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Linear-in-Temperature Resistivity and Planckian Dissipation Arise in a Stochastic Quantization Model of Cooper Pairs

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Abstract: We suppose that a Cooper pair (CP) will experience a damping force exerted by the condensed matter. A Langevin equation of a CP in two dimensional condensed matter is established. Following a method similar to Nelson's stochastic mechanics, generalized Schrödinger equation of a CP in condensed matter is derived. If the CPs move with a constant velocity, then the corresponding direct current (DC) electrical conductivity can be calculated. Therefore, a Drude like formula of resistivity of CPs is derived. We suppose that the damping coefficient of CPs in two dimensional cuprate superconductors is a linear function of temperature. Then the resistivity and scattering rate of CPs turn out to be also linear-in-temperature. The origin of linear-in-temperature resistivity and Planckian dissipation in cuprate superconductors may be the linear temperature dependence of the damping coefficient of CPs.

Keywords: Planckian resistivity; Planckian dissipation; Cooper pair; strange metal; stochastic mechanics

1. Introduction

It is known that the resistivity ρ of the normal states of cuprate superconductors obeys the following relationship [1–4]

$$\rho = \rho_0 + AT, \tag{1}$$

where ρ_0 is the residual resistivity, A is a coefficient independent of temperature, T is temperature.

These phenomena of linear temperature dependence of resistivity are found in numerous strongly correlated electron systems, such as the heavy fermion compounds [1,3], transition metal oxides [1,4,5], iron pnictides [1], magic angle twisted bilayer graphene, organic metals [1] and conventional metals [1], often in connection with unconventional superconductivity. Sometimes this linear-in-temperature resistivity is called Planckian resistivity [2]. When superconductivity is destroyed by a high magnetic field, the recovered normal state still obeys this law of linear-in-temperature resistivity in the low temperature region [4]. In most of the heavy fermion materials, the linear-in-temperature resistivity appears when they have been tuned by some external parameter to create a low-temperature continuous phase transition which is referred to a quantum critical point (QCP) [1]. Thus, the linear temperature dependence of resistivity are often associated with quantum criticality. The linear-in-temperature resistivity of *LSCO* with different gradients, different doping dependencies and different origins appears not only at high temperature but also at low temperature [6].

Strange metal behavior refers to a linear temperature dependence of the electrical resistivity [1,3]. A unified theory of this scaling law (1) in different strange metals is still an open problem [2,3].

Before the discovery of quantum mechanics, a successful formula of resistivity of metals is proposed in the Drude model ([7], p. 7). Shortly after the discovery of quantum mechanics, Sommerfeld improved the Drude model. In the Sommerfeld model, the following Drude formula of resistivity of metals can be derived approximately based on quantum theoty ([7], p. 251)

$$\rho = \frac{m^*}{ne^2\tau}, \tag{2}$$

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where n is the number dendity of electrons, e is the electric charge of an electron, m^* is the effective mass of an electron, τ is the relaxation time of an electron.

If the transport scattering rate $1/\tau$ is linear-in-temperature and is the only temperature-dependent quantity in Equation (2), then the scaling law (1) of resistivity can be derived directly. Thus, a clue to study the scaling law (1) is to investigate the relaxation time τ in the Drude formula (2).

The Drude formula (2) is valid only for charge carriers which obey the Fermi-Dirac distribution. Experiments have shown that the dominant charge carriers in $YBa_2Cu_3O_{7-\delta}$ (YBCO) film are Cooper pairs (CPs) [8]. Since CPs are not Fermions, the Drude formula (2) may be not valid in the normal states of cuprate superconductors. Thus, an interesting question is that whether a similar formula for the resistivity of the normal states of cuprate superconductors exists.

According to the Heisenberg uncertainty principle, a local equilibration time of any many-body quantum system cannot be faster than the following Planckian time τ_p [3]

$$\tau_p = \frac{\hbar}{k_B T}, \tag{3}$$

where h is the Plank constant, $\hbar = h/2\pi$, k_B is the Boltzmann constant.

This timescale τ_p is associated with quantum criticality and known to bound the validity of a Boltzmann description of transport [9]. τ_p is suggested to be the lower bound of the phase coherence time in quantum critical systems [9]. τ_p is also known to control the electronic dynamics of the cuprate strange metal [9]. Thus, an idea is that the relaxation time τ of CP in cuprate superconductors may be proportional to the Planckian time τ_p, i.e., $\tau = \alpha_0 \tau_p$, where α_0 is a dimensionless parameter. Indeed, experiments have shown that the scattering rate $1/\tau$ in the region of the temperature-linear resistivity of a wide range of metals, including heavy fermion, oxide [4,5,8], pnictide, organic metals and conventional metals, can be written as [1]

$$\frac{1}{\tau} = \frac{\alpha_0 k_B T}{\hbar}, \tag{4}$$

where $\alpha_0 \approx 1$.

Equation (4) shows that the relaxation time τ is approximately equal to the Planckian time τ_p, i.e., $\tau \approx \tau_p$. The case of $\alpha_0 \approx 1$ is referred to the Planckian dissipation [3]. It is surprising that the linear-in-temperature scattering rate $1/\tau$ and the behaviors of Planckian dissipation in these materials (except the conventional metals) can be seen down to low temperatures with appropriate tuning by magnetic field, chemical composition or hydrostatic pressure [1]. It is suggested that there may be a fundamental principle governing the transport of CPs [8].

If Equation (4) and Equation (2) are valid in the normal states of cuprate superconductors, then Equation (1) may be derived. In this manuscript we focus on this clue and try to derive the scaling law (1).

2. Stochastic Mechanics of a Cooper Pair in Two Dimensional Condensed Matter

In order to explain the energy quantization of atoms, E. Schrödinger proposes the following equation for a non-relativistic particle moving in a potential [10]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi + U(\mathbf{r})\psi, \tag{5}$$

where t is time, $\mathbf{r}$ is a point in space, $\psi(\mathbf{r}, t)$ is the wave function, m is the mass of the particle, $U(\mathbf{r})$ is the potential, h is the Plank constant, $\hbar = h/2\pi$, $\nabla^2 \equiv \partial^2/\partial r_1^2 + \partial^2/\partial r_2^2 + \partial^2/\partial r_3^2$ is the Laplace operator in a Cartesian coordinate $\{r_1, r_2, r_3\}$.

The Schrödinger equation (5) is a fundamental assumption in non-relativistic quantum mechanics [10]. Although the Schrödinger equation can be used to describe some non-relativistic

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quantum phenomena, the origin of quantum phenomena remains an unsolved problem in physics for more than 100 years [11,12]. Although the axiomatic system of quantum mechanics was firmly established, the interpretation of quantum mechanics is still open [11,12]. There exist some paradoxes in quantum mechanics [13–16], for instance, the paradox of reduction of a wave packet and the paradox of the Schrödinger cat.

Fényes proposed an interpretation of quantum mechanics based on a Markov process. Fényes' work was developed by Weizel and discussed by Kershaw [17]. According to Luis de Broglie [18], the success of the probabilistic interpretation of $|\psi|^2$ inspired Einstein to speculate that the probability $|\psi|^2$ is generated by a kind of hidden Brownian motions of particles. This kind of hidden motions was called quasi-Brownian motions by Luis de Broglie [18].

If the quantum phenomena stem from the stochastic motions of particles, then we may establish a more fundamental and more powerful theory of quantum phenomena other than quantum mechanics. The Schrödinger equation may no longer be a basic assumption and may be derived in this new theory. Indeed, E. Nelson [19] derived the Schrödinger equation by means of theory of stochastic processes based on the assumption that every particle with mass m in vacuum is subject to Brownian motion with diffusion constant $\hbar/2m$.

Inspired by Nelson's stochastic mechanics [19–28], we propose a theoretical derivation of the Schrödinger equation based on Newton's second law and a mechanical model of vacuum [29].

Recently, monolayer crystals of the high-temperature superconductor $Bi_2Sr_2CaCu_2O_{8+\delta}$ (Bi-2212) was obtained by a fabrication process [30]. The superconductivity, the pseudogap, charge order and the Mott state at various doping concentrations of the monolayer Bi-2212 reveals that the phases are indistinguishable from those in the bulk [30]. Monolayer Bi-2212 displays the fundamental physics of cuprate superconductors [30]. Therefore, monolayer copper oxides is a platform for studying high-temperature superconductivity in two dimensions. Thus, we focus on two dimensional condensed matters.

Modern experiments, for instance, the Casimir effect [31,32], have shown that vacuum is not empty. Thus, we suppose that there is a damping force exerted on each particle by vacuum [29]. For a microscopic particle moving in vacuum, we have the following relation [29]

$$\hbar = \frac{2k_0 T_0}{\eta_0}, \tag{6}$$

where k_0 is a constant similar to the Boltzmann constant k_B, T_0 is the temperature of the $\Omega(0)$ substratum in the vicinity of the particle in vacuum [33], η_0 is a damping coefficient related to vacuum.

It is known that a CP in condensed matter may be scattered by ions, electrons, phonon, etc. In the Drude theory of metals, the effect of individual electron collisions is approximately treated by introducing a damping force into the equation of motion of an electron ([7], p. 11). Following the Drude theory, we suppose that a CP in a condensed matter will experiences not only a damping force exerted by vacuum but also an additional damping force exerted by the condensed matter. We introduce a two dimensional Cartesian coordinate system $\{r_1, r_2\}$ which is attached to the condensed matter. We suppose that the two dimensional velocity $\mathbf{v} = d\mathbf{r}/dt$ of the CP exists. Applying Newton's second law, the motion of a CP may be described by the following Langevin equation [34]

$$\begin{aligned} m_c \frac{d^2\mathbf{r}}{dt^2} &= -\eta_0 m_c \mathbf{v} - \eta_1 m_c \mathbf{v} \\ &\quad -\eta_2 m_c \frac{d^2\mathbf{r}}{dt^2} + \mathbf{F}(\mathbf{r}, t) + ¸(t), \end{aligned} \tag{7}$$

where m_c is the mass of the CP, η_1 is a damping coefficient related to the condensed matter, η_2 is a quasi-inertial force coefficient, ¸(t) is a two dimensional random force and $\mathbf{F}(\mathbf{r}, t)$ is a two dimensional external force field.

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We introduce the following definitions

$$-\eta_0 m_c \mathbf{v} - \eta_1 m_c \mathbf{v} \;=\; -\eta_0 m_d \mathbf{v}, \tag{8}$$
$$m_c \frac{d^2\mathbf{r}}{dt^2} + \eta_2 m_c \frac{d^2\mathbf{r}}{dt^2} \;=\; m_q \frac{d^2\mathbf{r}}{dt^2}, \tag{9}$$

where m_d is the damping mass of the CP, m_q is the quasi-inertial mass of the CP.

Equation (8) and Equation (9) can be written as

$$m_d \;=\; \frac{\eta_0 + \eta_1}{\eta_0} m_c, \tag{10}$$
$$m_q \;=\; (1 + \eta_2) m_c, \tag{11}$$

Using Equation (8) and Equation (9), Equation (7) can be written as

$$m_q \frac{d^2\mathbf{r}}{dt^2} = -\eta_0 m_d \mathbf{v} + \mathbf{F}(\mathbf{r}, t) + \boldsymbol{\xi}(t). \tag{12}$$

Let $\xi_i(t)$ be the ith component of the random force $\boldsymbol{\xi}(t)$, i.e., $\boldsymbol{\xi}(t) = (\xi_1(t), \xi_2(t))$.

Assumption 1. *Assume that the force field $\mathbf{F}(\mathbf{r}, t)$ is a continuous function of $\mathbf{r}$ and t. Inspired by the Ornstein-Uhlenbeck theory [35,36] of Brownian motion, we suppose that the random force $\boldsymbol{\xi}$ exerted on the CP by the condensed matter is a two-dimensional Gaussian white noise [37–40] and the variance $E(\xi_i^2)$ of the ith component of $\boldsymbol{\xi}$ is [29]*

$$E(\xi_i^2(t)) \equiv \sigma_i^2 = 2\eta_0 m_d k_0 T_\omega, \tag{13}$$

where $\sigma_i > 0$, $i = 1, 2$, k_0 is a parameter similar to the Boltzmann constant, T_ω is the temperature of the $\Omega(0)$ substratum [33] in the location of the condensed matter.

For convenience, we introduce the following notation

$$D_1 = \frac{\sigma_1^2}{2} = \eta_0 m_d k_0 T_\omega. \tag{14}$$

The Gaussian white noise $\boldsymbol{\xi}$ is the generalized derivative of a Wiener process $\mathbf{Q}(t)$ [37,39]. We can write formally [37–40]

$$\boldsymbol{\xi}(t) = \frac{d\mathbf{Q}(t)}{dt}, \tag{15}$$

where $\mathbf{Q}(t)$ is a two-dimensional Wiener process with a diffusion constant D_1.

The mathematically rigorous form of Equation (12) is the following stochastic differential equations based on Itô stochastic integral [36]

$$\begin{cases} d\mathbf{r}(t) = \mathbf{v}(t)dt, \\ m_q d\mathbf{v}(t) = -\eta_0 m_d \mathbf{v}(t)dt + \mathbf{F}(\mathbf{r}, t)dt + d\mathbf{Q}(t), \end{cases} \tag{16}$$

where $\mathbf{r}(t)$, $\mathbf{v}(t)$, $\mathbf{F}(\mathbf{r}, t)$ and $\mathbf{Q}(t), t \geq 0$ are stochastic processes on a probability space $(\Omega, \mathcal{F}, P)$, $d\mathbf{Q}(t)$ are independent of all of the $\mathbf{r}(s)$, $\mathbf{v}(s)$, with $s \leq t$, $\mathbf{r}(0) = \mathbf{r}_0$, $\mathbf{v}(0) = \mathbf{v}_0$.

The microstate of the CP at time t is defined by the random vector $(\mathbf{r}(t), \mathbf{v}(t))$ [29].

Proposition 1. *Suppose that Equations (13) are valid and the force field $\mathbf{F}(\mathbf{r}, t) : R^2 \times R_+ \to R^2$ satisfy a global Lipschitz condition, that is, for some constant C_0,*

$$|\mathbf{F}(\mathbf{r}_1, t) - \mathbf{F}(\mathbf{r}_2, t)| \leq C_0 |\mathbf{r}_1 - \mathbf{r}_2|, \tag{17}$$

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for all $\mathbf{r}_1$ *and* $\mathbf{r}_2$ *in* R^2*, where* R^2 *denotes the two-dimensional Descartes space,* R_+ *denotes the set of positive real numbers. Then, at a time scale of an observer very large compare to the relaxation time* τ_c*,*

$$\tau_c \equiv \frac{m_q}{\eta_0 m_d}, \tag{18}$$

the solution $\mathbf{r}(t)$ *of the Langevin equation Equation (16) converges to the solution* $\mathbf{y}(t)$ *of the following Smoluchowski equation Equation (20) with probability one uniformly for t in compact subintervals of* $[0, \infty)$ *for all* $\mathbf{v}_0$*, i.e.,*

$$\lim_{1/\tau_c \to \infty} \mathbf{r}(t) = \mathbf{y}(t), \tag{19}$$

where $\mathbf{y}(t)$ *is the solution of the following Smoluchowski equation*

$$d\mathbf{y}(t) = \mathbf{b}(\mathbf{y}, t)dt + d\mathbf{w}(t), \tag{20}$$

where $\mathbf{y}(0) = \mathbf{r}_0$*,* $\mathbf{w}(t)$ *is a two-dimensional Wiener process with a diffusion constant* D_3 *defined by*

$$D_3 = \frac{k_0 T_\omega}{\eta_0 m_d}. \tag{21}$$

A proof of Proposition 1 can be found in the Appendix A. Following similar methods in Ref. [29], a Schrödinger like equation (A56) and Equation (A22) can be derived, refers to Appendix B.

Putting Equation (A22) into Equation (A56), we have the following result.

Proposition 2. *The Schrödinger like equation Equation (A56) reduces to the following Schrödinger like equation*

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_w}\nabla^2\psi + U(\mathbf{r})\psi, \tag{22}$$

where m_w *is the wave mass defined by Equation (A21).*

From Equation (10), in vacuum the damping mass m_d reduces to the mass m_c of a CP. In vacuum, T_ω reduces to T_0. Therefore, in vacuum the wave mass m_w defined by Equation (A21) reduces to m_c. Thus, the Schrödinger like equation (22) in the condensed matter is a generalization of the Schrödinger equation (5) in vacuum.

3. Calculation of Direct Current (DC) Electrical Conductivity

If there is no external magnetic field and the external electric field **E** is a constant vector field, then the Langevin equation (12) can be written as

$$m_q \frac{d^2\mathbf{r}}{dt^2} = -\eta_0 m_d \mathbf{v} + e_c \mathbf{E} + \text{¸}(t), \tag{23}$$

where e_c is the electric charge of the CP.

If the mean velocity $\mathbf{v}$ of the CP is high enough such that $d\mathbf{v}/dt = 0$, then we call this velocity as drift velocity and denotes it as $\mathbf{v}_d$. Thus, if the observer look at the CP for a time long enough comparing to the relaxation time τ_c, then he will observe the long time averaged quantities of the Langevin equation (23). Since $(d\mathbf{v}/dt)|_{\mathbf{v}=\mathbf{v}_d} = 0$ and $E\text{¸}(t) = 0$, the long time averaged form of the Langevin equation (23) can be written as ([7], p. 7; [41], p. 16)

$$\mathbf{v}_d = \frac{e_c \mathbf{E}}{\eta_0 m_d}. \tag{24}$$

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The current density $\mathbf{j}$ corresponding to the drift velocity $\mathbf{v}_d$ is ([7], p. 7; [41], p. 16)

$$\mathbf{j} = n_c e_c \mathbf{v}_d, \tag{25}$$

where n_c is the number density of CPs.

Putting Equation (24) and Equation (25), we have

$$\mathbf{j} = \frac{n_c e_c^2}{\eta_0 m_d} \mathbf{E}. \tag{26}$$

Equation (25) can be written as ([7], p. 7)

$$j_i = \sum_j \sigma_{ij} E_j, \tag{27}$$

where j_i is the ith component of the current density $\mathbf{j}$, E_j is the jth component of the electric field $\mathbf{E}$, σ_{ij} is the conductivity tensor which can be written as

$$\sigma_{ij} = \sigma \delta_{ij}, \tag{28}$$

where

$$\sigma = \frac{n_c e_c^2}{\eta_0 m_d}, \tag{29}$$

δ_{ij} is the Kronecker symbol

It is known that the resistivity of the normal states of cuprate superconductors exhibits strong anisotropy ([42], p. 190). Thus, Equation (28) may be only valid for the plane conductivity ρ_{ab} of two dimensional cuprate superconductors and not valid for bulk cuprate. Noticing $\rho_{ab} = 1/\sigma$ and Equation (29), we have

$$\rho_{ab} = \frac{\eta_0 m_d}{n_c e_c^2}. \tag{30}$$

Using Equation (18), Equation (30) can also be written as

$$\rho_{ab} = \frac{m_q}{n_c e_c^2 \tau_c}. \tag{31}$$

4. Linear Temperature Dependence of Resistivity in the Normal States of Cuprate Superconductors

Using Equation (10), Equation (30) can be written as

$$\rho_{ab} = \frac{(\eta_0 + \eta_1) m_c}{n_c e_c^2}. \tag{32}$$

Inspired by Equation (32), we speculate that the origin of the linear-in-temperature resistivity of the strange metals may be the linear temperature dependence of the damping coefficient $\eta_0 + \eta_1$. Thus, we introduce the following assumption.

Assumption 2. *Suppose that the following relationship is valid in the strange metal states of two dimensional cuprate superconductors*

$$\frac{\eta_0 + \eta_1}{\eta_0} = b_0 \frac{T}{T_0}. \tag{33}$$

where b_0 ia a parameter to be determined.

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If $T = T_0$, then $\eta_1 = 0$ [29]. Thus, $b_0 = 1$. Using Equation (33), Equation (32) can be written as

$$\rho_{ab} = A_1 T, \tag{34}$$

where

$$A_1 = \frac{\eta_0 m_c}{n_c e_c^2 T_0}. \tag{35}$$

Noticing Equation (6), Equation (34) can also be written as

$$\rho_{ab} = A_2 T, \tag{36}$$

where

$$A_2 = \frac{2k_0 m_c}{\hbar n_c e_c^2}. \tag{37}$$

If A_2 is independent of temperature T, then Equation (34) shows that the plane resistivity ρ_{ab} is a linear function of temperature T. Thus, Equation (1) is derived based on the stochastic quantization model of CP in two dimensional cuprate superconductors.

A prediction of Equation (36) is that if $T = 0$, then $\rho_{ab} = 0$, i.e., the residual resistivity $\rho_0 = 0$.

5. Linear Temperature Dependence of Scattering Rate of CPs

Using Equation (10) and Equation (11), Equation (18) can be written as

$$\frac{1}{\tau_c} = \frac{\eta_0 + \eta_1}{1 + \eta_2}. \tag{38}$$

According to Equation (38), the origin of the linear-in-temperature scattering rate $1/\tau_c$ of the cuprate strange metals may be the linear temperature dependence of the damping coefficient $\eta_0 + \eta_1$. Noticing Equation (33), Equation (38) can be written as

$$\frac{1}{\tau_c} = c_1 T, \tag{39}$$

where

$$c_1 = \frac{\eta_0}{T_0(1 + \eta_2)}. \tag{40}$$

Noticing Equation (6), Equation (39) can also be written as

$$\frac{1}{\tau_c} = c_2 \frac{k_B T}{\hbar}, \tag{41}$$

where

$$c_2 = \frac{2k_0}{k_B(1 + \eta_2)}. \tag{42}$$

If c_1 is independent of temperature T, then Equation (39) shows that the scattering rate $1/\tau_c$ is a linear function of temperature T.

If we suppose that $c_2 \approx 1$, then Equation (4) is derived. Thus, we may say that CPs in the cuprate strange metals are undertaking the Planckian dissipation [3].

6. Conclusion

The origin of the linear-in-temperature resistivity of the normal state of hole-doped cuprate superconductors is a unsolved problem. Inspired by the Drude formula of resistivity, we speculate that the transport scattering rate of CPs in the normal states of cuprate superconductors may be linear-in-temperature. Thus, a clue to explain the linear-in-temperature scaling law of resistivity in

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strange metal states of cuprate superconductors is to seek a Drude like formula of resistivity and investigate the relaxation time of CP dynamics. We suppose that a CP in a condensed matter will experiences not only a damping force exerted by vacuum but also an additional damping force exerted by the condensed matter. Thus, a Langevin equation of a CP in two dimensional condensed matter is established. Following a similar method of Nelson's stochastic mechanics, generalized Schrödinger equation in condensed matter is derived. If CPs move with a constant velocity, then the electrical current density corresponding to the drift velocity can be calculated. Therefore, a Drude like formula of resistivity of CPs is derived. The damping coefficient of CPs in two dimensional cuprate superconductors is supposed to be a linear function of temperature. Thus, the plane resistivity and scattering rate of CPs turn out to be also linear functions of temperature.

Appendix A. Proof of Proposition 1

We introduce the following definitions

$$\beta = \frac{\eta_0 m_d}{m_q}, \tag{A1}$$

$$\mathbf{K}(\mathbf{r}, t) = \frac{\mathbf{F}(\mathbf{r}, t)}{m_q}, \tag{A2}$$

$$\mathbf{B}(t) = \frac{\mathbf{Q}(t)}{m_q}. \tag{A3}$$

Then, $\mathbf{B}(t)$ is a two-dimensional Wiener process with a diffusion constant D_2 [36]

$$D_2 = \frac{D_1}{m_q^2} = \frac{\eta_0 m_d k_0 T_\omega}{m_q^2} = \frac{\beta k_0 T_\omega}{m_q}. \tag{A4}$$

Using Equations (A1-A3), Equation (16) can be written as

$$\begin{cases} d\mathbf{r}(t) = \mathbf{v}(t)dt, \\ d\mathbf{v}(t) = -\beta\mathbf{v}(t)dt + \mathbf{K}(\mathbf{r}, t)dt + d\mathbf{B}(t). \end{cases} \tag{A5}$$

We introduce the following definitions

$$\mathbf{b}(\mathbf{r}, t) = \frac{\mathbf{K}(\mathbf{r}, t)}{\beta}, \tag{A6}$$

$$\mathbf{w}(t) = \frac{\mathbf{B}(t)}{\beta}. \tag{A7}$$

Noticing Equation (A1), $\mathbf{w}(t)$ is a two-dimensional Wiener process with a diffusion constant D_3 [36]

$$D_3 = \frac{D_2}{\beta^2} = \frac{k_0 T_\omega}{\eta_0 m_d}. \tag{A8}$$

Using Equation (A6-A7), Equation (A5) can be written as

$$\begin{cases} d\mathbf{r}(t) = \mathbf{v}(t)dt, \\ d\mathbf{v}(t) = -\beta\mathbf{v}(t)dt + \beta\mathbf{b}(\mathbf{r}, t)dt + \beta d\mathbf{w}(t). \end{cases} \tag{A9}$$

Let $\mathbf{r}(t)$ be the solution of Equation (A9) with $\mathbf{r}(0) = \mathbf{r}_0, \mathbf{v}(0) = \mathbf{v}_0$. According to Equation (17), the functions $\mathbf{b}(\mathbf{r}, t) : R^2 \times R_+ \to R^2$ also satisfies a global Lipschitz condition. For a time scale of an observer very large compare to the relaxation time $\tau_c \equiv 1/\beta$, he concludes that β can be regarded as infinity, i.e., $\beta \to +\infty$. Applying Nelson's Theorem 10.1 ([43], p. 59), the solution $\mathbf{r}(t)$ of the Langevin equation Equation (A9) converges to the solution $\mathbf{y}(t)$ of the Smoluchowski equation Equation (20) with probability one uniformly for t in compact subintervals of $[0,\infty)$ for all $\mathbf{v}_0$. □

Appendix B. Generalized Schrödinger equation of a CP in two dimensional condensed matter

Noticing the asymmetry in time t, we can introduce the following Langevin equation [19]

$$\begin{cases} d\mathbf{r}(t) = \mathbf{v}(t)dt, \quad \mathbf{r}(0) = \mathbf{r}_0 \\ md\mathbf{v}(t) = -f\mathbf{v}(t)dt + \mathbf{F}(\mathbf{r},t)dt + d\mathbf{Q}_*(t), \end{cases} \tag{A10}$$

where $d\mathbf{N}_*(t)$ are independent of all of the $\mathbf{r}(s)$, $\mathbf{v}(s)$, with $s \geq t$, $\mathbf{v}(0) = \mathbf{v}_0$.

We define the following mean forward derivative $D\mathbf{y}(t)$ and the mean backward derivative $D_*\mathbf{y}(t)$ [19]

$$D\mathbf{y}(t) = \lim_{\triangle t \to 0+} E_t\left[\frac{\mathbf{y}(t+\triangle t) - \mathbf{y}(t)}{\triangle t}\right], \tag{A11}$$

$$D_*\mathbf{y}(t) = \lim_{\triangle t \to 0+} E_t\left[\frac{\mathbf{y}(t) - \mathbf{y}(t-\triangle t)}{\triangle t}\right], \tag{A12}$$

where E_t denotes the conditional expectation given the state of the system at time t.

We also have another Smoluchowski equation [19]

$$d\mathbf{y}(t) = \mathbf{b}_*(\mathbf{y},t)dt + d\mathbf{w}_*(t), \tag{A13}$$

where $\mathbf{w}_*(t)$ has the same properties as $\mathbf{w}(t)$ except that the $d\mathbf{w}_*(t)$ are independent of the $\mathbf{y}(s)$ with $s \geq t$.

Based on Equation (A11-A12), we have [19]

$$D\mathbf{y}(t) = \mathbf{b}(\mathbf{y},t), \tag{A14}$$

$$D_*\mathbf{y}(t) = \mathbf{b}_*(\mathbf{y},t). \tag{A15}$$

For the probability density $\rho(\mathbf{y},t)$ of $\mathbf{y}$, we have the following forward Fokker-Planck equation and the backward Fokker-Planck equation [19]

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{b}) + D_3\nabla^2\rho, \tag{A16}$$

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{b}_*) - D_3\nabla^2\rho, \tag{A17}$$

where $\nabla\cdot \equiv \partial/\partial r_1 + \partial/\partial r_2$ is the divergence operator in the two dimensional Cartesian coordinate $\{r_1, r_2\}$, $\nabla^2 \equiv \partial^2/\partial r_1^2 + \partial^2/\partial r_2^2$ is the two dimensional Laplace operator.

We introduce the definitions of current velocity $\mathbf{v}_1(t)$ and osmotic velocity $\mathbf{u}_1(t)$ [19]

$$\mathbf{v}_1 = \frac{1}{2}(\mathbf{b} + \mathbf{b}_*), \tag{A18}$$

$$\mathbf{u}_1 = \frac{1}{2}(\mathbf{b} - \mathbf{b}_*). \tag{A19}$$

The current velocity $\mathbf{v}_1(t)$ is the deterministic part of the total velocity $\mathbf{b}(t)$ of the CP. The osmotic velocity $\mathbf{u}_1(t)$ is the stochastic part of the total velocity $\mathbf{b}(t)$. The non-zero osmotic velocity $\mathbf{u}_1(t)$ is a difference between stochastic mechanics deterministic mechanics [29].

We have the following result [19]:

$$\mathbf{u}_1 = D_3\frac{\nabla\rho}{\rho} = D_3\nabla(\ln\rho). \tag{A20}$$

We introduce the following definition of wave mass.

Definition A1. *The wave mass of the particle is defined by*

$$m_w \equiv \frac{T_0}{T_\omega} m_d. \tag{A21}$$

Using Equation (6), Equation (A21) and Equation (21), we have

$$D_3 = \frac{\hbar}{2m_w}. \tag{A22}$$

Similar to the method of Ref. [19], we introduce the following definition of osmotic potential R_1

$$m_w \mathbf{u}_1 = \nabla R_1, \tag{A23}$$

where the osmotic potential R_1 is defined by

$$R_1 \triangleq m_w D_3 \ln \rho. \tag{A24}$$

We introduce the definition of the mean second derivative $\mathbf{a}(t)$ of the stochastic process $\mathbf{y}(t)$ [19]

$$\mathbf{a}(t) = \frac{1}{2} D D_* \mathbf{y}(t) + \frac{1}{2} D_* D \mathbf{y}(t). \tag{A25}$$

Applying a similar method of E. Nelson [19], we can derive the following Proposition A1 [29].

Proposition A1. *The current velocity field $\mathbf{v}_1(\mathbf{r}, t)$ and the osmotic velocity field $\mathbf{u}_1(\mathbf{r}, t)$ satisfy the following coupled equations:*

$$\begin{aligned} \frac{\partial \mathbf{v}_1}{\partial t} &= \frac{\mathbf{F}}{m_w} - (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 + (\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 \\ &\quad + D_3 \nabla^2 \mathbf{u}_1, \qquad \text{(A26)} \\ \frac{\partial \mathbf{u}_1}{\partial t} &= -D_3 \nabla(\nabla \cdot \mathbf{v}_1) - \nabla(\mathbf{v}_1 \cdot \mathbf{u}_1). \qquad \text{(A27)} \end{aligned}$$

Similar to the deterministic newtonian mechanics, we can also introduce the following concept of deterministic momentum field $\mathbf{p}_d(\mathbf{r}, t)$ and stochastic momentum field $\mathbf{p}_s(\mathbf{r}, t)$ of the Brownian particle:

$$\mathbf{p}_d(\mathbf{r}, t) = m_w \mathbf{v}_1(\mathbf{r}, t), \tag{A28}$$

$$\mathbf{p}_s(\mathbf{r}, t) = m_w \mathbf{u}_1(\mathbf{r}, t). \tag{A29}$$

Proposition A2. *If there exists a functions $S_1(\mathbf{r}, t)$ such that*

$$\mathbf{p}_d = \nabla S_1, \tag{A30}$$

then, the deterministic momentum field $\mathbf{p}_d(\mathbf{r}, t)$ and stochastic momentum field $\mathbf{p}_s(\mathbf{r}, t)$ of the Brownian particle satisfy the following equations

$$\begin{aligned} \frac{\partial \mathbf{p}_d(t)}{\partial t} &= \mathbf{F} - \frac{1}{2m_w} \nabla(\mathbf{p}_d^2) + \frac{1}{2m_w} \nabla(\mathbf{p}_s^2) \\ &\quad + D_3 \nabla^2 \mathbf{p}_s, \qquad \text{(A31)} \\ \frac{\partial \mathbf{p}_s(t)}{\partial t} &= -D_3 \nabla^2 \mathbf{p}_d - \frac{1}{m_w} \nabla(\mathbf{p}_d \cdot \mathbf{p}_s). \qquad \text{(A32)} \end{aligned}$$

Proof of Proposition A2. We have the following equations in field theory:

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}), \tag{A33}$$

$$\nabla^2\mathbf{a} = \nabla(\nabla\cdot\mathbf{a}) - \nabla\times(\nabla\times\mathbf{a}), \tag{A34}$$

$$\nabla\times(\nabla\varphi) = 0, \tag{A35}$$

where **a** and **b** are arbitrary vectors, φ is an arbitrary scalar function.

Using Equation (A23), Equation (A30), Equation (A33) and Equation (A35), we have

$$\frac{1}{2}\nabla(\mathbf{v}_1^2) = (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1, \tag{A36}$$

$$\frac{1}{2}\nabla(\mathbf{u}_1^2) = (\mathbf{u}_1\cdot\nabla)\mathbf{u}_1. \tag{A37}$$

Using Equation (A30), Equation (A34) and Equation (A35), we have

$$\nabla^2\mathbf{v}_1 = \nabla(\nabla\cdot\mathbf{v}_1). \tag{A38}$$

Putting Equation (A36-A37) into Equation (A26) and using Equation (A23) and Equation (A30), we obtain Equation (A31). Putting Equation (A38) into Equation (A27) and using Equation (A23) and Equation (A30), we obtain Equation (A32). □

We may call the functions $S_1(\mathbf{r},t)$ defined in Equation (A30) as the current potential. The current potential $S_1(\mathbf{r},t)$ is not uniquely defined by the deterministic momentum field $\mathbf{p}_d(\mathbf{r},t)$. For instance, let $S_1' = S_1 + c_0$, where c_0 is an arbitrary constant. Then, we also have $\nabla(S_1') = \mathbf{p}_d$.

Theorem A1. *If there exist two functions $U(\mathbf{r})$ and S_1 such that*

$$\mathbf{F}(\mathbf{r},t) = -\nabla U(\mathbf{r}), \tag{A39}$$

$$\mathbf{p}_d = \nabla S_1, \tag{A40}$$

then, the generalized Hamilton's principal function

$$W_1 \triangleq S_1 - iR_1 \tag{A41}$$

satisfies the following generalized Hamilton-Jacobi equation

$$\begin{aligned} -\frac{\partial W_1}{\partial t} &= \frac{1}{2m_w}(\nabla W_1)^2 + U(\mathbf{r}) \\ &\quad -iD_3\nabla^2 W_1 + \theta_1(t) + i\theta_2(t), \end{aligned} \tag{A42}$$

where $\theta_1(t)$ and $\theta_2(t)$ are two unknown real functions of t.

Proof of Theorem A1. We multiply Equation (A31) with -1 and then plus Equation (A32) multiplied by i. Thus, we obtain

$$\begin{aligned} -\frac{\partial(\mathbf{p}_d - i\mathbf{p}_s)}{\partial t} &= -\mathbf{F} + \frac{1}{2m_w}\nabla[(\mathbf{p}_d - i\mathbf{p}_s)^2] \\ &\quad -iD_3\nabla^2(\mathbf{p}_d - i\mathbf{p}_s). \end{aligned} \tag{A43}$$

We introduce the following definition

$$\mathbf{p} \triangleq \mathbf{p}_d - i\mathbf{p}_s, \quad i^2 = -1. \tag{A44}$$

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Thus, Equation (A43) becomes

$$-\frac{\partial \mathbf{p}}{\partial t} = -\mathbf{F} + \frac{1}{2m_w}\nabla(\mathbf{p}^2) - iD_3\nabla^2\mathbf{p}. \tag{A45}$$

We may regard the function S_1 and R_1 as the deterministic part and stochastic part of a generalized Hamilton's principal function S defined by

$$W_1 \triangleq S_1 - iR_1, \quad i^2 = -1. \tag{A46}$$

Putting Equation (A23) and Equation (A40) into Equation (A44) and using Equation (A46), we have

$$\mathbf{p} = \nabla W_1. \tag{A47}$$

Putting Equation (A47) into Equation (A45), we obtain

$$-\frac{\partial(\nabla W_1)}{\partial t} = -\mathbf{F} + \frac{1}{2m_w}\nabla[(\nabla W_1)^2] - iD_3\nabla^2(\nabla W_1). \tag{A48}$$

Noticing $\mathbf{F} = -\nabla U(\mathbf{r})$, Equation (A48) becomes

$$-\frac{\partial\nabla(W_1)}{\partial t} = \nabla V + \frac{1}{2m_w}\nabla[(\nabla W_1)^2] - iD_3\nabla^2(\nabla W_1). \tag{A49}$$

Equation (A49) can be written as

$$\nabla\left[\frac{\partial W_1}{\partial t} + U(\mathbf{r}) + \frac{1}{2m_w}\nabla W_1^2 - iD_3\nabla^2(\nabla W_1)\right] = 0. \tag{A50}$$

Integration of Equation (A50) gives

$$\begin{aligned} -\frac{\partial W_1}{\partial t} &= U(\mathbf{r}) + \frac{1}{2m_w}\nabla W_1^2 - iD_3\nabla^2(\nabla W_1) \\ &\quad +\theta_1(t) + i\theta_2(t), \end{aligned} \tag{A51}$$

where $\theta_1(t)$ and $\theta_2(t)$ are two unknown real functions of t. □

The generalized Hamilton's principal function $W_1 \triangleq S_1 - iR_1$ is not uniquely defined by $\mathbf{p}_d$. The reason is that $\mathbf{p}_d = \nabla S_1$. Thus, S_1 is not uniquely defined by $\mathbf{p}_d$.

Similar to Bohr's correspondence principle, we may also introduce the following correspondence principle in stochastic mechanics.

Assumption 3. *If the diffusion constant D_3 is small enough, i.e., $D_3 \to 0$, then, the generalized Hamilton-Jacobi equation Equation (A42) in stochastic mechanics becomes identical to the following Hamilton-Jacobi equation in classical mechanics [44]*

$$-\frac{\partial W}{\partial t} = \frac{1}{2m}(\nabla W)^2 + U(\mathbf{r}), \tag{A52}$$

where $W(\mathbf{r}, t)$ is a real function called Hamilton's principal function, $U(\mathbf{r})$ is a potential.

Theorem A2. *Suppose that the assumptions Equation (A39 - A40) are valid. Then, the generalized Hamilton's principal function W_1 satisfies the following generalized Hamilton-Jacobi equation*

$$-\frac{\partial W_1}{\partial t} = \frac{1}{2m_w}(\nabla W_1)^2 + U(\mathbf{r}) - iD_3\nabla^2 W_1. \tag{A53}$$

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Proof of Theorem A2 Let $D_3 = 0$. Then, from Equation (A20), we have $\mathbf{u}_1 = 0$. Thus, from Equation (A29), we have $\mathbf{p}_s = 0$. Then, from Eq.A23, R_1 is a constant. Thus, Equation (A42) can be written as

$$-\frac{\partial S_1}{\partial t} = \frac{1}{2m_w}(\nabla S_1)^2 + U(\mathbf{r}) + \theta_1(t) + i\theta_2(t), \tag{A54}$$

According to Assumption 3, Equation (A54) should be identical to the Hamilton-Jacobi equation Equation (A52). Thus, we obtain $\theta_1(t) = 0$ and $\theta_2(t) = 0$. □

Similar to the Hamiltonian mechanics [44], we introduce the following definition of wave function

$$\psi(\mathbf{r}, t) = \exp\left[\frac{iW_1(\mathbf{r}, t)}{2m_w D_3}\right]. \tag{A55}$$

The generalized Hamilton's principal function W_1 is not uniquely defined by the deterministic momentum field $\mathbf{p}_d$. Therefore, the wave function $\psi(\mathbf{r}, t)$ defined by Equation (A55) is not uniquely defined by $\mathbf{p}_d$.

Theorem A3. *The wave function $\psi(\mathbf{r}, t)$ satisfies the following Schrödinger like equation*

$$i\frac{\partial \psi}{\partial t} = -D_3\nabla^2\psi + \frac{1}{2m_w D_3}U(\mathbf{r})\psi. \tag{A56}$$

Equation (A56) is equivalent to the generalized Hamilton-Jacobi equation Equation (A53).

Proof of Theorem A3. From the definition Equation (A55), we have

$$W_1(\mathbf{r}, t) = \frac{2m_w D_3}{i}\ln\psi(\mathbf{r}, t). \tag{A57}$$

Putting Equation (A57) into Equation (A53), we obtain a Schrödinger like equation Equation (A56). Conversely, putting Equation (A55) into Equation (A56), we obtain the generalized Hamilton-Jacobi equation Equation (A53). □

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