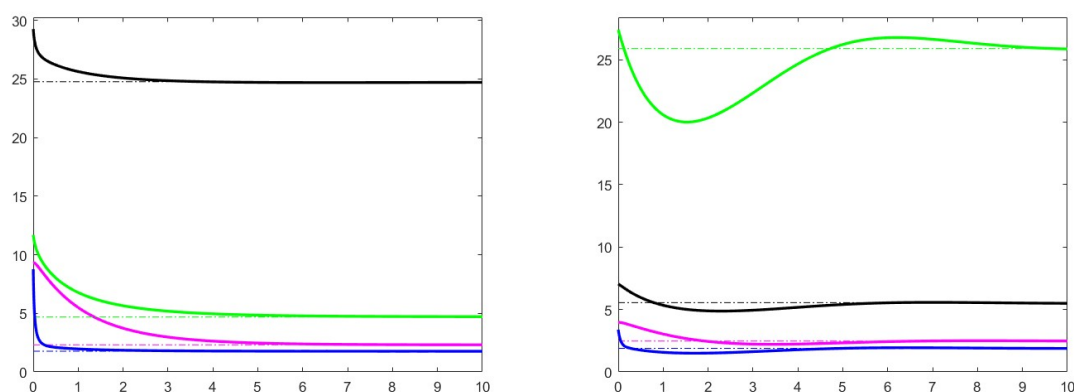


Figure 13. Stable critical points with positive population, $\delta = 0.3$. On the left: case ii.8) with $P_{12} = (2.3138, 4.6972, 24.7626, 1.7676)$ stable and within the cone for $\gamma\lambda - k = 0.3 > 0$ obtained with $k = 0.7, \lambda = 10, \gamma = 0.1$, eigenvalues: $\rho_k = -0.7454, -8.5064, -0.6207, -0.1805$. On the right: case iii.3) with $P_{12} = (2.4700, 25.8987, 5.5504, 1.8869)$, stable and within the cone for $\gamma\lambda - k = 0.1 > 0$ obtained with $k = 0.9$. Eigenvalues: $\rho_k = -12.4196, -0.4555 \pm 0.6858i, -0.0239$.

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